JOURNAL OF DIFFERENTIAL EQUATIONS 20, 28–36 (1976)

One-to-Oneness for Linear Retarded Functional Differential Equations

JACK K. HALE*

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

AND

WALDYR M. OLIVA†

Instituto de Matematica e Estatistica, Universidade de Sao Paulo, Brasil

Received August 19, 1974

The purpose of this paper is to show that the set of linear nonautonomous retarded functional differential equations such that the solution operator is one-to-one is a dense subset of all linear equations. We also show by example that this set is not too large.

1. PRELIMINARIES

Let R^n be an n-dimensional Euclidean space with norm $|\cdot|$, $I = [-r, 0]$, $r \geqslant 0$, and $C(I, R^n)$ be the Banach space of all continuous functions $\phi: I \to R^n$ with the norm $\|\phi\| = \sup_{\theta \in I} |\phi(\theta)|$. Let $\mathfrak{A}$ be the Banach space of all linear continuous functions $A: C(I, R^n) \to R^n$ with the usual norm $\|A\| = \sup_{\|\phi\|=1} |A(\phi)|$.

Consider now the Banach space $\mathfrak{B}$ of all $n \times n$ matrices η of normalized functions of bounded variation on I, $\eta = (\eta_{ij})$, with

$$\|\eta\| = \max_{1 \leqslant i \leqslant n} \left[\sum_{j=1}^{n} V(\eta_{ij}, I) \right]$$

* This research was supported in part by the National Science Foundation under contract GP-28931X2, and in part by the U.S. Army Research Office, Durham, under contract DA-ARO-D-31-124-73-G-130.

† Research supported in part by Fundação de Amparo a Pesquisa do Estado de Sao Paulo (FAPESP), Brasil, Proc. no. 11-74/603.

where $V(\eta_{ij}, I)$ is the variation of the normalized function $\eta_{ij}: I \to R$. The classical Riesz representation theorem [1] defines a topological linear isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$; to each $A \in \mathfrak{A}$ corresponds an $\eta \in \mathfrak{B}$ such that $\| A \| \leqslant \| \eta \|$ and $A\phi = \int_{-r}^{0} d\eta(\theta)\,\phi(\theta)$ for all $\phi \in C(I, R^n)$. Also, there is a constant $k > 0$ such that $\| \eta \| \leqslant k \| A \|$ for all A.

Let $C^1(R, \mathfrak{A})$ be the space of C^1-mappings from R to $\mathfrak{A}$ with the C^1 uniform topology on compact sets of R. Given $\epsilon > 0$ and any compact set $K \in R$, a (K, ϵ) neighborhood of $L \in C^1(R, \mathfrak{A})$ is the set $v(L, K, \epsilon)$ of all $H \in C^1(R, \mathfrak{A})$ such that

$$\| H(t) - L(t)\| + \| \dot{H}(t) - \dot{L}(t)\| < \epsilon \qquad \text{for any} \quad t \in K.$$

The symbol $\dot{H}(t)$ denotes the derivative of $H(t)$ with respect to t. The topological linear isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ considered above shows that if $H(t)\phi = \int_{-r}^{0} d\nu(t, \theta)\,\phi(\theta)$, $H \in C^1(R, \mathfrak{A})$, then the map $\nu: t \in R \to \nu(t, \cdot) \in \mathfrak{B}$ is differentiable and $\dot{H}(t)\phi = \int_{-r}^{0} d\dot{\nu}(t, \theta)\,\phi(\theta)$.

Also, let $C^o(R, \mathfrak{A})$ be the space of continuous mappings from R to $\mathfrak{A}$ with the uniform topology on compact subsets of R.

DEFINITION 1.1. One says that $L: R \to \mathfrak{A}$ has *smoothness on the measure* if there exists a scalar function $\gamma(t, s)$ continuous for $t \in R$, $s \in [0, r]$, $\gamma(t, 0) = 0$, such that if $L(t)\phi = \int_{-r}^{0} d\eta(t, \theta)\,\phi(\theta)$, then

$$\left| \lim_{h \to 0^+} \int_{-r+h}^{-r+s} d\eta(t, \theta)\,\phi(\theta) \right| \leqslant \gamma(t, s) \| \phi \|$$

for any $t \in R$, $0 < s \leqslant r$. If the matrix $A(t; L) = \eta(t, -r^+) - \eta(t, -r)$ is nonsingular at $t = \sigma$, one says that L is *atomic at* $-r$ *at* σ [2]. If $A(t; L)$ is nonsingular on a set $K \subset R$, one says L is *atomic* at $-r$ *on* K.

LEMMA 1.1. *If $L \in C^o(R, \mathfrak{A})$, then L has smoothness on the measure.*

Proof. If $L(t)\phi = \int_{-r}^{0} d\eta(t, \theta)\,\phi(\theta)$, then $\|L(t)\| \leqslant \| \eta(t, \cdot)\|$. Since $L \in C^o(R, \mathfrak{A})$, for any $t \in R$, $\epsilon > 0$, there is a $\delta > 0$ such that $\| \eta(t, \cdot) - \eta(\tau, \cdot)\| < \epsilon$ if $| t - \tau | < \delta$. This means that for any $[a, b] \subset I$ and any $i, j = 1, 2,..., n$,

$$V(\eta_{ij}(t, \cdot) - \eta_{ij}(\tau, \cdot); [a, b]) < \epsilon \qquad \text{if} \quad | t - \tau | < \delta. \tag{1.1}$$

For a fixed i and $0 < s \leqslant r$, let

$$\mu_i(t, s) = \sum_{j=1}^{n} V(\eta_{ij}(t, \cdot), [-r^+, -r + s]).$$

From (1.1), we have $\mu_i(t, s)$ is continuous in t uniformly with respect to s. Also, $\mu_i(t, s)$ is nondecreasing in s, $\mu_i(t, s) \to 0$ as $s \to 0$.

In R^2, consider the set $\{(s, y), y = \mu_i(t, s), s \in (0, r]\}$, and its closed convex hull $\Gamma(t)$. Let $\gamma_i(t, s) = \sup\{y: (s, y) \in \Gamma(t)\}$. Then $\gamma_i(t, s)$ is continuous in t uniformly with respect to s. Also, for each fixed t, it is continuous in s with $\gamma_i(t, s) \to 0$ as $s \to 0$. If we define $\gamma_i(t, 0) = 0$, then $\gamma_i(t, s)$ is jointly continuous in t, s. If $\gamma(t, s) = \max_i \gamma_i(t, s)$, then γ satisfies the conditions of the lemma and the proof is complete.

2. Linear Retarded Functional Differential Equations (LRFDE)

For any $L: R \to \mathfrak{A}$ a LRFDE for a function x taking a subset of R into R^n is defined as

$$\dot{x}(t) = L(t)\, x_t, \tag{2.1}$$

where, for each $t \in R$, x_t is an element of $C(I, R^n)$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in I$. A solution of (2.1) through $(\sigma, \phi) \in R \times C(I, R^n)$ is a continuous function x defined on an interval $[\sigma - r, \sigma + \beta)$, for some $\beta > 0$ such that $x_\sigma = \phi$ and x satisfies (2.1) on $[\sigma, \beta)$. A solution through (σ, ϕ) will be denoted by $x(\sigma, \phi)$ and we let $T(t, \sigma)\phi = x_t(\sigma, \phi)$. The operator $T(t, \sigma)$ will be called the *solution operator*.

The following lemma is a special case of [2, Theorem 6.1].

Lemma 2.1. *If $L \in C^0(R, \mathfrak{A})$ is atomic at $-r$ at σ and there is an $0 < \alpha < r$ such that $\dot{\phi}(\theta)$ is continuous for $\theta \in [-\alpha, 0]$, $\dot{\phi}(0) = L(\sigma, \phi)$, then there is an $\bar{\alpha}$, $0 < \bar{\alpha} \leqslant \alpha$ and a unique solution x of* (2.1) *on* $[\sigma - r - \bar{\alpha}, \sigma]$ *through* (σ, ϕ).

The meaning of this lemma is the unique backward continuation of the solution of (2.1) defined by the initial condition (σ, ϕ). It is easy to see that if L is atomic at $-r$ for any $t \in R$, then the solution operator $T(t, \sigma)$ is one-to-one.

Our objective is to understand what happens when $\det A(t; L) = 0$ for some t. As we shall see below, the set of $L \in C^1(R, \mathfrak{A})$ for which the solution operator is one-to-one is dense in $C^1(R, \mathfrak{A})$. To prove this result we need to make two types of approximations of L; the first one on the jump $A(t; L)$ and the second one on the measure η.

We shall also see that for any compact set $K \subset R$, the set of $L \in C^1(K, \mathfrak{A})$ for which the solution operator is one-to-one is not open. Here, $C^1(K, \mathfrak{A})$ is the space of C^1 functions from K to $\mathfrak{A}$ with the C^1 uniform topology.

3. Perturbation of the Jump Matrix

Let $\mathfrak{M}^n$ be the Banach space of dimension n^2 of all $n \times n$ real matrices with the usual norm. If $A(t, L)$ is the jump matrix of (2.1) with $L \in C^1(R, \mathfrak{A})$, then the mapping $t \to A(t, L)$ is a C^1-mapping from R to $\mathfrak{M}^n$. For any open set $\mathcal{O} \subset \mathfrak{M}^n$, consider $C^1(R, \mathcal{O})$ with the C^1 uniform topology on compact sets of R. The following result will be needed.

LEMMA 3.1. *For any integer $k \geqslant 0$, the set $L_k{}^n$ of all $n \times n$ matrices of rank k is a smooth submanifold of $\mathfrak{M}^n$ of dimension $n^2 - (n - k)^2$. Also, the set of all $n \times n$ matrices with zero determinant is the finite (closed) union $\bigcup_{0 \leqslant k \leqslant n-1} L_k{}^n$.*

Recall that a set $\mathfrak{R}$ in a topological space X is called residual if there exists a countable set S of open dense subsets $\mathfrak{R}_n \subset X$ such that $\mathfrak{R} \supset \bigcap_{n \in S} \mathfrak{R}_n$. Since $C^1(R, \mathcal{O})$ is a Baire space, any residual subset $\mathfrak{R}$ is dense. We say $A \in C^1(R, \mathfrak{M}^n)$ is transversal to $L_k{}^n$ if, whenever the curve $\dot{A}(t)$ intersects $L_k{}^n$, then $A(t)$ and $L_k{}^n$ span the whole space at the point of intersection. The following special case of Thom's transversality theorem [3] is needed.

LEMMA 3.2. *For any integer k the set of $A \in C^1(R, \mathfrak{M}^n)$ such that A is transversal to $L_k{}^n$ is a residual subset of $C^1(R, \mathfrak{M}^n)$.*

Let $D = \det\colon \mathfrak{M}^n \to R$ be the determinant function. A regular point $M \in \mathfrak{M}^n$ is a matrix where the derivative $D'(M)$ is surjective. Let $\mathfrak{M}^n_{n-1} = \{A \in M^n\colon \operatorname{rank} A \geqslant n - 1\}$. It is clear that $\mathfrak{M}^n_{n-1}$ is open in $\mathfrak{M}^n$ since the determinant function is continuous in the coefficients of the matrix.

LEMMA 3.3. *The set of regular points of D is the open set $\mathfrak{M}^n_{n-1}$.*

Since the finite intersection of residual sets is a residual set, we may apply Lemma 3.2 $(n - 1)$-times to obtain a residual set in $C^1(R, \mathfrak{M}^n)$ such that any A in this set is transversal to $L_0{}^n, L_1{}^n, \ldots, L^n_{n-1}$. Since $\dim L_k{}^n = n^2 - (n - k)^2$, it follows that $A(t) \notin L_k{}^n$, $0 \leqslant k \leqslant n - 2$, for any $t \in R$. Therefore, $A(t) \in \mathfrak{M}^n_{n-1}$ for all t and A is transversal to the manifold L^n_{n-1}. Then, by Lemma 3.3, the graph of the determinant map $t \to (t, D(A(t)))$ is transversal in R^2 to the t-axis.

We may now prove

PROPOSITION 3.1. *For any $L \in C^1(R, \mathfrak{A})$, $\epsilon > 0$, there is an $\tilde{L} \in C^1(R, \mathfrak{A})$, $\| \tilde{L} - L \| < \epsilon$ such that the jump matrix $A(t; \tilde{L})$ of $\tilde{L}$ is in $\mathfrak{M}^n_{n-1}$ for all t, $A(\cdot, \tilde{L})$ is transversal to L^n_{n-1}, and the set of $t \in R$ such that $A(t; \tilde{L}) \in L^n_{n-1}$ is a zero dimensional submanifold containing only a finite number of points in each compact set.*

Proof. From the remarks preceding the statement of the proposition there is an $\bar{A}$ as close to $A(\cdot; L)$ as desired in $C^1(R, \mathfrak{M}^n)$ such that $\bar{A}$ satisfies all the properties stated in the proposition. Let $B(t) = \bar{A}(t) - A(t; L)$ and $\tilde{L}(t)\phi =$

$L(t)\phi + B(t)\phi(-r)$. Then $A(t, \tilde{L}) = \bar{A}(t)$. If $\bar{A}$ is close to $A(\cdot; L)$ in $C^1(R, \mathfrak{M}^n)$, then it is easy to see that $\tilde{L}$ is close to L in $C^1(R, \mathfrak{A})$, and the proof is complete.

As a corollary to the proof of the above result, we have

COROLLARY 3.1. *Let* $\mathscr{L} = \{L \in C^1(R, \mathfrak{A})$: *there is a countable set* U *of points with no finite accumulation point such that for any* $t \notin U$, *there is an* $\epsilon > 0$ *with* $T(t, t - \epsilon; L)$ *one-to-one*$\}$. *Then* $\mathscr{L}$ *is a residual set in* $C^1(R, \mathfrak{A})$.

This remark is interesting for the following reason. Consider the subset $\mathscr{D}(R) \subset C^1(R, \mathfrak{A})$ with the following property. For any $L \in \mathscr{D}(R)$, there is an integer N such that the corresponding measure $\eta(t, \theta)$ has at most N discontinuities in θ for all $t \in R$ and, in addition, $\eta(t, \theta)$ is a step function in θ. For any compact set $K \subset R$, $\mathscr{D}(K)$ is defined in a similar way.

COROLLARY 3.2. *The set of* L *in* $\mathscr{D}(R)$ *for which the solution operator is one-to-one is residual. Also, for any compact set* $K \subset R$, *the corresponding set is open in* $\mathscr{D}(K)$.

Proof. The remarks preceding Proposition 3.1 imply that there is a residual set of $\bar{A} \in C^1(R, \mathfrak{M}^n)$ such that $\bar{A}(t) \in \mathfrak{M}^n_{n-1}$ for all t, $\bar{A}$ is transversal to L^n_{n-1} and the set of $t \in R$ such that $\bar{A}(t) \in L^n_{n-1}$ is a countable set U containing only a finite number of points in each compact set. Let L be any element of $\mathscr{D}(R)$ with $A(t; L) = \bar{A}(t)$. The set of all such L is a residual set in $\mathscr{D}(R)$. Also, for any $t \notin U$, there is an $\epsilon > 0$ such that $T(t, t - \epsilon; L)$ is one-to-one.

To discuss $t \in U$, we need some notation. Let $\theta_{ij}(t)$ be any point of discontinuity of $\eta_{ij}(t, \theta)$ with jump $\xi_{ij}(t) \neq 0$. We wish to show there is an open interval containing t such that each $\theta_{ij}(t)$ is constant in this interval. In the proof of Lemma 1.1, it was shown that for any $\epsilon > 0$, there is a $\delta > 0$ such that for any closed interval $J \subset I$ and containing $\theta_{ij}(t)$,

$$V(\eta_{ij}(t, \cdot) - \eta_{ij}(\tau, \cdot); J) < \epsilon$$

if $| t - \tau | < \delta$. Choose $\epsilon < | \xi_{ij}(t)|$ and suppose there is a τ with $| t - \tau | < \delta$ such that $\eta_{ij}(\tau, \theta)$ is continuous at $\theta = \theta_{ij}(t)$. Then there exists a J such that $\eta_{ij}(\tau, \theta)$ is continuous in θ on J. But, then

$$V((\eta_{ij}(t, \cdot) - \eta_{ij}(\tau, \cdot)); J) = | \xi_{ij}(t)| > \epsilon,$$

which is a contradiction. Consequently, $\theta_{ij}(\tau)$ is constant for $| t - \tau | < \delta$.

Returning to $\eta(t, \theta)$, this means there are a finite number of numbers $0 \leqslant r_1 < r_2 < \cdots < r_k = r$, $k \leqslant N$, such that the LRFDE is given by

$$\dot{x}(\tau) = \sum_{j=1}^{k-1} B_j(\tau)\, x(\tau - r_j) + \bar{A}(\tau)\, x(\tau - r) \tag{3.1}$$

for $| t - \tau | < \delta$, where the B_j are C^1 matrix functions.

From the properties of $\bar{A}(\tau)$, there is δ_1, $0 < \delta_1 < \delta$, such that $\bar{A}(\tau)$ is nonsingular for $t - \delta_1 \leqslant \tau < t$. Therefore, the solution of (3.1) is given by

$$x(\tau - r) = \bar{A}^{-1}(\tau) \left[\dot{x}(\tau) - \sum_{j=1}^{k-1} B_j(\tau)\, x(\tau - r_j)\right].$$

This shows there is a unique backward solution of this equation on $t - \delta_1 \leqslant \tau \leqslant t$ corresponding to the initial data $(t, 0)$. Since the system is linear, it is sufficient to consider such initial data. Therefore, $T(t, \sigma; L)$ is one-to-one even when $t \in U$. This completes the proof of the first part of the corollary.

To prove the second part, one repeats the complete argument using the fact that Thom's transversality theorem gives the openess for compact K since the union $\bigcup_{0 \leqslant k \leqslant n-1} L_k{}^n$ is closed by Lemma 3.1.

To complete this section, we give an example to show that the above perturbation on the jump matrix is not enough to guarantee uniqueness of backward continuation.

EXAMPLE 3.1. Consider the scalar equation

$$\dot{x}(t) = -(t/2)\, x(t-1) - \int_{-1}^{0} x(t+\theta)\, d\theta.$$

In terms of our general notation $n = 1$, $r = 1$,

$$L(t)\phi = -(t/2)\, \phi(-1) - \int_{-1}^{0} \phi(\theta)\, d\theta,$$

$A(t; L) = -t/2$. The function $A(t; L)$ is nonzero except at $t = 0$. Choose a point $t_0 = -\epsilon$, $\epsilon > 0$, and consider the function $z(t) = -t - 1$ for $t \in [-1 - \epsilon, -1]$ and $z(t) = 0$ for $t \in [-1, -\epsilon]$. The function $x(\tau) = z(\tau)$ for $\tau \in [-1 - \epsilon, -\epsilon]$ and $x(\tau) = 0$ for $\tau \in [-\epsilon, 0]$ is a solution of the given equation. In fact, one observes that

$$\begin{aligned} 0 &= -(t/2)\, x(t-1) - \int_{-1}^{0} x(t+\theta)\, d\theta \\ &= -(t/2)(-t) - \int_{-1}^{-1-t} (-t - \theta - 1)\, d\theta \\ &= (t^2/2) + \int_{0}^{t} (-t + \xi)\, d\xi. \end{aligned}$$

Thus, $x(\tau)$ satisfies the equation and $x(\tau) \equiv 0$ for $\tau \geqslant 0$. This shows that we do not have uniqueness of the backward extension with initial data $(\sigma, \phi) = (0, 0)$. In fact, $x(\tau)$ and the zero function are two distinct solutions.

The phenomena in Example 3.1 have been independently observed by Lillo [4], where a more complete discussion of the backward continuation problem is also given.

4. The Main Result

Our purpose in this section is to prove

THEOREM 4.1. *Suppose $L \in C^1(R, \mathfrak{A})$ is given. Then, in any neighborhood of L in $C^1(R, \mathfrak{A})$, there is an $\tilde{L}$ such that the solution operator $T(t, \sigma; L)$ of the* LRFDE $\dot{x}(t) = L(t)\, x_t$ *is one-to-one.*

Proof. If $L(t)\phi = \int_{-r}^{0} d\eta(t, \theta)\phi(\theta)$, choose $0 < \rho < r$ and define $L_\rho \in C^1(R, \mathfrak{A})$ by

$$L_\rho(t)\phi = A(t; L)\, \phi(-r) + \int_{-r+\rho}^{0} d\eta(t, \theta)\, \phi(\theta).$$

Our first task is to show that L_ρ is close to L if ρ is sufficiently small.

From Lemma 1.1, both L and $\dot{L}$ have smoothness on the measure. Consequently, there are $\gamma(t, \rho)$, $\bar{\gamma}(t, \rho)$ continuous for $t \in R$, $0 \leqslant \rho \leqslant r$, vanishing at $\rho = 0$ such that for any $\rho > 0$, $\phi \in C(I, R^n)$,

$$\left| \lim_{h \to 0} \int_{-r+h}^{-r+\rho} d\eta(t, \theta)\, \phi(\theta) \right| \leqslant \gamma(t, \rho)\, \| \phi \|,$$

$$\left| \lim_{h \to 0} \int_{-r+h}^{-r+\rho} d\dot{\eta}(t, \theta)\, \phi(\theta) \right| \leqslant \bar{\gamma}(t, \rho)\, \| \phi \|.$$

On the other hand, these expressions show that

$$\| L_\rho(t) - L(t) \| + \| \dot{L}_\rho(t) - \dot{L}(t) \| \leqslant \gamma(t, \rho) + \bar{\gamma}(t, \rho)$$

for all t, ρ.

Suppose K is a compact set in R and $\epsilon > 0$ is given. Then there is a $\rho_0 > 0$ such that $\gamma(t, \rho) + \bar{\gamma}(t, \rho) < \epsilon$ if $t \in K$, $0 \leqslant \rho < \rho_0$. The estimate above shows that L_ρ belongs to the (K, ϵ) neighborhood of L if $0 \leqslant \rho < \rho_0$. This shows L_ρ is close to L if ρ is sufficiently small.

From Proposition 1, we may assume $A(t; L)$ has a countable number of points where $D(A(t; L)) = 0$ and these points have no finite accumulation point. Our next objective is to show that under these conditions, $T(t, \sigma; L_\rho)$

is one-to-one for $0 < \rho \leqslant r$. The reason for this is basically the same as stated in Corollary 3.2, but we give the complete proof.

Consider the equation

$$\dot{x}(t) = L_\rho(t)\, x_t = A(t; L)\, x(t-r) + \int_{-r+\rho}^{0} d\eta(t, \theta)\, x(t+\theta).$$

Since the system is linear, we may assume the initial data (σ, ϕ) are $(\sigma, 0)$. From the properties of $A(t; L)$, there is a $\delta > 0$ such that $A(t, L)$ is nonsingular for $\sigma - \delta \leqslant t < \sigma$. Therefore, the solution of the above equation is

$$x(t-r) = A^{-1}(t; L)\left[\dot{x}(t) - \int_{-r+\rho}^{0} d\eta(t, \theta)\, x(t+\theta)\right].$$

Since $-r + \rho \leqslant \theta \leqslant 0$ and $\sigma - \delta \leqslant t < \sigma$, we have $(\sigma - r) + (\rho - \delta) \leqslant t + \theta < \sigma$. This shows that $x(t - r) = 0$ for $\sigma - \delta \leqslant t < \sigma$ and the solution through $(\sigma, 0)$ has a unique backward extension. This completes the proof of the theorem.

5. A Counterexample

Theorem 4.1 shows that the set of equations for which the solution operator is one-to-one is dense in $C^1(R, \mathfrak{A})$. We are not able to prove or disprove that it is residual. The purpose of this section is to show that even on a compact set $K \subset R$, the set of one-to-one maps is not open in $C^1(K, \mathfrak{A})$, in contrast to the situation encountered in the proof of Corollary 3.2.

To see this, suppose $n = 1$, $r = 1$, and consider the equation

$$\dot{x}(t) = -(t/2)\, x(t-1) + \int_{-1^+}^{0} d\eta(t, \theta)\, x(t+\theta).$$

As we have seen in the proof of Theorem 4.1, we can approximate this system by

$$\dot{y}(t) = -(t/2)\, y(t-1) + \int_{-1+\rho}^{0} d\eta(t, \theta)\, y(t+\theta),$$

with $\rho > 0$ such that the solution operator is one-to-one.

We now show that we can find another equation arbitrarily close to this one for which the solution operator is not one-to-one. The approximates can be made to involve θ only in the interval $[-1, -1 + \rho/2]$ so that the unperturbed equation may be taken as

$$\dot{y}(t) = -(t/2)\, y(t-1) \overset{\text{def}}{=} L(t)\, y_t\,.$$

Choose a sequence of positive $\epsilon_j \to 0$ as $j \to \infty$ and $0 < \delta_j < \epsilon_j$. Define a C^1-function $\alpha_j: R \to R$ so that α_j is nonincreasing on $(-\infty, -1]$, nondecreasing on $(-1, \infty)$ such that $\alpha_j(s) = 0$ for $-\infty < s \leqslant -1 - \epsilon_j$, $\alpha_j(s) = -\epsilon_j$ for $-1 - \delta_j \leqslant s \leqslant -1$, $\alpha_j(s) = 0$ for $-1 + \epsilon_j - \delta_j \leqslant s < \infty$, and $\int_{-1}^{0} |\alpha_j(t + \theta)| \, d\theta \leqslant \epsilon_j$ for all $t \in R$. Now consider the equation

$$\begin{aligned} \dot{z}(t) &= -(t/2)\, z(t-1) + \int_{-1}^{0} \alpha_j(t+\theta)\, z(t+\theta)\, d\theta \\ &= -(t/2)\, z(t-1) + \int_{t-1}^{t} \alpha_j(s)\, z(s)\, ds \stackrel{\text{def}}{=} L_j(t)\, z_t\,. \end{aligned}$$

Consider the functions $\psi_j(t) = (-t-1)\,\epsilon_j$ for $t \in [-\delta_j - 1, -1]$, $\psi_j(t) = 0$ for $t \in [-1, -\delta_j]$. Let $\sigma_j = -\delta_j$ and consider the initial value problem (σ_j, ϕ_j) with $\phi_j = \psi_{j_{\sigma_j}}$. If $t \in [-\delta_j, 0]$, then

$$\begin{aligned} \dot{z}(t) &= -(t/2)(-t\epsilon_j) - \int_{t-1}^{t} \epsilon_j z(s)\, ds \\ &= \left[(t^2/2) - \int_{t-1}^{-1} (-s-1)\, ds\right] \epsilon_j \\ &= 0. \end{aligned}$$

Consequently, $z(t) = 0$ on $[-\delta_j, 0]$ and $z_o = 0$. Since $z = 0$ is also a solution, we see that the solution operator corresponding to α_j is not one-to-one for any j. This proves the assertion, since $\|L(t) - L_j(t)\| < \epsilon$ for all t and j large enough.

References

1. F. Riesz and B. Sz-Nagy, "Functional Analysis," Ungar, New York, 1955.
2. J. K. Hale, "Functional Differential Equations," Springer-Verlag, New York/Berlin, 1971.
3. R. Thom, Quelques properties globales des varietes differentiables, *Comment. Math. Helv.* **28** (1954), 71–86.
4. J. Lillo, Backward continuation of retarded functional differential equations, *J. Differential Equations*, to appear.