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Orlando Stanley Juriaans

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Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20570 - Ag. Jardim Paulistano
01452-990 - São Paulo - Brasil

Abstract

Several special cases of the conjectures of Bovdi and Zassenhaus are proved. We also deal with special cases of the following conjecture: let α be a torsion unit of the integral group ring $\mathbb{Z}G$ and m the smallest positive integer such that $\alpha^m \in G$; then, m is a divisor of the expoent of the quotient group $G/Z(G)$ provided this expoent is finite.

Introduction

Let G be a group and let $V\mathbb{Z}G$ be the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. Given an element $x = \sum x(g)g \in \mathbb{Z}G$ we set

$$T^{(k)}(x) = \sum_{g \in G(k)} x(g) \ ,$$

called the k-generalized trace of x. Here $G(k) = \{g \in G : o(g) = k\}$. We also set

$$\tilde{x}(g) = \sum_{h \sim g} x(h) \ ,$$

where $\sim$ denotes conjugacy.

Bovdi proved the following [2, Lemma 1.1]:

Lemma 1: If p is a prime, $x \in V\mathbb{Z}G$ and $o(x) = p^n$, then $T^{(p^n)}(x) \equiv 1 (\mathrm{mod}\ p)$ and $T^{(p^i)}(x) \equiv 0 (\mathrm{mod}\ p)$ for $i < n$.

Also, he conjectured that actually

$$T^{(p^n)}(x) = 1 \quad \text{and} \quad T^{(p^i)}(x) = 0 \quad \text{for} \quad i < n\ .$$

In [2] Bovdi's Conjecture is proved for nilpotent metabelian groups.

I - The conjecture of Bovdi

Let G be a group and m, n positive integers. We shall say that G is (m, n)-absorvent if the subgroup $H = \langle g \in G : o(g) | m^n \rangle$ has expoent less than or equal to m^n. If G is (m, n)-absorvent for all pairs (m, n) then G is called absorvent.

1.1. Lemma

a. If G is absorvent then $T(G)$ is a subgroup of G.

b. The following groups are absorvent:

i) Abelian groups;

ii) Regular p-groups;

iii) Q_{2^n}, the generalized quaternion group of order 2^n.

Proof

a. Let $g, h \in T(G)$, $o(g) = m$, $o(h) = n$ and $k = mn$. Since G is $(k, 1)$ absorvent, we see that $gh \in T(G)$.

b. Part (i) is obvious, part (ii) is a consequence of [9, Th. 3.14] and (iii) is a consequence of [1, 5.3.5]. □

Our main results in this section are the following.

Theorem A: Bovdi's conjecture holds for any solvable group G such that every sylow subgroup of G is abelian.

Theorem B: Let G be a finite group such that, for every prime p, if $p \mid |G|$ then $p^4 \nmid |G|$. Then, Bovdi's Conjecture holds for G. □

We start by proving the following

1.2. Lemma: Let G be a finite group, p a prime and $T \lhd G$ a p'-group, i.e., $p \nmid |T|$. Let $\psi : \mathbb{Z}G \to \mathbb{Z}G/T$ be the natural projection; let $\alpha \in V\mathbb{Z}G$ be such that $o(\alpha) = p^n$ and set $\beta = \psi(\alpha)$. Then $T^{(p^j)}(\alpha) = T^{(p^j)}(\beta)$.

Proof: Set:

$$\begin{aligned} S &= \{g \in G : o(g) = p^k m \ , \ \ p \nmid m \ , \ \ g^{p^k} \in T\} \\ S_1 &= \{g \in S : o(g) = p^k m \ , \ \ m \neq 1\} \end{aligned}$$

Note that, if $g \in G$ is a p-element, then $o(\psi(g)) = o(g)$. If $g \in G$ is not a p-element then $\tilde{\alpha}(g) = 0$ by [5, Th. 7]. Using these facts, we have that:

$$T^{(p^j)}(\beta) = \sum_{o(\psi(g))=p^j} \alpha(g) = \sum_{g\in S} \alpha(g) = \sum_{o(g)=p^j} \alpha(g) + \sum_{g\in S_1} \alpha(g) = \sum_{o(g)=p^j} \alpha(g) = T^{(p^j)}(\alpha)$$

□

We shall also need the following result.

1.3. Lemma: Let H be an abelian p-sylow subgroup of a finite solvable group G. Then one of the following holds:

i) $H \lhd G$;

ii) G has a normal p'-subgroup.

Proof: Suppose H is not normal in G. In this case, Fit (G) is not a p-group. For, if this was so, then by [7, 5.4.4], we would have $H \leq C_G(\mathrm{Fit}\,(G)) = Z(\mathrm{Fit}\,(G)) \leq H$ and hence $H = \mathrm{Fit}\,(G)$ would be normal in G.

Now choose a prime $q \neq p$ and set $T \in \mathrm{SYL}_q(\mathrm{Fit}\,(G))$. □

1.4. Lemma: Let G be a finite group and $\alpha \in V\mathbb{Z}G$ an element such that $o(\alpha) = p^n$. If G is (p,k)-absorvent for all $k \leq n$ then

$$T^{(p^j)}(\alpha) = \delta_{n_j} .$$

Proof: Let $H_k = \langle g \in G : o(g) | p^k \rangle$. Then $H_k \lhd G$ and $H_k = \{g \in G : o(g) | p^k\}$ since G is (p,k)-absorvent.

Consider the projection $\mathbb{Z}G \to \mathbb{Z}G/H_k$ and let β be the image of α. Then, by Berman's Theorem [10, III-1.3] $\beta(1) \in \{0,1\}$. But $\beta(1) = \sum_{g \in H_k} \alpha(g) = \sum_{0 \leq j \leq k} T^{(p^j)}(\alpha)$. Hence $\sum_{0 \leq j \leq k} T(\alpha)^{(p^j)} \in \{0,1\}$ for all $0 \leq k \leq n$.

So there is a unique index j_0, $0 \leq j_0 \leq n$, such that $T^{(p^{j_0})}(\alpha) \neq 0$. By Lemma 1 $j_0 = n$.

□

As a consequence we have

Corollary: Bovdi's conjecture holds for groups which are absorvent.

Theorema A is a consequence of the following result.

Theorem C: Let G be a finite solvable group and $\alpha \in V\mathbb{Z}G$ an element of order p^n. Suppose that there exists a subgroup $H \in \mathrm{syl}_p(G)$ which is abelian. Then $T^{(p^j)}(\alpha) = \delta_{n_j}$.

Proof: Let G be a least counterexample. We shall derive a contradiction.

By Lemma 1.3, either $H \lhd G$ or G has a normal p'-subgroup. Note that if $H \lhd G$ then, since H is an abelian p-sylow subgroup, G is (p,k)-absorvent for all $k \leq n$. Hence, by Lemma 1.4, we have that $T^{(p')}(\alpha) = \delta_{n_j}$.

If G has a normal p'-subgroup then by the minimality of G and Lemma 2 we have $T^{(p')}(\alpha) = \delta_{n_j}$. So we have obtained a contradiction. □

Theorem B is a consequence of the following result.

Theorem D: Let G be a finite solvable group and denote by L the smallest non trivial term of the lower central series of G. Suppose that if p is a prime such that $p \mid |L|$ then $p^4 \nmid |G|$. Then, Bovdi's Conjecture holds for G.

Proof: Let G be a least counter example and assume that $\alpha \in V\mathbb{Z}G$ with $o(\alpha) = p^n$ is an element which does not satisfy Bovdi's Conjecture. Let $F = \mathrm{Fit}\,(G)$ be the Fitting subgroup of G. Then by Lemma 1.2, F is a p-group. Also $p \mid |L|$ by [7, 5.4.4] and Lemma 1.2. Hence, by hypoteses, $p^4 \nmid |G|$. In fact, we claim that $p^3 \mid |G|$. For, if $p^3 \nmid |G|$, then the p-Sylow subgroups of G are abelian and Theorem C would now give a contradiction.

Let $H \in \mathrm{syl}_p(G)$ be normal. Then, by [7, 5.3.5] and Lemmas 1.1 and 1.4, we must have $p = 2$ and $H = D_4$, which is not absorvent. Since F is a 2-group we have, by [7, 5.4.4], that $C_G(H) = Z(H)$, the centre of H which is isomorphic to C_2. So, by [8, 3.2.3], G/C_2 has a monomorphic image in $\mathrm{Aut}\,(D_4)$. Now, by [7, p. 141], $\mathrm{Aut}\,(D_4)$ is a 2-group and hence G is nilpotent. A contradiction since Zassenhaus' Conjecture holds for nilpotent groups [11, Th. 1].

So, we may assume that H is not normal and hence $|F| = p$ or p^2. If $|F| = p$, then, by [7, 5.4.4], we have that $F \subset Z(H)$ and hence $H \subset C_G(F)$, which is a contradiction by [7, 5.4.4]. So, we may also assume $|F| = p^2$. Let $\mathrm{Frat}\,(G)$ be the Frattini subgroups of G; then $\mathrm{Fit}\left(\frac{G}{\mathrm{Frat}\,G}\right) = \frac{\mathrm{Fit}\,(G)}{\mathrm{Frat}\,G}$. Since G is solvable, $\mathrm{Frat}\,G < \mathrm{Fit}\,G$ and if $\mathrm{Frat}\,G \neq 1$ then $G/\mathrm{Frat}\,G$ would have abelian p-Sylow subgroups and so, by [7, 5.4.4], $\frac{\mathrm{Fit}\,G}{\mathrm{Frat}\,G}$ is a p-sylow and thus $\mathrm{Fit}\,G \in \mathrm{syl}_p(G)$ which is a contradiction. Hence $\mathrm{Frat}\,(G) = 1$. Now [7, 5.2.13] and [8, 9.3.7] imply that there exists a subgroup $Y < H$ such that $H = F \times Y$

Note that by [7, 5.2.15] we have that $F \cong C_p \times C_p$.

We shall discuss separately two cases.

Case 1. $p \neq 2$.

If $\exp(H) = p$, then we are done, since, in this case, $\alpha(1) + T^{(p)}_{(\alpha)} = 1$, by [5, Th. 7].

If $\exp(H) = p^2$ then $Y \subseteq F$. Contradicting the fact that $H = \times Y$.

Case 2. $p = 2$.

Since $F \cong C_2 \times C_2$ we must have $H = D_4$. By [8, 3.23], $\frac{G}{C_G(F)}$ has a monomorphic image in $\mathrm{Aut}\,(F) \cong S_3$. Hence, by [7, 5.4.4], we have that $|G| \bigm| 24$.

Since G can not be nilpotent we must have $|G| = 24$. Using the classification of these groups in [1, p. 160], we have that either G has a normal 3-Sylow subgroup or $G \cong S_4$. If G has a normal 3-sylow we can apply Lemma I.2. If $G \cong S_4$ then, by [3] Zassenhaus' Conjecture holds for G. This give us a final contradiction.

□

Our results can also be used to prove the following:

Corollary: Bovdi's Conjecture holds for the following groups:

1. Frobenius groups with an absorvent kernel;
2. Groups which contain a normal absorvent subgroup of prime power index;
3. Groups of the form $G = HX$, where $H \triangleleft G$ is absorvent, and $(|H|, |X|) = 1$.
4. Groups G such that G' is nilpotent and if $p \bigm| |G'|$ then a p-Sylow of G is absorvent.

Remark: The corollary is proved using Lemma I.2 and [5, 2.10].

The results still hold if we suppose that G is p-constrained for some prime p.

II - The Conjecture of Zassenhaus

This conjecture says that if $\alpha \in V\mathbb{Z}G$ is a torsion unit then α is conjugated in QG to an element of G. In [5] this is proved to be equivalent to the following: there is an element $g \in G$, unique up to conjugacy such that $\tilde{\alpha}(g) \neq 0$.

We shall prove the following results.

Theorem A: Let G be a finite group such that, for every $g \in G - Z(G)$, the set $g.G'$ is precisely the conjugacy class of g. Then the conjecture of Zassenhaus holds for G.

Theorem B: Let $G = \langle a \rangle \times X$, where $o(a) = p^n$, p is a prime integer and X is an abelian p'-group. Then, Zassenhaus' Conjecture holds for G.

We shall need the following preliminary result.

2.1. Lemma: Let $H \leq Z(G)$ and let $\psi : \mathbb{Z}G \to \mathbb{Z}(G/H)$ be the natural projection. Let $\alpha \in V\mathbb{Z}G$ be such that $(o(\alpha), |H|) = 1$, and set $\beta = \psi(\alpha)$. Then, $\alpha \sim g_0$, in QG, if and only if $\beta \sim \bar{g}_0$ in $Q(G/H)$.

Proof: We just have to prove the converse. Denote,

$$\begin{aligned} S_g &= \{t^{-1}gt\theta : \theta \in H \ , \ \ t \in G\} \\ S'_g &= \{t^{-1}gt\theta : \theta \in H \ , \ \ \theta \neq 1 \ , \ \ t \in G\} \end{aligned}$$

As we remarked at the beginning, it will suffice to show that $\tilde{\alpha}(g) \in \{0, 1\}$, for every $g \in G$.

We discuss two cases.

Case 1. $(o(g), |H|) \neq 1$

By our hypoteses, there is a prime $p | o(g)$ and $p \not| \, o(\alpha)$. Hence, by [5, Th. 2.7], we have that $\tilde{\alpha}(g) = 0$.

Case 2. $(o(g), |H|) = 1$.

Denote by $\bar{g} = \psi(g)$ and by $[\bar{g}]$ its conjugacy class. By [5, Th. 2.5] we have that $\tilde{\beta}(\bar{g}) \in \{0, 1\}$. Now $\bar{h} \in [\bar{g}]$ if and only if $h \in S_g$. Since $H \leq Z(G)$ $(o(g), |H|) = 1$ we have that $(o(h), |H|) = 1$ if and only if $h \in [g]$. Hence,

$$\tilde{\beta}(g) = \sum_{h \in S_g} \alpha(h) = \tilde{\alpha}(g) + \sum_{h \in S'_g} \alpha(h) = \tilde{\alpha}(g) \; .$$

Where the last equality follow from case 1 and the fact that S'_g is a normal subset.

Proof of Theorem A: Let $\alpha = \sum \alpha(g)g$. Choose $g_1, \ldots, g_n \in G$ such that $g_i g_j^{-1} \notin G'$, if $i \neq j$, and supp $(\alpha) \subseteq \bigcup_{i=1}^{t} g_i G'$. Then we can write $\alpha = \sum_i \sum_{t \in g_i G'} \alpha(t)t$.

Since G/G' is abelian, looking at the image of α in $\mathbb{Z}(G/G')$ we see, by [2, Lemma 2.8], that there is a unique $g_0 \in sup_p(\alpha)$ such that

$$\sum_{t \in g_0 G'} \alpha(t) \neq 0 \; .$$

By our hypotheses, $\sum_{t \in g_0 G'} \alpha(t) = \tilde{\alpha}(g_0)$. Hence there is a unique g_0, up to conjugacy, such that $\tilde{\alpha}(g_0) \neq 0$. □

Proof of Theorem B: Let G be a least counterexample and let $\alpha \in V\mathbb{Z}G$, $o(\alpha) = m$ be an element which is not conjugated, in QG, to a group element. We shall derive a contradiction.

By [5, Th. 2.10] we have that $p | o(\alpha)$. By Whitcomb Argument [10, III-5.3] there is a unique pair of elements $g \in \langle a \rangle$, $x \in X$, such that $\alpha - gx \in \Delta G \Delta G'$.

Since, $p | o(\alpha)$ [5, Lemma 5.1] implies that $x \in Z(G)$. Let $\beta = x^{-1}\alpha$, then $o(\beta) = o(g) = p^n$, for some $n \geq 0$.

We now discuss two cases.

Case 1. $X \cap Z(G) \neq 1$.

Choose $H \subset X \cap Z(G)$, $H \neq 1$. By the minimality of G and Lemma 2.1, $\beta \sim g$ in QG and so $\alpha \sim gx$ in QG.

Case 2. $X \cap Z(G) = 1$.

In this case, X has an isomorphic copy in $\mathrm{Aut}(\langle a \rangle)$ and hence X is cyclic. By [6] Zassenhaus' Conjecture holds.

So we have established a contradiction.

□

III - Another conjecture of Bovdi

The following conjecture, as far as we know, is also due to Bovdi.

Conjecture: Let $n = \exp(G/Z(G))$. If $\alpha \in V\mathbb{Z}G$ is a torsion unit and m is the smallest positive integer such that $\alpha^m \in G$, then m divides n.

Our principal result in this section is the following:

Theorem A: Let G be a finite group, $\alpha \in V\mathbb{Z}G$, a torsion unit, and $\psi : \mathbb{Z}G \rightarrow \mathbb{Z}(G/Z(G))$ the natural projection. Set $\beta = \psi(\alpha)$ and let m be the smallest positive integer such that $\alpha^m \in G$. If there exists an element $g \in G$ such that $o(\beta) = o(\psi(g))$, then $m | \exp(G/Z(G))$.

Proof: Let $k = o(\beta)$. Then, by hypotesis, $k | \exp(G/Z(G))$. Also, $\alpha^k - 1 \in \Delta(G, Z(G))$. So $\alpha^k = g + \theta$, $g \in Z(G)$, $\theta \in \Delta G \Delta Z(G)$. Hence, by Berman's Theorem, $\alpha^k = g \in G$. By the minimality of m we must have that $m|k$. Hence, $m| \exp(G/Z(G))$.

□

Remarks

1. By [3, Lemma 2.8], we may substitute finite by nilpotent assume also that $\exp(G/Z(G))$ is finite.

2. If G is metabelian but not finite, the conjecture holds if we assume that $G/Z(G)$ is finite and $U(1 + \Delta G \Delta ZG)$ is torsion free.

3. If $\exp(G/Z(G))$ is finite and $U(1+\Delta G \Delta ZG)$ is torsion free, then Bovdi's Conjecture implies this one.

The same proof applies to prove these remarks.

As a consequence we have

Corollary: The conjecture holds for:

1. Metabelian groups;
2. Nilpotent groups;
3. p-elements when $U(1 + \Delta G \Delta Z(G))$ is torsion free.

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