RT-MAP-8107

EMBEDDING OF TREES

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A sequence $P=(v_1,v_2,\ldots,v_n)$ of distinct vertices of VG is called a *path* if $v_iv_{i+1} \in EG$ for all $1 \le i \le n-1$. The *length* of this path is n-1. If, furthermore, $v_nv_1 \in EG$ then we have a *cycle* of length n which will be denoted by $[v_1,v_2,\ldots,v_n]$. A cycle of length $|VG|$ is called *hamiltonian*.

A vertex v of a graph G is called a *cut-vertex* of G if the number of components of the graph obtained from G by removing the vertex v, is greater than the number of components of G.

An *isomorphism* between two graphs G and G' is a one-to-one mapping ϕ: VG $\longrightarrow$ VG' such that $xy \in EG$ if and only if $\phi(x)\phi(y) \in EG'$. If ϕ is an isomorphism between G and a subgraph of G' then we say that ϕ is an *embedding* of G in G'. In this case G is said to be *embeddable* in G'.

The following result, which can be easily proved, will be used in the next two sections.

LEMMA 1 - *If* G *is a graph with* $|EG| > |VG|(k-1)/2$ *then* G *has a vertex* v *such that* $d(v) \ge k$.

SECTION 1

In this section we prove the validity of the conjecture by Erdős and Sós [1] when $k = |VG| - 1$.

The following definition and lemma will be used in the next theorem.

Given a subgraph T' of a tree T, we say that T' is a *pending tree* of T if $|VT'| = 3$, $|ET'| = 2$ and there is a vertex $v \in VT'$ such that any edge in $ET \setminus ET'$ incident to VT' is incident to v. In some cases we use the notation v-*pending tree*.

LEMMA 2 - *If* T *is a tree with at least 2 edges then* T *has a pending tree.*

PROOF - Let $P=(v_1,v_2,\ldots,v_n)$ be a path of maximum length in T. Clearly, $n \geq 3$. If $d(v_{n-1}) = 2$ then

$$T' = (\{v_{n-2},v_{n-1},v_n\},\{v_{n-2}v_{n-1},v_{n-1}v_n\})$$

is a pending tree of T. Otherwise, there is a vertex $x \in VT, x \neq v_i$ $(1 \leq i \leq n)$ such that $v_{n-1}x \in ET$.

Observe that $d(x)=1$, otherwise T would have a path of length greater than $n-1$, a contradiction. In this case,

$$T' = (\{v_{n-1},x,v_n\},\{v_{n-1}x,v_{n-1}v_n\})$$

is a pending tree of T.

∎

THEOREM 1 - *Let* G *be a graph with* $|VG| = k+1$ *and* T *a tree with* k *edges. If* $|EG| > |VG|(k-1)/2$ *then* T *is embeddable in* G.

PROOF - By induction on k.

If G is a complete graph the result is obvious. Suppose therefore that G is not complete.

It is easy to see that the theorem is true for $k \leq 2$.

By the induction hypothesis, given a tree $\hat{T}$ with j edges, $j < k$, and a graph $\hat{G}$ such that

$$|V\hat{G}| = j+1 \quad \text{and} \quad |E\hat{G}| > |V\hat{G}|(j-1)/2,$$

$\tilde{T}$ is embeddable in $\hat{G}$.

Let us define a graph G' and a tree T', from G and T, respectively.

(a) Definition of G':

Since G is not complete, there is a vertex $x \in VG$ such that $d(x) < k$.

By lemma 1, there is a vertex $y \in VG$ such that $d(y) \geq k$. As $d(v) \leq k$ for all $v \in VG$, it follows that $d(y)=k$.

Let G' be the graph obtained from G by removing the vertices x and y.

Observe that:

i) $|VG'| = k-1$

ii) $|EG'| = |EG| - d(x) - d(y) + 1 \geq |EG| - 2k + 2 > (k-1)(k-3)/2$
$= |VG'|(k-3)/2$

(b) Definition of T':

By lemma 2, T has a v-pending tree $\tilde{T}$.

Let T' be the tree obtained from T by removing the vertices of $V\tilde{T} \setminus \{v\}$.

Observe that:

i) $|VT'| = k-1$

ii) $|ET'| = k-2$

By the induction hypothesis, there is an embedding ϕ' of T' in G'.

Let us define now an embedding ϕ of T in G.

First consider the case were $\tilde{T}$ = ({v,u,w},{vu,uw}).

Let ϕ: VT $\longrightarrow$ VG such that

$$\phi(z) = \begin{cases} y \text{ if } z=u \\ x \text{ if } z=w \\ \phi'(z), \text{ otherwise.} \end{cases}$$

Since ϕ' is an embedding of T' in G' and the edges xy and $\phi(v)y \in EG \setminus EG'$, it follows that ϕ is an embedding of T in G.

Consider now the case where $\tilde{T}$ = ({v,u,w},{vu,vw}).

Let ϕ: VT $\longrightarrow$ VG such that:

$$\phi(z) = \begin{cases} y \text{ if } z = v \\ x \text{ if } z = w \\ \phi'(v) \text{ if } z = u \\ \phi'(z) \text{ otherwise.} \end{cases}$$

As d(y)=k, the restriction of ϕ to VT' is an embedding of T' in the subgraph obtained from G by removing the vertices x and $\phi'(v)$.

Since the edges yx and $y\phi'(v) \in EG \setminus EG'$, it follows that ϕ is an embedding of T in G.

■

SECTION 2

In this section we prove that under the assumptions of Erdös and Sós'conjecture, G contains any tree with k edges that has a path of length greater than k-2.

First, we present two results due to G.A. Dirac [2], which will be used in the proof of the next proposition.

LEMMA 3 - *Let* G *be a graph. IF* $d(v) \geq k \geq 2$ *for all* $v \in VG$, *and if* $|VG| \leq 2k$, *then* G *has a hamiltonian cycle.*

LEMMA 4 - *Let* G *be a graph with no cut-vertex. If* $d(v) \geq k \geq 2$ *for all* $v \in VG$, *and if* $|VG| \geq 2k$ *then* G *contains a cycle of length at least* 2k.

The proposition below is a particular case of a result due to P. Erdös & T. Gallai [3].

PROPOSITION - *Let* $G=(VG,EG)$ *be a connected graph containing no cycle of length greater than* $k-1$, $k \geq 3$. *Then* $|EG| \leq (|VG|-1)(k-1)/2$.

PROOF - By induction on $|VG|$. If $|VG|=1$ the result is obvious. If $1 < |VG| < k$ then $|EG| \leq (|VG|-1)|VG|/2 = (|VG|-1)(k-1)/2$.

Suppose $|VG| \geq k$. We will consider two cases:

i) G has a cut-vertex.

Let $G'_1, G'_2, \ldots, G'_p$ $(p \geq 2)$ be the components obtained from G by removing a cut-vertex v. Let G_i, $1 \leq i \leq p$, be graphs such that:

$$VG_i = VG'_i \cup \{v\}$$

$$EG_i = EG'_i \cup \{wv \in EG \text{ such that } w \in VG'_i\}$$

Let $|VG_i| = n_i$, $1 \leq i \leq p$. Then $n_1+n_2+\ldots+n_p-p+1 = |VG|$ and

$n_i < |VG|$, $1 \le i \le p$. Since each G_i does not contain cycles of length greater than k-1, by the induction hypothesis:

$$|EG| = |EG_1| + |EG_2| + \dots + |EG_p| \le$$

$$\le ((n_1-1) + (n_2-1) + \dots + (n_p-1))(k-1)/2 = (|VG|-1)(k-1)/2.$$

ii) G has no cut-vertex.

We show that G has a vertex v such that $d(v) \le (k-1)/2$. Suppose that such a vertex does not exist. We have the cases:

1) $k = 2j$

In this case $d(x) \ge j$ for all $x \in VG$.

Since $|VG| \ge 2j$, applying lemma 4, we conclude that G has a cycle of length at least k, a contradiction.

2) $k = 2j+1$.

In this case $d(x) \ge j+1$ for all $x \in VG$.

Since $|VG| \ge 2j+1$ either $|VG| = 2j+1$ or $|VG| \ge 2j+2$. In both cases we obtain a contradiction, using lemma 3 and lemma 4, respectively.

Let G_1 be the graph obtained from G by removing the vertex v.

Since G_1 does not contain cycles of length greater than k-1 by the induction hypothesis, $|EG_1| \le (|VG|-2)(k-1)/2$. Thus

$$|EG| = d(v) + |EG_1| \le (k-1)/2 + (|VG|-2)(k-1)/2 = (|VG|-1)(k-1)/2.$$

The proof of the proposition is concluded.

■

COROLLARY - *Given a connected graph* G, *if* $|EG| > |VG|(k-1)/2$, $k \ge 3$, *then* G *contains a cycle of length at least k.*

THEOREM 2- *Let* G *be a graph such that* $|EG| > |VG|(k-1)/2$. *Then* G *contains any tree with* k *edges that has a path of length greater than* k-2.

PROOF - By contradiction.

Let G' be a graph with the smallest number of vertices such that $|EG'| > |VG'|(k-1)/2$ and G' does not satisfy the theorem.

Suppose G' is not connected. Let $G'_1, G'_2, \ldots, G'_p$, $p \geq 2$, be the components of G'. If $|EG'_i| \leq |VG'_i|(k-1)/2$ for all $1 \leq i \leq p$, then $|EG'| \leq |VG'|(k-1)/2$. Therefore for some i, $1 \leq i \leq p$, $|EG'_i| > |VG'_i|(k-1)/2$. But this contradicts the choice of G'. Thus G' is connected.

If $0 \leq k < 3$ the theorem is trivially true. Suppose therefore that $k \geq 3$. By the corollary, G' contains a cycle $C = [v_1, v_2, \ldots v_n]$ such that $n \geq k$.

Case 1- C *is not hamiltonian.*

Since G' is connected, there is a vertex $w \in VG'$ not in C, w adjacent to a vertex of C. In this case, it is immediate that G' contains any tree with k edges that has a path of length greater than k-2. Contradiction.

Case 2- C *is hamiltonian.*

By lemma 1, C contains a vertex v such that $d(v) \geq k$. But then $n \geq k+1$.

Let T be a tree with k edges that has a path of length greater than k-2.

If T is a path, clearly T is embeddable in G', a contradiction.

If T is not a path then T has a path $P = (x_1, x_2, \ldots, x_k)$ which contains a vertex x_i, $1 < i < k$, such that $d(x_i) = 3$.

It is easy to find an embedding ϕ' of P in C such that $\phi'(x_i) = v$.

Since $d(v) \geq k$, there is a vertex w in C which does not belong to the image of ϕ' and such that $wv \in EG'$.

Therefore, from ϕ' we can obtain an embedding ϕ of T in G', a contradiction.

■

REMARK - It is easy to verify that the conjecture is true for $k \leq 5$.

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