DEPARTAMENTO DE MATEMÁTICA APLICADA

Relatório Técnico

RT-MAP- 9903

Complete Asymptotic Expansion of
$$\sum_{k=1}^{N-1} k^{\alpha}(N-k)^{\beta} \text{ as } N \to \infty$$

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Agosto de 1999

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Complete Asymptotic Expansion of $\sum_{k=1}^{N-1} k^\alpha (N-k)^\beta$ as $N \to \infty$

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We show, for all complex α, β, that as $N \to \infty$

$$\sum_{k=1}^{N-1} k^\alpha (N-k)^\beta = S_N(\alpha,\beta) = S_N(\beta,\alpha) \simeq$$

$$\frac{\alpha!\beta!}{(\alpha+\beta+1)!} N^{\alpha+\beta+1} + \sum_{j\geq 0} \binom{\alpha}{j} (-1)^j N^{\alpha-j} \zeta(-\beta-j) \tag{1}$$

$$+ \binom{\beta}{j} (-1)^j N^{\beta-j} \zeta(-\alpha-j)$$

where $z! = \Gamma(z+1)$ and ζ is Riemann's zeta function. Basic properties of Γ and ζ, as well as the Bernoulli numbers and Euler-Maclaurin formula used bellow, may be found in *M.Abramowitz and I. Stegun*, Handbook of Mathematical Functions (National Bureau of Standards, 1964; J. Wiley, 1972). Here we only note $\zeta(p) = \sum_1^\infty n^{-p}$ if $Re\ p > 1$ and $\zeta(p) - \frac{1}{p-1}$ extends to an entire analytic function with value $\gamma = -\Gamma'(1)$ for $p = 1$.

More explicitly, if m is any positive integer and α, β are in $\mathbb{C}\backslash\{integers \leq$

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$-m-2\}$, then as $N \to \infty$

$$S_N(\alpha,\beta) - \left\{ \frac{\alpha!\beta!}{(\alpha+\beta+1)!} N^{\alpha+\beta+1} + \sum_{j=0}^{m} \binom{\alpha}{j} (etc.) \right\} = \tag{2}$$

$$= O(N^{Re\ \alpha-m-1} + N^{Re\ \beta-m-1})$$

and the estimate is uniform for α, β in any compact set of $\mathbf{C} \setminus \{integers \leq -m-2\}$. We exclude integers $\leq -m-2$ so that if one (or both) of α,β is a negative integer and $\alpha!\beta!$ is singular, the sum in (2) also includes a singular ζ (undefined "$\zeta(1)$") and the singularities cancel. Despite appearances, the left-side of (2) is analytic in α, β.

When α or β (or both) is a negative integer, we use the limit form of (1).

If $\alpha = -m$ is a negative integer and β is not an integer,

$$S_N(-m,\beta) \simeq (-1)^{m-1} \binom{\beta}{m-1} N^{1+\beta-m} \left(logN + \sum_{1}^{N-1} \frac{1}{j} - \psi(2+\beta-m) \right)$$

$$+ \sum_{j \geq 0} \binom{-m}{j} (-1)^j N^{-m-j} \zeta(-\beta-j) + \sum_{j\geq 0} \binom{\beta}{j} (-1)^{j-1} N^{\beta-j} \zeta(m-j) \tag{3}$$

where $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$, "log" denotes the natural logarithm and we omit the obvious troublemaker in the second sum ($j = m-1$ gives undefined $\zeta(m-j)$). As usual, an empty sum $(\sum_1^0 \ldots)$ is zero. The regular part of (1), omitting any term with undefined "$\zeta(1)$" or "$z!$", when z is a negative integer, will be denoted by "(reg)". Thus the second line of (3) is "(reg)".

If $\alpha = -m$ and also β is an integer, k or $-k$,

$$S_N(-m,k) \simeq (-1)^{m-1} \binom{k}{m-1} N^{1+k-m} \left(logN + \gamma + \sum_{1}^{m-1} \frac{1}{j} - \sum_{1}^{1+k-m} \frac{1}{j} \right) + (reg) \tag{4}$$

when $k \geq m-1$;

$$S_N(-m,k) \simeq (-1)^{k-1} \frac{k!(m-k-2)!}{(m-1)!(k-1)!} N^{1+k-m} + (reg) \tag{5}$$

when $0 \leq k \leq m-2$;

$$S_N(-m,-k) \simeq \frac{(m-k-2)!}{(m-1)!(k-1)!} N^{1-k-m} \left(2logN + 2\gamma + \sum_1^{m-1} \frac{1}{j} \right. \\ \left. + \sum_1^{k-1} \frac{1}{j} - 2 \sum_1^{m+k-2} \frac{1}{j} \right) + (reg) \tag{6}$$

when $k \geq 1$.

We will see many terms $N^{\alpha+\beta-k}$ ($k = integer \geq 0$) in the calculation, but they all disappear from the final result (1). Since the calculation is rather long, we first do it formally and then with remainder estimates.

If $N = 2n+1$, $S_N(\alpha,\beta) = \sum_{k=1}^{n} k^\alpha (N-k)^\beta + k^\beta (N-k)^\alpha$; but for $N = 2n$, the term with $k = n = N-k$ would be counted twice in this sum. Let $\varepsilon_N = 1$ if N is odd, $\varepsilon_N = 0$ if N is even; then

$$S_N(\alpha,\beta) + (1-\varepsilon_N)\left(\frac{N}{2}\right)^{\alpha+\beta} = \sum_{k=1}^{n} k^\alpha (N-k)^\beta + k^\beta (N-k)^\alpha = \\ \sum_{j=0}^{\infty} \binom{\beta}{j} (-1)^j N^{\beta-j} \sum_{k=1}^{n} k^{\alpha+j} + \binom{\alpha}{j} (-1)^j N^{\alpha-j} \sum_{k=1}^{n} k^{\beta+j} \tag{7}$$

where $n = \dfrac{N-\varepsilon_N}{2} = \dfrac{N}{2}$ or $\dfrac{N-1}{2}$, whichever is an integer.

The Euler-Maclaurin formula (details later) gives, if $p \neq -1$,

$$\sum_{k=1}^{n} k^p \simeq \zeta(-p) + \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \sum_{k\geq 1} \frac{B_{2k}}{2k} \binom{p}{2k-1} n^{p-2k+1}$$

$$\sum_{k=1}^{n} (k+\frac{1}{2})^p \simeq (2^{-p}-1)\zeta(-p) + \frac{(n+\frac{1}{2})^{p+1}}{p+1} + \frac{1}{2}(n+\frac{1}{2})^{p+1} \\ + \sum_{k\geq 1} \frac{B_{2k}}{2k} \binom{p}{2k-1} (n+\frac{1}{2})^{p-2k+1}$$

as $n \to \infty$. When $N = 2n+1$, we use $\sum_1^n k^p = 2^{-p} \sum_1^{2n+1} k^p - \sum_0^n (k+\frac{1}{2})^p$, so we combine the odd and even cases as:

$$\begin{aligned}\sum_{k=1}^{n} k^p \simeq \; & \zeta(-p) + \frac{(N/2)^{p+1}}{p+1} + \frac{1-\varepsilon_N}{2}\left(\frac{N}{2}\right)^p + \\ & \sum_{k\ge 1} \frac{B_{2k}}{2k}\binom{p}{2k-1} N^{p-2k+1} \cdot \{1 + \varepsilon_N(2^{1-2k}-2)\}\end{aligned} \tag{8}$$

Formal substitution of (8) in (7) yields

$$S_N(\alpha,\beta) + (1-\varepsilon_N)\left(\frac{N}{2}\right)^{\alpha+\beta} \simeq \sum_{-1}^{\infty} C_k N^{\alpha+\beta-k}$$

$$+ \sum_{j\ge 0} (-1)^j \binom{\alpha}{j} N^{\alpha-j}\zeta(-\beta-j) + (-1)^j \binom{\beta}{j} N^{\beta-j}\zeta(-\alpha-j)$$

where

$$C_{-1} = \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{\alpha}{j} \frac{2^{-j-\beta-1}}{j+\beta+1} + \binom{\beta}{j} \frac{2^{-j-\alpha-1}}{j+\alpha+1} \right\}$$

$$C_0 = \frac{1-\varepsilon_N}{2} \sum_{j=0}^{\infty} (-1)^j \left\{ \binom{\alpha}{j} 2^{-j-\beta} + \binom{\beta}{j} 2^{-j-\alpha} \right\}$$

and for $k \ge 1$, $C_{2k} = 0$, $C_{2k-1} = \dfrac{B_{2k}}{2k}\{1 + \varepsilon_N(2^{1-2k}-2)\}\, Q_{2k-1}(\alpha,\beta)$,

$$Q_k(\alpha,\beta) = \sum_{j\ge 0} (-1)^j \left\{ \binom{\alpha}{j}\binom{\beta+j}{k} 2^{-j-\beta+k} + \binom{\beta}{j}\binom{\alpha+j}{k} 2^{-j-\alpha+k} \right\}.$$

It is easy to see that $C_0 = (1-\varepsilon_N)2^{-\alpha-\beta}$, and that C_{-1} is analytic in α, β if neither is a negative integer. If $\alpha > -1$ and $\beta > -1$,

$$\begin{aligned} C_{-1} &= \sum_{j\ge 0} (-1)^j \binom{\alpha}{j} \int_0^{\frac{1}{2}} t^{j+\beta} dt + (-1)^j \binom{\beta}{j} \int_{\frac{1}{2}}^{1} (1-t)^{j+\alpha} dt \\ &= \int_0^1 t^\beta (1-t)^\alpha dt = \frac{\alpha!\beta!}{(\alpha+\beta+1)!}. \end{aligned}$$

Thus $C_{-1} = \dfrac{\alpha!\beta!}{(\alpha+\beta+1)!}$ for all α,β in $\mathbb{C} \setminus \{-1,-2,-3,\cdots\}$.

Using $\left|\binom{z}{k}\right| \leq \frac{1}{k!}\prod_{0}^{k-1}(|z|+i) = (-1)^k\binom{-|z|}{k}$ we see, for $|t| < \frac{1}{4}$, the double series $\sum_{k\geq 0} t^k Q_k(\alpha,\beta)$ is absolutely convergent with the sum

$$\left(\frac{1}{2}+t\right)^\alpha \left(\frac{1}{2}-t\right)^\beta + \left(\frac{1}{2}+t\right)^\beta \left(\frac{1}{2}-t\right)^\alpha,$$

an even function of t. Thus all odd coefficients vanish and all $C_{2k-1} = 0$.

This gives , formally, the result (1) - at least when α, β are in $\mathbb{C} \setminus \{-1,-2,-3,\cdots\}$. Now we argue with more care, and with remainder estimates.

Let m be some fixed positive integer, and for $p \neq -1$, define

$$\begin{aligned} \widehat{z}_N(p,m) = & \sum_{k=1}^{n} k^p - \left(\frac{(N/2)^{p+1}}{p+1} + \frac{1-\varepsilon_N}{2}\left(\frac{N}{2}\right)^p + \right. \\ & \left. + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{p}{2k-1}\left(\frac{N}{2}\right)^{p-2k+1}\{1+\varepsilon_N(2^{1-2k}-2)\}\right) \end{aligned} \tag{9}$$

where $n = N/2$ or $(N-1)/2$, whichever is an integer. (We estimate $\widehat{z}_N(p,m)$ later, for large N and possibly large $|p|$.) Substitution of (9) in (7) yields: for α, β in $\mathbb{C} \setminus \{-1,-2,-3,\cdots\}$

$$\begin{aligned} S_N(\alpha,\beta) = & \left\{\frac{\alpha!\beta!}{(\alpha+\beta+1)!}N^{\alpha+\beta+1} \right. \\ & \left. + \sum_{j=0}^{m}\binom{\alpha}{j}(-1)^j N^{\alpha-j}\zeta(-\beta-j) + \binom{\beta}{j}(-1)^j N^{\beta-j}\zeta(-\alpha-j)\right\} = \\ & = \sum_{j=0}^{m}\binom{\alpha}{j}(-1)^j N^{\alpha-j}\left(\widehat{z}_N(\beta+j,m) - \zeta(-\beta-j)\right) + (sym) \\ & + \sum_{j=m+1}^{\infty}\binom{\alpha}{j}(-1)^j N^{\alpha-j}\widehat{z}_N(\beta+j,m) + (sym), \end{aligned} \tag{10}$$

where "$+(sym)$" means to add the corresponding term with α and β interchanged. Our estimate for $\widehat{z}_N(p,m)$ will show (if $|\alpha|, |\beta| \leq m-1$) the right-side of (10) has modulus $O(N^{Re\ \alpha + Re\ \beta - 2m+1} + N^{Re\ \alpha - m-1} + N^{Re\ \beta - m-1})$ as $N \to \infty$ (or, $O(N^{Re\ \alpha - m - 1} + N^{Re\ \beta - m-1})$ as $N \to \infty$, if $\min\{Re\ \alpha, Re\ \beta\} \leq m-2$) uniformly for allowed α, β. (The "m" used here may be different from that of (2) above.)

The exact Euler-Maclaurin formula for a $\mathbf{C}^{2m+1}$ function $f : [a,b] \to \mathbf{C}$, where a, b are integers with $a < b$, is:

$$\sum_{k=a}^{b} f(k) = \int_a^b f + \frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(x)\right]_{x=a}^{b} +$$

$$+ \int_a^b \frac{\overline{B}_{2m+1}(x)}{(2m+1)!} f^{(2m+1)}(x)dx$$

where $\overline{B}_k(x+1) = \overline{B}_k(x)$ for all x, $B_k = \overline{B}_k(0^+)$, and for $|t| < 2\pi$,

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{\overline{B}_k(x)}{k!}, \quad \text{in } 0 < x < 1,$$

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k B_k}{k!}.$$

With $f(x) = x^p$ ($p \neq -1$), $a = 1$, $b = n$, we have

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{p}{2k-1} n^{p-2k+1} + z_n(p,m) \tag{11}$$

where

$$z_n(p,m) = \frac{1}{2} - \frac{1}{p+1} - \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{p}{2k-1} + \int_1^n \overline{B}_{2m+1}(x)\binom{p}{2m+1} x^{p-2m-1}dx.$$

If $Re\ p < 2m$, $z_n(p,m)$ converges as $n \to \infty$:

$$z_n(p,m) = z_\infty(p,m) - \int_n^\infty \overline{B}_{2m+1}(x)\binom{p}{2m+1} x^{p-2m-1}dx$$

and $p \mapsto z_\infty(p,m)$ is analytic for $p \neq -1$, $Re\ p < 2m$. If $Re\ p < -1$, $z_\infty(p,m) = \zeta(-p)$, so equality holds for all $p \neq -1$ with $Re\, p < 2m$. (Formula (11) may be used to compute $\zeta(-p)$.)

If $p \neq -1$, $Re\ p < 2m$,

$$|z_n(p,m) - \zeta(-p)| \leq \left|\binom{p}{2m+1}\right| \sup |\overline{B}_{2m+1}(\cdot)| \frac{n^{Re\ p-2m}}{2m - Re\ p};$$

for any p with $|p+1| \geq 1$,

$$|z_n(p,m)| \leq C_m(1+|p|)^{2m+1}(1+n^{Re\ p-2m+1})$$

where the constant C_m depends only on m. (Here we use the crude estimate: $\int_1^n x^s dx \leq 1 + n^{s+2}$ for any real s and $n \geq 1$.)

Similarly with $f(x) = (x+1/2)^p$

$$\begin{aligned}\textstyle\sum_0^n (k+1/2)^p = \quad & \tilde{z}_n(p,m) + \tfrac{(n+1/2)^{p+1}}{p+1} + \tfrac{1}{2}(n+1/2)^p \\ & + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{p}{2k-1}(n+1/2)^{p-2k+1}\end{aligned}$$

where

$$\left|\tilde{z}_n(p,m) - \zeta(-p)(2^{-p}-1)\right| \leq \left|\binom{p}{2m+1}\right| \sup |\overline{B}_{2m+1}(\cdot)| \frac{(n+1/2)^{Re\ p-2m}}{2m - Re\ p},$$

for $Re\ p < 2m$, and for $Re\ p \geq -m$ with $|p+1| \geq 1$,

$$|\tilde{z}_n(p,m)| \leq \tilde{C}_m(1+|p|)^{2m+1}(1+(n+1/2)^{Re\ p-2m+1})$$

for a constant $\tilde{C}_m$ depending only on m.

When N is even, $\hat{z}_N(p,m) = z_{N/2}(p,m)$; when N is odd, $\hat{z}_N(p,m) = 2^{-p}z_N(p,m) - \tilde{z}_{\frac{N-1}{2}}(p,m)$.

For $Re\ p < 2m$

$$|\hat{z}_N(p,m) - \zeta(-p)| \leq 2\left|\binom{p}{2m+1}\right| \sup |\overline{B}_{2m+1}(\cdot)| \frac{(N/2)^{Re\ p-2m}}{2m - Re\ p}; \tag{12}$$

for $Re\ p \geq -m$ with $|p+1| \geq 1$,

$$|\widehat{z}_N(p,m)| \leq \widehat{C}_m(1+|p|)^{2m+1}(1+(N/2)^{Re\ p-2m+1}) \tag{13}$$

where $\widehat{C}_m = 2^m C_m + \widetilde{C}_m$ depends only on m. Also, for $p \in \mathbb{C}$, $\left|\binom{p}{2m+1}\right| / (1+|p|)^{2m+1}$ is bounded by a constant depending on m.

Now suppose $\alpha,\ \beta \in \mathbb{C} \setminus \{-1,-2,-3,\cdots\}$ and $|\alpha|,\ |\beta| \leq m-1$. If $p = \alpha + j$ or $\beta + j$ for integers $j \geq 0$, then $Re\ p \geq -m+1$ for all j, $Re\ p \leq 2m-1$ for $0 \leq j \leq m$, and for $j \geq m+1$, $|p+1| \geq j+2-m \geq 3$. Substitution of (1) and (3) in (10) shows the right-side of (10) has modulus $O(N^{Re\ \alpha + Re\ \beta - 2m+1} + N^{Re\ \alpha - m-1} + N^{Re\ \beta - m-1})$ as $N \to \infty$, uniformly for allowed α, β. Thus the asymptotic expansion is proved even when α or β (or both) is a negative integer.

RELATÓRIOS TÉCNICOS DO DEPARTAMENTO DE MATEMÁTICA APLICADA

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