Some results on extension of maps and applications

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Dedicated to Professor Gilberto Loibel, in memorian.

This paper concerns extension of maps using obstruction theory under a
non-classical viewpoint. It is given a classification of homotopy classes of maps and
as an application it is presented a simple proof of a theorem by Adachi about
equivalence of vector bundles. Also it is proved that, under certain conditions, two
embeddings are homotopic up to surgery if and only if the respective normal bundles
are *SO*-equivalent.

1. Introduction

The classification of embeddings between manifolds is an interesting problem in
topology which have been studied extensively. This paper is devoted to classification
of embeddings between manifolds, up to surgery.

Two embeddings $f, g \colon M \to N$ between manifolds are homotopic up to surgery
on N if it is possible to make a finite number of surgeries on N outside of the images

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of f and g obtaining a new manifold N' and maps $f', g' \colon M \to N'$ such that f' and g' are homotopic.

Consider the case where M and N are closed orientable C^∞ manifolds of dimensions m and n, respectively, and $m \leqslant (n-2)/2$. Then we ask on which conditions f and g are homotopic up to surgery on N? In [3], the authors showed that if $M = \mathbb{S}^m$ or N is a π-manifold then f and g are homotopic up to surgery on N.

In this paper, we prove the following theorem.

THEOREM 1.1. *If M and N are two closed orientable manifolds of dimensions m and n, respectively, with $m \leqslant (n-2)/2$, such that:*

(a) *$H^{4k}(M; \mathbb{Z})$ are free groups, for all $k \geqslant 1$,*

(b) *$H^{8k+1}(M; \mathbb{Z}_2) = H^{8k+2}(M; \mathbb{Z}_2) = 0$, for all $k \geqslant 1$,*

then two embeddings $f, g \colon M \to N$ are homotopic up to surgery on N if and only if the normal bundles ν_f and ν_g are equivalent as orientable vector bundles.

Also, as an application of this theory we give a simple proof of a nice result of Adachi [1], about equivalence of vector bundles.

This paper is organized as follows: in §2, basic and essentially known results of homotopy theory are considered, and in §3, applications of the theory are given.

2. Homotopy of maps

Let X be a CW-complex and Y an n-simple CW-complex, for some $n \geqslant 1$. For an abelian group G, let $\mathcal{K} = K(G, n)$ be an Eilenberg–MacLane space and $i_n \in H^n(\mathcal{K}; G)$ an n-characteristic element.

Given $u \in H^n(Y; G)$, by [5, 8.1.10] there exists a map $\varphi \colon Y \to \mathcal{K}$ such that $\varphi^*(i_n) = u$. The induced homomorphism $\varphi_* \colon \pi_n(Y) \to \pi_n(\mathcal{K})$ gives a coefficient homomorphism

$$\varphi_u^p \colon H^p(X; \pi_n(Y)) \to H^p(X; \pi_n(\mathcal{K})), \text{ for all } p. \tag{2.1}$$

THEOREM 2.1. *Let $f, g \colon X^{(n+1)} \to Y$ be homotopic maps over $X^{(n-1)}$ satisfying $f^*(u) = g^*(u)$. If φ_u^n is a monomorphism, then f and g are homotopic maps over $X^{(n)}$.*

Proof. Let $H \colon X^{(n-1)} \times I \to Y$ be a homotopy between the restricted maps $f_{n-1} = f|X^{(n-1)}$ and $g_{n-1} = g|X^{(n-1)}$, and consider $d(f_n, H, g_n) \in H^n(X; \pi_n(Y))$ the obstruction to the extension of H to a homotopy between $f_n = f|X^{(n)}$ and $g_n = g|X^{(n)}$.

Since $\varphi_u^n(d(f_n, H, g_n)) = d(\varphi \circ f_n, \varphi \circ H, \varphi \circ g_n)$ and $f^*(u) = g^*(u)$, then $(\varphi \circ f)^*(i_n) = (\varphi \circ g)^*(i_n)$. Hence, $d(f_n, H, g_n) = 0$ and so the result follows. $\square$

THEOREM 2.2. *Given a map $f \colon X^{(n+1)} \to Y$ and $\alpha \in H^n(X^{(n+1)}; G)$, there exists a map $g \colon X^{(n+1)} \to Y$ such that $g^*(u) = \alpha$ and $g|X^{(n-1)} = f|X^{(n-1)}$ if and only if $\alpha - f^*(u)$ belongs to the image of φ_u^n.*

Proof. Let $c\colon X^{(n+1)} \to Y$ be a constant map. Suppose that there exists $g\colon X^{(n+1)} \to Y$ as stated in theorem 2.2. Then, one has $\varphi_u^n(d(f_n, g_n)) = d(c, \varphi \circ g_n) - d(c, \varphi \circ f_n) = \alpha - f^*(u)$.

Conversely if $\alpha - f^*(u)$ belongs to the image of φ_u^n, let $\beta \in H^n(X; \pi_n(Y))$ be such that $\varphi_u^n(\beta) = \alpha - f^*(u)$.

It follows then the existence of $g\colon X^{(n+1)} \to Y$ such that $d(f_n, g_n) = \beta$ and $g^*(u) = \varphi_u^n(d(f_n, g_n)) + d(c, \varphi \circ f_n) = \alpha$. $\qquad\square$

THEOREM 2.3. *Consider* $f\colon X^{(n)} \to Y$ *and* $\alpha \in H^n(X^{(n+1)}; G)$. *If* φ_u^n *is an epimorphism and* φ_u^{n+1} *is a monomorphism, then there exists* $g\colon X^{(n+1)} \to Y$ *such that* $g|X^{(n-1)} = f|X^{(n-1)}$ *and* $g^*(u) = \alpha$.

Proof. Let $o(f) \in H^{n+1}(X; \pi_n(Y))$ be the obstruction to the extension of f_{n-1} to $(n+1)$-skeleton. Since $\varphi_u^{n+1}(o(f)) = o(\varphi \circ f) = 0$ and φ_u^{n+1} is a monomorphism we have that $o(f) = 0$. Then, there exists an extension of f_{n-1} to $X^{(n+1)}$ and so, applying theorem 2.2 to it, the proof is finished. $\qquad\square$

PROPOSITION 2.4. *Assuming that* Y *is n-simple for all* $n \leqslant \dim X$, $H^i(X; \pi_i(Y)) = 0$ *for* $i \neq n$, $H^{i+1}(X; \pi_i(Y)) = 0$ *for* $i > n$, *and* φ_u^n *is a monomorphism, then the map* $E\colon [X, Y] \to H^n(X; G)$ *defined by* $E([f]) = f^*(u)$ *is injective and* $\operatorname{im} E = \operatorname{im} \varphi_u^n \approx H^n(X; \pi_n(Y))$.

Proof. Since $H^i(X; \pi_i(Y)) = 0$, for $i < n$, any two maps $f, g\colon X \to Y$ are homotopic over the $(n-1)$-skeleton. If $f^*(u) = g^*(u)$ and φ_u^n is a monomorphism, it follows from theorem 2.1 that f and g are homotopic over the n-skeleton. Since $H^i(X; \pi_i(Y)) = 0$ for $i > n$, then f and g are homotopic and so E is injective.

Given $f\colon X \to Y$ then f is homotopic to a constant map over the $(n-1)$-skeleton. It follows from theorem 2.2 that $E([f]) = f^*(u)$ belongs to the image of φ_u^n.

If $\alpha \in H^n(X; G) = H^n(X^{(n+1)}; G)$, by theorem 2.2 there exists $h\colon X^{(n+1)} \to Y$ such that $h^*(u) = \alpha$ and h is constant over the $(n-1)$-skeleton. Since $H^{i+1}(X; \pi_i(Y)) = 0$ for $i > n$, there exists $f\colon X \to Y$ with $f^*(u) = \alpha$. Hence $\operatorname{im} E = \operatorname{im} \varphi_u^n$. $\qquad\square$

THEOREM 2.5. *Suppose that* Y *is n-simple, for all* $n \leqslant \dim X$ *and* $J = \{n : H^n(X; \pi_n(Y)) \neq 0\} \neq \emptyset$. *For each* $n \geqslant 1$, *let* G_n *be an abelian group,* $u_n \in H^n(Y; G_n)$ *and* $\varphi_{u_n}^n\colon H^n(X; \pi_n(Y)) \to H^n(X; G_n)$ *as in* (2.1). *If* $\varphi_{u_n}^n$ *is a monomorphism, then the function*

$$E\colon [X, Y] \to \prod_{n \in J} H^n(X; G_n)$$

given by $E([f]) = \prod_{n \in J} f^*(u_n)$ *is injective (where* $\prod$ *denotes the Cartesian product).*

Proof. Let p be the least positive integer such that $H^p(X; \pi_p(Y)) \neq 0$. Since $H^i(X; \pi_i(Y)) = 0$, $i < p$, any two maps $f, g\colon X \to Y$ are homotopic over $X^{(p-1)}$.

Suppose $f^*(u_p) = g^*(u_p)$. Since $\varphi_{u_p}^p$ is a monomorphism, it follows from theorem 2.1 that f and g are homotopic over $X^{(p)}$ and so applying theorem 2.1 successively, the result follows. $\qquad\square$

Now, we present some interesting consequences of theorem 2.5.

Let $\mathcal{K} = K(H_n(Y), n)$ be an Eilenberg–MacLane space. Then, the map $\varphi \colon Y \to \mathcal{K}$ such that $\varphi^*(i_n) = u_n$ induces the Hurewicz homomorphism

$$\varphi_* = \mathfrak{X}(Y) \colon \pi_n(Y) \to \pi_n(\mathcal{K}) = H_n(Y).$$

Let us consider

$$\mathfrak{X}_n^p(Y) = \varphi_{u_n}^p \colon H^p(X; \pi_n(Y)) \to H^p(X; H_n(Y))$$

the coefficient homomorphism as defined in (2.1).

Let $u_n \in H^n(Y; H_n(Y))$ be an element such that $\langle u_n, x \rangle = x$, for all $x \in H_n(Y)$.

THEOREM 2.6. *If $\mathfrak{X}_n^n(Y)$ is a monomorphism for all $n \in J$ then the function E is injective.*

Proof. Let $\mathcal{K} = K(H_n(Y), n)$ and observe that the commutative diagram

$$
\begin{array}{ccc}
\pi_n(Y) & \xrightarrow{\;\mathfrak{X}(Y)\;} & H_n(Y) \\
{\scriptstyle \varphi_*}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_*} \\
\pi_n(\mathcal{K}) & \xrightarrow[\;\mathfrak{X}(\mathcal{K})\;]{} & H_n(\mathcal{K})
\end{array}
$$

induces for all n the commutative diagram

$$
\begin{array}{ccc}
H^n(X; \pi_n(Y)) & \xrightarrow{\;\mathfrak{X}_n^n(Y)\;} & H^n(X; H_n(Y)) \\
{\scriptstyle \varphi_{u_n}^n}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}^n} \\
H^n(X; \pi_n(\mathcal{K})) & \xrightarrow[\;\mathfrak{X}_n^n(\mathcal{K})\;]{} & H^n(X; H_n(\mathcal{K})).
\end{array}
$$

Then, the result follows from theorem 2.5. $\square$

COROLLARY 2.7. *Suppose $\mathfrak{X}_n^n(Y)$ is a monomorphism and $H_{n-1}(X)$ a free group, for all $n \in J$. Then, two maps $f, g \colon X \to Y$ are homotopic if and only if for each $n \in J$ the induced homomorphisms $f_*, g_* \colon H_n(X) \to H_n(Y)$ are equal.*

If Y is $(n-1)$-connected ($\pi_1(Y)$ abelian if $n = 1$) and $u \in H^n(Y; \pi_n(Y))$ is the element corresponding to the inverse of the Hurewicz homomorphism

$$\mathfrak{X}(Y) \colon \pi_n(Y) \to H_n(Y)$$

then $\varphi_* \colon \pi_n(Y) \to \pi_n(\mathcal{K})$ is the identity map, where $\varphi \colon Y \to \mathcal{K}$ is such that $\varphi^*(i_n) = u$. It follows that φ_u^p is an isomorphism for all X and p. This is true if u is any n-characteristic element.

Let Y and Z be topological spaces and $\varphi\colon Y \to Z$ a map. The following commutative diagram

$$
\begin{array}{ccc}
\pi_n(Y) & \xrightarrow{\ \varphi_*\ } & \pi_n(Z) \\
\Big\downarrow{\scriptstyle\mathfrak{X}(Y)} & & \Big\downarrow{\scriptstyle\mathfrak{X}(Z)} \\
H_n(Y) & \xrightarrow[\varphi_*]{} & H_n(Z)
\end{array}
$$

induces for all topological space X, the commutative diagram

$$
\begin{array}{ccc}
H^p(X;\pi_n(Y)) & \xrightarrow{\ \varphi^p\ } & H^p(X;\pi_n(Z)) \\
\Big\downarrow{\scriptstyle\mathfrak{X}_n^p(Y)} & & \Big\downarrow{\scriptstyle\mathfrak{X}_n^p(Z)} \\
H^p(X;H_n(Y)) & \xrightarrow[\varphi^p]{} & H^p(X;H_n(Z)).
\end{array}
$$

THEOREM 2.8. *Let X and Y be CW-complexes and Y i-simple, for all $i \leqslant \dim X$. If $\mathfrak{X}_i^i(Z) \circ \varphi^i \colon H^i(X;\pi_i(Y)) \to H^i(X;H_i(Z))$ are monomorphisms, then φ induces an injective map $\varphi_\#\colon [X,Y] \to [X,Z]$ defined by $\varphi_\#([f]) = [\varphi \circ f]$.*

Proof. Consider $\mathcal{K} = K(H_i(Z),i)$, $u_Z^i \in H^i(Z;H_i(Z))$ for all i and $E'\colon [X,Z] \to \prod_i H^i(X;H_i(Z))$ defined by $E'([\varphi \circ f]) = \prod_i(\varphi \circ f)^*(u_Z^i)$.

Let us define $E\colon [X,Y] \to \prod_i H^i(X;H_i(Z))$ by $E([f]) = \prod_i f^*(u_Y^i)$ where $u_Y^i = \varphi^*(u_Z^i)$. So we have the commutative diagram

$$
\begin{array}{ccc}
[X,Y] & \xrightarrow{\ \ E\ \ } & \prod_i H^i(X;H_i(Z)) \\
{\scriptstyle\varphi_\#}\Big\downarrow & \nearrow{\scriptstyle E'} & \\
[X,Z] & &
\end{array}
$$

and since $\mathfrak{X}_i^i(Z) \circ \varphi^i$ are monomorphisms for all i, it follows from theorem 2.5 that E is injective and so, $\varphi_\#$ is injective as well. $\qquad\square$

3. Applications

Given an O-stable vector bundle $\xi \to B$ where O is the stable orthogonal group, let $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$ and $p_k(\xi) \in H^{4k}(B;\mathbb{Z})$ be the Stiefel–Whitney and Pontrjagin classes of ξ, respectively.

Let $\gamma \to BO$ be the O-universal bundle and let $w_i = w_i(\gamma)$ and $p_k = p_k(\gamma)$ be the universal Stiefel–Whitney and Pontrjagin classes, respectively.

Write $\pi_n = \pi_n(BO)$, $\mathcal{K}_n = K(\pi_n, n)$ and let $i_n \in H^n(\mathcal{K}_n; \pi_n)$ be the element corresponding to the inverse of Hurewicz homomorphism.

Also, consider the maps $\varphi_n \colon BO \to \mathcal{K}_n$ for $n \geqslant 1$ satisfying

$$\varphi_n^*(i_n) = \begin{cases} w_n, & n = 1, 2, \\ p_k, & n = 4k, \ k \geqslant 1, \end{cases}$$

and denote by $\varphi_\# \colon \pi_n(BO) \to \pi_n(\mathcal{K}_n)$ the homomorphism induced by φ_n, which induces the coefficient homomorphism

$$\varphi_n^n \colon H^n(X; \pi_n(BO)) \to H^n(X; \pi_n),$$

for any topological space X.

LEMMA 3.1. *Let X be a topological space satisfying:*

1. *if $x \in H^{4k}(X; \mathbb{Z})$, $x \neq 0$, and $k \geqslant 1$, then $(2k-1)!a_k x \neq 0$, where*

$$a_k = \begin{cases} 1, & \text{for } k \text{ even,} \\ 2, & \text{for } k \text{ odd.} \end{cases}$$

2. $H^{8j+1}(X; \mathbb{Z}_2) = H^{8j+2}(X; \mathbb{Z}_2) = 0$, *for each $j \geqslant 1$.*

Under these conditions, the coefficient homomorphism φ_n^n is a monomorphism for $n = 1, 2$ or $n = 4k$, with $k \geqslant 1$.

Proof. For $n = 1, 2$ the proof is straightforward since $\pi_n(BO) = \mathbb{Z}_2$.

For $n = 4k$, with $k \geqslant 1$, the homomorphism $\varphi \colon \pi_n(BO) \to \mathbb{Z}$ is given by $\varphi(\alpha) = (2k-1)!a_k$, where α is the generator of $\pi_n(BO)$ (see [1]).

Consider the long exact sequence

$$\cdots \to H^{n-1}(X; H) \xrightarrow{\beta} H^n(X; \mathbb{Z}) \xrightarrow{\varphi^n} H^n(X; \mathbb{Z}) \to H^n(X; H) \to \cdots$$

where $H = \mathbb{Z}/\operatorname{im}\varphi$ and β is the Bockstein operator. We see that there are no non-zero elements of $H^n(X; \mathbb{Z})$ belonging to the image of β, because if there is such an element we would have $(2k-1)!a_k x = 0$.

Then, φ^n are monomorphisms, for $n = 4k$ with $k \geqslant 1$. $\qquad\square$

THEOREM 3.2 Adachi. *Let X be a CW-complex satisfying conditions 1 and 2 of lemma 3.1 and let ξ_1 and ξ_2 be two O-stable vector bundles over X. Then, ξ_1 and ξ_2 are equivalent if and only if $w_1(\xi_1) = w_1(\xi_2)$, $w_2(\xi_1) = w_2(\xi_2)$ and $p_k(\xi_1) = p_k(\xi_2)$, for every $k \geqslant 1$.*

Proof. Let $f_1, f_2 \colon X \to BO$ be the classifying maps for ξ_1 and ξ_2. Thus,

$$f_1^*(w_i) = w_i(\xi_1) = w_i(\xi_2) = f_2^*(w_i), \ i = 1, 2,$$
$$f_1^*(p_k) = p_k(\xi_1) = p_k(\xi_2) = f_2^*(p_k), \ k \geqslant 1,$$

and since $H^n(X; \pi_n) = 0$ for $n \neq 1, 2$ and $n \neq 4k$, $k \geqslant 1$ and φ^n is a monomorphism in these dimensions, it follows from theorem 2.5 that f_1 is homotopic to f_2. $\qquad\square$

REMARK 1. Let us observe that in the Adachi's proof [**1**, theorem 5], using results on the characteristic classes, a mapping F was explicitly constructed by the skeletonwise induction as being the homotopy between f_1 and f_2, with $f_1, f_2 \colon X \to BO$ the classifying maps for ξ_1 and ξ_2.

Let $f \colon \mathbb{S}^p \to N^n$ be an embedding with trivial normal bundle. Consider N' the manifold obtained by surgery on N along the embedding f. In this case, we say that a surgery of type $(p+1, n-p)$ was done on N'.

From [**4**], one has the following:

PROPOSITION 3.3. *If N is a compact orientable manifold then it is possible to obtain N' by surgeries of type $(p+1, n-p)$ on N, with $p \leqslant (n-2)/2$, such that the map $\varphi_i \colon \pi_i(N') \to \pi_i(BSO(n))$ induced by the classifying map for the tangent bundle of N' is injective for $i \leqslant (n-2)/2$.*

Let now $\varphi_{N\#} \colon [M, N] \to [M, BSO(n)]$ be the map defined by $\varphi_{N\#}([f]) = [\varphi_N \circ f]$, where $\varphi_N \colon N \to BSO(n)$ is the classifying map for the tangent bundle of N.

Next, we prove the main result of the paper.

Proof of Theorem 1.1. Let N' be the manifold obtained by surgery on N (of type $(p+1, n-p)$ with $p \leqslant (n-2)/2$ outside of the images of f and g, such that $\varphi_i \colon \pi_i(N') \to \pi_i(BSO(n))$, the induced of the classifying map of N', is injective for $i \leqslant (n-2)/2$ (see proposition 3.3). Moreover, since N' is 1-parallelizable (N' is orientable), by [**4**, theorem 3] we get that N' is 1-connected, and consequently N' is i-simple for all i.

Now, we show that the coefficient homomorphism

$$\mathfrak{X}_i^i(BSO(n)) \circ \varphi^i \colon H^i(M; \pi_i(N')) \to H^i(M; H_i(BSO(n))),$$

induced by φ_i and the Hurewicz homomorphism $\mathfrak{X}_i(BSO(n))$, is a monomorphism for $i \leqslant \dim M = m \leqslant (n-2)/2$ (cf. [**2**, theorem 3.1]).

If $i = 8k+1$ or $8k+2$, for $k \geqslant 1$, with $i \leqslant (n-2)/2$, then $\pi_i(BSO(n)) = \mathbb{Z}_2$ and since φ_i is injective it follows that $\pi_i(N')$ is either the trivial group or $\mathbb{Z}_2$. If $\pi_i(N') = \mathbb{Z}_2$, since $H^{8k+1}(M; \mathbb{Z}_2) = H^{8k+2}(M; \mathbb{Z}_2) = 0$ for $k \geqslant 1$ by assumption, it follows that $\mathfrak{X}_i^i(BSO(n)) \circ \varphi^i$ is a monomorphism.

For $n \geqslant 6$, observe that $\mathfrak{X}_2(BSO(n))$ and φ_2 are isomorphisms because $\pi_1(BSO(n)) = 0$ and $\pi_2(BSO(n)) = \mathbb{Z}_2$.

Consider $h \colon \mathbb{S}^{4k} \to BSO(n)$ a generator of $\pi_{4k}(BSO(n))$ and p_k a Pontrjagin class of $h^*(\gamma_n)$. Then, we obtain that $h^*(p_k) = (-1)^{k+1}(2k-1)!a_k s$, where s is a generator of $H^{4k}(\mathbb{S}^{4k})$ (see [**1**]) and a_k is defined in lemma 3.1(1).

Further, $h_*([\mathbb{S}^{4k}]) \neq 0$, where $[\mathbb{S}^{4k}] \in H_{4k}(\mathbb{S}^{4k})$ is the fundamental class of $\mathbb{S}^{4k}$. Thus, if β is a non-zero element of $\pi_{4k}(BSO(n))$ then $\mathfrak{X}_{4k}(\beta)$ is a multiple of $h_*([\mathbb{S}^{4k}])$, from which it follows that the Hurewicz homomorphism

$$\mathfrak{X}_{4k} \colon \pi_{4k}(BSO(n)) \to H_{4k}(BSO(n))$$

is a monomorphism, for $k \geqslant 1$.

Next, if $i = 4k$ with $i \leqslant (n-2)/2$, we have $\pi_i(BSO(n)) = \mathbb{Z}$, $H_{4k-1}(M)$ is torsion free and then φ^{4k} and $\mathfrak{X}^{4k}$ are monomorphisms. So, we conclude that $\mathfrak{X}^{4k} \circ \varphi^{4k}$ are monomorphisms also, for $4k \leqslant (n-2)/2$. For other k, the groups $\pi_i(BSO(n))$ are trivial.

By theorem 2.8, we conclude that $\varphi \colon N' \to BSO(n)$ induces the injective map

$$\varphi_{\#} \colon [M, N'] \to [M, BSO(n)].$$

In addition, since $\varphi_N \circ f \colon M \to BSO(n)$ classifies $f^*(\tau N)$, the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\ f,g\ } N \xrightarrow{\ \varphi_N\ } & BSO(n) \\
& \searrow_{f',g'} \quad \nearrow_{\varphi} & \\
& N' &
\end{array}
$$

guarantees that $\nu_f \simeq \nu_g \Leftrightarrow \varphi_N \circ f \simeq \varphi_N \circ g \Leftrightarrow \varphi \circ f' \simeq \varphi \circ g' \Leftrightarrow f' \simeq g'$. $\qquad \square$

Acknowledgements

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