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# Minimal Legendrian submanifolds of $S^{2n+1}$ and absolutely area-minimizing cones

Vincent Borrelli and
Claudio Gorodski

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**Abstract**

We use minimal Legendrian submanifolds in spheres to construct examples of absolutely area-minimizing cones and we prove two results about Legendrian 2-tori in $S^5$.

## 1 Introduction

A Legendrian submanifold of a $(2n+1)$-dimensional contact manifold is an $n$-dimensional integral submanifold of the contact distribution. In this paper we consider the unit sphere $S^{2n+1}$ equipped with the contact structure inherited from its standard embedding in $\mathbb{C}^{n+1}$. To an arbitrary immersion $f : M \to S^{2n+1}$ one can associate its cone map $\hat{f} : M \times (0, +\infty) \to \mathbb{R}^{2n+2}$ where $\hat{f}(x,t) = tf(x)$ and we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. It turns out that $f$ is a Legendrian immersion if and only if $\hat{f}$ is a Lagrangian immersion with respect to the standard symplectic structure of $\mathbb{R}^{2n+2}$. We also note that the normal vectors to $f$ in $S^n$ at $x$ are exactly the normal vectors to $\hat{f}$ at $(x,t)$ for $t > 0$. Furthermore, if the second fundamental form of $f$ in $S^n$ with respect to a normal vector $\nu$ in $x$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the second fundamental form of $\hat{f}$ with respect to $\nu$ in $(t,x)$ has eigenvalues $t\lambda_1, \ldots, t\lambda_n, 0$. It follows that $f$ is a minimal immersion if and only if $\hat{f}$ is a minimal immersion.

The manifold of all Lagrangian planes in $\mathbb{R}^{2n+2}$ is called the *Lagrangian Grassmannian*, and it is easy to see that this manifold is diffeomorphic to the homogeneous space $\mathbf{U}(n+1)/\mathbf{O}(n+1)$. To an arbitrary Lagrangian immersion $g : P \to \mathbb{R}^{2n+2}$ one can now naturally associate a *Gauss map* $G(g) : P \to \mathbf{U}(n+1)/\mathbf{O}(n+1)$. Consider the composition

$$P \xrightarrow{G(g)} \mathbf{U}(n+1)/\mathbf{O}(n+1) \xrightarrow{\det^2_{\mathbb{C}}} S^1. \tag{1}$$

The pull-back of the volume form of $S^1$ under (1) is called the *Maslov form* $\mu(g)$ of $g$, and the class $[\mu(g)]$ it defines is called the *Maslov class* of $g$. A result of Morvan ([8]) states that the Maslov form of $g$ is given by $\mu(g) = -\frac{n+1}{\pi}\langle JH, \cdot\rangle$, where $H$ is the mean curvature vector of $g$, $J$ the complex structure in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ and $\langle \cdot, \cdot \rangle$ denotes the real inner product. As a corollary, a Lagrangian immersion is minimal if and only if it has a vanishing Maslov form.

Notice that a Lagrangian immersion has a vanishing Maslov form precisely when the tangent planes in any two distinct points differ by an element of $\mathbf{SU}(n+1)$. In [4], Section III, Harvey and Lawson define a $(n+1)$-dimensional submanifold of $\mathbb{R}^{2n+2}$ to be *special Lagrangian* if the tangent planes in any two distinct points differ by an element of $\mathbf{SU}(n+1)$. As an application of the theory of calibrations introduced in the same paper, they show that special Lagrangian submanifolds have the property of being absolutely area-minimizing. In particular, we conclude from the above discussion that *the cone over a compact minimal Legendrian submanifold of $S^{2n+1}$ is absolutely area-minimizing.* It is interesting to notice that the same conclusion can be drawn from Proposition 2.17 in [4], which in fact contains a restatement of Morvan's formula (namely, equation (2.19)).

In the first part of this paper we use this conclusion to construct two series of examples of absolutely area-minimizing cones. The first series generalizes Corollary 3.21 and Example 3.22 in [4]. The second series unifies the proofs of the absolute area-minimality of some examples in [1, 2].

In the second part of the paper we prove two results about Legendrian 2-tori in $S^5$. The first one is a weak form of the converse to the fact that any minimal Legendrian 2-torus of $S^5$ has a vanishing Maslov class.

**Theorem 1.** – *Let $f: T^2 \to S^5$ be a Legendrian immersion. Then $f$ is regularly homotopic through Legendrian immersions to a minimal Legendrian embedding if and only if $[\mu(\hat{f})] = 0$.*

We then consider minimal Legendrian tori of $S^5$ under the additional assumption that the metric is flat.

**Theorem 2.** – *Let $K \subset \mathbf{SU}(3)$ be a maximal torus subgroup and consider its natural action by Legendrian automorphisms on $S^5 \subset \mathbb{C}^3$. Then each principal orbit is a flat minimal Legendrian 2-torus in $S^5$. Conversely, every flat minimal Legendrian 2-torus in $S^5$ is given that way (hence, it is a hexagonal torus and it is unique up to an element in $\mathbf{SU}(3)$).*

Kenmotsu proved in [6] that a flat minimal 2-torus in $S^5$ is an orbit of a torus subgroup of $\mathbf{SO}(6)$. Our result says that a flat minimal Legendrian 2-torus in $S^5$ is an orbit of a torus subgroup of $\mathbf{SU}(3)$. In particular, there are infinitely many congruence classes of flat minimal 2-tori in $S^5$, but only one congruence class of flat minimal Legendrian 2-tori in $S^5$.

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## 2 Examples of minimal Legendrian submanifolds of $S^{2n+1}$

In this section we present examples of compact homogeneous minimal Legendrian submanifolds of $S^{2n+1}$. The cones over these submanifolds are area-minimizing. Although some of the examples are already known, they are here presented from a new and unified perspective.

## 2.1 Twisted normal bundles

In [4] Harvey and Lawson defined a submanifold $M$ of $S^n$ to be *austere* if the set of eigenvalues of the Weingarten operator of $M$ with respect to any normal vector is invariant under multiplication by $-1$. Then they showed that for any compact austere submanifold $M$ of $S^{2n+1}$, the cone over

$$\Phi : M \times S^1 \to S^{2n+1}, \qquad \Phi(x, e^{i\theta}) = (\cos\theta\, x, \sin\theta\, \nu_x),$$

where $\nu_x$ ranges over all unit vectors normal to $M$ in $S^{2n+1}$ at $x$, is special Lagrangian.

Note that the map $\Phi$ fails to be an immersion for $\sin\theta = 0$ if the codimension of $M$ in $S^{2n+1}$ is bigger than one, and it fails to be an immersion for $\cos\theta = 0$ if 0 is an eigenvalue of the Weingarten operator of $M$ with respect to some unit normal. Therefore we will consider the following variation of this construction. Let $M$ be a submanifold of the unit sphere $S^n$ and $NM$ be the normal bundle of $M$ in $S^n$. We denote by $\widetilde{M}_+$, $\widetilde{M}_-$ two copies of $NM^{\leq\frac{\pi}{2}} = \{\nu \in NM : |\nu| \leq \pi/2\}$ and by $\widetilde{M}$ the double of $NM^{\leq\frac{\pi}{2}}$ i. e. $\widetilde{M} = \widetilde{M}_+ \cup \widetilde{M}_- / \partial\widetilde{M}_+ \sim \partial\widetilde{M}_-$. Note that if $M$ is a closed hypersurface of $S^n$ then $\widetilde{M}$ is diffeomorphic to the product $M \times S^1$. Define

$$\Psi : \widetilde{M} \to S^{2n+1}, \qquad \Psi(\nu_x) = \begin{cases} (\epsilon\cos|\nu_x|\, x, \sin|\nu_x|\, \frac{\nu_x}{|\nu_x|}) & \text{if } \nu_x \neq 0, \\ (\epsilon x, 0) & \text{if } \nu_x = 0, \end{cases}$$

where $\epsilon = \pm 1$ according to whether $x \in \widetilde{M}_\pm$. It is easy to see that $\Psi$ has the same image as $\Phi$.

**Proposition 1.** – *If 0 is never an eigenvalue of the Weingarten operator of $M$ with respect to any unit normal, then $\Psi : \widetilde{M} \to S^{2n+1}$ is a Legendrian immersion. If in addition $M$ is austere, then $\Psi$ is minimal.*

**Proof.** – We will first compute the differential of $\Psi$ at a unit normal vector $\nu_{x_0}$. Near $x_0$ we choose an orthonormal frame field $e_1, \ldots, e_p$ and a normal frame field $\nu_1, \ldots, \nu_q$ where $p + q = n$ and $e_1, \ldots, \nu_q$ is positively oriented. Without loss of generality, we may assume that $(\nabla\nu_k)^N_{x_0} = 0$ where $()^N$ denotes the orthogonal projection onto the normal space $N_{x_0}M$. We can also take $e_1, \ldots, e_p$ to be the eigenvectors of the Weingarten operator $A^{\nu_{x_0}}$, namely $A^{\nu_{x_0}}(e_k) = \lambda_k e_k$. Upon a linear change of coordinates we may further assume that at $x_0$ the vectors $x_0, e_1, \ldots, \nu_q$ give the standard basis of $\mathbb{R}^{n+1} \supset S^n \supset M$.

If $\nu_{x_0} \neq 0$ we may assume $\nu_1 = \frac{\nu_{x_0}}{|\nu_{x_0}|}$. The differential of $\Psi$ at $\nu_{x_0}$ is then given by

$$\Psi_*(e_j) = (\epsilon\cos|\nu_{x_0}|\, e_j, -\lambda_j \sin|\nu_{x_0}|\, e_j) \quad \text{for } 1 \leq j \leq p \text{ and}$$

$$\Psi_*(\nu_j) = \begin{cases} (-\epsilon\sin|\nu_{x_0}|\, x_0, \cos|\nu_{x_0}|\, \frac{\nu_{x_0}}{|\nu_{x_0}|}) & \text{if } j = 1, \\ (0, \sin|\nu_{x_0}|\, \frac{\nu_j}{|\nu_{x_0}|}) & \text{if } 2 \leq j \leq q. \end{cases}$$

If $\nu_{x_0} = 0$ we have

$$\Psi_*(e_j) = (\epsilon e_j, 0) \quad \text{for } 1 \leq j \leq p \text{ and}$$

$$\Psi_*(\nu_j) = (0, \nu_j) \quad \text{for } 1 \leq j \leq q.$$

In these formulae the fields $e_j$, $\nu_j$ are evaluated at $x_0$.

The above calculation immediately shows that $\Psi$ is an immersion (if $\lambda_j \neq 0$ for $1 \leq j \leq p$) and Legendrian (since $\langle \Psi_*(v), J\Psi(\nu_{x_0})\rangle = \langle \Psi_*(v), J\Psi_*(v')\rangle = 0$ where each of $v$, $v'$ is one of $e_1, \ldots, \nu_q$ and $J : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is given by $J(x,y) = (-y,x)$). Next we compute the Maslov form of the cone over $\Psi$. For this purpose, we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$ and $J$ as the multiplication by $i$, and we represent $G(\hat{\Psi})(\nu_{x_0})$ by a unitary matrix which we specify by its column vectors and then compute the squared determinant. If $\nu_{x_0} \neq 0$ the matrix has as columns the unit vectors

$$\begin{array}{lll} \epsilon\cos|\nu_{x_0}|\, x_0 + i\sin|\nu_{x_0}|\nu_1, & \frac{\epsilon\cos|\nu_{x_0}| - i\lambda_1\sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_1^2\sin^2|\nu_{x_0}|} e_1, \ldots, \frac{\epsilon\cos|\nu_{x_0}| - i\lambda_p\sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_p^2\sin^2|\nu_{x_0}|} e_p, \\ -\epsilon\sin|\nu_{x_0}|\, x_0 + i\cos|\nu_{x_0}|\,\nu_1, & i\nu_2, \ldots, i\nu_q. \end{array}$$

Therefore

$$(\det)^2 G(\hat{\Psi})(\nu_{x_0}) = \epsilon^2 i^{2q}\left[\prod_{j=1}^{p} \frac{(\epsilon\cos|\nu_{x_0} - i\lambda_j\sin|\nu_{x_0}|)^2}{\cos^2|\nu_{x_0}| + \lambda_j^2\sin^2|\nu_{x_0}|}\right](\det)^2(x_0, e_1, \ldots, e_p, \nu_1, \nu_2, \ldots \nu_q).$$

If $M$ is austere this expression equals $(-1)^q$ (because

$$(\epsilon\cos|\nu_{x_0}| - i\lambda\sin|\nu_{x_0}|)(\epsilon\cos|\nu_{x_0}| + i\lambda\sin|\nu_{x_0}|) = \cos^2|\nu_{x_0}| + \lambda^2\sin^2|\nu_{x_0}|$$

for $\lambda \in \mathbb{R}$). If $\nu_{x_0} = 0$ we get:

$$(\det)^2(\epsilon x_0, \epsilon e_1, \ldots, \epsilon e_p, i\nu_1, \ldots, i\nu_q) = \epsilon^{2p} i^{2q}(\det)^2(x_0, e_1, \ldots, e_p, \nu_1, \ldots, \nu_q) = (-1)^q.$$

Thus $(\det)^2 G(\hat{\Psi})$ is a constant map. This implies that the Maslov form of $\hat{\Psi}$ vanishes, hence $\Psi$ is minimal. □

## 2.2 Twisted normal bundles of isoparametric submanifolds

We shall now apply the construction discussed in section 2.1 to certain leaves of certain isoparametric foliations of spheres. Let $M^{n-1}$ be an *isoparametric hypersurface* of $S^n$ i. e. a hypersurface with constant principal curvatures. It is known that $M$ is a level set of a homogeneous polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies the so-called Cartan-Münzner equations ([9]). In particular, one can assume $M$ is compact from the beginning. Moreover, the level sets of $F$ are connected and the regular ones correspond exactly to the parallel hypersurfaces of $M$, which are also isoparametric; the one-parameter family $\mathcal{F}$ thus obtained is called an *isoparametric family*. There are precisely two singular levels for $F$; these are minimal submanifolds of $S^n$ and in fact the focal manifolds of the isoparametric family. Denote the principal curvatures of $M$ by $\lambda_1 < \ldots < \lambda_g$ with multiplicities $m_1, \ldots, m_g$. Then $m_k = m_{k+2}$ ($k$ mod $g$). In particular: there are at most two distinct multiplicities, which we denote by $(m_1, m_2)$; moreover $m_1 = m_2$ if $g$ is odd. We have also that $g \in \{1,2,3,4,6\}$, the degree of $F$ is $g$ and the distance between the focal manifolds is $\pi/g$. Now the orbit space $S^n/\mathcal{F} \ni M_t$ is isometric to the closed interval $[0,\pi/g] \ni t$ and the focal manifolds correspond to the boundary points $\{0, \pi/g\}$. Finally, the principal curvatures of $M_t$ are $\lambda_k(t) = \cot(t + (k-1)\pi/g)$ for $k = 1, \ldots, g$, and the principal curvatures of the focal manifold $M_0$ (resp. $M_{\pi/g}$) with respect to any unit normal are $\lambda_k(0)$ for $k = 2, \ldots, g$ (resp. $\lambda_k(\pi/g)$ for $k = 1, \ldots, g-1$). Next there are two possibilities.

1. If $g$ is odd then the focal manifolds are congruent under the antipodal map of $S^n$. Since $m_1 = m_2$, they are compact austere submanifolds of $S^{n+1}$, and since they have no principal curvature equal to zero, the construction of section 2.1 yields an example of a minimal Legendrian embedding of $\widetilde{M}$ into $S^{2n+1}$. It is easy to use the foliation of $S^n$ by parallel hypersurfaces to show that $\widetilde{M}$ is diffeomorphic to the sphere $S^n$.

2. If $g$ is even *and* the $m_1 = m_2$ then the isoparametric hypersurface $M_{t/2g}$ is a compact austere submanifold of $S^{n+1}$ all of whose principal curvatures are nonzero. Therefore the construction of section 2.1 yields an example of a minimal Legendrian immersion of $\widetilde{M}$ into $S^{2n+1}$. It is easy to use the fact that $M_{t/2g}$ is invariant under the antipodal map of $S^n$ to show that this immersion induces a minimal Legendrian embedding of $\widetilde{M}/\mathbf{Z}_2 \simeq M_{t/2g} \times_{\mathbf{Z}_2} S^1$ into $S^{2n+1}$ (the nontrivial element of $\mathbf{Z}_2$ acts on each factor of the product $M_{t/2g} \times S^1$ by the antipodal map).

From the known examples of isoparametric foliations of $S^n$, we gather below the ones with $m_1 = m_2 = m$. These are all homogeneous, namely the orbital foliation induced on $S^n$ by an orthogonal representation $\rho$ of a compact connected Lie group $G$ on $\mathbb{R}^{n+1}$.

g=1: Here $m = n - 1$. The foliation is by umbilic hypersurfaces, namely the hyperspheres parallel to the equator. The focal manifolds are points and $\widetilde{M}$ is a totally geodesic totally real $S^n \subset S^{2n+1}$.

g=2: Here $n$ is odd and $m = \frac{n-1}{2}$. The foliation is by hypersurfaces parallel to the Clifford torus $M_{\pi/4} = S^{\frac{n-1}{2}}(1/\sqrt{2}) \times S^{\frac{n-1}{2}}(1/\sqrt{2}) \subset S^n$, and $\widetilde{M}/\mathbf{Z}_2$ is diffeomorphic to $(S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}}) \times_{\mathbf{Z}_2} S^1$. In particular we get a minimal Legendrian immersion

$$S^1 \times S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}} \to S^{2n+1}.$$

g=3: Here $m = 1, 2, 4, 8$. Let $G$ to be group of $3 \times 3$ $\mathbf{F}$-unitary matrices, where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbb{H}$ (quaternions), $\mathbf{O}$ (octonions), according to whether $m = 1, 2, 4, 8$. Note that $G = \mathbf{O}(3)$, $\mathbf{U}(3)$, $\mathbf{Sp}(3)$, $\mathbf{F}_4$, respectively. Let $V$ be the real vector space of $3 \times 3$ traceless $\mathbf{F}$-Hermitean matrices. Note that $\dim V = 3m + 2$. Now $\rho : G \to \mathbf{O}(\mathbf{R}^{3m+2})$ is conjugation. The focal manifolds are the so called standard embeddings of the projective planes, namely $\mathbf{R}P^2$, $\mathbf{C}P^2$, $\mathbb{H}P^2$ and $\mathbf{O}P^2$. The construction gives minimal Legendrian embeddings

$$S^4 \to S^9, \quad S^7 \to S^{15}, \quad S^{13} \to S^{27}, \quad S^{25} \to S^{51}.$$

It is easily verified that these embeddings are not congruent to the standard embedding $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{C}^{n+1}$ of the unit sphere.

g=4: Here $m = 1, 2$. If $m = 1$ then the representation $\rho : \mathbf{U}(2) \to \mathbf{O}(\mathbb{R}^6)$ is on the space of $2 \times 2$ complex symmetric matrices and given by $\rho(A)X = AXA^t$. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{U}(2)/\mathbf{Z}_2$ and $\widetilde{M}/\mathbf{Z}_2$ is diffeomorphic to $(\mathbf{U}(2)/\mathbf{Z}_2) \times_{\mathbf{Z}_2} S^1$. Since $\mathbf{U}(2)$ is diffeomorphic to $S^1 \times S^3$, in particular we get a minimal Legendrian immersion

$$S^1 \times S^1 \times S^3 \to S^{11}.$$

If $m = 2$ then $\rho : \mathrm{Sp}(2) \to \mathrm{O}(\mathbb{R}^{10})$ is the adjoint representation. The isoparametric hypersurfaces are diffeomorphic to $\mathrm{Sp}(2)/T^2$ where $T^2 \subset \mathrm{Sp}(2)$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $\mathrm{Sp}(2)/T^2 \times_{\mathbb{Z}_2} S^1$.

g=6: Here $m = 1, 2$. If $m = 1$ then $\rho$ is the only almost faithful real irreducible 8-dimensional representation of $\mathrm{SO}(4)$. The isoparametric hypersurfaces are diffeomorphic to $\mathrm{SO}(4)/\mathbb{Z}_2$ and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(\mathrm{SO}(4)/\mathbb{Z}_2) \times_{\mathbb{Z}_2} S^1$. Since $\mathrm{SO}(4)$ is finitely covered by $S^3 \times S^3$, in particular we get a minimal Legendrian immersion

$$S^1 \times S^3 \times S^3 \to S^{15}.$$

If $m = 2$ then $\rho : \mathrm{G}_2 \to \mathrm{O}(\mathbb{R}^{14})$ is the adjoint representation. The isoparametric hypersurface are diffeomorphic to $\mathrm{G}_2/T^2$ where $T^2 \subset \mathrm{G}_2$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $\mathrm{G}_2/T^2 \times_{\mathbb{Z}_2} S^1$.

## 2.3 Orbits of isotropy representations of Hermitian symmetric spaces

We shall now show that certain orbits of the isotropy representations of certain Hermitian symmetric spaces give rise to examples of Legendrian submanifolds in spheres. The results quoted without proof can all be found in [5] and [7]. Let $X$ be an Hermitian symmetric space of the compact type, $G$ its connected group of isometries, $K$ the isotropy subgroup. So $X = G/K$, $X$ is a product of compact irreducible Hermitian symmetric spaces, and $G$ is the product of the corresponding simple groups of isometries. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of the Lie algebra $\mathfrak{g}$ of $G$ into the $\pm 1$-eigenspaces of the involution, which will be denoted by $\sigma$. Here $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is naturally identified with the tangent space of $X$ at the base point. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. We choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$ (this is possible, since $\mathfrak{k}$ is of maximal rank in $\mathfrak{g}$). Then $\mathfrak{h}^c$ is a Cartan subalgebra in $\mathfrak{g}^c$. The roots of $\mathfrak{g}^c$ which are also roots of $\mathfrak{k}^c$ are called compact roots. Introduce an ordering in the system of roots. To each root $\alpha$ we define the coroot $H_\alpha \in \mathfrak{h}^c$ by the rule $\alpha(H) = (H_\alpha, H)$ for all $H \in \mathfrak{h}^c$ (here $(\cdot,\cdot)$ denotes the Cartan-Killing form of $\mathfrak{g}^c$) and we associate a root vector $X_\alpha$ so that

$$(X_\alpha - X_{-\alpha}), \quad i(X_\alpha + X_{-\alpha}) \in \mathfrak{g} \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{2H_\alpha}{|\alpha|^2}.$$

There exists an element $Z$ in the center of $\mathfrak{k}$ such that, for every noncompact positive root $\alpha$, $[Z, X_\alpha] = -iX_\alpha$ and $[Z, X_{-\alpha}] = iX_{-\alpha}$. The restriction of $\mathrm{ad}(Z)$ to $\mathfrak{p}$ is a complex structure, which will be denoted by $J$. By means of this $J$, we view $\mathfrak{p}$ as $\mathbb{C}^{\dim X}$. The roots $\alpha$, $\beta$ are called strongly orthogonal if $\alpha \pm \beta$ are not roots. There exists a set $\Delta$ of strongly orthogonal noncompact positive roots such that the real subspace $\mathfrak{a}$ spanned by the $i(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta$, is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.

Define the Cayley transform $c = \exp \frac{\pi}{4} \sum_{\alpha \in \Delta} i(X_\alpha + X_{-\alpha}) \in G$ and let $\tau = \mathrm{ad}(c)^2$, viewed as an automorphism of $\mathfrak{g}$. Then $\tau^4 = 1$ and the following are equivalent (see [7], Remark 2, p. 286 and Proposition 4.4, p. 273):

- the noncompact dual $X^*$ of $X$ is of tube type;

- $\tau(Z) = -Z$;
- $\tau\sigma = \sigma\tau$;
- $\tau^2 = 1$.

*Hereafter we shall assume that the Hermitian symmetric space $X^*$ is of tube type*, that is, holomorphically equivalent to the tube over a self-dual cone. Since $\tau$ is an involution of $\mathfrak{g}$ which preserves $\mathfrak{k}$ and $\mathfrak{p}$, we have the splittings $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$, $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$ into the $\pm 1$-eigenspaces of $\tau$. Now $\mathrm{ad}(c)$ preserves $\mathfrak{k}^+$ and $\mathfrak{p}^+$, and interchanges $\mathfrak{k}^-$ with $\mathfrak{p}^-$, and $J$ interchanges $\mathfrak{p}^+$ with $\mathfrak{p}^-$. Note that $Z \in \mathfrak{k}^-$ and define $p = \mathrm{ad}(c)Z \in \mathfrak{p}^-$. Let $\rho : K \to \mathbf{U}(\mathfrak{p})$ be the linear isotropy representation of $X$ and consider the $K$-orbit through $p$. The tangent space $T_pK(p)$ is

$$\begin{aligned}
[\mathfrak{k}, p] &= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}, Z] \\
&= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^+, Z] + \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^-, Z] \\
&= \mathrm{ad}(c)[\mathfrak{k}^+, Z] + \mathrm{ad}(c)\, J \,\mathrm{ad}(c)^{-1}\mathfrak{k}^- \\
&= 0 + \mathrm{ad}(c)\, J\, \mathfrak{p}^- \\
&= \mathrm{ad}(c)\, \mathfrak{p}^+ \\
&= \mathfrak{p}^+.
\end{aligned}$$

Now $[\mathfrak{k}, p]$ is a Lagrangian subspace of $\mathfrak{p}$. Let $\mathfrak{k}'$ be the Cartan-Killing orthogonal complement of $Z$ in $\mathfrak{k}$ and let $K'$ be the analytic subgroup of $K$ corresponding to $\mathfrak{k}'$. Since $[\mathfrak{k}, p] = [\mathfrak{k}', p] \oplus \mathbb{R}[Z, p] = [\mathfrak{k}', p] \oplus \mathbb{R}Jp$, we have that $[\mathfrak{k}', p] \oplus \mathbb{R}p$ is also a Lagrangian subspace of $\mathfrak{p}$. So the tangent space $T_pK'(p)$ is a Legendrian subspace of $\mathfrak{p}$. At this point, we assume that $X$ is irreducible. Then $\rho(K') \subset \mathbf{SU}(\mathfrak{p})$, so that $K'(p)$ is a Legendrian submanifold and its Maslov form vanishes. Hence $K'(p)$ is a compact minimal Legendrian submanifold of the unit sphere in $\mathfrak{p} \cong \mathbb{C}^{\dim X}$. According to [7], there are five types of irreducible compact Hermitian symmetric spaces $X$ whose noncompact duals are of tube type. These are listed in the table below, which also describes in each case the diffeomorphic type $M$ of $K'(p)$.

| $X$ | $M \subset S^{\dim X - 1}$ |
|---|---|
| $\mathbf{SO}(n+2)/\mathbf{SO}(2) \times \mathbf{SO}(n)$ $(n \geq 3)$ | $S^{n-1} \subset S^{2n-1}$ |
| $\mathbf{SU}(2n)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$ $(n \geq 2)$ | $\mathbf{SU}(n) \subset S^{2n^2-1}$ |
| $\mathbf{SO}(4n)/\mathbf{U}(2n)$ $(n \geq 2)$ | $\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$ |
| $\mathbf{Sp}(n)/\mathbf{U}(n)$ $(n \geq 3)$ | $\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$ |
| $\mathbf{E}_7/\mathbf{SO}(2)\mathbf{E}_6$ | $\mathbf{E}_6/\mathbf{F}_4 \subset S^{53}$ |

The first four minimal Legendrian embeddings in the table are given as follows: $S^{n-1} \subset S^{2n-1}$ is totally real; $\mathbf{SU}(n) \subset S^{2n^2-1}$ comes from the standard inclusion of $\mathbf{SU}(n)$ into the $n \times n$ complex matrices; $\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$ is the $\mathbf{SU}(n)$-orbit $\{AA^t : A \in \mathbf{SU}(n)\}$ in the space of $n \times n$ complex symmetric matrices; $\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$ is the $\mathbf{SU}(2n)$-orbit $\{A \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} A^t : A \in \mathbf{SU}(2n)\}$ in the space of $2n \times 2n$ complex skew-symmetric matrices.

## 2.4 Yet another example

Consider the 4-dimensional complex irreducible representation of SU(2), namely $\rho : \mathrm{SU}(2) \to \mathrm{SU}(4)$ ($\rho$ can be realized as the natural action of SU(2) on the space of homogeneous complex polynomials of degree 3 in two variables $z_1$, $z_2$). Upon suitable choices of basis elements, we can write

$$\rho\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha\beta^2 & \beta^3 & \sqrt{3}\alpha^2\beta \\ \sqrt{3}\alpha\bar{\beta}^2 & \alpha\bar{\alpha}^2 - 2\bar{\alpha}\beta\bar{\beta} & \sqrt{3}\bar{\alpha}^2\beta & \beta\bar{\beta}^2 - 2\alpha\bar{\alpha}\bar{\beta} \\ -\bar{\beta}^3 & -\sqrt{3}\bar{\alpha}^2\bar{\beta} & \bar{\alpha}^3 & \sqrt{3}\bar{\alpha}\bar{\beta}^2 \\ -\sqrt{3}\alpha^2\bar{\beta} & 2\alpha\bar{\alpha}\beta - \beta^2\bar{\beta} & \sqrt{3}\bar{\alpha}\beta^2 & \alpha^2\bar{\alpha} - 2\alpha\beta\bar{\beta} \end{pmatrix},$$

where $\alpha$, $\beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Therefore the induced representation on the Lie algebra level maps the basis elements of the Lie algebra $\mathfrak{su}(2)$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

respectively into

$$\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -2 \\ 0 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 2 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 0 & 0 & 0 & i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 2i \\ 0 & i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 2i & 0 & 0 \end{pmatrix}.$$

Let $M$ be the orbit through the point $p = (1, 0, 1, 0) \in \mathbb{C}^4$, which is the quotient of $S^3$ by a cyclic subgroup of order 3. The easily seen identities

$$\langle Xp, iYp\rangle = \langle Xp, ip\rangle = 0$$

for $X, Y \in \mathfrak{su}(2)$ show that the tangent space of $M$ at $p$ is a Legendrian subspace. Since $\rho[\mathrm{SU}(2)] \subset \mathrm{SU}(4)$, it follows that $M$ is a Legendrian submanifold of $S^7$ with vanishing Maslov form. Therefore it is minimal and the cone over $M$ is absolutely area-minimizing. Observe that this example is not the twisted normal bundle over a compact minimal surface in $S^3$ because of its topology.

# 3 Legendrian 2-tori in $S^5$

In this section we prove Theorems 1 and 2 which were stated in the introduction.

### 3.1 The proof of Theorem 1

Let $\lambda$ be the Liouville form. We denote by $I_{Leg}(T^2, S^5)$ the space of Legendrian immersions of $T^2$ into $S^5$ and by $\mathrm{Mono}_\lambda(TT^2, TS^5)$ the space of bundle monomorphisms $\sigma : TT^2 \to TS^5$ such that $\lambda_{|\sigma(F)} = 0$ and $d\lambda_{|\sigma(F)} = 0$ for each fiber $F$ of $TT^2$. (These two spaces are provided with the compact-open topology). The $h$-principle for Legendrian immersions implies that the map:

$$\begin{array}{ccc} I_{Leg}(T^2, S^5) & \longrightarrow & \mathrm{Mono}_\lambda(TT^2, TS^5) \\ g & \longmapsto & dg, \end{array}$$

is a weak homotopy equivalence.

**Lemma.** – *$\pi_0(I_{Leg}(T^2, S^5)) \simeq H^1(T^2, \mathbf{Z})$ and the identification is given by half the Maslov class.*

**Proof of the lemma.** – A 2-frame of $T_xT^2$ is said to be Legendrian if the plane it spans is Legendrian. The space of all Legendrian 2-frames can be identified with $\mathbf{GL}(3, \mathbf{C})$. Let $R = (e_1, e_2)$ be a global section of the frame bundle $\pi : FT^2 \to T^2$. There is a bijection:

$$\Psi : \ \mathrm{Mono}_\lambda(TT^2, TS^5) \longrightarrow C^0(T^2, \mathbf{GL}(3, \mathbf{C})), \quad \sigma \longmapsto \Psi(\sigma) = f_\sigma,$$

where $f_\sigma$ is the map $x \mapsto (\sigma(\pi(R_x)), \sigma(e_1(x)), \sigma(e_2(x)))$. Thus

$$\pi_0(I_{Leg}(T^2, S^5)) \simeq [T^2, \mathbf{GL}(3, \mathbf{C})] \simeq [T^2, \mathbf{U}(3)],$$

since $\mathbf{U}(3)$ is a deformation retract of $\mathbf{GL}(3, \mathbf{C})$. Consider the projection $\det : \mathbf{U}(3) \to \mathbf{U}(1)$ and let $u$ be the pull back of the (normalized) volume form $w$ of $\mathbf{U}(1)$. The class $u$ is a generator of $H^1(\mathbf{U}(3), \mathbf{Z})$ and, since $\pi_1(\mathbf{U}(3)) = \mathbf{Z}$, $\pi_2(\mathbf{U}(3)) = 0$, the degree map:

$$\begin{array}{cccc} \deg : & [T^2, \mathbf{U}(3)] & \longrightarrow & H^1(T^2, \mathbf{Z}) \\ & f & \longmapsto & f^*u \end{array}$$

is a bijection (see [10], for example). Let $p : \mathbf{U}(3) \to \mathbf{U}(3)/\mathbf{O}(3)$ be the projection. The Gauss map of $\hat{g}$ at $(x, 0)$ is given by

$$G(\hat{g})(x, 0) = p \circ r \circ (g(x), dg(e_1(x)), dg(e_2)(x)) = p \circ r \circ f_{dg}(x, 0),$$

where $r : \mathbf{GL}(3, \mathbf{C}) \to \mathbf{U}(3)$ is a retraction. Hence

$$[\mu(\hat{g})] = (\det{}^2 \circ G(\hat{g}))^* w = f_{dg}^* \circ r^* \circ p^* \circ (\det{}^2)^* w.$$

It is readily checked that $p^* \circ (\det^2)^* w = 2u$ and, of course, $r^* = id$ so

$$[\mu(\hat{g})] = 2 f_{dg}^* u.$$

Thus $\pi_0(I_{Leg}(T^2, S^5))$ is classified by half the Maslov class. □

The proof of the Theorem 1 is a consequence of this lemma combined with the fact that there actually exists a minimal Legendrian embedding of the 2-torus into $S^5$.

## 3.2 The proof of Theorem 2

Since any two maximal tori are conjugate, we may assume that $K$ is the diagonal subgroup of $\mathbf{SU}(3)$. The principal $K$-orbits are clearly 2-tori, minimal by holomorphicity of the $K$-action on $\mathbb{C}^3$, flat by their own homogeneity, and Legendrian by a simple computation (since these orbits are homogeneous under a subgroup of $\mathbf{SU}(3)$, it is enough to check that the tangent space at one single point is Legendrian).

Conversely, we want to show that a given flat minimal Legendrian immersion $x : T^2 \to S^5$ must be obtainable as above. For this purpose we shall use the method of moving frames of É. Cartan. The space of unitary frames $(e_0, e_1, e_2)$ of $\mathbb{C}^3$ can be naturally identified with the Lie group $\mathbf{U}(3)$. Then we have

$$de_\alpha = \sum_\beta \omega_\alpha^\beta e_\beta, \qquad \omega_\beta^\alpha + \bar{\omega}_\alpha^\beta = 0,$$

where the $\omega_\alpha^\beta$ $(0 \le \alpha, \beta \le 2)$ give a basis for the Maurer-Cartan forms on $\mathbf{U}(3)$. We write

$$\omega_\beta^\alpha = \varphi_\beta^\alpha + i\psi_\beta^\alpha,$$

where $\varphi_\beta^\alpha$ and $\psi_\beta^\alpha$ are real valued forms.

Consider the fibering $\mathbf{U}(3) \to S^5$, $(e_0, e_1, e_2) \mapsto e_0$. We seek to construct a distinguished global lift $\tilde{x} : S^5 \to \mathbf{U}(3)$ of $x$ which is adapted to its geometry and then look at the pull-back of the Maurer-Cartan forms under $\tilde{x}$, or, what amounts to the same, look at the Maurer-Cartan forms on $x(T^2)$.

We then only consider frames $(e_0, e_1, e_2)$ such that $e_0 = x$ and $e_1$, $e_2$ are tangent vectors to $x(T^2)$ at $x$. Since $x$ is Legendrian,

$$\psi_0^\alpha = \langle dx, ie_\alpha \rangle = 0 \qquad \text{for } \alpha = 0, 1, 2,$$

where $\langle \cdot, \cdot \rangle$ denotes the real inner product in $\mathbb{C}^3 \cong \mathbb{R}^6$. Moreover, since $T^2$ is globally flat, $\tilde{x}$ can be constructed so that $(e_1, e_2)$ is parallel. Therefore we also have that

$$\varphi_2^1 = \langle de_2, e_1 \rangle = 0$$

and then

$$d\varphi_0^\alpha = 0 \qquad \text{for } \alpha = 1, 2, \tag{2}$$

because $d\varphi_0^\alpha(e_1, e_2) = -\varphi_0^\alpha([e_1, e_2]) = -\varphi_0^\alpha(de_2(e_1)) = -\langle de_2(e_1), e_\alpha \rangle = -\varphi_2^\alpha(e_1) = 0$. Now we can write (on $x(T^2)$):

$$\begin{cases} dx &= \quad \varphi_0^1 e_1 + \varphi_0^2 e_2, \\ de_1 &= -\varphi_0^1 x + i\psi_1^1 e_1 + i\psi_1^2 e_2, \\ de_2 &= -\varphi_0^2 x + i\psi_1^2 e_1 + i\psi_2^2 e_2. \end{cases} \tag{3}$$

Exterior differentiation of the first equation in (3) and use of the other equations combined with (2) yields

$$\begin{aligned} \psi_1^1 \wedge \varphi_0^1 + \psi_2^1 \wedge \varphi_0^2 &= 0, \\ \psi_1^2 \wedge \varphi_0^1 + \psi_2^2 \wedge \varphi_0^2 &= 0. \end{aligned}$$

By Cartan's lemma we can put

$$\psi^\alpha_\beta = \sum_\gamma h^\alpha_{\beta\gamma}\varphi^\gamma_0,$$

where $\alpha$, $\beta$, $\gamma = 1, 2$ and $h^\alpha_{\beta\gamma} = h^\gamma_{\beta\alpha}$. As the immersion is minimal we also have that

$$h^\alpha_{11} + h^\alpha_{22} = 0 \qquad \text{for } \alpha = 1, 2.$$

Set $a = -h^1_{11}$, $b = h^1_{12}$. Now we have

$$\begin{aligned} \psi^1_1 &= -a\varphi^1_0 + b\varphi^2_0 = -\psi^2_2, \\ \psi^1_2 &= b\varphi^1_0 + a\varphi^2_0 = \psi^2_1. \end{aligned} \tag{4}$$

At this juncture it is convenient to complexify $\mathbb{R}^6$. We write $\mathbb{C}^6 = \mathbb{R}^6 \oplus j\mathbb{R}^6$ and linearly extend the complex structure of $\mathbb{R}^6$ to $\mathbb{C}^6$ so that $i$ and $j$ "commute". Set

$$\begin{aligned} E &= e_1 + je_2, \\ \Phi &= \varphi^1_0 + j\varphi^2_0, \\ \Psi &= \psi^1_1 + j\psi^2_1. \end{aligned}$$

Equations (3) now can be summarized as

$$\left\{ \begin{aligned} dx &= \tfrac{1}{2}(\bar{\Phi}E + \Phi\bar{E}), \\ dE &= -\Phi x + i\Psi\bar{E}. \end{aligned} \right. \tag{5}$$

Taking exterior derivatives of (5) and using (2) gives that

$$d\Phi = \bar{\Phi} \wedge \Psi = d\Psi = \Psi \wedge \bar{\Psi} + \frac{1}{2}\Phi \wedge \bar{\Phi} = 0. \tag{6}$$

Let $z$ be a holomorphic coordinate on $T^2$, so that we can write

$$\Phi = \lambda(z)dz, \qquad \lambda \neq 0.$$

Define the complex valued function $f(z) = a(z) + jb(z)$ on $T^2$. We shall show that $f$ is a holomorphic function.

This is to be a consequence of the above equations. In fact, (4) says that

$$\Psi = -\bar{f}\bar{\Phi}, \tag{7}$$

and then, exterior differentiation of (7) combined with (6) gives that

$$df \wedge dz = 0,$$

which proves our thesis. Since $T^2$ is a compact manifold, it follows that $f$ is constant. It also follows from (7) and the last of the equations in (6) that

$$|f|^2 = \frac{1}{2}. \tag{8}$$

So far the vector $E$ has been defined up to the transformation

$$E \mapsto E^* = e^{j\tau} E,$$

where $\tau$ is real and *constant.* Under such a change, $\Phi$ transforms according to

$$\Phi \mapsto \Phi^* = e^{j\tau}\Phi,$$

so, inspection of the equation

$$dE = -\Phi x - i\bar{f}\bar{\Phi}\bar{E}$$

shows that $f$ is changed according to

$$f \mapsto f^* = e^{-3j\tau} f.$$

Choose $\tau$ in such a way that $f(z)$ is real and positive, and hence identically equal to $1/\sqrt{2}$ by (8). Of course, the corresponding $E$ is now defined up to multiplication by $e^{\frac{2\pi j}{3}}$. Anyhow, the structural equations (5) read

$$\begin{cases} dx &= \frac{1}{2}(\bar{\Phi}E + \Phi\bar{E}), \\ dE &= -\Phi x - \frac{i}{\sqrt{2}}\bar{\Phi}\bar{E}. \end{cases}$$

It follows from this and a fundamental theorem of Lie (see [3], (1.3), p. 780) that the element of $\mathbf{U}(3)$ that fixes a given point in $x(T^2)$ and rotates by $\frac{2\pi}{3}$ in the tangent plane to $x(T^2)$ at that point induces to an isometry of $x(T^2)$. In particular the isotropy subgroup of $T^2$ in the induced metric at any point contains a cyclic group of order 3. Therefore the conformal class of the induced metric on $T^2$ is fixed as being flat hexagonal.

We next go back to equations (3) which now can be rewritten as

$$\begin{cases} dx &= \varphi_0^1 e_1 + \varphi_0^2 e_2, \\ de_1 &= -\varphi_0^1 x - \frac{i}{\sqrt{2}}\varphi_0^1 e_1 + \frac{i}{\sqrt{2}}\varphi_0^2 e_2, \\ de_2 &= -\varphi_0^2 x + \frac{i}{\sqrt{2}}\varphi_0^2 e_1 + \frac{i}{\sqrt{2}}\varphi_0^1 e_2. \end{cases} \tag{9}$$

Define $u = de_2(e_2) = -x + \frac{i}{\sqrt{2}}e_1$ and observe that $du(e_2) = -\frac{3}{2}e_2$. It follows that the integral curves of $e_2$ are planar circles of radius $||de_2(e_2)||^{-1} = \sqrt{2/3}$, and hence, the induced metric on $T^2$ is completely determined. Equations (9) now specify the Maurer-Cartan forms on $x(T^2)$ and then it follows from the above quoted theorem of Lie that we have rigidity, namely any two flat minimal Legendrian immersions of the 2-torus into $S^5$ are $\mathbf{U}(3)$-congruent. As the $\mathbf{U}(3)$-action on $S^5$ is ineffective, it is clear that the congrunce may be taken to be in $\mathbf{SU}(3)$. Now our immersion $x : T^2 \to S^5$ is $\mathbf{SU}(3)$-congruent to a $K$-orbit as above. It follows that $x(T^2)$ is an orbit of a $\mathbf{SU}(3)$-conjugate of $K$ and this finishes the proof of the theorem.

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