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C. Biasi
L.A.C. Ladeira
S.M.S. Godoy

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A REMARK ON THE AREA OF A SURFACE

C. BIASI, L.A.C. LADEIRA AND S.M.S. GODOY

ABSTRACT. In this paper we present a formula for the computation of the area of a surface, which is a kind of curvilinear Fubini's formula (Theorem 2). Such a formula relates the area of a surface with the integral of the *period map* of an ordinary differential equation on that surface. Some applications are given. We use technics of differential equations but this result can also be proved by using the coarea theorem.

RESUMO. Neste trabalho apresentamos uma fórmula para o cálculo da área de uma superfície, a qual é uma espécie de fórmula de Fubini curvilínea (Teorema 2). Tal fórmula relaciona a área da superfície com a integral de uma "aplicação período" de uma equação diferencial ordinária sobre a superfície. Algumas aplicações são dadas.Usamos técnicas de equações diferenciais, mas este resultado pode também ser provado usando-se o teorema da coarea.

Key words and phrases. differential equation, manifold, period, area, coarea formula.

1. AREA OF A SURFACE AND PERIOD MAP

Let M be an oriented 2-dimensional Riemannian manifold of class C^1. Let $f\colon M \to \mathbb{R}$ be a C^1 function whose derivative at a point x is denoted by $f'(x)\colon T_xM \to \mathbb{R}$, where T_xM is the tangent space of M at x. There exists a unique vector $g(x) \in T_xM$ such that $f'(x) \cdot v = \langle g(x), v\rangle$, for all $v \in T_xM$, where $\langle\,,\,\rangle$ denotes the inner product in T_xM.

Let $u(x)$ be the vector field obtained by rotating $g(x)$ by an angle of $\frac{\pi}{2}$ radians. Observe that the vectors $u(x)$ are tangent to the level curves of f, so the vectors $g(x)$ are orthogonal to the level curves of f. Denote this rotation by Φ; then $u(x) = \Phi(g(x))$ and $\Phi\colon T_xM \to T_xM$ is a linear operator such that: (i) $\Phi^2 = -I$, (ii) $\langle v, \Phi v\rangle = 0$, (iii) $\|\Phi(v)\| = \|v\|$, (iv) $\{v, \Phi(v)\}$ is a positive basis, if $v \neq 0$.

Consider the differential equation

$$\dot{x} = u(x)\,. \tag{1}$$

For every regular value $y \in f(M)$ the set $f^{-1}(y)$ is invariant by Eq. (1), that is, each solution of Eq. (1) with initial condition in $f^{-1}(y)$ is contained in M and describes a

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piece of the level curve $f^{-1}(y)$. We also observe that we have uniqueness of solutions for the initial value problem

$$\begin{aligned} \dot{x} &= u(x) \\ x(0) &= x_0 \,. \end{aligned} \tag{2}$$

In fact, a local solution of (2) is contained in the level curve $f(x) = c$, $(c = f(x_0))$ such that $\nabla f(x_0) \neq 0$. Suppose x, y are two solutions of (2). Since $\dot{x}(0) = u(x_0) = \dot{y}(0)$, the maps $x(t)$ and $y(t)$ are local parameterizations of the level curve $f^{-1}(c)$ through the point x_0. Thus there is a change of variable of class C^1, $s = h(t)$, $h(0) = c$, such that $x(t) = y(s)$. By the chain rule, $\dot{x}(t) = \dot{h}(t)\,\dot{y}(h(t))$, and therefore $u(x(t)) = \dot{h}(t)\,u(y(h(t))) = \dot{h}(t)\,u(x(t))$, which implies $\dot{h}(t) = 1$, and hence $h(t) = t$, for all t. This shows the uniqueness of solutions of (2).

Definition: We define the *period map* $p\colon \mathbb{R} \to [0,\infty) \cup \{+\infty\}$ of (1) in the following way: for each regular value $y \in f(M)$, there exists a finite or countable union of disjoint intervals $J_y = \cup_\lambda J_\lambda \subset \mathbb{R}$ and a solution $x\colon J_y \to M$ of (1) (since the set J_y may be disconnected, we mean solution in an extended sense: the orbit is a union of orbits in the usual sense) such that x is $1-1$ and $x(J_y) = f^{-1}(y)$. Then we define $p(y) = \sum_\lambda l(J_\lambda)$, where $l(J_\lambda)$ is the length of J_λ.
For each $y \in f(M)$ that is not a regular value of f, we put $p(y) = +\infty$; and if $f^{-1}(y) = \emptyset$, we define $p(y) = 0$.

We remark that the period function at a point z corresponds to the time elapsed by the solution to traverse the orbit $f^{-1}(z)$. When this orbit is periodic, $p(z)$ is the minimal period of the orbit.

We recall that an extended real valued function q is said to be *lower semi continuous* at a point z_0 such that $q(z_0) < \infty$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(z_0) - \varepsilon < q(z)$, whenever $|z - z_0| < \delta$. It is lower semi continuous at a z_0 such that $q(z_0) = \infty$ if, for any L, there is a $\delta > 0$ such that $q(z) > L$, whenever $|z - z_0| < \delta$.

Theorem 1. *The period function p is lower semi continuous.*

Proof: We shall only present the proof for the case $p(z_0) < \infty$. Since $p(z_0) < \infty$, the series $\sum_\lambda l(J_\lambda)$ is convergent. Given $\varepsilon > 0$, there exists n such that $\sum_{i=n+1}^{\infty} l(J_i) < \varepsilon$. Using the theorem on local form of submersions (see [7]) and the continuity of solutions of differential equations with respect to initial conditions we can conclude the existence of a finite family of closed intervals $\{Q_0\,, Q_1\,, \ldots, Q_n\}$, $Q_i \subset J_i\ \forall i$, satisfying $l(J_i) - l(Q_i) < \varepsilon/n$, and exists $\bar{\delta} > 0$ such that is $|x_0 - \bar{x}_0| < \bar{\delta}$, the solution $x(t, \bar{x}_0)$ is defined at least on Q_i. Hence

$$p\bar{z}_0 \geq \sum_{i=1}^{n} l(Q_i) > \sum_{i=1}^{n} [l(J_i) - \varepsilon] > [p(z_0) - \varepsilon] - \varepsilon = p(z_0) - 2\varepsilon.$$

Remark 1: Let $p\colon \mathbb{R} \to [0,\infty) \cup \{+\infty\}$ be a given lower semi continuous function and $M = \{(t,z) \in \mathbb{R}^2, 0 < t < p(z)\}$. Since M is open, then M is a Riemannian manifold with the induced metric of $\mathbb{R}^2$. If $f\colon M \to \mathbb{R}$ is given by $f(t,z) = z$, the correspondent period map for this function f is p.

Remark 2: The period map is Lebesgue measurable, but may not be Riemann integrable, even if $f(M)$ and p are bounded. Let $A \subset (0,1)$ be a closed set with positive measure and empty interior. Consider the open set $M = ((0,1) - A) \times (-2,2) \cup (A \times (-1,1))$ and the map $p(x)$ defined by $p(x) = 2$ if $x \in A$ and $p(x) = 4$ if $x \notin A$. The function p is discontinuous in A and since the measure of A is not zero, then p is not Riemann integrable.

In the proof of Lemma 2 below we need the following equality, related to Schwarz inequality, whose proof is immediate.

Lemma 1. *Let u, v vectors in a 2-dimensional Hilbert space, and let $\tilde{u}$ be a vector that is orthogonal to u such that $\|\tilde{u}\| = \|u\|$. Then*

$$\|u\|^2\|v\|^2 = \langle u, v\rangle^2 + \langle \tilde{u}, v\rangle^2.$$

Lemma 2. *Let M be a 2-dimensional oriented Riemannian manifold with boundary, and let $f\colon M \to \mathbb{R}$ be a C^1 map such that:*

(a) f and $f|_{\partial M}$ have no critical points;

(b) For any $z \in f(M)$, there exists an interval $[r,s]$, $r < z < s$, such that

$f^{-1}[r,s]$ is compact in M;

(c) For any $z \in f(M)$, the set $D(z) = \{x \in M : f(x) < z\}$ has finite area $A(z)$.

For each $x \in M$, let $\tilde{A}(x) = A(f(x))$ and $\tilde{p}(x) = p(f(x))$. Then:

(i) $A'(z) = p(z), \quad \forall z \in f(M)$

(ii) $\tilde{A}'(x) = \tilde{p}(x)\, f'(x), \quad \forall x \in M$.

Proof: Since we can work in each connected component of

$f^{-1}(z)$, we shall assume that $f^{-1}(z)$ is connected, for all z. Let $z \in J$, then $f^{-1}(z)$ is an orbit which we will parameterize by z.

For any x_0 let $z_0 = f(x_0)$ and consider the initial value problem

$$\varphi'(z) = \frac{g(\varphi(z))}{\|g(\varphi(z))\|^2}$$
$$\varphi(z_0) = x_0.$$

Since we work in each chart of M we can assume that there exists a z_0 such that the orbit of the corresponding solution $\varphi(z)$ intercepts every level curve of f, that is, the function $\varphi(z)$ is defined for all $z \in J$. Then we have

$$\langle\, g(\varphi(z))\,, \varphi'(z)\,\rangle = 1.$$

Integrating both sides, we get $f(\varphi(z)) = z + c$.

Since $f(\varphi(z_0)) = f(x_0) = z_0$, we have $z_0 = f(x_0) = f(\varphi(z_0)) = z_0 + c$, which implies $c = 0$. Therefore $f(\varphi(z)) = z$, $\forall z \in J$.

Now consider the solutions $x(t,z)$ of the differential equation $\dot{x} = u(x)$ such that $x(0,z) = \varphi(z)$; then clearly $f(x(t,z)) = z, \forall z$. The map $x(t,z)$ parameterizes M; If $W = \{(t,z)\ :\ 0 < t < p(z),\ z \in J\}$ then $x\colon W \to M$ is a diffeomorphism.We claim that if

$E = \|\frac{\partial x}{\partial t}\|^2$, $G = \|\frac{\partial x}{\partial z}\|^2, F = \langle\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z}\rangle$, then $EG - F^2 = 1$, for all $t\,, z$.

In fact, from $f(\varphi(z)) = z$, $\forall z \in J$, is follows that $f(x(t,z)) = z$, $\forall z \in J$, for $x(0,z) = \varphi(x)$ and $x(t,z)$ is a level curve of f. Note that since $\Phi^2 = -I$, $\frac{\partial x}{\partial t} = -\Phi\frac{\partial x}{\partial z}$ implies $-\Phi\big(\frac{\partial x}{\partial z}\big) = g$. Hence

$1 = \langle\, g(x)\,, \frac{\partial x}{\partial z}\,\rangle = \langle -\Phi\big(\frac{\partial x}{\partial t}\big), \frac{\partial x}{\partial z}\rangle$

and it follows from Lemma 1 that

$$1 = \langle\, g(x)\,, \frac{\partial x}{\partial z}\,\rangle^2 = \langle\Phi\big(\frac{\partial x}{\partial z}\big), \frac{\partial x}{\partial t}\rangle^2 = \|\frac{\partial x}{\partial t}\|^2\|\frac{\partial x}{\partial z}\|^2 - \langle\frac{\partial x}{\partial t}, \frac{\partial x}{\partial t}\rangle^2$$
$$= EG - F^2.$$

From the Differential Geometry we have

$$\text{Area}\,(M) = \int_W \sqrt{EG - F^2}\, dt\, dz = \int_W dt\, dz = \int_{z_0}^{z_1} p(z)\, dz.$$

Therefore $A(z) = A(z_0) + \int_{z_0}^{z} p(\tilde{z})d\tilde{z}$, and then $A'(z) = p(z)$. This shows (i). The proof of (ii) follows from (i) and the chain rule.

Lemma 3. *Let M be a compact oriented Riemannian manifold with boundary, and f be a C^1 map with no critical points. Then*

$$\text{Area}\,(M) = \int p(z)\,dz.$$

Proof: Denote by CV the set of critical values of $f|_{\partial M}$. Since M is compact, CV is a closed set. Since f has no critical points, Sard's Theorem implies $m(CV) = 0$.

Then $\mathbb{R}\backslash CV$ is a countable union (possibly finite) of open intervals, that is, $\mathbb{R}\backslash CV = \cup I_n$, where I_n is an open interval for all n. Since the result holds in each I_n, Lemma 2 implies

$$\text{Area}\,(f^{-1}(I_n)) = \int_{I_n} p(x)\,dx.$$

Then, since the area of $f^{-1}(CV)$ is zero (because f has no critical points), we have

$$\begin{aligned}\int_{\mathbb{R}} p(z)\,dz &= \int_{\mathbb{R}\backslash CV} p(z)\,dz = \sum_n \int_{I_n} p(z)\,dz \\ &= \sum_n \text{Area}\,(f^{-1}(I_n)) = \text{ Area}\,(M).\end{aligned}$$

Theorem 2. *Let M be an oriented Riemannian manifold without boundary, and let f be a C^1 map without critical points. Then*

$$\text{Area}\,(M) = \int p(z)\,dz.$$

Proof: We can write $M = \cup M_n$, a countable union, in which $M_n \subset \overset{\circ}{M}_{n+1}$ and each M_n is a compact manifold. Let the p_n be the period map in M_n. By Lemma 3, we have

$$\text{Area}\,(M_n) = \int p_n(z)\,dz.$$

Note that $p(z) = \lim_{n\to\infty} p_n(z)$, the sequence $(p_n(z))$ is bounded by $p(z)$ and $p_{n+1}(z) \geq p_n(z)$. By Lebesgue Dominated Convergence Theorem, we have

$$\text{Area}\,(M) = \lim_{n\to\infty} \text{Area}\,(M_n) = \lim_{n\to\infty} \int p_n(z)\, dz = \int p(z)\, dz.$$

Remark 3: Since each p_n is continuous and p is not Riemann integrable, the convergence cannot be uniform.

If the set of critical points of f has positive measure, we have the following result, whose proof is immediate.

Corollary 1. *Let M be an oriented Riemannian manifold without boundary, and let f be a C^1 map. Then*

$$\text{Area}\,(M) = \int p(z)\, dz + \text{Area}\,(C_f),$$

where C_f is the set of critical points of f.

Remark 4: If the differential equation (1) is

$\dot{x} = \lambda(x)u(x),\quad \lambda(x) \neq 0$, then, $A'(z) = \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$, where $\tilde{p}(z)$ is the period function for the map $\lambda(x)u(x)$. In fact, we have

$$1 = \langle\, g(x)\,, \frac{\partial x}{\partial z}\,\rangle = \langle\, -\Phi(u(x))\,, \frac{\partial x}{\partial t}\,\rangle = \frac{1}{\lambda(x)}\langle -\Phi\Big(\frac{\partial x}{\partial t}\Big), \frac{\partial x}{\partial z}\rangle.$$

Since

$\|\Phi(\frac{\partial x}{\partial t})\| = \|\frac{\partial x}{\partial t}\|$ and $\Phi(\frac{\partial x}{\partial t})$ is orthogonal to $\frac{\partial x}{\partial t}$, we have:

$$\Big\|\frac{\partial x}{\partial t}\Big\|^2 \Big\|\frac{\partial x}{\partial z}\Big\|^2 = \langle\, \tfrac{\partial x}{\partial t},\, \tfrac{\partial x}{\partial z}\rangle^2 + \langle\, \Phi(\tfrac{\partial x}{\partial t}),\, \tfrac{\partial x}{\partial z}\,\rangle^2 = \langle\, \tfrac{\partial x}{\partial t},\, \tfrac{\partial x}{\partial z}\,\rangle^2 + (\lambda(x))^2.$$

Then, $EG - F^2 = \lambda^2(x)$; hence

$A(z) - A(z_0) = \int_{z_0}^{z} d\bar{z} \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$. Therefore, $A'(z) = \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$.

2. Examples

1) Consider the cone $M = \{(x,y,z) \in \mathbb{R}^3 : x^2 = y^2 + z^2\}$ and the function $f\colon M \to \mathbb{R}$ defined by $f(x,y,z) = x$.

Consider the zero level curve of f given by $\phi(x,y,z) = x^2 - y^2 - z^2 = 0$ and let $P = (x,y,z)$ be a point of M. The tangent plane of M at P is $T_PM = \{v = (v_1, v_2, v_3) : \langle\, \nabla\phi(x,y,z),\, v\,\rangle = 2xv_1 - 2yv_2 - 2zv_3 = 0\}$.

For $v \in T_P M$ we have $f'(P).v = \langle \nabla f(P), v\rangle = \langle (1,0,0),(v_1,v_2,v_3)\rangle = v_1$. A vector $g(P) \in T_P M$ so that $\langle g(P), v\rangle = f'(P).v = v_1$ is

$$g(P) = \Big(\frac{y^2+z^2}{x^2+y^2+z^2}, \frac{y\,x}{x^2+y^2+z^2}, \frac{z\,x}{x^2+y^2+z^2}\Big).$$

The vector $u(P)$ is orthogonal to both $\nabla\phi(P)$ and $g(P)$, and $|u(P)| = |g(P)|$. Then $u(P) = \big(0, \frac{z}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}\big)$.

The differential equation is

$$\text{(3)} \qquad \begin{aligned} \dot P &= u(P) \\ P(0) &= (x_0, y_0, z_0) \in M\,. \end{aligned}$$

The solution of (3) is $(x(t), y(t), z(t)) = (x_0, x_0\cos(\frac{1}{x_0\sqrt{2}}t),\ x_0\sin(\frac{1}{x_0\sqrt{2}}t))$.

Therefore, the period function is $p(x) = 2\sqrt{2}\pi x$ and hence the area of M is

$$A = 2\sqrt{2}\int_0^h \pi x dx = \sqrt{2}\pi h^2,$$

which coincides with the known formula for the area of the cone.

2) As another example, consider the nonlinear second order scalar differential equation

$$\text{(4)} \qquad \ddot z + 2z^3 = 0.$$

It can be written as a 2-dimensional differential equation

$$\text{(5)} \qquad \dot x = u(x),$$

where $x = (x_1\,, x_2)$, $u(x_1\,, x_2) = (x_2\,, -2x_1^3)$. The orbits of (5) in the $(x_1\,, x_2)-$plane are the level curves of the *total energy* function $f(x,y) = (x_2^2 + x_1^4)/2$. For each $h > 0$, the level curve

$$\text{(6)} \qquad x_1^4 + x_2^2 = 2h\,,$$

is a periodic orbit of (5) in the phase plane $(x_1\,, x_2)$ that intersects the x_1-axis at the points $(-a, 0)$ and $(a, 0)$, where $a^4 = 2h$. Due to the symmetry of (6), this orbit is symmetric with respect to both x_1- and x_2- axes. Each corresponding solution has maximum amplitude a. By solving Eq. (6) for $x_2 = \frac{dx_1}{dt}$ and then integrating from 0 to a over a quarter of orbit we get the period $p(h)$ of this orbit as $p(h) = \frac{C}{(2h)^{1/4}}$, where $C = 4\int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{\sqrt{\pi}\Gamma(1/4)}{\Gamma(3/4)} \simeq 1.31103$. Note that $p(h) \to \infty$ as $h \to 0^+$, and $p(h) \to 0$ as $h \to \infty$. Then Theorem 2 implies that the area A of the annulus between the orbits corresponding to the energy levels h and k, $0 \le k < h$, is given by

$$A = \int_k^h p(\lambda)\, d\lambda = C \int_k^h \lambda^{-1/4}\, d\lambda = \frac{4C}{3}(h^{3/4} - k^{3/4}).$$

In particular, the area enclosed by the orbit corresponding to the energy level $h = a^4/2$ is

$$A = \frac{4C}{3} h^{3/4} = \frac{4C}{3\sqrt[4]{2}} a^3.$$

Similar computations yield analogous conclusions for the more general equation

$$\ddot{x} + x^{2n+1} = 0.$$

3. An Application

We now present a type of Cavalieri's principle. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$, be maps without critical points and p_1, p_2 the respective "period functions".

Suppose that $p_1(z) = p_2(z)$ almost everywhere. Then, given $z_0, z_1 \in \mathbb{R}$, we have Area $(f_1^{-1}(z_0, z_1))$ =Area $(f_2^{-1}(z_0, z_1))$.

In particular, Area (M_1) = Area (M_2).

4. A Coarea Formula

As another application, we give a proof of a particular case of the famous coarea theorem (see, [5]). We remark that, conversely, Theorem 2 can be proved from the coarea formula, by doing convenient identifications, but we do not intend to do it here.

Theorem 3. *Let M be an open subset of $\mathbb{R}^2$ and $f : M \to \mathbb{R}$ a C^1 function without critical points. Then $\int_M J(f(x))\, dx = \int \mu_1(f^{-1}(z))\, dz$, where $\mu_1(f^{-1}(z))$ is 1-dimensional Hausdorff measure of $f^{-1}(z)$ and $J(f(x))$ is the norm of the gradient of f.*

Remark 5: Note that in this case $\mu_1(f^{-1}(z))$ is the length of $f^{-1}(z)$.

The proof of Theorem 3 is a consequence of the lemmas below.

Lemma 4. *Suppose $Jf(x) = c$. Then* $\int_M J(f(x))\,dx = \int \mu_1(f^{-1}(z))\,dz$.

Proof: We have $c\,\text{Area}\,(M) = c\int_M dx = \int_M \tilde{c}\,dx = \int_M Jf(x)\,dx.$

But we also know by Theorem 2, that

$$c\,\text{Area}\,(M) = c\int p(z)\,dz = \int c\,p(z)\,dz = \int l(f^{-1}(z))\,dz = \int \mu_1(f^{-1}(z))\,dz,$$

and the conclusion follows.

Lemma 5. *Suppose that $Jf(x)$ satisfies $c - \epsilon \le J(f(x)) \le c + \epsilon$, where $0 < \epsilon < c$.*

Then,

$$|\int_M J(f(x))\,dx - \int \mu_1(f^{-1}(z))\,dz| \le \frac{2\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz. \tag{7}$$

Proof: We have

$$\int_M (c-\epsilon)\,dx \le \int_M Jf(x)\,dx \le \int_M (c+\epsilon)\,dx.$$

Hence,

$$(c-\epsilon)\ \text{Area}\,(M) \le \int_M Jf(x)\,dx \le (c+\epsilon)\ \text{Area}\,(M).$$

We also have

$$\mu_1(f^{-1}(z)) = \int_0^{p(z)} \|\dot{x}\|\,dt \le (c+\epsilon)p(z) \quad \text{and} \quad \mu_1(f^{-1}(z)) \ge (c-\epsilon)p(z)$$

and these inequalities imply

$$(c-\epsilon)\int p(z)\,dz = \frac{c-\epsilon}{c+\epsilon}\int (c+\epsilon)\,p(z)\,dz \ge \frac{c-\epsilon}{c+\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

and

$$(c+\epsilon)\int p(z)\,dz = \frac{c+\epsilon}{c-\epsilon}\int (c-\epsilon)\,p(z)\,dz \le \frac{c+\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

Therefore,

$$\frac{c-\epsilon}{c+\epsilon}\int \mu_1(f^{-1}(z))\,dz \le \int_M J(f(x))\,dx \le \frac{c+\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

and from these inequalities we get (7).

Lemma 6. *Let M be a bounded open subset of $\mathbb{R}^2$ and f a C^1 function on $\overline{M}$. Suppose that there is $c > 0$ such that $Jf(x) \ge c$, $\forall x \in M$. Then,*

$$\int_M J(f(x))\,dx = \int \mu_1(f^{-1}(z))\,dz.$$

Proof: Suppose $\epsilon > 0$ is given. Since $J(f(x))$ is continuous, there exist open sets $M_1, M_2, \ldots, M_n$ such that, $M - \bigcup_{i=1}^n M_i$ has measure zero and $c_i - \epsilon \le J(f(x)) \le c_i + \epsilon$, $\forall x \in M_i$, $i = 1, \ldots, n$, where $c_i = J(f(x_i))$ for some $x_i \in M_i$.

Then, denoting $I_i = f(M_i)$, $i = 1, \ldots, n$, we have

$$\begin{aligned}\int_M J(f(x))\,dx &= \int_{\cup M_i} J(f(x))\,dx = \sum_{i=1}^n \int_{M_i} J(f(x))\,dx\\ &= \sum_{i=1}^n \int_{I_i} \mu_1(f^{-1}(z))\,dz + \sum_{i=1}^n \Big(\int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1(f^{-1}(z))\,dz\Big)\\ &= \int_{I_i} \mu_1(f^{-1}(z))\,dz + \sum_{i=1}^n \Big(\int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1(f^{-1}(z))\,dz\Big).\end{aligned}$$

From Lemma 2 we have:

$$\begin{aligned}\sum_{i=1}^n \Big|\int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1 f^{-1}(z)\,dz\Big| &\le \sum_{i=1}^n \frac{2\epsilon}{c_i-\epsilon}\int_{I_i} \mu_1(f^{-1}(z))\,dz\\ &= \frac{2\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz.\end{aligned}$$

Since the integral $\int \mu_1 f^{-1}(z)\,dz$ is finite and $\epsilon > 0$ is arbitrary, we get:

$$\int_M Jf(x)\,dx = \int \mu_1(f^{-1}(z))\,dz.$$

5. An inequality for the area of a surface

In the application bellow we get some estimates for the area of the surface using a result due to Lasota and Yorke [8] (see also [1]). The Riemmanian manifold that we consider is a Riemannian sub manifold in a Hilbert space. We recall the following result:

Lemma 7. *Let $x(t)$ be a nontrivial p-periodic solution of the equation $x'(t) = \mu(x(t))$. Assume that μ is Lipschitzian with Lipschitz constant L on the orbit of x. Then $Lp \geq 2\pi$.*

Assume $f\colon M \to \mathbb{R}$ is C^1 and has precisely n critical values $a_1, ..., a_n$ in $(a, b) = f(M)$. For any $z \in I_j = (a_j, a_{j+1})$, $j = 0, 1, ..., n$, where $a_0 = a, a_n = b$, let n_j be the number of corresponding periodic orbits (we recall that for any z, $f^{-1}(z)$ consists of finitely many periodic orbits).

Then,

$$L \text{ Area}\,(M) = L \int p(z)\,dz = \sum_{i=1}^{n} \int_{I_i} Lp(z)\,dz \geq 2\pi \sum_{i=1}^{n} n_i l(I_i).$$

Therefore,

$$\text{Area}\,(M) \geq \frac{2\pi}{L} \sum_{i=1}^{n} n_i l(I_i).$$

Thus we have proved:

Theorem 4. *Suppose $f : M \to \mathbb{R}$ is a C^1 map such that f' is Lipschitzian with Lipschitz constant L. If f has precisely n critical values, then (using the above notation),*

$$\text{Area}\,(M) \geq \frac{2\pi}{L} \sum_{i=1}^{n} n_i l(I_i).$$

In particular, if f has no critical values then,

$$\text{Area}\,(M) \geq \frac{2\pi}{L}(b - a).$$

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Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo- Campus de São Carlos, Caixa Postal- 668, 13560-970 São Carlos- SP

E-mail address: biasi@icmc.sc.usp.br, ladeira@icmc.sc.usp.br, smsgodoy@icmc.sc.usp.br

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