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R. Costa and L. S. Ikemoto Murakami

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ON IDEMPOTENTS AND ISOMORPHISMS OF MULTIPLICATION ALGEBRAS OF BERNSTEIN ALGEBRAS

R. Costa and L.S. Ikemoto Murakami

Departamento de Matemática, Instituto de Matemática e Estatística,
Universidade de São Paulo,
Caixa Postal 66281, São Paulo - SP, 05315–970, Brazil
E-mail address: rcosta@ime.usp.br, ikemoto@ime.usp.br

Abstract. The purpose of this paper is twofold. First of all we characterize Bernstein algebras having a nilpotent kernel using idempotents of their multiplication algebras. Secondly, we describe some properties of Bernstein algebras which are preserved by isomorphisms of their multiplication algebras.

1. Introduction

The study of multiplication algebras of Bernstein algebras began in [2] and [3]. This paper is a continuation of these two ones. We recall that a *baric algebra* over a field F is a pair (A, ω), where A is a not necessarily associative algebra over F and ω is a character of A, that is, $\omega: A \to F$ is a homomorphism and $\omega \neq 0$; in this case ω is called the *weight function* of (A, ω). Moreover, $N = \ker \omega$ is a two-sided ideal of A of codimension 1. A baric algebra (A, ω) is *Bernstein* if A is commutative and $(x^2)^2 = \omega(x)^2 x^2$, for all $x \in A$. From now on we consider only finite dimensional algebras over fields of characteristic not 2 and with at least 5 elements. Given a Bernstein algebra (A, ω), the set of its nonzero idempotents is $Ip(A) = \{x^2 : \omega(x) = 1\}$ and given $e \in Ip(A)$, we have the Peirce decomposition relative to this idempotent,

$$A = Fe \oplus U_e \oplus V_e, \tag{1}$$

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where $U_e = \{x \in N : 2xe = x\}$ and $V_e = \{x \in N : xe = 0\}$. Unless necessary, we omit the subscript e in U_e and V_e. The relations

$$U^2 \subseteq V,\ UV \subseteq U,\ V^2 \subseteq U,\ UV^2 = 0 \tag{2}$$

as well as the identities, for $u \in U$ and $v \in V$,

$$u^3 = 0,\ u(uv) = 0,\ uv^2 = 0,\ u^2(uv) = 0,\ u^2v^2 = 0,\ (uv)^2 = 0 \tag{3}$$

hold in A. The dimensions of U and V in (1) are invariant under change of idempotents, and the pair $(1 + \dim U, \dim V)$, which therefore is well defined, is called the *type* of A. If A satisfies $A^2 = A$ then we say that A is *nuclear*. This is equivalent to $U^2 = V$ in (1). The subspace $L = \{u \in U : uU = 0\}$ is an ideal of A, independent of the idempotent (see, for instance, [5, Th. 3.4.19]). We will denote by I the smallest ideal of A such that N/I is nilpotent. Since N/L is nilpotent, we have $I \subseteq L$. When $U^2 = 0$ in (2), A is called *exceptional* and when $UV = V^2 = 0$, A is *normal*. These conditions are independent of the idempotent in A. Other characterizations of these algebras can be found in [5] or [7], as well as the basic theory of Bernstein algebras. These algebras were defined in the seventies as a model for studing Hardy-Weinberg law in Genetics.

Given a Bernstein algebra $A = Fe \oplus U \oplus V$, its multiplication algebra $M(A)$ is the subalgebra of $\mathrm{End}(A)$ generated by the operators L_x, $x \in A$, defined by $L_x(a) = xa$, for all $a \in A$. This algebra is baric, with weight function $\widetilde{\omega}$ defined on the generators by $\widetilde{\omega}(L_x) = \omega(x)$, for all $x \in A$. The function $\widetilde{\omega}$ is called the *weight function on* $M(A)$ *induced* by ω. We have $\ker \widetilde{\omega} = (N : A) = \{\sigma \in M(A) : \sigma(A) \subseteq N\}$. The operator $2L_e^2 - L_e$ is an idempotent of weight 1 and so we have the decomposition

$$M(A) = F(2L_e^2 - L_e) \oplus (N : A). \tag{4}$$

Moreover, $4L_e - 4L_e^2 \in (N : A)$ is also an idempotent and every $\sigma \in M(A)$ is decomposed in the form

$$\sigma = \alpha(2L_e^2 - L_e) + \beta(4L_e - 4L_e^2) + \theta,$$

with $\alpha, \beta \in F$ and $\theta \in \widetilde{N}$, where $\widetilde{N}$ is the ideal of $M(A)$ generated by $\{L_x : x \in N\}$. Observe that $\widetilde{N} \subseteq (N : A)$.

If $A = Fe \oplus U \oplus V$ is a Bernstein algebra then $M(A)$ has the decomposition relative to the idempotent $2L_e^2 - L_e$,

$$M(A) = F(2L_e^2 - L_e) \oplus \widetilde{U} \oplus \widetilde{V}, \tag{5}$$

where $\widetilde{U} = \{\psi_x : x \in U \oplus U^2\}$, here $\psi_x(e) = x$ and $\psi_x(N) = 0$, and $\widetilde{V} = \{\sigma \in (N : A) : \sigma(e) = 0\}$. These subspaces satisfy the following relations:

$$\widetilde{U}^2 = 0,\ \widetilde{U}\widetilde{V} = 0,\ \widetilde{V}\widetilde{U} \subseteq \widetilde{U},\ \widetilde{V}^2 \subseteq \widetilde{V}. \tag{6}$$

If $U \neq 0$, we can refine the decomposition presented in (5) using the idempotent $4L_e - 4L_e^2 \in \widetilde{V}$. We have $\widetilde{V} = \widetilde{V}_{11} \oplus \widetilde{V}_{10} \oplus \widetilde{V}_{01} \oplus \widetilde{V}_{00}$, such that if $\sigma \in \widetilde{V}$ then

$$\sigma \in \widetilde{V}_{11} \;\Leftrightarrow\; \sigma(U) \subseteq U \text{ and } \sigma(V) = 0; \tag{7}$$
$$\sigma \in \widetilde{V}_{10} \;\Leftrightarrow\; \sigma(U) = 0 \text{ and } \sigma(V) \subseteq U; \tag{8}$$
$$\sigma \in \widetilde{V}_{01} \;\Leftrightarrow\; \sigma(U) \subseteq V \text{ and } \sigma(V) = 0; \tag{9}$$
$$\sigma \in \widetilde{V}_{00} \;\Leftrightarrow\; \sigma(U) = 0 \text{ and } \sigma(V) \subseteq V. \tag{10}$$

We have two characterizations of normal Bernstein algebras.

Lemma 1. *Let A be a Bernstein algebra. Are equivalent:*

(i) *A is normal;*
(ii) *$V_{10} = 0$;*
(iii) $M(A) = F(2L_e^2 - L_e) \oplus \widetilde{U} \oplus F(4L_e - 4L_e^2) \oplus \{L_u - 2L_eL_u : u \in U\}$.

For every integer $k \geq 1$ and $\sigma \in \widetilde{N}^k$, the image of σ is contained in N^k. In particular, we have the following relevant fact:

Theorem 1. *The ideal N of A is nilpotent if and only if the ideal $\widetilde{N}$ of $M(A)$ is nilpotent.*

All the above facts about multiplication algebras of Bernstein algebras were proved in [2,3].

In the following $A = Fe \oplus U \oplus V$ will be a Bernstein algebra of type $(1+r, s)$, $L = \{u \in U : uU = 0\}$ and $I \subseteq L$ the smallest ideal of A such that N/I is nilpotent, where $N = U \oplus V$. Recall that the symbol ψ_x denotes a linear mapping in $\widetilde{U}$ and hence, we are assuming that $x \in U \oplus U^2$.

2. Idempotents in the multiplication algebra

After the description of some properties of idempotents in multiplication algebras, we will have a characterization of Bernstein algebras such that $N = \ker \omega$ is nilpotent. We will see, initially, how is the Peirce decomposition of idempotents in $M(A)$, for A Bernstein. First, we note that the weight of an idempotent in $M(A)$ is either 0 or 1.

Proposition 1. (i) *An element $\sigma \in M(A)$ is an idempotent of weight* 1 *(that is, $\widetilde{\omega}(\sigma) = 1$) if and only if $\sigma = 2L_e^2 - L_e + \psi_x + \theta$, with $\theta \in \widetilde{V}$ an idempotent and $x \in \ker\theta \cap (U \oplus U^2)$.*

(ii) *An element $\sigma \in (N : A)$ is an idempotent if and only if $\sigma = \psi_x + \theta$, where $\theta \in \widetilde{V}$ is an idempotent and $x \in \operatorname{Im}\theta$.*

PROOF: Let σ be an element in $M(A)$ of weight 1. According to (5) we can write $\sigma = 2L_e^2 - L_e + \psi_x + \theta$, with $\psi_x \in \widetilde{U}$ and $\theta \in \widetilde{V}$. Then σ is an idempotent if and only if $2L_e^2 - L_e + \psi_x + \theta\psi_x + \theta^2 = 2L_e^2 - L_e + \psi_x + \theta$. This equality is equivalent to $\theta\psi_x = 0$ and $\theta^2 = \theta$ since $\theta\psi_x$ is in $\widetilde{U}$. So σ is an idempotent if and only if θ is an idempotent and $\theta(x) = 0$. The proof for (ii) is similar. ■

By the previous proposition, the description of idempotents in $M(A)$ can be restricted to the description of idempotents in $\widetilde{V}$. The elements in $\widetilde{V}$ can be written in the form $\alpha(4L_e - 4L_e^2) + \theta$, where $\alpha \in F$ and $\theta \in \widetilde{V} \cap \widetilde{N}$ since $\widetilde{V} = F(4L_e - 4L_e^2) + (\widetilde{V} \cap \widetilde{N})$. Recall that the idempotent $4L_e - 4L_e^2$ is zero if and only if $U = 0$.

Lemma 2. *If N is nilpotent, then the nonzero idempotents in $\widetilde{V}$ have the form $4L_e - 4L_e^2 + \theta$, with $\theta \in \widetilde{V} \cap \widetilde{N}$.*

PROOF: If $U = 0$, then $4L_e - 4L_e^2 = 0$ and hence $\widetilde{V} \subseteq \widetilde{N}$ is nilpotent by Theorem 1. Therefore $\widetilde{V}$ has not idempotents distinct from zero, in this case. Next, we will assume that $U \neq 0$. Then $4L_e - 4L_e^2 \notin \widetilde{N}$ since $\widetilde{N}$ is nilpotent, and hence $\widetilde{V} = F(4L_e - 4L_e^2) \oplus (\widetilde{V} \cap \widetilde{N})$. Thus, for σ an idempotent element in $\widetilde{V}$, we have that $\sigma = \alpha(4L_e - 4L_e^2) + \theta$, with $\theta \in \widetilde{V} \cap \widetilde{N}$. Then, because $\widetilde{V} \cap \widetilde{N}$ is an ideal of $\widetilde{V}$, the relation $\sigma^2 = \sigma$ implies $\alpha^2 = \alpha$, that is, $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$ then $\sigma = \theta \in \widetilde{N}$; so $\sigma = 0$, because $\widetilde{N}$ is nilpotent. In this way, if σ is a nonzero idempotent, we must have $\alpha = 1$. ■

Lemma 3. *For all $\sigma \in M(A)$ and $v \in V$, $v \neq 0$, we have $\sigma(v) \neq v$.*

PROOF: Let $\sigma \in M(A)$ satisfying $\sigma(v) = v$ for some $v \in V$. Recall that σ has a decomposition $\sigma = \alpha(2L_e^2 - L_e) + \beta(4L_e - 4L_e^2) + \theta$ with $\theta \in \widetilde{N}$. Thus $v = \sigma(v) = \theta(v)$ and hence $v = \theta^k(v)$, for all k. Then, because the elements of $\widetilde{N}^k$ have their images in N^k, we obtain that $v = \theta^k(v) \in N^k \subseteq I \subseteq L \subseteq U$ for some $k \geq 1$, that is, $v \in V \cap U = \{0\}$. ■

Corollary 1. *Every idempotent in $(N : A)$ has rank less than or equal to r.*

PROOF: Suppose that there is an idempotent $\sigma \in (N : A)$ of rank greater than r. By definition of $(N : A)$, we have that $\mathrm{Im}\,(\sigma) \subseteq N$ and hence there is $0 \neq v \in \mathrm{Im}\,(\sigma) \cap V$, what is a contradicition, by Lemma 3. ■

Corollary 2. *Every idempotent in $M(A)$ has rank less than or equal to $r+1$.*

Corollary 3. *They are equivalent:*

(i) $s > 0$;

(ii) *$M(A)$ is an algebra without unity.*

Proposition 2. *If N is nilpotent then all the non-zero idempotents of $\widetilde{V}$ have rank r.*

PROOF: By Lemma 2, an idempotent $\sigma \in \widetilde{V}$ can be decomposed as $\sigma = 4L_e - 4L_e^2 + \theta$ with $\theta \in \widetilde{V} \cap \widetilde{N}$. If rank of σ is less than r, then $\dim(\ker \sigma) > s$ and hence there exists $0 \neq u \in \ker \sigma \cap U$. So $0 = \sigma(u) = (4L_e - 4L_e^2)(u) + \theta(u) = u + \theta(u)$ and therefore $\theta(u) = -u$, contrary to the nilpotency of θ. Thus, the rank of σ is greater than or equal to r. Finally, using Lemma 3 we conclude that the rank of σ is necessarily r. ■

Now, from Proposition 1 and Proposition 2 we obtain the following fact.

Corollary 1. *If N is nilpotent then*

(i) *the idempotents of $M(A)$ of weight 1 have rank* 1 *or* $1+r$;

(ii) *the idempotents of $(N : A)$ have rank r.*

PROOF: If σ is an idempotent of weight 1 then, by (i) of Proposition 1, $\sigma = 2L_e^2 - L_e + \psi_x + \theta$, where $\theta \in \widetilde{V}$ is an idempotent and $x \in \ker \theta$. If $\theta = 0$ then $\sigma = 2L_e^2 - L_e + \psi_x$ has rank 1. If $\theta \neq 0$ then, by the Proposition 2, $\mathrm{rk}\,(\theta) = r$ and consequently $\mathrm{rk}\,(\sigma) = 1 + r$. In the case that σ is an idempotent in $(N : A)$, we have, by (ii) of Proposition 1, that $\sigma = \psi_x + \theta$, with $x \in \mathrm{Im}\,\theta$ and $\theta^2 = \theta$. So $\mathrm{rk}\,(\sigma) = \mathrm{rk}\,(\theta) = r$. ■

We intend to classify Bernstein algebras having nilpotent kernel using idempotents in their multiplication algebras. For this, it will be necessary to describe the idempotents in $\widetilde{N}$.

Proposition 3. *If $\sigma \in \widetilde{N}$ is an idempotent then* $\mathrm{Im}\,\sigma \subseteq I \subseteq L$.

PROOF: Let $\overline{A} = A/I$ and $\pi \colon M(A) \to M(\overline{A})$ be the epimorphism given by $\pi(L_x) = L_{\overline{x}}$, where $\overline{x} = x + I$. Obviously $\overline{N} = \{\overline{x} : x \in N\}$ is nilpotent. Thus, from Theorem 1 we get that $\widetilde{\overline{N}}$ is also nilpotent. Then, because $\pi(\widetilde{N}) = \widetilde{\overline{N}}$, for every idempotent $\sigma \in \widetilde{N}$, we have $\pi(\sigma) = 0$, that is $\mathrm{Im}\,\sigma \subseteq I$. ■

Corollary 1. *If* $4L_e - 4L_e^2 \in \widetilde{N}$ *then* A *is exceptional.*

Thus we have proved that if A is non-exceptional or N is nilpotent with $r > 0$, then

$$M(A) = F(2L_e^2 - L_e) \oplus F(4L_e - 4L_e^2) \oplus \widetilde{N}. \tag{11}$$

Theorem 2. *Let* $A = Fe \oplus N$ *be Bernstein of type* $(1+r, s)$. *If* N *is nilpotent then all the idempotents in* $(N : A)$ *have rank* r. *The converse is true if* A *is non exceptional.*

PROOF: If N is nilpotent then, by Corollary 1 of Proposition 2, idempotent elements of $(N : A)$ have rank r. For the converse we will assume that N is not nilpotent. In view of the Theorem 1, $\widetilde{N}$ is not nilpotent and hence there is an idempotent $\sigma \in \widetilde{N}$, $\sigma \neq 0$. Now by Proposition 3 we have that $\operatorname{Im}\sigma \subseteq I$. Then, because I is a proper subspace of U, we have that the rank of σ is less than r. ■

Now we will see other properties of idempotents in $M(A)$.

Proposition 3 gives us a characterization of idempotents in $\widetilde{N}$. From this result, we can obtain more information about these elements:

Proposition 4. *An element* $\sigma \in \widetilde{N}$ *is an idempotent if and only if* $\sigma = \sigma_{11} + \sigma_{10}$, *where* $\sigma_{11} \in \widetilde{V}_{11}$ *is an idempotent and* $\sigma_{10} \in \widetilde{V}_{10}$ *is such that* $\operatorname{Im}(\sigma_{10}) \subseteq \operatorname{Im}(\sigma_{11})$.

PROOF: If $\sigma \in \widetilde{N}$ is an idempotent then, by Proposition 3, its image is contained in $I \subseteq L$. Then $\sigma \in \widetilde{V}_{11} \oplus \widetilde{V}_{10}$. Decompose $\sigma = \sigma_{11} + \sigma_{10}$, with $\sigma_{11} \in \widetilde{V}_{11}$ and $\sigma_{10} \in \widetilde{V}_{10}$. From $\sigma^2 = \sigma$ we obtain $\sigma_{11}^2 + \sigma_{11}\sigma_{10} = \sigma_{11} + \sigma_{10}$, that is, $\sigma_{11}^2 = \sigma_{11}$ and $\sigma_{11}\sigma_{10} = \sigma_{10}$. It follows that σ_{11} is an idempotent and $\operatorname{Im}(\sigma_{10}) \subseteq \operatorname{Im}(\sigma_{11})$. The reverse is trivial. ■

3. ON ISOMORPHISMS OF MULTIPLICATION ALGEBRAS

Here we will consider the following problem: "If A and A' are Bernstein algebras such that $M(A)$ and $M(A')$ are isomorphic and A has the property P then does A' also have the property P?" We will derive some properties of Bernstein algebras that are preserved by isomorphisms of their multiplication algebras.

We prove initialy (Theorem 3) that all isomorphisms of multiplication algebras of Bernstein algebras are baric, that is, they preserve the induced weight function $\widetilde{\omega}$ already defined.

Lemma 4. *Let (B,ω) and (B',ω') be two baric algebras such that there is an isomorphism $\Psi : B \to B'$. Then:*

(i) *Ψ is a baric isomorphism if and only if $\Psi(\ker\omega) \subseteq \ker\omega'$;*

(ii) *if there is $c \in B$ such that both c and $\Psi(c)$ have weight 1 and $\Psi((Fc)\ker\omega)$ is contained in $\ker\omega'$ then Ψ is a baric isomorphism.*

PROOF:

(i) As Ψ is an isomorphism, $\Psi(\ker\omega)$ has codimension 1 in B'. Thus if $\Psi(\ker\omega) \subseteq \ker\omega'$ then $\Psi(\ker\omega) = \ker\omega'$. Moreover, if $c \in B$ is such that $\omega(c) = 1$ then $\omega'(\Psi(c)) = \omega'(\Psi(c^2)) = \omega'(\Psi(c)^2)$, that is, $\omega'(\Psi(c)) = 0$ or 1. But $\Psi(c) \notin \ker\omega'$. Therefore, $\omega'(\Psi(c)) = 1$. Using the decomposition $B = Fc \oplus \ker\omega$, we conclude that $\omega' \circ \Psi = \omega$, that is, Ψ is baric. The converse is trivial.

(ii) We have that elements of $\Psi(Fc)\Psi(\ker\omega)$ have weight 0. Since $\Psi(c)$ has weight 1, we have $\Psi(\ker\omega) \subseteq \ker\omega'$. By part (i) above, Ψ is a baric isomorphism. ■

Lemma 5. *The following relation holds:*

$$\bigcap_{f\in Ip(A)} \widetilde{V}_f = \{\sigma \in M(A) : \sigma(A^2) = 0\}.$$

PROOF: Let σ in the intersection of V_f, $f \in Ip(A)$. For each $u \in U_e$ we have $\sigma(e+u+u^2) = 0$, since $\sigma \in \widetilde{V}_f$, where $f = e+u+u^2$. As $\sigma(e) = 0$, we have $\sigma(u+u^2) = 0$. We now take the element $-u \in U_e$ and obtain $\sigma(-u+u^2) = 0$, that is, $\sigma(u) = 0 = \sigma(u^2)$, for all $u \in U_e$. Therefore $\sigma(A^2) = \sigma(Fe \oplus U_e \oplus U_e^2) = 0$. On the other hand, if $\sigma(A^2) = 0$ then $\sigma(e+u+u^2) = 0$, for all $u \in U$. Thus $\sigma \in \widetilde{V}_f$, for all $f \in Ip(A)$. ■

Theorem 3. *Every isomorphism between any two multiplication algebras of Bernstein algebras is baric.*

PROOF: Suppose that the isomorphism Φ: $M(A) \to M(A')$ is not baric. Let e and e' be two fixed idempotents in A and A' respectively. Using the inverse isomorphism, that is not baric too, and (ii) of Lemma 4, we have that $P = \Phi^{-1}(\{2L_{e'}^2 - L_{e'} + \psi_x : x \in U_{e'} \oplus U_{e'}^2\})$ is contained in $(N : A)$. Thus $\{0\} = \Phi((2L_e^2 - L_e)P) = \Phi(2L_e^2 - L_e)\Phi(P)$, and hence $0 = [\Phi(2L_e^2 - L_e)](2L_{e'}^2 - L_{e'} + \psi_{x'})(e') = [\Phi(2L_e^2 - L_e)](e' + x')$, for all $x' \in U_{e'} \oplus U_{e'}^2$. Consequently, $[\Phi(2L_e^2 - L_e)]((A')^2) = 0$ but this is impossible since $\Phi(2L_e^2 - L_e)$ is a non-zero idempotent and $\mathrm{Im}\,[\Phi(2L_e^2 - L_e)] \subseteq (A')^2$. ■

We have seen in Proposition 1 (i) how are the idempotents of weight 1 in multiplication algebras of Bernstein algebras. The next result shows us how are the idempotents that might be the image by an isomorphism of the operator $2L_e^2 - L_e$.

Proposition 5. *Let $A = Fe \oplus U \oplus V$ and $A' = Fe' \oplus U' \oplus V'$ be Bernstein and $\Phi : M(A) \to M(A')$ an isomorphism. Then for every $x \in U \oplus U^2$ there exists $x' \in U' \oplus U'^2$ such that $\Phi(2L_e^2 - L_e + \psi_x) = 2L_{e'}^2 - L_{e'} + \psi_{x'}$.*

PROOF: Let x be an element in $U \oplus U^2$ and consider the idempotent $\sigma = 2L_e^2 - L_e + \psi_x$. In view of the Proposition 1 we have that $\Phi(\sigma)$ is of the form $2L_{e'}^2 - L_{e'} + \psi_{x'} + \theta'$, with $\theta' \in \widetilde{V}'$ and idempotent. Since Φ is a baric isomorphism, there is $\theta \in (N : A)$ such that $\Phi(\theta) = \theta'$. Then $0 = (2L_e^2 - L_e + \psi_x)\theta$ forces that $0 = \Phi(2L_e^2 - L_e + \psi_x)\theta) = (2L_{e'}^2 - L_{e'} + \psi_{x'} + \theta')\theta' = \theta'^2 = \theta'$. Consequently, we have proved that $\theta' = 0$. ■

Remark 1. In view of the Proposition 5, if $U' \neq 0$ then there are idempotents of weight 1 in $M(A')$ that cannot be the image of $2L_e^2 - L_e$ by isomorphisms. For instance, $2L_{e'}^2 - L_{e'} + \psi_{x'} + 4L_{e'} - 4L_{e'}^2$, with $x' \in U'^2$.

Proposition 6. *If A and A' are Bernstein and Φ: $M(A) \to M(A')$ is an isomorphism then $\Phi(\widetilde{U}) = \widetilde{U}'$.*

PROOF: Let now x be an element in $U \oplus U^2$ and consider $\psi_x \in \widetilde{U}$. Then, by Proposition 5, $\Phi(\psi_x) = \Phi((2L_e^2 - L_e + \psi_x) - (2L_e^2 - L_e)) = \Phi(2L_e^2 - L_e + \psi_x) - \Phi(2L_e^2 - L_e) \in \widetilde{U}'$. Therefore $\Phi(\widetilde{U}) \subseteq \widetilde{U}'$. In a similar way, using the isomorphism Φ^{-1} we obtain that $\Phi^{-1}(\widetilde{U}') \subseteq \widetilde{U}$. So $\Phi(\widetilde{U}) = \widetilde{U}'$. ■

In view of the Proposition 6, if $A = Fe \oplus U_e \oplus V_e$ and $A' = Fe \oplus U_{e'} \oplus V_{e'}$ are Bernstein algebras such that $M(A)$ and $M(A')$ are isomorphic with Φ as isomorphism, then we have the following results.

Corollary 1. *The algebras A^2 and A'^2 have the same dimension.*

Corollary 2. *We have that $\Phi(F(2L_e^2 - L_e) \oplus \widetilde{U}) = F(2L_{e'}^2 - L_{e'}) \oplus \widetilde{U}'$.*

Proposition 7. *Let Φ: $M(A) \to M(A')$ be an isomorphism. Then, for every $\sigma \in M(A)$, we have that*

$$\operatorname{rk}(\sigma\big|_{A^2}) \leq \operatorname{rk}(\Phi(\sigma)\big|_{A'^2}) \leq \operatorname{rk}(\Phi(\sigma)).$$

PROOF: Let $\sigma = \alpha(2L_e^2 - L_e) + \psi_x + \theta \in M(A)$, $\psi_x \in \widetilde{U}$, $\theta \in \widetilde{V}$. If $\alpha \neq 0$ or $x \neq 0$, $\operatorname{rk}(\alpha(2L_e^2 - L_e) + \psi_x) = 1$. So we have $\operatorname{rk}(\sigma\big|_{A^2}) = 1 + \operatorname{rk}(\theta\big|_{A^2})$ or $\operatorname{rk}(\sigma\big|_{A^2}) =$

$\mathrm{rk}\,(\theta|_{A^2})$. If $\mathrm{rk}\,(\sigma|_{A^2}) = 1+\mathrm{rk}\,(\theta|_{A^2}) = 1+m$ then there are $x_1, \ldots, x_m \in U\oplus U^2$ such that $\{\sigma(e), \sigma(x_1), \ldots, \sigma(x_m)\}$ is a linearly independent set, that is, for all $(\lambda_0, \lambda_1, \ldots, \lambda_m) \in F^{m+1}\setminus\{0\}$, we have $0 \neq \lambda_0\sigma(e)+\lambda_1\sigma(x_1)+\cdots+\lambda_m\sigma(x_m) = (\lambda_0\sigma(2L_e^2 - L_e) + \lambda_1\sigma\psi_{x_1} + \cdots + \lambda_m\sigma\psi_{x_m})(e)$. Applying the isomorphism Φ and recalling that $\Phi(\widetilde{U}) = \widetilde{U}'$ and $\Phi(2L_e^2 - L_e) = 2L_{e'}^2 - L_{e'} + \psi_{x'}$, for some $x' \in \widetilde{U}' \oplus \widetilde{U}'^2$, we get:

$$\lambda_0\Phi(\sigma)(2L_{e'}^2 - L_{e'} + \psi_{x'}) + \lambda_1\Phi(\sigma)\psi_{x_1'} + \cdots + \lambda_m\Phi(\sigma)\psi_{x_m'} \neq 0,$$

with $\psi_{x_i'} = \Phi(\psi_{x_i})$. This operator is in $F(2L_{e'}^2 - L_{e'}) \oplus \widetilde{U}'$. Then, evaluating on e',

$$\lambda_0\Phi(\sigma)(e' + x') + \lambda_1\Phi(\sigma)(x_1') + \cdots + \lambda_m\Phi(\sigma)(x_m') \neq 0,$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_m) \in F^{m+1} \setminus \{0\}$, that is,

$$\{\Phi(\sigma)(e' + x'), \Phi(\sigma)(x_1'), \ldots, \Phi(\sigma)(x_m')\}$$

is a linearly independent set. Consequently $\mathrm{rk}\,(\sigma|_{A^2}) \leq \mathrm{rk}\,(\Phi(\sigma)|_{A'^2})$. If $\mathrm{rk}\,(\sigma|_{A^2}) = \mathrm{rk}\,(\theta|_{A^2})$, there are $x_1, \ldots, x_m \in U\oplus U^2$ such that $\{\sigma(x_1), \ldots, \sigma(x_m)\}$ is linearly independent. The proof that $\mathrm{rk}\,(\sigma|_{A^2}) \leq \mathrm{rk}\,(\Phi(\sigma)|_{A'^2})$ in this case is similar to the previous one. ∎

Corollary 1. *If $\Phi\colon M(A) \to M(A')$ is an isomorphism and A is a nuclear Bernstein algebra then* $\mathrm{rk}\,(\sigma) \leq \mathrm{rk}\,(\Phi(\sigma))$*, for all $\sigma \in M(A)$.* ∎

Corollary 2. *If $\Phi\colon M(A) \to M(A')$ is an isomorphism and A, A' are nuclear Bernstein algebras then* $\mathrm{rk}\,(\sigma) = \mathrm{rk}\,(\Phi(\sigma))$*, for all $\sigma \in M(A)$.* ∎

Proposition 8. *If $A = Fe \oplus U \oplus V$ and $A' = Fe' \oplus U' \oplus V'$ are Bernstein with isomorphic multiplication algebras then* $\dim U = \dim U'$.

PROOF: Let $\dim U = r$ and $\dim U' = r'$. The idempotent $4L_e - 4L_e^2$ has rank r and $(4L_e - 4L_e^2)|_V = 0$. By Proposition 7, $\mathrm{rk}\,(\Phi(4L_e - 4L_e^2)) \geq r$. On the other hand, the idempotent $\Phi(4L_e - 4L_e^2) \in (N' : A')$ has rank less or equal to r', by Proposition 1. Then $r \leq \mathrm{rk}\,(\Phi(4L_e - 4L_e^2)) \leq r'$. In a similar way, using the inverse isomorphism, we obtain $r' \leq r$. Consequently, $\dim U = r = r' = \dim U'$. ∎

Corollary 1. *If A and A' are Bernstein with isomorphic multiplication algebras then A^2 and A'^2 have the same type.*

Corollary 2. *If A and A' are Bernstein with isomorphic multiplication algebras, then A is exceptional if and only if A' is exceptional.*

The Corollary 2 shows us a class of Bernstein algebras that is preserved by isomorphisms of their multiplication algebras. We will see in the following that the same property holds for normal algebras and for algebras with nilpotent kernel that are not exceptional. In the next lemma, the baric algebras are not necessarily commutative.

Lemma 6. *Let $(B,\omega),(B',\omega')$ be arbitrary baric algebras, $\Phi\colon B \to B'$ a baric isomorphism and $x_0' \in \operatorname{ann}(\ker\omega')$. The linear application $f_{x_0'}\colon B \to B'$ defined by $f_{x_0'}(x) = \Phi(x) + \omega(x)x_0'$ is an isomorphism if and only if $x_0' = \Phi(e)x_0' + x_0'\Phi(e)$, for some element $e \in B$ of weight 1.*

PROOF: Let $e \in B$ be an element of weight 1 and consider $e' = \Phi(e)$. Then, for every $x, y \in B$ we have that

$$\begin{aligned} f_{x_0'}(x)f_{x_0'}(y) &= (\Phi(x)+\omega(x)x_0')(\Phi(y)+\omega(y)x_0') \\ &= \Phi(x)\Phi(y)+\omega(y)\Phi(x)x_0'+\omega(x)x_0'\Phi(y) \\ &= \Phi(xy)+\omega(y)\omega(x)e'x_0'+\omega(x)x_0'\omega(y)e' \\ &= \Phi(xy)+\omega(x)\omega(y)(e'x_0'+x_0'e') \\ &= \Phi(xy)+\omega(xy)(e'x_0'+x_0'e'). \end{aligned}$$

Thus $f_{x_0'}(xy) = f_{x_0'}(x)f_{x_0'}(y)$ for all $x, y \in B$ if and only if $x_0' = \Phi(e)x_0' + x_0'\Phi(e)$. ■

In the case of Bernstein algebras, x_0' must be an element of $U' \cap \operatorname{ann}\ker\omega'$. For multiplication algebras of Bernstein algebras, $B = M(A)$ and $B' = M(A')$, it is enough that $x_0' \in \widetilde{U}' \cap \operatorname{ann}(N' : A')$, that is, $x_0' \in \{\psi_{x'} \colon x' \in \operatorname{ann}A'\}$.

Consider now an isomorphism $\Phi\colon M(A) \to M(A')$, where A is a Bernstein algebra and A' is a normal Bernstein algebra. Let $e \in A$ and $e' \in A'$ be idempotent elements. We know that $\Phi(2L_e^2 - L_e)$ is of the form $2L_{e'}^2 - L_{e'} + \Psi_{x'}$, with $x' \in U' \oplus U'^2$. If $x' = u' + v'$, with $u' \in U'$ and $v' \in U'^2$, then we have that

$$f_{x_0'}(2L_e^2 - L_e) = 2L_{e_0'}^2 - L_{e_0'}$$

where $x_0' = \psi_{U'^2 - v'}$ and e_0' is the idempotent $e' + u' + U'^2$. Thus, the following porposition is proved.

Proposition 9. *If A and A' are Bernstein algebras with isomorphic multiplication algebras and A' is normal, then there exist $\Phi\colon M(A) \to M(A')$ isomorphism and idempotents $e \in A$ and $e' \in A'$ such that $\Phi(2L_e^2 - L_e) = 2L_{e'}^2 - L_{e'}$.*

Proposition 10. *If A and A' are Bernstein with isomorphic multiplication algebras and A' is normal then so is A.*

PROOF: We have $M(A) = F(2L_e^2 - L_e) \oplus \widetilde{U} \oplus \widetilde{V}_{11} \oplus \widetilde{V}_{10} \oplus \widetilde{V}_{01} \oplus \widetilde{V}_{00}$ and $M(A') = F(2L_{e'}^2 - L_{e'}) \oplus \widetilde{U}' \oplus F(4L_{e'} - 4L_{e'}^2) \oplus \{L_u - 2L_eL_u \colon u \in U'\}$. Let's prove that $\widetilde{V}_{10} = 0$. By the previous corollary, we may assume that $\Phi \colon M(A) \to M(A')$ is an isomorphism such that $\Phi(2L_e^2 - L_e) = 2L_{e'}^2 - L_{e'}$. Thus Φ preserves the first Peirce decomposition, that is, $\Phi(\widetilde{U}) = \widetilde{U}'$ and $\Phi(\widetilde{V}) = \widetilde{V}'$. Let $\sigma \in \widetilde{V}_{10}$. Since $\sigma^2 = 0$, we have $\Phi(\sigma) \in \widetilde{V}' \cap \widetilde{N}' = \widetilde{V}'_{01}$. Then $\Phi(\sigma) = \Phi((4L_e - 4L_e^2)\sigma) = \Phi(4L_e - 4L_e^2)\Phi(\sigma) \in (\widetilde{V}'_{11} \oplus \widetilde{V}'_{01})\widetilde{V}'_{01} = 0$. Therefore $\sigma = 0$, that is, $V_{01} = 0$. Consequently A is normal. ■

The next example (obtained by J.C. Gutiérrez F.) shows that we may have non isomorphic normal nuclear Bernstein algebras with the same type and isomorphic multiplication algebras.

Example 1. Let $A = \langle e, u_1, u_2, v \rangle$ and $A' = \langle e', u'_1, u'_2, v' \rangle$ be Bernstein algebras with the following non-zero products:

$$A: \quad e^2 = e \quad eu_1 = \tfrac{1}{2}u_1 \quad eu_2 = \tfrac{1}{2}u_2 \quad u_1^2 = v \quad u_2^2 = v$$

$$A': \quad e'^2 = e' \quad e'u'_1 = \tfrac{1}{2}u'_1 \quad e'u'_2 = \tfrac{1}{2}u'_2 \quad u'_1u'_2 = v$$

Both A and A' are normal and nuclear, have the same type and isomorphic multiplication algebras; however they are not isomorphic if considered as algebras over the field of real numbers.

Proposition 11. *Let A and A' be non-exceptional Bernstein algebras with isomorphic multiplication algebras. Then N is nilpotent if and only if N' is nilpotent.*

PROOF: Let A and A' be non-exceptional Bernstein algebras with isomorphic multiplication algebras, and consider $\Phi \colon M(A) \to M(A')$ an isomorphism. If N is nilpotent then $\widetilde{N}$ is also nilpotent and by (11) we have that $\Phi(\widetilde{N}) \subseteq \widetilde{N}'$. So $\Phi(\widetilde{N}) = \widetilde{N}'$. Consequently, $\widetilde{N}'$ is also nilpotent and this forces that N' is nilpotent. The converse is obtained in an analogous way, using the isomorphism Φ^{-1}. ■

The hypothesis that A is not exceptional in the previous proposition is necessary, according to the following example.

Example 2. Let $A = Fe \oplus U \oplus V$ be the Bernstein algebra with bases $\{u_1, u_2\}$ of U and $\{v_1, v_2\}$ of V and the non-zero products in $N = U \oplus V$

	u_1	u_2	v_1	v_2
u_1				
u_2				
v_1			u_1	u_2
v_2			u_2	

This algebra is exceptional, $M(A) = F(2L_e^2 - L_e) \oplus \langle \psi_{u_1}, \psi_{u_2} \rangle \oplus F(4L_e - 4L_e^2) \oplus \langle L_{v_1}, L_{v_2} \rangle$ and $\widetilde{N} = \langle L_{v_1}, L_{v_2} \rangle$ is nilpotent.

We now consider the Bernstein algebra $A' = F_{e'} \oplus U' \oplus V'$ with non-zero products

	u_1'	u_2'	v_1'	v_2'
u_1'			u_1'	
u_2'			u_2'	
v_1'	u_1'	u_2'		
v_2'				

where $\{u_1', u_2'\}$ and $\{v_1', v_2'\}$ are bases of U' and V', respectively. We observe that $M(A') = F(2L_{e'}^2 - L_{e'}) \oplus \langle \psi_{u_1'}, \psi_{u_2'} \rangle \oplus F(4L_{e'} - 4L_{e'}^2) \oplus \langle 2L_e L_{u_1'} - \frac{1}{2}\psi_{u_1'}, 2L_{e'} L_{u_2'} - \frac{1}{2}\psi_{u_2'} \rangle$ is isomorphic to $M(A)$, under the application $L_e \mapsto L_{e'}$, $\psi_{u_i} \mapsto \psi_{u_i'}$; $L_{v_i} \mapsto 2L_e L_{u_i'} - \frac{1}{2}\psi_{u_i'}$, $i = 1, 2$, but $\widetilde{N}' = F(L_{v_1'}) \oplus \widetilde{V}_{10}' = F(4L_{e'} - 4L_{e'}^2) \oplus \widetilde{V}_{10}'$ is not nilpotent.

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Rua do Matão, 1010 - Cidade Universitária
Caixa Postal 66281 - CEP 05315-970