Nonlinear Differ. Equ. Appl. (2017) 24:27

1021-9722/17/030001-23
published online May 22, 2017
DOI 10.1007/s00030-017-0451-0

Nonlinear Differential Equations and Applications NoDEA

Stability of standing waves for NLS-log equation with δ-interaction

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Abstract. We study analytically the orbital stability of the standing waves with a peak-Gausson profile for a nonlinear logarithmic Schrödinger equation with δ-interaction (attractive and repulsive). A major difficulty is to compute the number of negative eigenvalues of the linearized operator around the standing wave. This is overcome by the perturbation method, the continuation arguments, and the theory of extensions of symmetric operators.

Mathematics Subject Classification. Primary 35Q51, 35J61, Secondary 47E05.

Keywords. Nonlinear Schrödinger equation, Orbital stability, Standing wave, δ-interaction, Self-adjoint extension, Deficiency indices, Semi-bounded operator.

1. Introduction

Bialynicki-Birula and Mycielski [14] built a model of nonlinear wave mechanics based on the following Schrödinger equation with a logarithmic non-linearity (NLS-log equation henceforth)

$$i\partial_t u + \Delta u + u\text{Log}|u|^2 = 0, \tag{1.1}$$

where $u = u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $n \geq 1$. This equation has been proposed in order to obtain a nonlinear equation which helped to quantify departures from the strictly linear regime, preserving in any number of dimensions some fundamental aspects of quantum mechanics, such as separability and additivity of total energy of noninteracting subsystems. The NLS-log equation admits applications to dissipative systems [32], quantum mechanics, quantum optics [15], nuclear physics [30], transport and diffusion phenomena (for example, magma

J. Angulo was supported partially by Grant CNPq/Brazil, N. Goloshchapova was supported by Fapesp under the Projects 2012/50503-6 and 2016/02060-9.

transport) [23], open quantum systems, effective quantum gravity, theory of superfluidity, and Bose–Einstein condensation (see [30,40] and the references therein). We refer to [16,18] for a study of existence and uniqueness of the solutions to the associated Cauchy problem in a suitable functional framework, as well as for a study of the asymptotic behavior of its solutions and their orbital stability.

In this paper we study the following nonlinear logarithmic Schrödinger equation with δ-interaction (NLS-log-δ henceforth) on the line

$$i\partial_t u - \mathcal{H}_\gamma^\delta u + u\mathrm{Log}|u|^2 = 0. \tag{1.2}$$

Here $u = u(t,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, $\gamma \in \mathbb{R}\backslash\{0\}$, and $\mathcal{H}_\gamma^\delta$ is the self-adjoint operator on $L^2(\mathbb{R})$ defined by

$$\begin{aligned} &\mathcal{H}_\gamma^\delta = -\frac{d^2}{dx^2},\\ &\mathrm{dom}(\mathcal{H}_\gamma^\delta) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) : f'(0+) - f'(0-) = -\gamma f(0)\}. \end{aligned} \tag{1.3}$$

The operator $\mathcal{H}_\gamma^\delta$ corresponds to the formal expression $l_\gamma^\delta = -\frac{d^2}{dx^2} - \gamma\delta$ (see [7] for details). Equation (1.2) can be viewed as a model of a singular interaction between nonlinear wave and an inhomogeneity. The delta potential can be used to model an impurity, or defect, localized at the origin. Formally the NLS-log-δ model can be described by the following problem

$$\begin{cases} i\partial_t u(t,x) + \partial_x^2 u(t,x) = -u(t,x)\mathrm{Log}|u(t,x)|^2, \quad x \neq 0, \quad t \in \mathbb{R},\\ \lim\limits_{x\to 0^+}[u(t,x) - u(t,-x)] = 0,\\ \lim\limits_{x\to 0^+}[\partial_x u(t,x) - \partial_x u(t,-x)] = -\gamma u(t,0),\\ \lim\limits_{x\to\pm\infty} u(t,x) = 0. \end{cases}$$

A similar formal model with power nonlinearity has been introduced in [21].

Our aim is to investigate an orbital stability of standing wave solutions $u(t,x) = e^{i\omega t}\varphi_{\omega,\gamma}$ for Eq. (1.2) with *peak-Gausson* profile

$$\varphi_{\omega,\gamma}(x) = e^{\frac{\omega+1}{2}} e^{-\frac{1}{2}(|x|+\frac{\gamma}{2})^2}. \tag{1.4}$$

The main stability result of this paper is the following.

Theorem 1.1. *Let $\gamma \neq 0$ and $\varphi_{\omega,\gamma}$ be defined by* (1.4)*. Let also $\widetilde{W}$ be defined by* (2.5)*. Then the following assertions hold.*

(i) *If $\gamma > 0$, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is orbitally stable in $\widetilde{W}$.*
(ii) *If $\gamma < 0$, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is orbitally unstable in $\widetilde{W}$.*
(iii) *The standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is orbitally stable in $\widetilde{W}_{\mathrm{rad}}$.*

The proof of Theorem 1.1 is based on the approach established by Grillakis et al. [28,29]. We prove the well-posedness of the Cauchy problem for NLS-log-δ equation on $\widetilde{W}$ in Sect. 3. For this purpose we use the idea of the proof of [17, Theorem 9.3.4]. Namely, we approximate the logarithmic nonlinearity by a Lipschitz continuous nonlinearities, construct a sequence of global solutions of the regularized Cauchy problem in $C(\mathbb{R}, H^1(\mathbb{R}))$, then we pass to the limit using standard compactness results, and finally we extract a subsequence which converges to the solution of limiting equation (1.2). Section 4 is devoted to the proof of Theorem 1.1. We emphasize that our stability approach does not use variational methods which are standard in the study of the stability of standing waves for the NLS with point defects (see [4,5,24,25,27]). In Sect. 4.1 we linearize NLS-log-δ equation around the peak-Gausson profile $\varphi_{\omega,\gamma}$ via the key functional $S_{\omega,\gamma} = E + (\omega+1)Q$. As a result we obtain two self-adjoint Schrödinger operators of harmonic oscillator type

$$\mathcal{L}_1^\gamma = -\frac{d^2}{dx^2} + \Big(|x| + \frac{\gamma}{2}\Big)^2 - 3, \quad \mathcal{L}_2^\gamma = -\frac{d^2}{dx^2} + \Big(|x| + \frac{\gamma}{2}\Big)^2 - 1.$$

Stability study requires investigation of the certain spectral properties of $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$ on the domain

$$\mathrm{dom}(\mathcal{L}_j^\gamma) = \big\{f \in \mathrm{dom}(\mathcal{H}_\gamma^\delta) : x^2 f \in L^2(\mathbb{R})\big\}, \quad j \in \{1,2\}.$$

The main difficulty is to count the number of negative eigenvalues of $\mathcal{L}_1^\gamma$. We propose two specific approaches to do this. For $\gamma > 0$ we give a novel approach based on the theory of extensions of symmetric operators of Krein-von Neumann. For $\gamma < 0$ we use the analytic perturbation theory and the classical continuation argument based on the Riesz-projection. At the end of the Sect. 4.2 we give the proof of Theorem 1.1.

Let us also mention that the extension theory was applied in [9] to investigate the stability of standing waves of the NLS equation with δ'-interaction on a star graph $\mathcal{G}$ (see also [2,3,12])

$$i\partial_t \mathbf{U} - \mathbf{H}_\lambda^{\delta'}\mathbf{U} + |\mathbf{U}|^{p-1}\mathbf{U} = 0,$$

where $\mathbf{H}_\lambda^{\delta'}$ is the self-adjoint operator on $L^2(\mathcal{G})$ defined for $\lambda \in \mathbb{R}\backslash\{0\}$ by

$$(\mathbf{H}_\lambda^{\delta'}\mathbf{U})(x) = \big(-u_j'(x)\big)_{j=1}^N, \quad x \neq 0,$$
$$\mathrm{dom}(\mathbf{H}_\lambda^{\delta'}) = \left\{\begin{array}{c} \mathbf{U} = (u_j)_{j=1}^N \in H^2(\mathcal{G}) : u_1'(0) = \cdots = u_N'(0), \\ \sum\limits_{j=1}^N u_j(0) = \lambda u_1'(0) \end{array}\right\}.$$

Notation We denote by $L^2(\mathbb{R})$ the Hilbert space equipped with the inner product $(u,v) := \mathrm{Re}\int_{\mathbb{R}} u(x)\overline{v(x)}dx$. Its norm is denoted by $\|\cdot\|_2$. By $H^1(\mathbb{R})$, $H^2(\mathbb{R}\backslash\{0\}) = H^2(\mathbb{R}_-) \oplus H^2(\mathbb{R}_+)$ we denote Sobolev spaces. We denote by $\widetilde{W}$ and $\widetilde{W}_{\mathrm{rad}}$ the weighted Hilbert spaces $\big\{f \in H^1(\mathbb{R}) : xf \in L^2(\mathbb{R})\big\}$ and $\big\{f \in H^1(\mathbb{R}) : xf \in L^2(\mathbb{R}), f(x) = f(-x)\big\}$ respectively.

Let $\mathcal{A}$ be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}$. The deficiency numbers of $\mathcal{A}$ are denoted by $n_\pm(\mathcal{A}) := \dim\ker(\mathcal{A}^* \mp i\mathcal{I})$, where $\mathcal{I}$ is the identity operator. The number of negative eigenvalues is denoted by $n(\mathcal{A})$ (counting multiplicities). The spectrum (resp. point spectrum) and the resolvent set of $\mathcal{A}$ are denoted by $\sigma(\mathcal{A})$ (resp. $\sigma_p(\mathcal{A})$) and $\rho(\mathcal{A})$.

The space dual to $\mathfrak{H}$ is denoted by $\mathfrak{H}'$, and $B(\widetilde{W}, \widetilde{W}')$ denotes the space of bounded operators from $\widetilde{W}$ to $\widetilde{W}'$.

2. Previous results and basic notions

For completeness of the exposition, below we will discuss key results on the standing waves of NLS-log equation. First, let us give a definition of the orbital stability. The basic symmetry associated to Eq. (1.2) is the phase-invariance (while the translation invariance does not hold due to the defect). Thus, the definition of stability takes into account only this type of symmetry and is formulated as follows.

Definition 2.1. Let X be a Hilbert space. For $\eta > 0$ let

$$U_\eta(\varphi_{\omega,\gamma}) = \Big\{ v \in X : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta}\varphi_{\omega,\gamma}\|_X < \eta \Big\}.$$

The standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is *(orbitally) stable* in X if for any $\epsilon > 0$ there exists $\eta > 0$ such that for any $u_0 \in U_\eta(\varphi_{\omega,\gamma})$, the solution $u(t)$ of (1.2) with $u(0) = u_0$ satisfies $u(t) \in U_\epsilon(\varphi_{\omega,\gamma})$ for all $t \in \mathbb{R}$. Otherwise, $e^{i\omega t}\varphi_{\omega,\gamma}$ is said to be *(orbitally) unstable* in X.

It is interesting to note that NLS-log equation (1.1) possesses standing-wave solutions $u(t,x) = e^{i\omega t}\varphi_\omega(x)$ of the Gaussian shape

$$\varphi_\omega(x) = e^{\frac{\omega+n}{2}} e^{-\frac{1}{2}|x|^2}$$

for any dimension n and any frequency ω (see [14]). The orbital stability properties of the Gaussian profile φ_ω in the relevant class

$$W(\mathbb{R}^n) = \{f \in H^1(\mathbb{R}^n) : |f|^2 \mathrm{Log}|f|^2 \in L^1(\mathbb{R}^n)\} \tag{2.1}$$

have been studied in [16]. Cazenave showed that standing waves with Gaussian profile are stable in $W(\mathbb{R}^n)$ under radial perturbations for $n \geq 2$. The proof of this result is based on the fact that the space $H^1_{\mathrm{rad}}(\mathbb{R}^n)$ is compactly embedded into $L^2(\mathbb{R}^n)$ for $n \geq 2$. Later Cazenave and Lions in [20, Remark II.3] showed that such standing waves are orbitallly stable on all $W(\mathbb{R}^n)$ for $n \geq 1$.

We also remark that Angulo and Hernandez in [10] showed (via the variational approach) the orbital stability of the ground states $\varphi_{\omega,\gamma}$ in the space $W(\mathbb{R})$ in the case of attractive δ-interaction ($\gamma > 0$). In should be noted that investigation of the orbital stability of $\varphi_{\omega,\gamma}$ in the case of repulsive δ-interaction ($\gamma < 0$) via constrained minimizer for the action or the energy functional is not applicable (see [10, Remark 4.5]).

Recently has been considered NLS-log equation with an external potential V satisfying specific conditions

$$i\partial_t u + \Delta u - V(x)u + u\mathrm{Log}|u|^2 = 0.$$

From the result of Ji and Szulkin in [33] it follows that there exist infinitely many profiles of standing wave $u(x,t) = e^{i\omega t}\varphi_\omega$ (see also [38]) for coercive V. Namely, the elliptic equation

$$-\Delta\varphi_\omega + (V(x) + \omega)\varphi_\omega = \varphi_\omega \mathrm{Log}|\varphi_\omega|^2 \tag{2.2}$$

has infinitely many solutions for $V \in C(\mathbb{R}^n, \mathbb{R})$ such that $\lim_{|x|\to\infty} V(x) = +\infty$. Moreover they showed the existence of a ground state solution (a nontrivial positive solution with least possible energy) for bounded potential such that

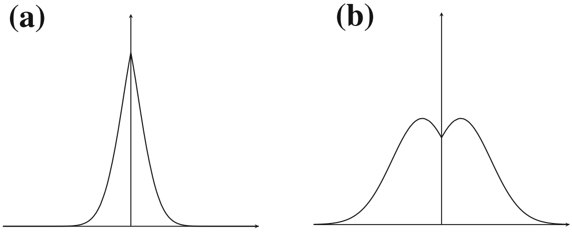

FIGURE 1. **a** $\varphi_{\omega,\gamma}(x)$ for $\gamma > 0$, **b** $\varphi_{\omega,\gamma}(x)$ for $\gamma < 0$

$\omega + 1 + V_\infty > 0$, in which $V_\infty := \lim\limits_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^n} V(x)$, and $\sigma(-\Delta + V(x) + \omega + 1) \subset (0, +\infty)$. For $V \equiv 0$ and $n \geq 3$, the authors in [22] showed the existence of infinitely many weak solutions to (2.2). Also they showed that the Gaussian profile φ_{-n} is nondegenerated, that is $\ker(L) = \mathrm{span}\{\partial_{x_i}\varphi_{-n} : i = 1, 2, \ldots, N\}$, where $Lu = -\Delta u + (|x|^2 - n - 2)u$ is the linearized operator for $-\Delta u - nu = u\mathrm{Log}|u|^2$ at φ_{-n}.

The main advantage of using the delta potential $V(x) = -\gamma\delta(x)$ is the existence of explicit expression (1.4) for the profile $\varphi_{\omega,\gamma}$ (see Fig. 1a, b) satisfying the equation

$$\mathcal{H}^\delta_\gamma \varphi + \omega\varphi - \varphi\mathrm{Log}|\varphi|^2 = 0. \tag{2.3}$$

This peak-Gausson profile is constructed from the known solution of (2.3) in the case $\gamma = 0$ on each side of the defect pasted together at $x = 0$ to satisfy the continuity and the jump condition $\varphi'(0+) - \varphi'(0-) = -\gamma\varphi(0)$ at $x = 0$. Moreover, the following result holds.

Theorem 2.2. *The set of all solutions to* (2.3) *is given by* $\{e^{i\theta}\varphi_{\omega,\gamma} : \theta \in \mathbb{R}\}$.

The proof of this theorem can be found in Appendix.

As it was announced in Theorem 1.1, our stability analysis is elaborated in the specific space $\widetilde{W}$. To explain the choice of this space let us introduce the following two basic functionals associated with Eq. (1.2):

- "charge" functional

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}} |u(x)|^2 dx,$$

- "energy" functional

$$E(u) = \tfrac{1}{2}||\partial_x u||_2^2 - \frac{1}{2}\int_{\mathbb{R}} |u(x)|^2\mathrm{Log}|u(x)|^2 dx - \frac{\gamma}{2}|u(0)|^2. \tag{2.4}$$

These functionals are continuosly differentiable in $W(\mathbb{R})$ defined by (2.1) (see [16]). Moreover, at least formally, E is conserved by the flow of (1.2). The use of the space $W(\mathbb{R})$ is mainly due to the fact that the functional E fails to be continuosly differentiable on $H^1(\mathbb{R})$ (see [16]). As we use the approch by Grillakis et al. [28,29], the functional E needs to be twice continuously differentiable at the $\varphi_{\omega,\gamma}$. To satisfy this condition we propose the "weighted space" (i.e., X coincides with $\widetilde{W}$ in Definition 2.1)

$$\widetilde{W} = H^1(\mathbb{R}) \cap L^2(x^2dx) = \big\{ f \in H^1(\mathbb{R}) : xf \in L^2(\mathbb{R}) \big\}. \tag{2.5}$$

In particular, the space $\widetilde{W}$ naturally appears in definition of the linearization of the second derivative of $S_{\omega,\gamma} = E + (\omega+1)Q$ at $\varphi_{\omega,\gamma}$. Note that, due to the inclusion $\widetilde{W} \subset W(\mathbb{R})$ (see Lemma 3.1 below), the functional E is continuously differentiable on $\widetilde{W}$.

3. The Cauchy problem in $\widetilde{W}$

In this section we prove the well-posedness of the Cauchy problem for (1.2) in the space $\widetilde{W}$. The idea of the proof is an adaptation of the proof of [17, Theorem 9.3.4]. The following lemma implies that Q and E are well-defined on $\widetilde{W}$.

Lemma 3.1. *Let $W(\mathbb{R})$ and $\widetilde{W}$ be the Banach spaces defined by*

$$W(\mathbb{R}) = \{ f \in H^1(\mathbb{R}) : |f|^2 \operatorname{Log}|f|^2 \in L^1(\mathbb{R})\},$$
$$\widetilde{W} = \big\{ f \in H^1(\mathbb{R}) : xf \in L^2(\mathbb{R}) \big\}.$$

Then $\widetilde{W} \subset W(\mathbb{R})$.

Proof. (1) It is easily seen that $\widetilde{W} \subset L^1(\mathbb{R})$. Indeed, for $f \in \widetilde{W}$ and $-\infty < a < 0 < b < \infty$ we have

$$\begin{aligned}\int_{\mathbb{R}} |f|dx &= \int_{-\infty}^{a} |f|dx + \int_a^b |f|dx + \int_b^\infty |f|dx \\ &= \int_{-\infty}^{a} |f|\cdot x \cdot \frac{1}{x}dx + \int_a^b |f|dx + \int_b^\infty |f|\cdot x\cdot\frac{1}{x}dx \\ &\le \Big(\int_{-\infty}^{a}(xf)^2dx\Big)^{\frac12}\Big(\int_{-\infty}^{a}\frac{1}{x^2}dx\Big)^{\frac12} + (b-a)\sup_{[a,b]}|f| \\ &\quad + \Big(\int_b^\infty (xf)^2dx\Big)^{\frac12}\Big(\int_b^\infty \frac{1}{x^2}dx\Big)^{\frac12} < \infty.\end{aligned}$$

(2) Let again $f \in \widetilde{W}$, then

$$\int_{\mathbb{R}} |f|^2|\operatorname{Log}|f||dx = \int_{\{x\in\mathbb{R}:|f|<1\}} |f|^2|\operatorname{Log}|f||dx + \int_{\{x\in\mathbb{R}:|f|\ge1\}} |f|^2|\operatorname{Log}|f||dx. \tag{3.1}$$

Note also that

$$|\operatorname{Log}|f|| < \frac{1}{|f|} \quad \text{for} \quad |f|<1, \quad \text{and} \quad |\operatorname{Log}|f|| < |f| \quad \text{for} \quad |f| \ge 1. \tag{3.2}$$

Since $f \in H^1(\mathbb{R})$, there exists $c > 0$ such that $|f| < 1$ for $\mathbb{R}\backslash[-c, c]$. Thus, from (3.1), (3.2), and the inclusion $\widetilde{W} \subset L^1(\mathbb{R})$ we get

$$\begin{aligned}\int_{\mathbb{R}} |f|^2 |\mathrm{Log}|f||dx &\le \int_{\{x\in\mathbb{R}:|f|<1\}} |f|dx + \int_{\{x\in\mathbb{R}:|f|\ge 1\}} |f|^3 dx \\ &\le \int_{\{x\in\mathbb{R}:|f|<1\}} |f|dx + 2c \sup_{[-c,c]} |f|^3 < \infty.\end{aligned}$$

The assertion is proved. □

The global well-posedness property of the Cauchy problem for (1.2) is ensured by the following theorem.

Theorem 3.2. *If $u_0 \in \widetilde{W}$, there is a unique solution $u(t)$ of (1.2) such that $u(t) \in C(\mathbb{R}, \widetilde{W})$ and $u(0, x) = u_0$. Furthermore, the conservation of energy and charge hold, i.e., for any $t \in \mathbb{R}$, we have*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0).$$

Moreover, if an initial data u_0 is even, the solution $u(t)$ is also even.

Proof. Generally we use an approach proposed in [18] with few natural modifications. The proof can be divided into three parts.

(1) We introduce the "reduced" Cauchy problem

$$\begin{cases} i\partial_t u_n - \mathcal{H}_\gamma^\delta u_n + u_n f_n(|u_n|^2) = 0, \\ u_n(0) = u_0. \end{cases} \tag{3.3}$$

Here $f_n(s) = \inf\{n, \sup\{-n, f(s)\}\}$ with $f(s) = \mathrm{Log} s$, $s > 0$. We define $F_n(s) = \int_0^s f_n(\sigma)d\sigma$. By Theorem 3.3.1 in [17], we imply that for any $u_0 \in H^1(\mathbb{R})$ there exists unique global solution u_n of (3.3) such that $u_n \in C(\mathbb{R}, H^1(\mathbb{R}))$ and $u_n(0) = u_0$. Moreover, the conservation of charge and energy hold, i.e., for all t

$$\begin{aligned}&||u_n(t)||_2 = ||u_0||_2, \quad E_n(u_n(t)) = E_n(u_0), \\ &E_n(u(t)) = \tfrac{1}{2}||\partial_x u(t)||_2^2 - \tfrac{1}{2}\int_{\mathbb{R}} F_n(|u(t,x)|^2)dx - \tfrac{\gamma}{2}|u(t,0)|^2.\end{aligned}$$

Indeed, we may check the assumptions of Theorem 3.3.1 in [17]. Note that f_n is Lipschitz continuous from $\mathbb{R}_+$ to $\mathbb{R}$. We also notice that $\mathcal{H}_\gamma^\delta$ defined in (1.3) satisfies $\mathcal{H}_\gamma^\delta \ge -m$, where $m = \gamma^2/4$ if $\gamma > 0$, and $m = 0$ if $\gamma < 0$. Thus, $A = -\mathcal{H}_\gamma^\delta - m$ is the self-adjoint negative operator in $X = L^2(\mathbb{R})$ on the domain $\mathrm{dom}(A) = \mathrm{dom}(\mathcal{H}_\gamma^\delta)$. Moreover, in our case the norm

$$||v||_{X_A}^2 = ||v'||_2^2 + (m+1)||v||_2^2 - \gamma|v(0)|^2$$

is equivalent to the usual $H^1(\mathbb{R})$-norm.

(2) The second step is analogous to Lemma 2.3.5 in [18]. In particular, it can be shown that there exists solution u of (1.2) in the sense of distributions (which appears to be weak-$*$ limit of solutions u_{n_k} of Cauchy problem (3.3)) such that conservation of charge holds. Further, conservation of energy $E(u)$ defined by (2.4) follows from its monotonicity. Thus, inclusion $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ follows from conservation laws.

(3) The last step is to show that the inclusion $xu_0 \in L^2(\mathbb{R})$ implies the inclusion $xu \in L^2(\mathbb{R})$. The proof of this fact repeats one of Lemma 7.6.2 from [19]. □

Remark 3.3. For the completeness of the exposition we remark that for $\gamma > 0$ the unitary group $G_\gamma(t) = e^{-it\mathcal{H}_\gamma^\delta}$ associated to Eq. (3.3) (or equivalently to (1.2)) is given explicitly by the formula (see [6,26])

$$G_\gamma(t)\phi(x) = e^{it\Delta}(\phi * \tau_\gamma)(x)\chi_+^0 + \left[e^{it\Delta}\phi(x) + e^{it\Delta}(\phi * \rho_\gamma)(-x)\right]\chi_-^0,$$

where

$$\rho_\gamma(x) = -\frac{\gamma}{2}e^{\frac{\gamma}{2}x}\chi_-^0, \quad \tau_\gamma(x) = \delta(x) + \rho_\gamma(x).$$

Here χ_+^0 and χ_-^0 denote the characteristic functions of $[0, +\infty)$ and $(-\infty, 0]$ respectively.

4. Proof of the main result

In this Section we prove Theorem 1.1. Initially we define key functional $S_{\omega,\gamma}$ associated with NLS-log-δ equation. Next we establish the relation between the second variation of $S_{\omega,\gamma}$ and the self-adjoint operators $\mathcal{L}_2^\gamma$ and $\mathcal{L}_1^\gamma$. Verifying the spectral properties of $\mathcal{L}_2^\gamma$ and $\mathcal{L}_1^\gamma$, we arrive at the assertions of Theorem 1.1. In our analysis we follow some ideas from [35].

4.1. Linearization of NLS-log-δ equation

We start introducing the key functional $S_{\omega,\gamma} = E + (\omega+1)Q$. It can be easily verified that the profile $\varphi_{\omega,\gamma}$ is a critical point of $S_{\omega,\gamma}$. Indeed, for $u, v \in \widetilde{W}$,

$$\begin{aligned} S'_{\omega,\gamma}(u)v &= \frac{d}{dt}S_{\omega,\gamma}(u+tv)|_{t=0} \\ &= \operatorname{Re}\left[\int_{\mathbb{R}} u'\overline{v'}dx - \int_{\mathbb{R}} u\overline{v}(\mathrm{Log}|u|^2 - \omega)dx - \gamma u(0)\overline{v(0)}\right]. \end{aligned}$$

Since $\varphi_{\omega,\gamma}$ satisfies (2.3), $S'_{\omega,\gamma}(\varphi_{\omega,\gamma}) = 0$.

In the approach by [29] crucial role is played by spectral properties of the linear operator associated with the second variation of $S_{\omega,\gamma}$ calculated at $\varphi_{\omega,\gamma}$. To express $S''_{\omega,\gamma}(\varphi_{\omega,\gamma})$ it is convenient to split $u, v \in \widetilde{W}$ into real and imaginary parts: $u = u_1 + iu_2,\ v = v_1 + iv_2$. Then we get

$$\begin{aligned} S''_{\omega,\gamma}(\varphi_{\omega,\gamma})(u,v) &= \int_{\mathbb{R}} u_1'v_1'dx - \int_{\mathbb{R}} u_1v_1(\mathrm{Log}|\varphi_{\omega,\gamma}|^2-\omega+2)dx - \gamma u_1(0)v_1(0) \\ &\quad + \int_{\mathbb{R}} u_2'v_2'dx - \int_{\mathbb{R}} u_2v_2(\mathrm{Log}|\varphi_{\omega,\gamma}|^2-\omega)dx - \gamma u_2(0)v_2(0) \\ &= \int_{\mathbb{R}} u_1'v_1'dx + \int_{\mathbb{R}} u_1v_1\left(\left(|x|+\tfrac{\gamma}{2}\right)^2-3\right)dx - \gamma u_1(0)v_1(0) \\ &\quad + \int_{\mathbb{R}} u_2'v_2'dx + \int_{\mathbb{R}} u_2v_2\left(\left(|x|+\tfrac{\gamma}{2}\right)^2-1\right)dx - \gamma u_2(0)v_2(0). \end{aligned}$$

Therefore, $S''_{\omega,\gamma}(\varphi_{\omega,\gamma})(u,v)$ can be formally rewritten as

$$S''_{\omega,\gamma}(\varphi_{\omega,\gamma})(u,v) = B_1^\gamma(u_1,v_1) + B_2^\gamma(u_2,v_2), \tag{4.1}$$

where

$$\begin{aligned} B_1^\gamma(f,g) &= \int_{\mathbb{R}} f'g'dx + \int_{\mathbb{R}} fg\left(\left(|x|+\tfrac{\gamma}{2}\right)^2-3\right)dx - \gamma f(0)g(0), \\ B_2^\gamma(f,g) &= \int_{\mathbb{R}} f'g'dx + \int_{\mathbb{R}} fg\left(\left(|x|+\tfrac{\gamma}{2}\right)^2-1\right)dx - \gamma f(0)g(0), \end{aligned} \tag{4.2}$$

and $\mathrm{dom}(B_j^\gamma) = \widetilde{W} \times \widetilde{W}, j \in \{1,2\}$. Note that the forms $B_j^\gamma, j \in \{1,2\}$, are bilinear bounded from below and closed. Therefore, by the First Representation Theorem (see [34, Chapter VI, Section 2.1]), they define operators $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$ such that for $j \in \{1,2\}$

$$\begin{aligned} &\mathrm{dom}(\mathcal{L}_j^\gamma) = \{v \in \widetilde{W} : \exists w \in L^2(\mathbb{R})\ s.t.\ \forall z \in \widetilde{W},\ B_j^\gamma(v,z) = (w,z)\}, \\ &\mathcal{L}_j^\gamma v = w. \end{aligned} \tag{4.3}$$

In the following theorem we describe the operators $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$ in more explicit form. We show that they are basically the harmonic oscillator operators with δ-interaction.

Theorem 4.1. *The operators $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$ determined in (4.3) are given by*

$$\mathcal{L}_1^\gamma = -\frac{d^2}{dx^2} + \left(|x| + \frac{\gamma}{2}\right)^2 - 3, \quad \mathcal{L}_2^\gamma = -\frac{d^2}{dx^2} + \left(|x| + \frac{\gamma}{2}\right)^2 - 1$$

on the domain $D_\gamma := \{f \in \mathrm{dom}(\mathcal{H}_\gamma^\delta) : x^2 f \in L^2(\mathbb{R})\}$.

Proof. Since the proof for $\mathcal{L}_2^\gamma$ is similar to the one for $\mathcal{L}_1^\gamma$, we deal with $\mathcal{L}_1^\gamma$. Let $B_1^\gamma = B^\gamma + B_1$, where $B^\gamma : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ and $B_1 : \widetilde{W} \times \widetilde{W} \to \mathbb{R}$ are defined by

$$B^\gamma(u,v) = (u',v') - \gamma u(0)v(0), \quad B_1(u,v) = (V_1^\gamma u, v),$$

and $V_1^\gamma(x) = \left(|x| + \frac{\gamma}{2}\right)^2 - 3$. We denote by $\mathcal{L}^\gamma$ (resp. $\mathcal{L}_1$) the self-adjoint operator on $L^2(\mathbb{R})$ associated (by the First Representation Theorem) with B^γ (resp. B_1). Thus,

$$\begin{aligned} &\mathrm{dom}(\mathcal{L}^\gamma) = \{v \in H^1(\mathbb{R}) : \exists w \in L^2(\mathbb{R})\ s.t.\ \forall z \in H^1(\mathbb{R}),\ B^\gamma(v,z) = (w,z)\}, \\ &\mathcal{L}^\gamma v = w. \end{aligned}$$

We claim that $\mathcal{L}^\gamma$ is a self-adjoint extension of the following symmetric operator

$$\mathcal{L}^0 = -\frac{d^2}{dx^2}, \quad \mathrm{dom}(\mathcal{L}^0) = \{f \in H^2(\mathbb{R}) : f(0) = 0\}.$$

Indeed, let $v \in \mathrm{dom}(\mathcal{L}^0)$ and $w = -v'' \in L^2(\mathbb{R})$. Then for every $z \in H^1(\mathbb{R})$ we have $B^\gamma(v,z) = (w,z)$. Thus, $v \in \mathrm{dom}(\mathcal{L}^\gamma)$ and $\mathcal{L}^\gamma v = w = -v''$. Hence, $\mathcal{L}^0 \subset \mathcal{L}^\gamma$, which yields the claim. Therefore, by [7, Theorem 3.1.1], there exists $\beta \in \mathbb{R}$ such that $\mathcal{L}^\gamma = -\Delta_\beta$, where

$$\begin{aligned} &-\Delta_\beta = -\frac{d^2}{dx^2},\\ &\mathrm{dom}(-\Delta_\beta) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) :\ f'(0+) - f'(0-) = \beta f(0)\}. \end{aligned}$$

Next we shall prove that $\beta = -\gamma$. Take $v \in \mathrm{dom}(\mathcal{L}^\gamma)$ with $v(0) \neq 0$, then we obtain

$$(\mathcal{L}^\gamma v, v) = (v'(0+) - v'(0-))v(0) + \|v'\|_2^2 = \|v'\|_2^2 + \beta|v(0)|^2,$$

which should be equal to $B^\gamma(v,v) = \|v'\|_2^2 - \gamma|v(0)|^2$. Therefore, $\beta = -\gamma$.

Again, by the First Representation Theorem,

$$\begin{aligned} &\mathrm{dom}(\mathcal{L}_1) = \{v \in \widetilde{W} : \exists w \in L^2(\mathbb{R})\ s.t.\ \forall z \in \widetilde{W},\ B_1(v,z) = (w,z)\},\\ &\mathcal{L}_1 v = w. \end{aligned}$$

Note that $\mathcal{L}_1$ is the self-adjoint extension of the following multiplication operator

$$\mathcal{L}_{0,1} f = V_1^\gamma f, \quad \mathrm{dom}(\mathcal{L}_{0,1}) = \{f \in H^2(\mathbb{R}) : V_1^\gamma f \in L^2(\mathbb{R})\}.$$

Indeed, for $v \in \mathrm{dom}(\mathcal{L}_{0,1})$ we have $v \in \widetilde{W}$, and we define $w = V_1^\gamma v \in L^2(\mathbb{R})$. Then for every $z \in \widetilde{W}$ we get $B_1(v,z) = (w,z)$. Thus, $v \in \mathrm{dom}(\mathcal{L}_1)$ and $\mathcal{L}_1 v = w = V_1^\gamma v$. Hence, $\mathcal{L}_{0,1} \subseteq \mathcal{L}_1$. Since $\mathcal{L}_{0,1}$ is self-adjoint, $\mathcal{L}_1 = \mathcal{L}_{0,1}$. The Theorem is proved. □

Remark 4.2. (i) We mention that $\widetilde{W}$ coincides with the natural domain of the bilinear forms B_1^γ and B_2^γ which additionally justifies the choice of the space $\widetilde{W}$ for investigation of the orbital stability.

(ii) It's worth mentioning that the operators $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$ coincide up to a constant term, namely, $\mathcal{L}_1^\gamma = \mathcal{L}_2^\gamma - 2$. This is a special feature of the logarithmic nonlinearity that clarifies and simplifies the spectral analysis implemented in the next Subsection.

(iii) We remark that it is also possible to propose an alternative proof of Theorem 4.1 avoiding the decomposition of the forms B_j^γ into the the sum of two forms, though it will require more extensive proof. Indeed, the self-adjoint operator $\mathcal{L}_1^\gamma$ associated with the form B_1^γ is a self-adjoint extension of the symmetric operator $\mathcal{L}_0$ defined in (4.11). By [36, Chapter IV,§14, Theorems 7 and 8], we get

$$\mathrm{dom}(\mathcal{L}_1^\gamma) = \left\{ \begin{array}{c} f : f = f_0 + cf_i + ce^{i\theta} f_{-i}, \\ f_0 \in \mathrm{dom}(\mathcal{L}_0), c \in \mathbb{C}, \theta \in [0, 2\pi) \end{array} \right\}, \tag{4.4}$$

where $f_{\pm i}$ are deficiency vectors, namely, $\ker(\mathcal{L}_0^* \mp i\mathcal{I}) = \operatorname{span}\{f_{\pm i}\}$. The deficiency vector f_i has the form (we note that $f_{-i} = \overline{f_i}$)

$$f_i(x) = \begin{cases} C_1 U\Big(-\frac{3+i}{2}, \sqrt{2}(x + \frac{|\gamma|}{2})\Big), & x > 0, \\ C_2 U\Big(-\frac{3+i}{2}, \sqrt{2}(x - \frac{|\gamma|}{2})\Big), & x < 0, \end{cases}$$

where C_1, C_2 are fixed constants that guarantee continuity of f_i at $x = 0$. The function $U\Big(-\frac{3+i}{2}, \cdot\Big)$ was found reducing the equation

$$-f''(x) + (|x| + \tfrac{\gamma}{2})^2 f(x) - (3+i) f(x) = 0,$$

via change of variables to the Weber equation (see (19.1.2) in [1])

$$g''(z) - (\tfrac{1}{4}z^2 - \tfrac{3+i}{2}) g(z) = 0.$$

This equation has the solution $U(a, z)$ (with $a = -\frac{3+i}{2}$) such that $\lim\limits_{|z|\to\infty} U(a, z) = 0$ (see (19.8.1) in [1] and also [11, Chapter 6]). In particular, $U(a, z)$ is given by (see (19.3.1), (19.3.3), (19.3.4) in [1])

$$U(a, z) = \frac{1}{2^{\xi}\sqrt{\pi}} \Big[\cos(\xi\pi)\Gamma(1/2 - \xi)\, y_1(a, z) - \sqrt{2}\sin(\xi\pi)\Gamma(1 - \xi)\, y_2(a, z)\Big],$$

where $\xi = \frac{1}{2}a + \frac{1}{4}$ and

$$y_1(a, z) = \exp(-z^2/4)\, {}_1F_1\left(\tfrac{1}{2}a + \tfrac{1}{4}; \tfrac{1}{2}; \frac{z^2}{2}\right),$$

$$y_2(a, z) = z \exp(-z^2/4)\, {}_1F_1\left(\tfrac{1}{2}a + \tfrac{3}{4}; \tfrac{3}{2}; \frac{z^2}{2}\right),$$

in which ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function (see [1, Chapter 13]). The function $U(a, z)$ is called *parabolic cylinder function.* Using the definition of $y_1(a, z)$ and $y_2(a, z)$, it can be shown (after laborious calculations) that the set (4.4) coincides with D_γ.

Next, we consider the form $S''_{\omega,\gamma}(\varphi_{\omega,\gamma}) : \widetilde{W} \times \widetilde{W} \to \mathbb{C}$ as a linear operator $\mathcal{H}_{\omega,\gamma} : \widetilde{W} \to \widetilde{W}'$. Our main stability result follows from the next theorem (see [29, Instability Theorem and Stability Theorem]).

Theorem 4.3. *Let $\gamma \neq 0$ and*

$$p_\gamma(\omega_0) = \begin{cases} 1, & \text{if } \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0, \ \text{at } \ \omega = \omega_0, \\ 0, & \text{if } \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0, \ \text{at } \ \omega = \omega_0. \end{cases}$$

Then the following assertions hold.

(i) *If $n(\mathcal{H}_{\omega_0,\gamma}) = p_\gamma(\omega_0)$, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is orbitally stable in $\widetilde{W}$.*

(ii) *If $n(\mathcal{H}_{\omega_0,\gamma}) - p_\gamma(\omega_0)$ is odd, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}$ is orbitally unstable in $\widetilde{W}$.*

Remark 4.4. Analogous result holds for the case of the space $\widetilde{W}_{\mathrm{rad}}$.

Due to [29], the proof of this theorem requires verification of *Assumptions 1,2,3.*

Assumption 1 and Assumption 2 obviously hold:

- well-posedness of equation (1.2) (Theorem 3.2),
- the existence of a smooth curve of peak standing-wave $\omega \to \varphi_{\omega,\gamma}$ (see (1.4)).

Checking *Assumption 3* in [29] is equivalent to the following Theorem.

Theorem 4.5. *Let $\gamma \neq 0$, then for any $\omega \in \mathbb{R}$ the following assertions hold.*

(i) *The operator $\mathcal{H}_{\omega,\gamma}$ has only a finite number of negative eigenvalues.*
(ii) *The kernel of $\mathcal{H}_{\omega,\gamma}$ coincides with* $\mathrm{span}\{i\varphi_{\omega,\gamma}\}$.
(iii) *The rest of the spectrum of $\mathcal{H}_{\omega,\gamma}$ is positive and bounded away from zero.*

This Theorem will be proved below. From (4.1) we can define formally

$$\mathcal{H}_{\omega,\gamma}u = \mathcal{L}_1^\gamma u_1 + i\mathcal{L}_2^\gamma u_2, \tag{4.5}$$

where $u_1 = \mathrm{Re}(u)$, $u_2 = \mathrm{Im}(u)$. In connection with Theorem 4.3 and Theorem 4.5 our aim is to investigate the following three spectral conditions associated to $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$:

- the operator $\mathcal{L}_2^\gamma$ has $\ker(\mathcal{L}_2^\gamma) = \mathrm{span}\{\varphi_{\omega,\gamma}\}$ and $\inf(\sigma(\mathcal{L}_2^\gamma)\backslash\{0\}) > \varepsilon > 0$;
- the operator $\mathcal{L}_1^\gamma$ has a trivial kernel for all $\gamma \in \mathbb{R}\backslash\{0\}$, and $\inf(\sigma(\mathcal{L}_1^\gamma) \cap \mathbb{R}_+) > \varepsilon > 0$, while $\sigma(\mathcal{L}_1^\gamma) \cap \mathbb{R}_- = \{\lambda_k\}_{k=1}^n$, where $n < \infty$;
- the number of negative eigenvalues of the operator $\mathcal{L}_1^\gamma$.

These three conditions will be studied in the next Subsection. The main difficulty is to count the number of negative eigenvalues of $\mathcal{L}_1^\gamma$. We use two specific approaches to do this. For $\gamma > 0$ we apply exclusively the theory of extensions of symmetric operators. In the case $\gamma < 0$, we consider $\mathcal{L}_1^\gamma$ as a real-holomorphic perturbation of the one-dimensional harmonic oscillator operator

$$\mathcal{L}_1^0 = -\frac{d^2}{dx^2} + x^2 - 3, \quad \mathrm{dom}(\mathcal{L}_1^0) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}. \tag{4.6}$$

Using the perturbation theory, we claim that the point spectrum of $\mathcal{L}_1^\gamma$ depends holomorphically on the spectrum of $\mathcal{L}_1^0$. In particular, for $\gamma < 0$ we show the equality $n(\mathcal{L}_1^\gamma) = 2$, while for $\gamma > 0$ we obtain $n(\mathcal{L}_1^\gamma) = 1$. Moreover, we show that $n(\mathcal{L}_1^\gamma) = 1$ in the space $\widetilde{W}_{\mathrm{rad}}$ for any $\gamma \in \mathbb{R}\backslash\{0\}$.

4.2. Spectral properties of $\mathcal{L}_1^\gamma$ and $\mathcal{L}_2^\gamma$

Below we discuss the spectral properties of $\mathcal{L}_j^\gamma$, $j \in \{1, 2\}$. Let us make few general observations. First, since

$$\lim_{|x|\to+\infty} \left(|x| + \frac{\gamma}{2}\right)^2 = +\infty,$$

the operators $\mathcal{L}_j^\gamma$, $j \in \{1, 2\}$, have a discrete spectrum, $\sigma(\mathcal{L}_j^\gamma) = \sigma_p(\mathcal{L}_j^\gamma) = \{\lambda_k^j\}_{k\in\mathbb{N}}$ (see [13, Chapter II]). In particular, we have the following distribution of the eigenvalues

$$\lambda_0^j < \lambda_1^j < \cdots < \lambda_k^j < \cdots ,$$

with $\lambda_k^j \to +\infty$ as $k \to +\infty$. Due to semi-boundedness of $V_\gamma^1 = (|x| + \frac{\gamma}{2})^2 - 3$ and $V_\gamma^2 = (|x| + \frac{\gamma}{2})^2 - 1$, we obtain that any nontrivial solution of the equation

$$\mathcal{L}_j^\gamma v = \lambda_k^j v, \qquad v \in \mathrm{dom}(\mathcal{L}_j^\gamma),$$

is unique up to a constant factor (see [13]). Therefore, each eigenvalue λ_k^j is simple. Moreover, the following Proposition holds.

Proposition 4.6. *Let* $\gamma \in \mathbb{R}\backslash\{0\}$. *Then* $\ker(\mathcal{H}_{\omega,\gamma}) = \mathrm{span}\{i\varphi_{\omega,\gamma}\}$.

Proof. Since $\varphi_{\omega,\gamma} \in D_\gamma$ and $\mathcal{L}_2^\gamma \varphi_{\omega,\gamma} = 0$, we obtain immediately $\ker(\mathcal{L}_2^\gamma) = \mathrm{span}\{\varphi_{\omega,\gamma}\}$. Now, suppose that $u \in \ker(\mathcal{L}_1^\gamma)$ and $u \neq 0$. It means that $u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\})$ and

$$-u'' + ((|x| + \tfrac{\gamma}{2})^2 - 3)u = 0, \;\; x \neq 0, \tag{4.7}$$

$$u'(0+) - u'(0-) = -\gamma u(0). \tag{4.8}$$

Consider (4.7) on $(0, \infty)$. Then, the fact that $\ker(\mathcal{L}_2^\gamma) = \mathrm{span}\{\varphi_{\omega,\gamma}\}$ implies

$$-\varphi_{\omega,\gamma}'' + ((|x| + \tfrac{\gamma}{2})^2 - 1)\varphi_{\omega,\gamma} = 0 \quad \text{on} \quad (0, \infty). \tag{4.9}$$

Differentiating (4.9), we obtain that $\varphi_{\omega,\gamma}'$ satisfies (4.7) on $(0, \infty)$. Since we look for $L^2(\mathbb{R})$-solution, every solution of (4.7) in $(0, \infty)$ is of the form $\mu\varphi_{\omega,\gamma}', \mu \in \mathbb{R}$ [13, Chapter II]. Analogously, every solution in $(-\infty, 0)$ is given by $\nu\varphi_{\omega,\gamma}', \nu \in \mathbb{R}$. Thus, the solution u of (4.7)-(4.8) has the form

$$u = \begin{cases} -\mu\varphi_{\omega,\gamma}', & x \in (-\infty, 0), \\ \mu\varphi_{\omega,\gamma}', & x \in (0, \infty). \end{cases}$$

Since $u \in H^1(\mathbb{R})$ and $\varphi_{\omega,\gamma}$ satisfies condition (4.8), we get

$$u(0) = -\mu\varphi_{\omega,\gamma}'(0-) = \mu\varphi_{\omega,\gamma}'(0+) = -\tfrac{\mu}{2}\gamma\varphi_{\omega,\gamma}(0).$$

On the other hand, the fact that $\varphi_{\omega,\gamma}$ satisfies (4.9) implies

$$u'(0\pm) = \pm\mu\varphi_{\omega,\gamma}''(0\pm) = \pm\mu(\tfrac{\gamma^2}{4} - 1)\varphi_{\omega,\gamma}(0).$$

Finally, using (4.8), we arrive at

$$2\mu(\tfrac{\gamma^2}{4} - 1)\varphi_{\omega,\gamma}(0) = \tfrac{\mu}{2}\gamma^2\varphi_{\omega,\gamma}(0).$$

This is a contradiction, therefore $\mu = 0$ and $u \equiv 0$. The equality $\ker(\mathcal{H}_{\omega,\gamma}) = \mathrm{span}\{i\varphi_{\omega,\gamma}\}$ follows from (4.5). □

Summarizing the above facts we arrive at *the proof of Lemma 4.5*.
The following result implies non-negativity of the operator $\mathcal{L}_2^\gamma$.

Proposition 4.7. *Let* $\gamma \in \mathbb{R}\backslash\{0\}$. *Then* $n(\mathcal{L}_2^\gamma) = 0$.

The proof of Proposition 4.7 follows from positivity of $\varphi_{\omega,\gamma}$ and the following generalization of the classical Sturm oscillation theorem to the case of point interaction (see [13]).

Lemma 4.8. *Let $V(x)$ be a real-valued continuous function on $\mathbb{R}$. Let also $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ be eigenfunctions of the operator*

$$L_V = -\frac{d^2}{dx^2} + V(x), \quad \mathrm{dom}(L_V) = \left\{f \in \mathrm{dom}(\mathcal{H}_\gamma^\delta) : L_V f \in L^2(\mathbb{R})\right\},$$

corresponding to the eigenvalues $\lambda_1 < \lambda_2$ respectively. Suppose that n_1 and n_2 are the number of zeroes of φ_1, φ_2 respectively. Then $n_2 > n_1$.

Proof. Suppose that $\varphi_1(a) = \varphi_1(b) = 0$ and $-\infty < a < 0 < b \leq \infty$, besides $\varphi_1(\infty) = 0$ is understood in the sense of limit. Let also $\varphi_1 > 0$ in (a, b). Then $\varphi_1'(a) > 0$ and $\varphi_1'(b) \leq 0$. The "equality" $\varphi_1'(b) = 0$ takes place only if $b = \infty$ since $\varphi_1 \in H^2(0, \infty)$. Suppose that φ_2 has no zeros in (a, b) and $\varphi_2 > 0$ in (a, b). Using the fact that φ_1, φ_2 are eigenfunctions of L_V, we arrive at

$$\begin{aligned} 0 &= \int_a^b (\varphi_1\varphi_2'' - \varphi_1''\varphi_2)dx + \int_a^b (\lambda_2 - \lambda_1)\varphi_1\varphi_2 dx \\ &= \int_a^{-0} \frac{d}{dx}(\varphi_1\varphi_2' - \varphi_1'\varphi_2)dx + \int_{+0}^b \frac{d}{dx}(\varphi_1\varphi_2' - \varphi_1'\varphi_2)dx + \int_a^b (\lambda_2 - \lambda_1)\varphi_1\varphi_2 dx \\ &= \left[\varphi_1\varphi_2' - \varphi_1'\varphi_2\right]_a^b + \left[\varphi_1'\varphi_2 - \varphi_1\varphi_2'\right]_{-0}^{+0} + \int_a^b (\lambda_2 - \lambda_1)\varphi_1\varphi_2 dx. \end{aligned} \tag{4.10}$$

Since $\varphi_1, \varphi_2 \in \mathrm{dom}(\mathcal{H}_\gamma^\delta)$, we get $[\varphi_1'\varphi_2 - \varphi_1\varphi_2']_{-0}^{+0} = 0$. Therefore, from (4.10) and initial assumptions it easily follows that

$$0 > [\varphi_1\varphi_2' - \varphi_1'\varphi_2]_a^b = \varphi_1'(a)\varphi_2(a) - \varphi_1'(b)\varphi_2(b) > 0,$$

which is a contradiction. Thus, φ_2 has at least one zero in (a, b). Analogously, we can prove that there exists $\xi \in (-\infty, a]$ such that $\varphi_2(\xi) = 0$. Thereby, between two finite zeroes of φ_1 there exists a zero of φ_2, and between the last finite zero of φ_1 and ∞ (between the first finite zero of φ_1 and $-\infty$ respectively) there is at least one zero of φ_2. The proof is completed. □

Remark 4.9. Note that from $\ker(\mathcal{L}_2^\gamma) = \mathrm{span}\{\varphi_{\omega,\gamma}\}$ and $\inf(\sigma(\mathcal{L}_2^\gamma)\backslash\{0\}) > \varepsilon > 0$ it follows that $n(\mathcal{H}_{\omega,\gamma}) = n(\mathcal{L}_1^\gamma)$.

The number of negative eigenvalues $n(\mathcal{L}_1^\gamma)$ for $\gamma > 0$
Below we will show the following result.

Proposition 4.10. *Let $\gamma > 0$, then $n(\mathcal{L}_1^\gamma) = 1$. Moreover, for $\mathcal{L}_1^\gamma$ restricted to $\widetilde{W}_{\mathrm{rad}}$ we also have $n(\mathcal{L}_1^\gamma) = 1$. In particular, the unique negative simple eigenvalue equals -2, and $\varphi_{\omega,\gamma}$ is the corresponding eigenfunction.*

The proof of this proposition relies on the theory of extension of symmetric operators. We start with two preliminary results.

Lemma 4.11. *The operator defined by*

$$\begin{aligned} \mathcal{L}_0 &= -\frac{d^2}{dx^2} + \left(|x| + \frac{\gamma}{2}\right)^2 - 3, \\ \mathrm{dom}(\mathcal{L}_0) &= \left\{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R}), f(0) = 0\right\} \end{aligned} \tag{4.11}$$

is a densely defined symmetric operator with equal deficiency indices $n_\pm(\mathcal{L}_0) = 1$.

Proof. First, we establish the scale of Hilbert spaces associated with the self-adjoint operator (see [8, Section I, §1.2.2])

$$\mathcal{L} = -\frac{d^2}{dx^2} + \left(|x| + \frac{\gamma}{2}\right)^2, \quad \mathrm{dom}(\mathcal{L}) = \{f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R})\}.$$

Define for $s \geq 0$ the space

$$\mathscr{H}_s(\mathcal{L}) = \left\{f \in L^2(\mathbb{R}) : \|f\|_{s,2} = \left\|(\mathcal{L} + \mathcal{I})^{s/2} f\right\|_2 < \infty\right\}.$$

The space $\mathscr{H}_s(\mathcal{L})$ with norm $\|\cdot\|_{s,2}$ is complete. The dual space of $\mathscr{H}_s(\mathcal{L})$ will be denoted by $\mathscr{H}_{-s}(\mathcal{L}) = \mathscr{H}_s(\mathcal{L})'$. The norm in the space $\mathscr{H}_{-s}(\mathcal{L})$ is defined by the formula

$$\|\psi\|_{-s,2} = \left\|(\mathcal{L} + \mathcal{I})^{-s/2} \psi\right\|_2.$$

The spaces $\mathscr{H}_s(\mathcal{L})$ form the following chain

$$... \subset \mathscr{H}_2(\mathcal{L}) \subset \mathscr{H}_1(\mathcal{L}) \subset L^2(\mathbb{R}) = \mathscr{H}_0(\mathcal{L}) \subset \mathscr{H}_{-1}(\mathcal{L}) \subset \mathscr{H}_{-2}(\mathcal{L}) \subset ....$$

Thus, the space $\mathscr{H}_2(\mathcal{L})$ coincides with the domain of the operator $\mathcal{L}$. The norm of the space $\mathscr{H}_1(\mathcal{L})$ can be calculated as follows

$$\begin{aligned} \|f\|_{1,2}^2 &= ((\mathcal{L} + \mathcal{I})^{1/2} f, (\mathcal{L} + \mathcal{I})^{1/2} f) \\ &= \int\limits_{\mathbb{R}} \left(|f'(x)|^2 + |f(x)|^2 + \left(|x| + \frac{\gamma}{2}\right)^2 |f(x)|^2\right) dx. \end{aligned}$$

Therefore, we have the embedding $\mathscr{H}_1(\mathcal{L}) \hookrightarrow H_1(\mathbb{R})$ and, by Sobolev embedding, $\mathscr{H}_1(\mathcal{L}) \hookrightarrow L^\infty(\mathbb{R})$. From the former remark we obtain that the δ-functional, $\delta : \mathscr{H}_1(\mathcal{L}) \to \mathbb{C}$ acting as $\delta(\psi) = \psi(0)$ belongs to $\mathscr{H}_1(\mathcal{L})' = \mathscr{H}_{-1}(\mathcal{L})$ and consequently $\delta \in \mathscr{H}_{-2}(\mathcal{L})$. Therefore, using [8, Lemma 1.2.3], it follows that the restriction $\mathcal{L}_0$ of the operator $\mathcal{L}$ to the domain

$$\mathrm{dom}(\mathcal{L}_0') = \{\psi \in \mathrm{dom}(\mathcal{L}) : \delta(\psi) = \psi(0) = 0\}$$

is a densely defined symmetric operator with equal deficiency indices $n_\pm(\mathcal{L}_0') = 1$. Next, since $\mathcal{B} = -3\mathcal{I}$ is a bounded operator, we have from [36, Chapter IV, Theorem 6] that the operators $\mathcal{L}_0'$ and $\mathcal{L}_0 = \mathcal{L}_0' + \mathcal{B}$ have the same deficiency indices. This finishes the proof of the Lemma. □

To investigate the number of negative eigenvalues of $\mathcal{L}_1^\gamma$ we will use the following abstract result (see [36, Chapter IV, §14]).

Proposition 4.12. *Let $\mathcal{A}$ be a densely defined lower semi-bounded symmetric operator (i.e., $\mathcal{A} \geq m\mathcal{I}$) with finite deficiency indices $n_\pm(\mathcal{A}) = k < \infty$ in the Hilbert space $\mathfrak{H}$. Let also $\widetilde{\mathcal{A}}$ be a self-adjoint extension of $\mathcal{A}$. Then the spectrum of $\widetilde{\mathcal{A}}$ in $(-\infty, m)$ is discrete and consists of at most k eigenvalues counting multiplicities.*

Remark 4.13. Proposition 4.12 holds for upper semi-bounded operator $\mathcal{A}$ ($\mathcal{A} \leq M\mathcal{I}$) and interval (M, ∞), respectively.

Proof of Proposition 4.10. Recall that $\mathcal{L}_1^\gamma$ is the self-adjoint extension of the symmetric operator $\mathcal{L}_0$ defined by (4.11) (see proof of Theorem 4.1 above). Lemma 4.11 implies the equality $n_\pm(\mathcal{L}_0) = 1$.

Next, since $\gamma > 0$ ($\varphi'_{\omega,\gamma} \neq 0$ for $x \neq 0$) we can verify that for $f \in \mathrm{dom}(\mathcal{L}_0)$ we have (see [5, Subsection 6.1])

$$-f'' + \left[\left(|x| + \frac{\gamma}{2}\right)^2 - 3\right] f = \frac{-1}{\varphi'_{\omega,\gamma}} \frac{d}{dx}\left[(\varphi'_{\omega,\gamma})^2 \frac{d}{dx}\left(\frac{f}{\varphi'_{\omega,\gamma}}\right)\right], \qquad x \neq 0. \tag{4.12}$$

Now using (4.12) and integrating by parts, we get

$$\begin{aligned}(\mathcal{L}_0 f, f) = &\int_{-\infty}^{0-} (\varphi'_{\omega,\gamma})^2 \left|\frac{d}{dx}\left(\frac{f}{\varphi'_{\omega,\gamma}}\right)\right|^2 dx \\ &+ \int_{0+}^{\infty} (\varphi'_{\omega,\gamma})^2 \left|\frac{d}{dx}\left(\frac{f}{\varphi'_{\omega,\gamma}}\right)\right|^2 dx + \left[f'\overline{f} - |f|^2 \frac{\varphi''_{\omega,\gamma}}{\varphi'_{\omega,\gamma}}\right]_{0-}^{0+}.\end{aligned} \tag{4.13}$$

The integral terms in (4.13) are nonnegative. Due to the condition $f(0) = 0$, non-integral term vanishes, and we get $\mathcal{L}_0 \geq 0$. Therefore, from Proposition 4.12 we obtain $n(\mathcal{L}_1^\gamma) \leq 1$. From the other hand,

$$\mathcal{L}_1^\gamma \varphi_{\omega,\gamma} = (\mathcal{L}_2^\gamma - 2)\varphi_{\omega,\gamma} = -2\varphi_{\omega,\gamma}, \tag{4.14}$$

since $\mathcal{L}_2^\gamma \varphi_{\omega,\gamma} = 0$. Thus, $n(\mathcal{L}_1^\gamma) = 1$. The second assertion of Proposition 4.10 follows from (4.14) and the fact that $\varphi_{\omega,\gamma}$ is even. □

The number of negative eigenvalues $n(\mathcal{L}_1^\gamma)$ **for** $\gamma < 0$

The analysis previously applied to calculate the number $n(\mathcal{L}_1^\gamma)$ was based essentially on the fact that $\varphi'_{\omega,\gamma}(x) \neq 0$ for $x \neq 0$ in the case $\gamma > 0$. For $\gamma < 0$ the function $\varphi'_{\omega,\gamma}(x)$ has exactly two zeroes $x = \pm\frac{\gamma}{2}$, and the formula (4.12) could not be applied. To study the case of negative γ we will use the theory of analytic perturbations for linear operators (see [34,37]).

The following lemma states the analyticity of the families of operators $\mathcal{L}_j^\gamma$, $j \in \{1, 2\}$.

Lemma 4.14. *As a function of* γ, $(\mathcal{L}_1^\gamma)$ *and* $(\mathcal{L}_2^\gamma)$ *are two real-analytic families of self-adjoint operators of type (B) in the sense of Kato.*

Proof. By Theorem 4.1 and [34, Theorem VII-4.2], it suffices to prove that the families of bilinear forms (B_1^γ) and (B_2^γ) defined in (4.2) are real-analytic of type (B). Indeed, it is immediate that they are bounded from below and closed. Moreover, the decomposition of B_1^γ into B^γ and B_1, implies that $\gamma \to (B_1^\gamma v, v)$ is analytic. The proof for the family (B_2^γ) is similar. □

In what follows we also use the following classical result about the harmonic oscillator operator (4.6) (see [13]).

Lemma 4.15. *Let operator* $\mathcal{L}_1^0$ *be defined by* (4.6). *Then the following assertions hold.*

(i) $\mathcal{L}_1^0$ *has two simple nonpositive eigenvalues: the first one is negative and the second one is zero.*

(ii) $\ker(\mathcal{L}_1^0) = \text{span}\{\varphi'_{\omega,0}\}$.
(iii) *The rest of the spectrum of* $\mathcal{L}_1^0$ *is positive.*

Indeed, the above Lemma follows from the known fact $\sigma(\mathcal{L}_1^0) = \{2n-2 : n = 0, 1, 2, \ldots\}$.

Proposition 4.16. *There exist* $\gamma_0 > 0$ *and two analytic functions* $\Pi : (-\gamma_0, \gamma_0) \to \mathbb{R}$ *and* $\Omega : (-\gamma_0, \gamma_0) \to L^2(\mathbb{R})$ *such that*

(i) $\Pi(0) = 0$ *and* $\Omega(0) = \varphi'_{\omega,0}$.
(ii) *For all* $\gamma \in (-\gamma_0, \gamma_0)$, $\Pi(\gamma)$ *is the simple isolated second eigenvalue of* $\mathcal{L}_1^\gamma$, *and* $\Omega(\gamma)$ *is the associated eigenvector for* $\Pi(\gamma)$.
(iii) γ_0 *can be chosen small enough to ensure that for* $\gamma \in (-\gamma_0, \gamma_0)$ *the spectrum of* $\mathcal{L}_1^\gamma$ *is positive, except at most the first two eigenvalues.*

Proof. Using the spectral structure of the operator $\mathcal{L}_1^0$ (see Lemma 4.15), we can separate the spectrum $\sigma(\mathcal{L}_1^0)$ into two parts $\sigma_0 = \{\lambda_1^0, 0\}$ and σ_1 by a closed curve Γ (for example, a circle), such that σ_0 belongs to the inner domain of Γ and σ_1 to the outer domain of Γ (note that $\sigma_1 \subset (\epsilon, +\infty)$ for $\epsilon > 0$). Next, Lemma 4.14 and analytic perturbations theory imply that $\Gamma \subset \rho(\mathcal{L}_1^\gamma)$ for sufficiently small $|\gamma|$, and $\sigma(\mathcal{L}_1^\gamma)$ is likewise separated by Γ into two parts, such that the part of $\sigma(\mathcal{L}_1^\gamma)$ inside Γ consists of a finite number of eigenvalues with total multiplicity (algebraic) two. Therefore, we obtain from the Kato–Rellich Theorem (see [37, Theorem XII.8]) the existence of two analytic functions Π, Ω defined in a neighborhood of zero such that the items (i), (ii) and (iii) hold. □

Below we will study how the perturbed second eigenvalue $\Pi(\gamma)$ changes depending on the sign of γ. For small γ we have the following result.

Proposition 4.17. *There exists* $0 < \gamma_1 < \gamma_0$ *such that* $\Pi(\gamma) < 0$ *for any* $\gamma \in (-\gamma_1, 0)$, *and* $\Pi(\gamma) > 0$ *for any* $\gamma \in (0, \gamma_1)$.

Proof. From Taylor's theorem we have the following expansions

$$\Pi(\gamma) = \beta\gamma + O(\gamma^2) \quad \text{and} \quad \Omega(\gamma) = \varphi'_{\omega,0} + \gamma\psi_0 + O(\gamma^2), \tag{4.15}$$

where $\beta \in \mathbb{R}$ ($\beta = \Pi'(0)$) and $\psi_0 \in L^2(\mathbb{R})$ (since $\psi_0 = \Omega'(0)$). The desired result will follow if we show that $\beta > 0$. We compute $(\mathcal{L}_1^\gamma \Omega(\gamma), \varphi'_{\omega,0})$ in two different ways.

From (4.15) we obtain

$$\Pi(\gamma)\Omega(\gamma) = \beta\gamma\varphi'_{\omega,0} + O(\gamma^2). \tag{4.16}$$

Since $\mathcal{L}_1^\gamma \Omega(\gamma) = \Pi(\gamma)\Omega(\gamma)$, it follows from (4.16) that

$$(\mathcal{L}_1^\gamma \Omega(\gamma), \varphi'_{\omega,0}) = \beta\gamma\|\varphi'_{\omega,0}\|_2^2 + O(\gamma^2). \tag{4.17}$$

Having $\varphi'_{\omega,0} \in \text{dom}(\mathcal{L}_1^\gamma)$ ($\varphi'_{\omega,0}(0) = 0$) and $\mathcal{L}_1^0 \varphi'_{\omega,0} = 0$, we obtain

$$\mathcal{L}_1^\gamma \varphi'_{\omega,0} = \mathcal{L}_1^0 \varphi'_{\omega,0} + (\gamma|x| + \tfrac{\gamma^2}{4})\varphi'_{\omega,0} = (\gamma|x| + \tfrac{\gamma^2}{4})\varphi'_{\omega,0}. \tag{4.18}$$

Since $\mathcal{L}_1^\gamma$ is self-adjoint, we obtain from (4.15) and (4.18) that

$$\begin{aligned}(\mathcal{L}_1^\gamma\Omega(\gamma),\varphi'_{\omega,0}) &= (\Omega(\gamma),\mathcal{L}_1^\gamma\varphi'_{\omega,0}) = (\varphi'_{\omega,0},(\gamma|x|+\tfrac{\gamma^2}{4})\varphi'_{\omega,0})+O(\gamma^2)\\ &= \gamma\int_{\mathbb{R}}|x||\varphi'_{\omega,0}(x)|^2dx+O(\gamma^2).\end{aligned} \tag{4.19}$$

Finally, combination of (4.17) and (4.19) leads to

$$\beta = \frac{\int_{\mathbb{R}}|x||\varphi'_{\omega}(x)|^2dx}{\|\varphi'_{\omega,0}\|_2^2}+O(\gamma). \tag{4.20}$$

From (4.20) it follows that $\beta>0$, and, therefore, assertion is proved. □

Now we can count the number of negative eigenvalues of $\mathcal{L}_1^\gamma$ for any γ using a classical continuation argument based on the Riesz-projection.

Proposition 4.18. *Let $\gamma\in\mathbb{R}\backslash\{0\}$. Then we have*

(i) $n(\mathcal{L}_1^\gamma)=2$ *for* $\gamma<0$.
(ii) $n(\mathcal{L}_1^\gamma)=1$ *for* $\gamma>0$.
(iii) $n(\mathcal{L}_1^\gamma)=1$ *for* $\mathcal{L}_1^\gamma$ *restricted to* $\widetilde{W}_{\mathrm{rad}}$.

Proof. Recall that $\ker(\mathcal{L}_1^\gamma)=\{0\}$ for $\gamma\neq0$. Let $\gamma<0$ and define γ_∞ by $\gamma_\infty=\inf\{r<0:\mathcal{L}_1^\gamma$ has exactly two negative eigenvalues for all $\gamma\in(r,0)\}$.

Proposition 4.17 implies that γ_∞ is well defined and $\gamma_\infty\in[-\infty,0)$. We claim that $\gamma_\infty=-\infty$. Suppose that $\gamma_\infty>-\infty$. Let $N=n(\mathcal{L}_1^{\gamma_\infty})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0\in\Gamma\subset\rho(\mathcal{L}_1^{\gamma_\infty})$, and all the negative eigenvalues of $\mathcal{L}_1^{\gamma_\infty}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to $\mathcal{L}_1^{\gamma_\infty}$. Indeed, for $f\in\mathrm{dom}(\mathcal{L}_1^{\gamma_\infty})$

$$(\mathcal{L}_1^{\gamma_\infty}f,f)=\int_{\mathbb{R}}((f')^2+V^1_{\gamma_\infty}f^2)dx-\gamma|f(0)|^2\geq-3\|f\|_2^2$$

since $V^1_{\gamma_\infty}(x)\geq-3$ for all x.

Next, from Lemma 4.14 it follows that there is $\epsilon>0$ such that for $\gamma\in[\gamma_\infty-\epsilon,\gamma_\infty+\epsilon]$ we have $\Gamma\subset\rho(\mathcal{L}_1^\gamma)$ and for $\xi\in\Gamma$, $\gamma\to(\mathcal{L}_1^\gamma-\xi)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $\gamma\to P(\gamma)$ given by

$$P(\gamma)=-\frac{1}{2\pi i}\int_\Gamma(\mathcal{L}_1^\gamma-\xi)^{-1}d\xi$$

implies that $\dim(\mathrm{Ran}\,P(\gamma))=\dim(\mathrm{Ran}\,P(\gamma_\infty))=N$ for all $\gamma\in[\gamma_\infty-\epsilon,\gamma_\infty+\epsilon]$. Next, by definition of γ_∞, there exists $r_0\in(\gamma_\infty,\gamma_\infty+\epsilon)$, and $\mathcal{L}_1^\gamma$ has exactly two negative eigenvalues for all $\gamma\in(r_0,0)$. Therefore, $\mathcal{L}_1^{\gamma_\infty+\epsilon}$ has two negative eigenvalues and $N=2$, hence $\mathcal{L}_1^\gamma$ has two negative eigenvalues for $\gamma\in(\gamma_\infty-\epsilon,0)$, which contradicts with the definition of γ_∞. Therefore, $\gamma_\infty=-\infty$. Similar analysis can be applied to the case $\gamma>0$. The last assertion was proved for $\gamma>0$ in Proposition 4.10. In the case $\gamma<0$ the statement follows from item (i), the fact that any eigenfunction of $\mathcal{L}_1^\gamma$ is either even or odd, and the Sturm oscillation result in Lemma 4.8. □

Remark 4.19. We note that the curve Γ above can be chosen independently of the parameter $\gamma \in \mathbb{R}$. Indeed, the relation $V_1^\gamma(x) \geq -3$ for any γ implies $\inf \sigma(\mathcal{L}_1^\gamma) \geq -3$. Thus, Γ can be chosen as the rectangle $\Gamma = \partial R$, in which

$$R = \{z \in \mathbb{C} : z = z_1 + iz_2, (z_1, z_2) \in [-4, 0] \times [-a, a], \text{ for some } a > 0\}.$$

Proof of Theorem 1.1. (i) Let $\gamma > 0$ and $E : \widetilde{W} \longrightarrow \mathbb{R}$ be the energy functional defined by (2.4). From [16, Lemma 2.6] (with $-\Delta$ substituted by $\mathcal{H}_\gamma^\delta$) and the continuous embedding $\widetilde{W} \hookrightarrow W(\mathbb{R})$ we deduce that $\mathcal{E}''(\varphi_{\omega,\gamma}) \in B(\widetilde{W}, \widetilde{W'})$, where $\mathcal{E}''(\varphi_{\omega,\gamma})$ is the operator associated with the form $E''(\varphi_{\omega,\gamma})(u, v)$. Using Proposition 4.10, positivity of $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2$, Remark 4.9, and Theorem 4.5 we arrive at item (i) in Theorem 4.3 which induces the orbital stability of $e^{i\omega t}\varphi_{\omega,\gamma}$ in $\widetilde{W}$.

(ii) Let $\gamma < 0$. From item (i) of Proposition 4.18, the positivity of $\partial_\omega ||\varphi_{\omega,\gamma}||_2^2$, and Theorem 4.5, we get item (ii) of Theorem 4.3 which implies the instability of $e^{i\omega t}\varphi_{\omega,\gamma}$ in $\widetilde{W}$.

(iii) Stability of $e^{i\omega t}\varphi_{\omega,\gamma}$ in $\widetilde{W}_{\text{rad}}$ follows from item (iii) of Propositon 4.18 and item (i) in Theorem 4.3. □

Appendix

In this Appendix we show the uniqueness of the peak-standing wave solution $\varphi_{\omega,\gamma}$ stated in Theorem 2.2. The proof is based on the ideas from [14, 24, 31, 39].

Proof of Theorem 2.2. We divide the proof in 3 steps. Let φ be a solution to (2.3).

(1) We show initially that if $\varphi \in H^2(\mathbb{R}_+)$ is a solution to

$$-\varphi'' + \omega\varphi - \varphi \text{Log}|\varphi|^2 = 0 \tag{4.21}$$

on $\mathbb{R}_+$, then $\varphi = e^{i\theta_+}e^{\frac{\omega+1}{2}}e^{-\frac{(x-x_+)^2}{2}}$, where $\theta_+, x_+ \in \mathbb{R}$. Indeed, writing $\varphi(x) = e^{i\theta(x)}\rho(x)$, where θ and ρ are real-valued functions, we obtain from equation (4.21)

$$-\rho'' + \rho(\omega + (\theta')^2) - \rho \text{Log}\rho^2 + i(\theta''\rho + 2\theta'\rho') = 0.$$

Thus, in order to make the imaginary part vanish, we get $\theta''\rho + 2\theta'\rho' = 0$, which implies $\rho^2\theta' \equiv const := C$. Next, since

$$|\varphi'|^2 = (\rho')^2 + (\theta')^2\rho^2 \geq (\theta')^2\rho^2 \geq 0$$

and $\lim_{x\to\infty} |\varphi'| = 0$, we get $\lim_{x\to\infty} (\theta')^2\rho^2 = C \lim_{x\to\infty} \theta' = 0$. Therefore, $\lim_{x\to\infty} \theta'$ exists. Now, since $|\varphi| = \rho$, we obtain $\lim_{x\to\infty} \rho^2 = 0$, and thus $C = 0$, which implies $\theta(x) \equiv const := \theta_+$. Thereby, $\varphi(x) = e^{i\theta_+}\rho(x)$, where ρ satisfies

$$-\rho'' + \omega\rho - \rho \text{Log}\rho^2 = 0, \quad x \in \mathbb{R}_+. \tag{4.22}$$

From [39, Theorem 1] it follows that $\rho(x) > 0$.

(2) Multiplying (4.22) by ρ' and integrating we arrive at

$$(\rho')^2 = (1+\omega)\rho^2 - \rho^2 \ln|\rho|^2 + K. \tag{4.23}$$

Since $\rho \in H^2(\mathbb{R}_+)$, we get $K = 0$. Therefore, integrating (4.23) we obtain

$$\rho(x) = e^{\frac{\omega+1}{2}} e^{-\frac{(x-x_+)^2}{2}}, \qquad x_+ \in \mathbb{R}.$$

Thus, we get $\varphi = e^{i\theta_+} e^{\frac{\omega+1}{2}} e^{-\frac{(x-x_+)^2}{2}}$ on $\mathbb{R}_+$. Analogously, we can show that the $H^2(\mathbb{R}_-)$-solution of (4.21) on $\mathbb{R}_-$ is given by

$$\varphi = e^{i\theta_-} e^{\frac{\omega+1}{2}} e^{-\frac{(x-x_-)^2}{2}}, \qquad \theta_-, x_- \in \mathbb{R}.$$

(3) From items (1)–(2) above we obtain that the solution to (4.21) on $\mathbb{R}\backslash\{0\}$ is given by

$$\varphi = \begin{cases} e^{i\theta_+} e^{\frac{\omega+1}{2}} e^{-\frac{(x-x_+)^2}{2}}, & x > 0, \\ e^{i\theta_-} e^{\frac{\omega+1}{2}} e^{-\frac{(x-x_-)^2}{2}}, & x < 0. \end{cases}$$

Next, our aim is to find explicitly $x_\pm$ and $\theta_\pm$. Let $f(s) = -\omega s + s\mathrm{Log}(s^2)$ and $F(s) = \int_0^s f(t)dt$. Multiplying (4.21) by φ' and integrating from 0 to R, we get as $R \to \infty$

$$\frac{1}{2}(\varphi'(0+))^2 + F(\varphi(0+)) = 0.$$

Similarly, we obtain

$$\frac{1}{2}(\varphi'(0-))^2 + F(\varphi(0-)) = 0.$$

Since φ need to be continuous at $x = 0$, we get $|\varphi'(0-)| = |\varphi'(0+)|$. Therefore, $|x_-| = |x_+|$ and again by continuity condition we obtain $\theta_- = \theta_+ =: \theta$. To conclude the proof we need to recall that φ satisfies jump condition $\varphi'(0+) - \varphi'(0-) = -\gamma\varphi(0)$, which yields $x_+ = -\frac{\gamma}{2}$ and $x_- = \frac{\gamma}{2}$. □

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Received: 7 August 2016.
Accepted: 3 May 2017.