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FAMILIES OF SURFACES: FOCAL SETS, RIDGES AND UMBILICS

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Abstract

In [8], [9] and [10] we have studied the flat geometry of surfaces and families of surfaces in 3-space, that is the geometry associated with contact between the surface and lines or planes. We consider in this paper the geometry associated with contact between surfaces and spheres in 3-space and give a list of all possible changes of the ridge and sub-parabolic sets which can occur in a generic 1-parameter family of surfaces. We also refine the work in [16] and provide a geometric characterisation of the simple singularities of the folding map.

RESUMO

Em artigos [8], [9], e [10] estudamos a geometria plana de famílias das superfícies no espaco euclidiano. Neste artigo estudamos a geometria esferica de famílias a 1-parâmetro das superfícies. Obtemos todas as mudanças génericas do conjunto *ridge* que é o conjunto dos pontos onde a função distância ao quadrado tem uma singularidade $A_{\geq 3}$.

Existe um resultado de dualidade de Bruce-Wilkinson [16] que relaciona as singularidades da função distância ao quadrado e da aplicação dobrá. Esta dualidade permite estudar a geometria plana da superfície focal e a discoberta do conjunto *sub-parabólico* na superfície que é a pre-imagem do conjunto parabólico da superfície focal. Obtemos todas as mudanças genéricas do conjunto *sub-parabólico* e damos uma caracterisação geometrica das singularidades simples da aplicação dobrá.

Families of surfaces: focal sets, ridges and umbilics

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1 Introduction

In [8], [9] and [10] we have studied the flat geometry of surfaces and families of surfaces in 3-space, that is the geometry associated with contact between the surface and lines or planes. This flat geometry provides information on the Gauss map and the dual of a surface, and the viewgraph in the viewsphere. Moreover a remarkable duality relates the two types of tangency [14], [5].

We shall consider in this paper the geometry associated with contact between surfaces and spheres in 3-space. This approach gives, for instance, information on the principal curvatures/directions of the points of the surface, the lines of curvature and the nature of umbilics on the surface (see [24]).

The contact between a surface and a sphere is described by the $\mathcal{K}$-classes of the distance squared function from the centre of the sphere. The locus of points on the surface where there is a sphere of curvature having an A_3 contact with the surface is labelled *ridge curve* by Porteous. Ridges are of great intrinsic interest geometrically and have recently attracted attention in computer vision (see for example [18], [19],[25], [26], [11]) as they provide robust features that can be marked on an evolving shape. Our interest in this paper is to determine all the possible transitions that can occur generically on the ridge set on a 1-parameter family of surfaces. This is motivated, for example, by the time dependent images that often arise in medical images. A comparative study of anatomical pictures of the same patient taken at different times may present changes on the ridges on an associated surface, and these changes may be of medical significance.

We shall give in §2 a list of all possible changes of ridges which can occur in a generic 1-parameter family of surfaces. The main tools employed will be transversality theorems concerning the so-called Monge-Taylor map and stratified Morse theory, or at least a version developed in [6].

Just as in the case of the flat geometry of surfaces there is a duality which has proved to be most illuminating [16]. The dual family of mappings is the family of folding maps parametrised by the oriented planes in 3-space. For a fixed plane,

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the associated folding map measures the local reflectional symmetry of the surface at its points. The key observation is that a surface is locally symmetric at a given point across planes which contain the normal to the surface at that point. Moreover this symmetry is most marked when those planes in addition contain a principal direction. (One can also view the family of folding maps as providing information on the contact between surfaces and 0-dimensional spheres.)

It turns out that the dual of the bifurcation set of the family of distance squared functions is the bifurcation set of the family of folding maps [16]. This result provides a powerful tool for exploring the geometry of the focal set, and in particular the flat geometry associated to it [16]. For instance, the locus of points on the surface whose images on the focal set are parabolic, called the *sub-parabolic line*, proved to be as important as the ridge. The sub-parabolic line can be characterised in several ways (see §3). Its generic structure is studied in [16], [23].

We give in §3, as in the case for ridges, all the possible changes of the sub-parabolic line that can occur in a generic 1-parameter family of surfaces.

In §4 we refine the work in [16, 29] and provide a geometric characterisation of the simple singularities of the folding map.

We would like to thank Richard Morris, both for providing software, the 'Liverpool Surface Modelling Package', which proved useful in our investigations, and for convincing us that one of our earlier models for transition did not occur.

As a general background for the singularity theory approach to differential geometry we suggest [28] and [24].

2 Ridges on Families of Surfaces

In what follows M denotes a smooth surface in $\mathbb{R}^3$. We consider the family of distance-squared functions on this surface, defined by

$$\begin{array}{rcl} d \ : \ M \times \mathbb{R}^3 & \to & \mathbb{R} \\ (p,u) & \mapsto & d(p,u) = ||p-u||^2. \end{array}$$

The map d is used to describe the contact between the surface M and 2-spheres in $\mathbb{R}^3$. (This approach is due to Thom but most of the basic work in this area was carried out by Porteous.) Clearly the sphere centred at u and with radius r, given by the equation $d(p,u) - r^2 = 0$, is tangent to the surface at some point p_0 precisely when $d(p_0,u) = r^2$, and the function $d(-,u) = d_u$ has a singularity at p_0. The contact is then described by the $\mathcal{K}$-equivalence class of the corresponding germ $d_u : M, p \to \mathbb{R}$.

There is a 1-parameter family of spheres tangent to M at p_0, namely those centred along the normal to M at p_0. At a general point along this normal the sphere and surface will have contact of type A_1.

Away from umbilic points, there are exactly two points on the normal where the sphere has an A_2 (or more degenerate) contact with the surface. These points correspond to the centres of curvature. As p_0 varies on the surface these centres of curvature sweep out the two sheets of the focal surface. (We shall adopt, as in [24], an arbitrary colouring to distinguish between sets associated to the two principal curvatures, say red for one and blue for the other.) There is a curve on the surface where one of the spheres of curvature has an A_3 contact. This occurs when the corresponding curvature function is extremal along its line of curvature. The A_3 points are called *ridge points* by Porteous; they are surface points corresponding to cuspidal edges on the focal set. The ridge points inherit the colour of their corresponding principal curvature, so we have blue ridges and red ridges on a surface. The ridge curve may be tangent to a line of curvature of the same colour. This happens when the sphere has an A_4 contact with the surface.

At a generic umbilic point, where all sectional curvatures are equal but non-zero, the distance squared function has a D_4-singularity. In this case the two centres of curvature coincide.

A transverality theorem in [20] shows that for a generic surface the distance squared function can only acquire the singularities described above ($A_k, 1 \leq k \leq 4$ and D_4), and that these are versally unfolded by this family. What we wish to do is to describe the way in which the various features (ridges, A_4 points etc) change in a generic family of embeddings. We shall describe in this section the transitions of the ridge curve in a generic 1-parameter family of surfaces.

Let M be a given surface. We will be interested in families of embeddings $f : M \times I \to \mathbb{R}^3$ where I is some open, connected and finite interval. So for each $t \in I$ the set $f_t(M)$ is an embedded surface in $\mathbb{R}^3$, and will be denoted by M_t. We first consider the contact singularities occurring generically in such families, beyond those listed above. To establish a list, and some other useful facts concerning unfoldings, we need a transversality theorem. See [28] for background information.

Theorem 2.1 *Let M be a compact surface, I an open interval. Let k be a positive integer and $\mathcal{S}$ an $\mathcal{A}$-invariant Whitney regular stratification of the multijet-space ${}_rJ^k(M, \mathbb{R})$. Let $Emb^\infty(M, I, \mathbb{R}^3)$ denote the open subset of the space of smooth mappings $f : M \times \mathbb{R} \to \mathbb{R}^3$ with $f_t : M \to \mathbb{R}^3$ an embedding for each t. Then the set of $f \in Emb^\infty(M, I, \mathbb{R}^3)$ with the jet-extension*

$$ {}_rj_1^k d \circ f : M^{(r)} \times \mathbb{R}^3 \times I \to {}_rJ^k(M, \mathbb{R}) $$

given by

$$ {}_rj_1^k d \circ f(p, t, u) = {}_rj^k(d_u \circ f_t)(p) $$

transverse to $\mathcal{S}$ is residual. (We can replace residual by open and dense if we ask for transversality over a compact subinterval J of I.)

Proof The proof is a consequence of Theorem 1.1 in [5]. The key fact enabling us to apply this result to the present situation is that the family of distance squared functions on the ambient space is $\mathcal{A}$-versal. This follows trivially from the definition.

The key consequence that we require is the following result.

Corollary 2.2 *Let M, I, J be as before (J a compact subinterval of I). Then for an open dense set of mappings f in $Emb^\infty(M, I, \mathbb{R}^3)$ there are finitely many points $\{t(1), \ldots, t(s)\}$ in the interval J such that*

(i) If $t \notin \{t(1), \ldots, t(s)\}$ then the only singularities of the distance squared functions d_u for the surface M_t are of type $A_{\leq 4}$ and D_4. Moreover these singularities are versally unfolded by the family $d \circ f_t : M \times \mathbb{R}^3 \to \mathbb{R}$.

(ii) If t is one of the $t(j)$ then either we have a singularity of type A_5 and D_5, or we have one listed in (i) above which is not versally unfolded by the family $d \circ f_{t(j)}$. All singularities (including these) are versally unfolded by the family $d \circ f : M \times \mathbb{R}^3 \times I \to \mathbb{R}$, defined by $(d \circ f)(p, t, u) = d_u(f_t(p)) = ||f_t(u) - u||^2$.

Proof The proof follows that of Corollary 3.2 in [8] for the family of height functions.

So if f is a generic family of embeddings then the spherical geometry of the family of surfaces $f_t(M) = M_t$ for $t \in J$, as determined by the families $d \circ f_t$, has finitely many catastrophic events. These are of two types: (i) some point of the surface $M_{t(j)}$ may have contact with its tangent sphere which is more degenerate than an A_4 or D_4 singularities (an A_5 or D_5) or (ii) the singularity may be of type $A_{\leq 4}$ and D_4 but not versally unfolded by the family d of distance squared functions alone. In case (ii) this means that the contact is one of the prescribed types, but the geometry associated to that singularity does not follow the standard model. We remark here that versality of the unfolding is *automatic* for $A_{\leq 3}$ singularities, so that the various possible changes corresponding to non-versal $A_{\leq 3}$ contact are ruled out.

The *full bifurcation* set of the family d consists of the centres from which the distance-squared function is not stable, *i.e.* not Morse. This can happen in two ways: either the function has a degenerate singularity (*i.e.* the corresponding point is a centre of curvature) or it may have two critical points at the same level. The latter corresponds to a bitangent sphere, and the locus of centres of such spheres is the *symmetry set.* For a generic surface the local form of the full bifurcation set is determined by the contact between the corresponding (bitangent) sphere and the surface, and one can describe the local structure of the corresponding subsets of the symmetry set. See [7].

Our principal concern is with the ridges, and it is of interest to note that these cannot be described using the transversality result above. Instead we use some more direct techniques from [4]. Of course we can expect that the singularities of the distance-squared family will have a bearing on the appearance of the ridge curve on the surface, but there are further ingredients.

We shall now describe the possible changes on the ridge set associated to the cases in (i) and those in (ii). We shall distinguish two cases depending on whether the point p_0 on the initial surface is an umbilic or not. (The associated changes on the focal set are descibed in [2].)

2.1 Changes on ridges at umbilic points

As in [4] let p be a point on a surface, and we choose a smooth unit normal vector field, and a unit tangent vector field in a neighbourhood U of p. This allows us to determine at each point near p an orthonormal set of co-ordinates, and at any point q we write our surface locally in 'Monge form' as the graph of a function f of two variables x and y in the given normal z-direction. Let V_3 denote the set of pairs (f_2, f_3) where f_2 is a quadratic form, and f_3 a cubic form in the variables x and y; we call V_3 a *Monge-Taylor space.* Taking the 3-jet of f we obtain a smooth map, the *Monge-Taylor map* $F : U \to V_3$. The set V_3 has a natural $SO(2)$-action given by change of (x, y) co-ordinates (a different choice of initial tangent vector field). A subset Z of V_3 which is of any geometric significance will be $SO(2)$-invariant. Moreover if Z is furnished with a Whitney regular stratification then for any generic M the map germ $M, p \to V_3$ will be transverse to the strata of Z. See [4] for details.

At an umbilic point all sectional curvatures coincide. At such a point we can write the surface locally in Monge form with terms up to order 3 in the form $\frac{\kappa}{2}(x^2 + y^2) + f_3(x, y)$ for some cubic form f_3. The cubic part of the relevant distance-squared function is simply a non-zero multiple of $f_3(x, y)$. The distance-squared function is of type D_4 if and only if this cubic form has three distinct roots. If all three are real it is an elliptic umbilic D_4^+ and we have an elliptic umbilic on the surface; if two of the roots are complex (conjugate) it is a hyperbolic umbilic D_4^-.

We are interested in the set of points on the surface where the distance-squared function has an $A_{\geq 3}$ (ridge point). It is proved in [4] that there is one smooth ridge through a generic hyperbolic umbilic and three (pairwise transverse) through an elliptic umbilic. It is also known that ridges change colour at an umbilic [24]. See Figure 1(ii). We wish to describe how these configurations change in a generic 1-parameter family.

We first note that any cubic form can be written as $Re(\alpha z^3 + \beta z^2 \bar{z})$ where α and β are complex numbers and $z = x + iy$. One can then show that any such cubic form is $SO(2)$-equivalent to one of the form

$$Re(z^3 + \beta z^2 \bar{z})$$

(or is $SO(2)$-equivalent to $Re(z^2\bar{z}) = x(x^2+y^2)$). So we can view the set of cubic forms as points in the $\beta-$plane. There are three subsets of this plane (at least) of interest. The first consists of the umbilics which are not of type D_4, *i.e.* those cubic forms with repeated roots. The second consists of those umbilics which are not versally unfolded by the family of distance-squared functions. (A short calculation shows that elliptic umbilics are automatically versally unfolded.) The third consists of umbilics at which two of the ridge lines (or more) are tangent. In [4] it is shown that these sets correspond to the following subsets of the w-plane (Figure 1(i)).

I: Umbilic not of type D_4: the set $\beta = 2e^{i\theta} + e^{-2i\theta}$.

II: D_4 singularity not versally unfolded: the set $|\beta| = 3$; this is also the locus of points where the Monge-Taylor map fails to be transverse to the D_4 stratum.

III: Tangent ridges: the set $|\beta| = 1$.

Figure 1 here

We know then when to expect changes on the ridges. We show below what these changes are.

Theorem 2.3 *In generic 1-parameter families the ridge lines of a surface change as follows in a neighbourhood of an umbilic:*

(I) Elliptic to hyperbolic change via a D_5 singularity (occurs across set I). See Figure 2(i).

(II) The birth of a pair of hyperbolic umbilics (occurs across set II). See Figure 2(ii).

(III) Tangent ridges at elliptic umbilic (occurs across set III). See Figure 2(iii).

In Figure 2, the umbilics are distinguished by heavy dots, and the models are well defined up to diffeomorphism of the surface.

Proof We shall explain this case in some detail, since the others are similar.

Given a point p on our surface and the germ of a family of embeddings $i : M \times \mathbb{R}, (p, 0) \to \mathbb{R}^3$ we obtain a family of Monge-Taylor maps $\tilde{F} : M \times \mathbb{R}, (p, 0) \to V_4$. (Of course we need to choose some corresponding family of orthogonal co-ordinates as in [4].) This will be transverse to the set of A_3-points, say Z, in V_4 for a generic family of embeddings. This essentially was the straightforward transversality result established in [4]. We then determine the diffeomorphism type of the inverse image $\tilde{F}^{-1}(Z)$ at $(p, 0)$. For a generic family of embeddings we expect the natural projection $\pi : \tilde{F}^{-1}(Z), (p, 0) \hookrightarrow M \times \mathbb{R}, (p, 0) \to \mathbb{R}$ to be generic, in the sense that it will be a stratified Morse function [6]. Usually however we can construct the module of vector fields on $M \times \mathbb{R}$ (locally just $\mathbb{R}^3$) tangent to to the germ $\tilde{F}^{-1}(Z)$. We can then make a classification of smooth functions $\mathbb{R}^3, 0 \to \mathbb{R}, 0$ up to diffeomorphisms in the source preserving $\tilde{F}^{-1}(Z)$. (We also often allow arbitrary changes of co-ordinate in the target.) We expect our projection to be a stable (or the least degenerate) germ in the classification. This is established by computing the conditions for the projection to be non-Morse (or non-stable etc.) and showing that the resulting set of embeddings can be avoided in 1-parameter families. In this paper we spare the reader the lengthy details. We note however that in one case, covered in §3, the map projection π turns out to be non-generic because of its geometric origins. We can, nevertheless, still identify the class of functions to which it belongs and complete the relevant classification.

We shall start with the transition III.

Since we wish to describe the ridges on M we will need the corresponding subset of V_3. For an elliptic umbilic we may suppose that $F(p)$ is given by

$$F(p) = \alpha(x^2 + y^2) + x(x + \beta_1 y)(x + \beta_2 y)$$

for some α, β_1, β_2 with $\alpha\beta_1\beta_2(\beta_1 - \beta_2) \neq 0$ (that is we can arrange for $F(p)$ to take this form by a choice of co-ordinates at p). We now wish to choose a transversal to the $SO(2)$-orbit through $F(p)$ to describe the local structure of the ridge set at $F(p)$.The set of such points in V_3 would then be locally diffeomorphic to the product of that in the transversal with $\mathbb{R}$. This transversal is given by

$$\{(f_2, f_3) = ((\alpha+a_0)x^2 + 2a_1xy + (\alpha+a_2)y^2), x(x+(\beta_1+b_1)y)(x+(\beta_2+b_2)y)\}.$$

The condition for an $A_{\geq 3}$ on the surface is that when one considers the 3-jet of the relevant distance-squared functions the quadratic part must be a perfect square and the cubic and quadratic parts must have a common factor. Taking the centre of the distance-squared function as $(0, 0, \gamma)$ this implies that the quadratic form

$$(1 - 2\gamma(\alpha + a_0))x^2 - 4\gamma a_1xy + (1 - 2\gamma(\alpha + a_2))y^2$$

is degenerate, and has a root in common with f_3. This gives three hypersurfaces with equations

$$a_1 = 0, \quad a_1(1 - (\beta_1 + b_1)^2) - (\beta_1 + b_1)(a_0 - a_2) = 0,$$

$$a_1(1 - (\beta_2 + b_2)^2) - (\beta_2 + b_2)(a_0 - a_2) = 0.$$

So the $A_{\geq 3}$-set, which we henceforth call the ridge-set, consists locally of three smooth hyperplanes meeting in the umbilic set given by $a_0 - a_2 = a_1 = 0$. This intersection is pairwise transverse at $F(p)$ if and only if in addition to the above conditions we have $(\beta_1\beta_2 + 1) \neq 0$. The condition in invariant terms means that the roots of the cubic f_3 are not orthogonal.

We now need to consider this orthogonal case, so in what follows $\beta_2 = -\beta_1^{-1}$. We need to analyse the ridge set near the point $F(p) = \alpha(x^2 + y^2) + x(x + \beta_1 y)(x + \beta_2 y)$. We change co-ordinates as follows: let $u_1 = a_0 - a_2$, $u_2 = a_1$, $u_3 = \beta_1 + b_1$, $u_4 = \beta_2 + b_2, u_5 = a_0 + a_2$, so that the three hypersurfaces are given as

$$u_2 = 0, \quad u_2(1 - u_3^2) - u_1u_3 = 0, \quad u_2(1 - u_4^2) - u_1u_4 = 0;$$

note that the u_5 term is absent. The last two equations are linear in u_1 and u_2, and the determinant of the coefficients is $(u_3 - u_4)(1 + u_3u_4)$. Since, near our base point, $u_3u_4(u_3 - u_4) \neq 0$ it is not difficult to see that the strata are given by:

$$u_1 = u_2 = 1 + u_3u_4 = 0; \quad u_1 = u_2 = 0, \ 1 + u_3u_4 \neq 0;$$

$$1 + u_3u_4 = u_2(1 - u_3^2) - u_1u_3 = 0, \ u_2 \neq 0$$

together with the remaining parts of the hypersurfaces and their complement.

This is clearly a Whitney regular stratification; indeed it is a locally smooth product stratification. So given a generic 1-parameter family we will obtain a three dimensional section, which will be transverse to this stratification (provided it is transverse to the 2-dimensional stratum given by $u_1 = u_2 = 1 + u_3u_4 = 0$), and any two such will yield a diffeomorphic intersection with the ridge set. Such a transversal is given by setting $u_4 = \beta_2$ and $u_5 = 0$. The resulting stratification is diffeomorphic to the configuration in (u, v, w)-space given by $uv(v - uw) = 0$.

We now need to consider the projection to the time parameter t. One would expect this to determine a stratified Morse function on this set. It is clear that the limits of the tangent spaces to the 1-dimensional strata at the base-point (the origin) are $Sp\{\partial/\partial u\}$ and $Sp\{\partial/\partial w\}$, while those of the 2-dimensional strata are $Sp\{\partial/\partial v, \partial/\partial w\}$, $Sp\{\partial/\partial u, \partial/\partial w\}$ and $Sp\{w\partial/\partial v + \partial/\partial u, u\partial/\partial v + \partial/\partial w\}$. Thus the germ of a submersion h at the origin is Morse if and only if $\partial h/\partial u(0)$ and $\partial h/\partial w(0)$ are both non-zero. It follows from [6] that there are two topological types of Morse function on this stratification with representatives $u + w$ and $u - w$. Indeed there is only one type, since reversing the signs of v and w preserves the stratification, and interchanges the two.

Actually it is not difficult to check that these are the correct pictures up to diffeomorphism. To do this we need to compute vector fields tangent to the given variety in (u, v, w)-space. First we note that there are two Euler type vector fields $u\partial/\partial u + v\partial/\partial v, v\partial/\partial v + w\partial/\partial w$. Using one of them, say the second, we see that we now only need to find fields which annihilate the defining equation. Writing such a field as a sum $a\partial/\partial u + b\partial/\partial v + c\partial/\partial w$ we obtain the equation

$$a(v^2 - 2uvw) + b(2uv - u^2w) + c(-u^2v) = 0.$$

From this we see that u divides a, and using the first Euler vector field we find that we may suppose $a = 0$. It follows that the set of all vector fields is generated by the Euler fields and $uv\partial/\partial v + (2v - uw)\partial/\partial w$. (Indeed it follows from [27] that this variety is a free divisor, *i.e.* that these vector fields generate the module of tangent fields freely.) It is now clear that the functions $u \pm w$ are stable in the sense described in [13], and our assertion follows quite easily.

We now turn to case I. For an umbilic of type D_5 the cubic form has a repeated root, and we may suppose that $F(p)$ is given by

$$F(p) = \alpha(x^2 + y^2) + x^2(x + \beta y)$$

for some α, β with $\alpha\beta \neq 0$ (that is we can arrange for $F(p)$ to take this form by a choice of co-ordinates at p). We now wish to choose a transversal to the $SO(2)$-orbit through $F(p)$ to describe the local structure of the ridge set at $F(p)$. This transversal is given by

$$\{(f_2, f_3) = ((\alpha + a_0)x^2 + 2a_1xy + (\alpha + a_2)y^2), (x^2 + b_1y^2)(x + (\beta + b_2)y))\}$$

The cubic has three factors, namely $(x \pm \sqrt{-b_1}y)$ and $(x + (\beta + b_2)y)$. Again one expects that the condition for an $A_{\geq 3}$ on the surface will yield three hypersurfaces:

$$a_1(1 - b_1) \pm \sqrt{-b_1}(a_0 - a_2) = 0, \quad a_1(1 - (\beta + b_2)^2) - (\beta + b_2)(a_0 - a_2) = 0.$$

However it is clearly better to put the first two expressions together, to obtain

$$a_1^2(1 - b_1)^2 + b_1(a_0 - a_2)^2 = 0.$$

So the ridge set consists again of these two hypersurfaces meeting in the umbilic set given by $a_0 - a_2 = a_1 = 0$. We change co-ordinates again: let $u_1 =$

$a_0 - a_2$, $u_2 = a_1$, $u_3 = b_1$, $u_4 = \beta + b_2$, $u_5 = a_0 + a_2$, so that the two hypersurfaces are given as

$$u_2^2(1-u_3)^2 + u_1^2 u_3 = 0, \quad u_2(1-u_4^2) - u_1 u_4 = 0.$$

The D_5 umbilics are given by $u_1 = u_2 = u_3 = 0$, and it is not difficult to determine the stratification. Again this is a locally smooth product, and a suitable transversal is given by taking $u_4 = \beta$ and $u_5 = 0$. Some elementary changes of co-ordinates brings the relevant set into the simpler form $u(v^2 - u^2 w) = 0$ in (u, v, w)-space. This is a Whitney umbrella together with a plane containing the line of self-intersection. The vector fields tangent to this variety are found as above to be generated by $u\partial/\partial u + v\partial/\partial v$, $v\partial/\partial v + 2w\partial/\partial w$, $u^2\partial/\partial v + 2v\partial/\partial w$, so again we have a free divisor, and the relevant stable function here is $u + w$. The sections are shown in Figure 2(i).

Transition II is similar, but easier.

Figure 2 here

2.2 Changes on ridges away from umbilic points

We now turn to the changes away from umbilics and write the 5-jet of the function f as $f_2 + f_3 + f_4 + f_5$, where each f_i is homogeneous of degree i in x and y. We can reduce f_2 using the $SO(2)$-action to the form $f_2 = a_0 x^2 + a_2 y^2$ and write $f_3 = \sum_{i=0}^{3} b_i x^{3-i} y^i$, $f_4 = \sum_{i=0}^{4} c_i x^{4-i} y^i$ and $f_5 = \sum_{i=0}^{5} d_i x^{5-i} y^i$.

Recall that a sphere of curvature, say that corresponding to the principal direction $(0,1)$, has an A_3, A_4 or $A_{\geq 5}$ contact with the surface M when the following holds:

A_3 : $b_3 = 0$; $b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) \neq 0$;

A_4 : $b_3 = 0$; $b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) = 0$, $b_1 b_2^2 - 2b_2 c_3(a_0 - a_2) + 4d_5(a_0 - a_2)^2 \neq 0$;

$A_{\geq 5}$: $b_3 = 0$; $b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) = 0$, $b_1 b_2^2 - 2b_2 c_3(a_0 - a_2) + 4d_5(a_0 - a_2)^2 = 0$.

For the calculations in this section we need to compute three vectors: the tangent to the $SO(2)$-orbit of the function f, and the two generators of the image of the tangent space by the differential of the Monge-Taylor map $F : M, p \to V_4$. A simple calculation gives the first vector as $v = y\frac{\partial f}{\partial x}(x,y) - x\frac{\partial f}{\partial y}(x,y)$, that is

$$\begin{aligned} v = \;& 2(a_0 - a_2)xy - b_1 x^3 + (3b_0 - 2b_2)x^2 y + (2b_1 - 3b_3)xy^2 + b_2 y^3 - c_4 x^4 \\ & + 2(2c_0 - c_2)x^3 y + 3(c_1 - c_3)x^2 y^2 + 2(c_2 - 2c_4)xy^3 + c_3 y^4. \end{aligned}$$

Following Proposition 2.2 in [4] the image of $dF(p)$ is generated by u_1 and u_2 with

$$\begin{aligned} u_1 = & \; j^4\{-f_{xx}(0,0)x - f_{xy}(0,0)y + f_x(x,y) - f_{xx}(0,0)f_x(x,y)f(x,y) - \\ & f_{xy}(0,0)f_y(x,y)f(x,y)\}, \\ = & \; 3b_0x^2 + 2b_1xy + b_2y^2 + 4(c_0 - a_0^3)x^3 + 3c_1x^2y + 2(c_2 - 2a_0^2a_2)xy^2 + c_3y^3 \\ & +5(d_0 - 2a_0^2b_0)x^4 + 4(d_1 - 2a_0^2b_1)yx^3 + 3(d_2 - 2b_2a_0^2 - 2a_0b_0a_2)x^2y^2 \\ & +2(d_3 - 2a_0^2b_3 - 2a_0a_2b_1)xy^3 + (d_4 - 2a_0a_2b_2)y^4 \end{aligned}$$

$$\begin{aligned} u_2 = & \; j^4\{-f_{xy}(0,0)x - f_{yy}(0,0)y + f_y(x,y) - f_{yy}(0,0)f_y(x,y)f(x,y) - \\ & f_{xy}(0,0)f_x(x,y)f(x,y)\} \\ = & \; b_1x^2 + 2b_2xy + 3b_3y^2 + c_1x^3 + 2(c_2 - 2a_0a_2^2)x^2y + 3c_3xy^2 + 4(c_4 - a_2^3)y^3 \\ & +(d_1 - 2a_0a_2b_1)x^4 + 2(d_2 - 2a_2^2b_0 - 2a_0a_2b_2)yx^3 \\ & +3(d_3 - 2b_1a_2^2 - 2a_0a_2b_3)x^2y^2 + 4(d_4 - 2a_2^2b_2)xy^3 + (d_5 - 2a_2^2b_3)y^4. \end{aligned}$$

We wish to find the condition that the Monge-Taylor map fails to be transverse to the $A_{\geq 3}$ set in V_4. Working in a transversal to the $SO(2)$ orbit given by $(a_0 + \overline{a_0})x^2 + (a_2 + \overline{a_2})y^2 + (f_3 + \overline{f_3}) + (f_4 + \overline{f_4})$ in V_4, where $\overline{f_3}$ (resp. $\overline{f_4}$) is a general cubic (resp. general quartic), the $A_{\geq 3}$-set is given by $b_3 + \overline{b_3} = 0$. The conditions for the tangent vectors to the $A_{\geq 3}$-stratum in the transversal, the tangent space to the orbit, and u_1 and u_2 to fail to span V_4 are

$$(a_0 - a_2)c_3 - b_1b_2 = 0 \;\text{ and }\; b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) = 0.$$

The last equation implies that we have an $A_{\geq 4}$ contact. The expression for the tangent to the ridge given in §4.5 shows that, away from umbilic and parabolic points, the conditions above are necessary and sufficient for the ridge curve to be singular. One can also show that the family of distance squared functions is no longer a versal unfolding of the A_4 singularity when $(a_0 - a_2)c_3 - b_1b_2 = 0$.

For a given generic 1-parameter family the Monge-Taylor map is transverse to the $A_{\geq 3}$ set, and in the product space $M \times I$ the ridge points form a surface. Projection to the time parameter yields two types of Morse sections of this surface as shown in Figure 3. The corresponding transitions on the focal surface are well known [2].

In the transversal to the $SO(2)$-orbit, the $A_{\geq 4}$ set is given by

$$\begin{gathered} b_3 + \overline{b_3} = 0, \\ (b_2 + \overline{b_2})^2 - 4(a_0 + \overline{a_0} - a_2 - \overline{a_2})(c_4 + \overline{c_4} - (a_2 + \overline{a_2})^3) = 0. \end{gathered}$$

This is a smooth set of codimension 2 in V_4. If we use the coefficients of the monomials as a set of coordinates in V_k, a calculation shows that the tangent space to the A_4 stratum in the space V_4 is given by the intersection of the kernels of the 1-forms

$$\begin{aligned} \xi_1 = & \; -b_2 \mathrm{d}a_1 + 2(a_0 - a_2)\mathrm{d}b_3 \\ \xi_2 = & \; (c_4 - a_2^3)\mathrm{d}a_0 - \tfrac{1}{2}(c_3 - \tfrac{b_1b_2}{(a_0 - a_2)})\mathrm{d}a_1 - ((c_4 - a_2^3) + 3a_2^2(a_0 - a_2))\mathrm{d}a_2 - \tfrac{b_2}{2}\mathrm{d}b_2 \\ & +(a_0 - a_2)\mathrm{d}c_4. \end{aligned}$$

The Monge-Taylor map fails to be transverse to the $A_{\geq 4}$ set if and only if there exists λ and μ with $\lambda u_1 + \mu u_2 \neq 0$ such that

$$\xi_1(\lambda u_1 + \mu u_2) = \xi_2(\lambda u_1 + \mu u_2) = 0$$

Since $\xi_1(u_2) = b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) = 0$ (A_4 singularity at the origin), the system has a non-zero solution when $\xi_1(u_1) = 0$ or $\xi_2(u_2) = 0$. We have

$$\begin{aligned}\xi_1(u_1) &= (a_0 - a_2)c_3 - b_1 b_2,\\ \xi_2(u_2) &= b_1 b_2^2 - 2b_2 c_3(a_0 - a_2) + 4d_5(a_0 - a_2)^2.\end{aligned}$$

Thus transversality fails at an A_4 in two ways: (i) non-transverse A_3 which is dealt with at the begining of this section, or (ii) at a singularity worse than A_4, *i.e.*, $A_{\geq 5}$. In the last case the Monge-Taylor map is generically transverse to the $A_{\geq 3}$ stratum and the ridge remains a smooth curve.

In a generic 1-parameter family, the Monge-Taylor map intersects the $A_{\geq 4}$ set in a smooth curve and the resulting transitions on the ridge at an A_5 are as shown in Figure 4.

Theorem 2.4 *In a generic 1-parameter family the ridge curve changes as follows away from umbilic points:*

(1). Morse transitions at a non-transverse A_3 (Figure 3). This occurs at A_4 singularity where the family of distance squared functions fails to be a versal unfolding.

(2). The birth of a pair of A_4 points on a smooth ridge at an A_5 transition (Figure 4).

Figure 3 here

Figure 4 here

In particular, ridges are created generically as a closed curve through a non-versal A_4 singularity of the distance squared function. Note that the parabolic set is also created as a closed curve through a non-versal A_3 singularity of the height function [8].

3 Sub-parabolic lines on families of surfaces

Let p_0 be a point on the surface M which is not an umbilic. The lines of curvature form an orthogonal net in a neighbourhood of p_0. We have seen in the previous section that the ridge is characterised as the set of points where a principal curvature is extremal along the associated lines of curvature. The locus of points where the principal curvature is extremal along the other line of curvature is also of great geometric importance. This locus is called the 'sub-parabolic line' in [16], [29], [23], and later the 'flexcord' in publications of Porteous. It can be characterised, via a computation of the first and second fundamental forms of the focal set, as the locus of points on the surface whose image is the parabolic curve on the focal set [17], [15]. A result in [23] also shows that the sub-parabolic line is the locus of points where the other lines of curvature are inflexional. However, as in the case of ridges, the sub-parabolic lines were first and are best described using singularity theory, and an associated family of maps [29].

We choose a coordinate system for the ambient space where p_0 is the origin and the normal at p_0 is along the z-direction. We associate to the plane $y = 0$ the 'folding map' of the ambient space, given by $g(x, y, z) = (x, y^2, z)$, and its restriction to the surface M. The induced map $g \circ i : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ is singular (a crosscap or worse) if and only if the folding plane $y = 0$ is a principal plane, that is one of the principal directions is along the y-axis. The symmetry of the surface with respect to the folding plane is described by the $\mathcal{A}$-class of the singularities of $g \circ i$. It is proved in [16] that for a general surface, only simple singularities of maps $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ whose $\mathcal{A}_e$-codimension is ≤ 3 occur. If we write the surface locally in Monge form $z = f(x, y)$ where the Taylor series for f starts

$$a_0x^2 + a_1xy + a_2y^2 + \textstyle\sum_{i=0}^{3} b_i x^{3-i}y^i + \sum_{i=0}^{4} c_i x^{4-i}y^i + \\ \textstyle\sum_{i=0}^{5} d_i x^{5-i}y^i + \sum_{i=0}^{6} e_i x^{6-i}y^i + \sum_{i=0}^{7} f_i x^{7-i}y^i + \ldots$$

then the singularities of $g \circ i$ are as follows. (See the following section for their normal forms.)

$$\begin{array}{rl}
\text{Crosscap}: & a_1 \neq 0; \\
B_1 = S_1: & a_1 = 0,\ b_1 \neq 0,\ b_3 \neq 0; \\
B_2: & a_1 = 0,\ b_1 \neq 0,\ b_3 = 0,\ 4b_1d_5 - c_3^2 \neq 0; \\
B_3: & a_1 = 0,\ b_1 \neq 0,\ b_3 = 0,\ 4b_1d_5 - c_3^2 = 0, \\
& 2b_1^3 f_7 - (2d_3d_5 + c_3c_5)b_1^2 + (c_3^2 d_3 - c_1c_3d_5)b_1 - c_1c_3^2 \neq 0; \\
S_2: & a_1 = 0,\ b_1 = 0,\ b_3 \neq 0,\ c_1 \neq 0; \\
S_3: & a_1 = 0,\ b_1 = 0,\ b_3 \neq 0,\ c_1 = 0,\ d_1 \neq 0; \\
C_3: & a_1 = 0,\ b_1 = 0,\ b_3 = 0,\ c_1 \neq 0,\ c_3 \neq 0.
\end{array}$$

Note that this is all for a folding in the plane $y = 0$; there is a 1-parameter family of such foldings in planes containing the normal to the surface at the given point. There is an interpretation of the above conditions in terms of the geometry of the focal set and the symmetry set [16].

S_1 general smooth point of focal set
S_2 parabolic smooth point of focal set
S_3 cusp of Gauss at smooth point of focal set
B_2 general cusp point of focal set
B_3 (cusp) point of focal set in closure of parabolic curve on symmetry set
C_3 intersection point of cuspidal-edge and parabolic curve on focal set.

It is clear then that the folding map has an S_2 singularity at p_0 if and only if this point is sub-parabolic, so the singularities of the folding map capture the flat geometry of the focal surface. This is due to the duality result we pointed out in the introduction and state below (Theorem 3.1). We can also see that the folding map recognises ridge points (which correspond to B_2-singularities) but consideration of the condition for an A_4-singularity of the distance squared function in 2.2 shows that the folding map does not distinguish such points on the ridge.

The folding map also reveals some fascinating geometry of the surface and its focal set at umbilics. At such points all directions are principal so folding the surface in any normal plane induces a map with a singularity of type cross-cap or worse. On the projective line $\mathbb{R}P^1$ of such directions, there may be 3 or 1 directions where the singularity is of type S_2 (resp. B_2), *i.e.* there are 3 or 1 sub-parabolic lines (resp. ridges) through an umbilic [16]. It turns out that the configuration of these sub-parabolic lines is closely related to that of the lines of curvature [16], [23].

Let G denote the natural 3-parameter family of folding maps, one for each oriented plane in 3-space [16]. Then,

Theorem 3.1 *[16] The bifurcation set of the family G is the dual of the union of the focal and symmetry sets of M. More precisely the local part of $\mathcal{B}(G)$ is dual to the focal set and the self tangency part is dual to the symmetry set.*

To study the changes on the sub-parabolic lines we can adapt Theorem 2.1 to deal with the family G (which is also $\mathcal{A}$ versal when considered on the ambient space) and obtain the following.

Corollary 3.2 *Let M, I, J be as in Theorem 2.1. Then for an open dense set of mappings f in $Emb^\infty(M, I, \mathbb{R}^3)$ there are finitely many points $\{t(1), \dots, t(s)\}$ in the interval J such that*

(i) If $t \notin \{t(1), \dots, t(s)\}$ then the only singularities of the folding map G_u for the surface M_t are of type $S_{\leq 3}$, $B_{\leq 3}$ and C_3. Moreover these singularities are versally unfolded by the family $G \circ f_t : M \times \mathcal{L} \to \mathbb{R}$, where $\mathcal{L}$ is the set of oriented planes in $\mathbb{R}^3$.

(ii) If t is one of the $t(j)$ then either we have an additional singularity of type S_4, B_4, C_4 or F_4, or we have one of the singularities above which fails to be versally unfolded by $g \circ f_t$. All singularities (including these) are versally unfolded by the family $G \circ f : M \times \mathcal{L} \times I \to \mathbb{R}$, defined by $(G \circ f)(p, t, u) = g_u(f_t(p))$.

We shall study, in this section, the transitions that can occur on sub-parabolic lines in generic 1-parameter families of surfaces. These arise in three situations: (i) away from ridge and umbilic points, (ii) at ridge points, or (iii) at umbilic points.

Away from umbilics and ridge points, the focal set is smooth. Following [8], one expects the parabolic set on the focal set of a family of surfaces to be generically smooth or to undergo:

(i) Morse transitions at a non-transverse cusp of Gauss of the focal set,

(ii) birth/annihilation of a pair of cusps of Gauss on a smooth parabolic curve, or

(iii) the transition corresponding to cone sections at a flat umbilic of the focal set.

The last transition is ruled out. If a surface were to give rise to a focal set with a flat umbilic then the second fundamental form of the focal set at the relevant centre of curvature would have to vanish. This form is not difficult to compute. Indeed assuming that we parametrise our surface locally by lines of curvature the coefficients of the second fundamental form of the focal set at the centre of curvature corresponding to the radius of curvature ρ_1 are

$$e = \pm\frac{\sqrt{G}}{\rho_1}\frac{\partial\rho_1}{\partial u_1}, \quad f = 0, \quad g = \pm\frac{\rho_1 G}{\rho_2^2\sqrt{F}}\frac{\partial\rho_2}{\partial u_1}$$

where E, F, G are the coefficients of the first fundamental form of the surface, u_2(resp. u_1)=constant are the lines of curvature with radius of curvature ρ_1 (resp. ρ_2) (see [17]). Assuming the focal set has a flat umbilic, we can deduce that $\frac{\partial\rho_1}{\partial u_1} = \frac{\partial\rho_2}{\partial u_1} = 0$ and that means in particular that we are at the centre of curvature corresponding to a ridge. But then the focal set is not smooth.

Corollary 3.3 *Away from umbilics and ridge points, the sub-parabolic line undergoes the following transitions in a generic 1-parameter family of surfaces.*

(i) Morse transitions at a non-transverse S_2-singularity of the folding map. This occurs when the family of folding maps fails to versally unfold the S_3 singularity.

(ii) Birth/annihilation of a pair of S_3-singularities on a smooth sub-parabolic line. This occurs at an S_4 singularity of the folding map.

Proof The proof follows from [8] and the fact that the S_k- singularities of the folding map correspond to the A_k singularities of the height function on the focal set, [16].

In [29], T. Wilkinson computed the conditions for the family of folding maps to fail to versally unfold an S_3-singularity. With the notation as before, this condition is, in addition to having an S_3 ($b_1 = c_1 = 0$),

$$2(a_0 - a_2)(c_2 - 2a_0a_2^2) - b_2(3b_0 - 2b_2) = 0.$$

The computation in Lemma 4.5 shows that these conditions are equivalent to the sub-parabolic line being singular. It is also equivalent to the Monge-Taylor map failing to be transverse to the S_2 stratum.

When the sub-parabolic line crosses the corresponding ridge, the folding map acquires a singularity of type $C_{\geq 3}$. In general the two curves meet transversally, but may become tangential in a generic 1-parameter family. The tangency is eliminated by a generic perturbation of the surface. This case is studied in detail in §4.3.

Suppose now that p_0 is an umbilic point. The configuration of the sub-parabolic lines depends only on the cubic part of the function f [16, 29], and if this cubic is represented as in §2 in the form $Re(z^3 + \beta z^2\bar{z})$ then changes on these curves occur in two ways:

1. On the hypocycloid $\beta = -3(2e^{i\theta} + e^{-2i\theta})$: three S_2's inside, one outside. This curve separates the *lemon* from *star* and *monstar* umbilics.

2. On the circle $|\beta| = 3$: S_2 is not versally unfolded by the family of folding maps. The circle separates the *star* from the *monstar*.

It is also shown in [29] that the stratification of the Monge-Taylor space V_3 by the equations defining the sub-parabolic lines coincides with that defined by the ridges after a certain transformation is applied. Indeed up to an orthogonal change of co-ordinates we can work with elements of the form $Re\{\gamma z^2 + rz\bar{z} + z^3 + \beta z^2\bar{z}\}$ where β, γ are complex and r is real. The map $(\beta, \gamma, r) \mapsto (-3\beta, \gamma, r)$ is a self diffeomorphism of this affine space which interchanges the $A_3(= B_2)$ and S_2 subsets. As a consequence we would expect that the changes on these sets will coincide with those of the ridge curves described in §2.1. This is true for the transitions across the hypercycloid $\beta = -3(2e^{i\theta} + e^{-2i\theta})$ in the space of umbilics. However there is an additional complication on the circle $|\beta| = 3$. This now corresponds to two different phenomena covered in II and III in §2, namely the sub-parabolic lines are not transverse and the Monge-Taylor map fails to be transverse to the D_4 stratum. The point here is that the failure of the transversality of ridges on the circle $|\beta| = 1$ translates to failure of transversality of sub-parabolic lines on $|\beta| = 3$, while failure of transversality of the Monge-Taylor map to the D_4 stratum remains $|\beta| = 3$.

Suppose given a generic family of surfaces $M \times I \to \mathbb{R}^3$. The Monge-Taylor map $F : M \times I \to V_3$ is transverse to the stratification of V_3. The relevant strata are the S_2 points and from our work in §2 we know that the pull back at a generic umbilic point corresponding to $|\beta| = 1$ is diffeomorphic to the set $uv(v - uw) = 0$. The set $u = v = 0$ corresponds to umbilics. The set $v = w = 0$ corresponds to points of the surface where there are two S_2 reflection planes.

The upshot is that in this case we still have to consider a function on the set diffeomorphic to $uv(v - uw) = 0$ at $(u, v, w) = (0, 0, 0)$. However the fibre of the relevant submersion is tangent to the closure of the stratum X_1 (respectively X_2) given by $v = 0$ (respectively $v = uw$) (see below). We start by making a classification of the relevant functions.

Let $X = \{(u, v, w) : uv(v - uw) = 0\}$. From §2.1, the set of all vector fields tangent to this variety is generated by the vector fields

$$e_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, e_2 = v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, e_3 = uv\frac{\partial}{\partial v} + (2v - uw)\frac{\partial}{\partial w}$$

We shall classify germs of submersions $h : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ (of low codimension) up to diffeomorphisms that preserve the variety X in the source and allow any smooth change of coordinates in the target. We denote the resulting group by $\mathcal{G}$. The tangent space to the $\mathcal{G}$-orbit of a germ h by this action is given

$$T\mathcal{G}.h = \mathcal{O}_3\langle e_1(h), e_2(h), e_3(h)\rangle + h^*(\mathcal{M}_1)$$

where $\mathcal{O}_3$ denotes the set of germs of functions $\mathbb{R}^3, 0 \to \mathbb{R}$ and $h^*(\mathcal{M}_1)$ the pull back of the maximal ideal $\mathcal{M}_1$. We use the complete transversal tools in [12] and proceed inductively on the jet level.

Let $h = au + bv + cw$ with a, b, c not all zero. We have $e_1(f) = au + bv$, $e_2(f) = bv + cw$ and $e_3(f) = buv + c(2v - uw)$.

• $c \neq 0$: we can use Mather's lemma (see [12]) to reduce the 1-jet to $au+cw$. If $a \neq 0$ then (using $\simeq$ to denote $\mathcal{G}$-equivalence), $h \simeq u + w$ which is 1-$\mathcal{G}$-determined. If $a = 0$ then a complete 2-transversal is given by $h = w + du^2$. This is equivalent to $w + u^2$ (for $d \neq 0$) which is 2-$\mathcal{G}$-determined.

• $c = 0$: then $h \simeq u + v$ if $ab \neq 0$. A complete 2-transversal is given by $h = u + v + dw^2$ which reduces, after a change of scale for $d \neq 0$, to $u + v \pm w^2$. This germ is 2-$\mathcal{G}$-determined.

• $c = b = 0$, $a \neq 0$: the 1-jet can be written in the form $h = u$, and a complete 2-transversal has the form $h = u + lv^2 + mvw + nw^2$. It is not difficult to show that this reduces to $h = u + v^2 \pm w^2$ which is 2-$\mathcal{G}$-determined, provided $n(m^2 - 4ln) \neq 0$.

• $c = a = 0$, $b \neq 0$: we write $h = v$ by scaling. A complete 2-transversal is given by $h = v + lu^2 + muw + nw^2$. If $l \neq 0$ we can reduce to $h = v + u^2 + muw + nw^2$. A complete 3-transversal for this 2-jet is given by $h = v + u^2 + muv + nw^2 + pw^3$. This is equivalent to $h = v + u^2 + muv + nw^2 + w^3$ when $p \neq 0$.

Calculations show that the 3-jet $h = v + u^2 + muv + nw^2 + w^3$ is 3-$\mathcal{G}$-determined provided

$$n(m^2 - 4n)((1+m)^2 - 4n) \neq 0.$$

(The term w^3 in h is actually topologically redundant, though we do not use this result here.) We now seek a recognition criterion for a germ of this type.

Lemma 3.4 *A germ $h : \mathbb{R}^3, X, 0 \to \mathbb{R}, 0$ is equivalent to $v + u^2 + muv + nw^2 + w^3$ if*
(1) the germ h is a submersion;
(2) the set $h = 0$ has A_1 contact with X_1 (given by $v = 0$) and X_2 (given by $v = uw$);
(3) in the case when the contacts are both hyperbolic (i.e. that is $\mathcal{K}$-equivalent to $x^2 - y^2$) the branches are pairwise transverse.

Proof Conditions (1) and (2) show that the linear part of h is of the form αv for $\alpha \neq 0$. In this case we can set $\alpha = 1$. The quadratic parts of the contact between $h = 0$ and X_i are then determined by $h(u, 0, w) = lu^2 + muw + nw^2 + O(3)$ say. These forms are $lu^2 + muw + nw^2$ and $lu^2 + (1+m)uw + nw^2$. The two quadratic forms are non-degenerate if $(m^2 - 4ln)((1+m)^2 - 4ln) \neq 0$, and they have no common roots if and only if $ln \neq 0$. It is clear from above that these conditions ensure that h is 3-$\mathcal{G}$ determined and hence equivalent to $v + u^2 + muv + nw^2 + w^3$.

We can now determine the intersection of the fibres of the function $v + u^2 + muv + nw^2 + w^3$ with the variety $uv(v - uw) = 0$ for values of a given (m, n) in each connected component of the set $n(m^2 - 4n)((1+m)^2 - 4n) = 0$. These are as in Figure 5.

It will be convenient to parametrise the cubics of the form $Re\{z^3 + \beta z^2 \bar{z}\}$ with $|\beta| = 3$ in a slightly different way.

Lemma 3.5 *Any such cubic form is $SO(2)$ equivalent to one of the form $ax^3 + by^3$.*

Proof Replace z by $ze^{i\theta}$. This yields $Re\{z^3e^{3i\theta} + \gamma z^2\bar{z}\}$ for some $\gamma = \gamma_1 + i\gamma_2$ with $|\gamma| = 3$. Expanding we find the x^2y and xy^2 terms are $-3\sin(3\theta) + \gamma_2$ and $-3\cos(3\theta) - \gamma_1$ which both vanish for the correct choice of θ.

We now wish to describe the stratification in the V_3 space more explicitly. It is rather difficult to determine equations for the X_i (S_2 strata); we can however find the tangent space at our base points to the closures of the X_1 and X_2 surfaces.

Lemma 3.6 *If a general point in V_3 is given as $a_0x^2 + a_1xy + a_2y^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3$ then the tangent space at $a_0(x^2+y^2) + b_0x^3 + b_3y^3 = F_0$ to the X_1 and X_2 surfaces is given by $a_1 = 0$.*

Proof These tangent spaces coincide with the sum of the tangent space to the umbilic D_4 and $2S_2$ strata. We now need only find suitable paths in these strata, differentiate and evaluate to prove the result. For the umbilic stratum the path $F_0 + tC(x,y)$, where C is an arbitrary cubic, shows that $Sp\{x^3, x^2y, xy^2, y^3\}$ is a subset of the tangent space. For the $2S_2$ stratum one can check that $p + tx^2$ and $p + ty^2$ are paths of the required type, and the result now follows.

The above result shows that the X_i are determined by equations of the form $a_1 + g_i(a,b) = 0$ where the g_i lie in $\mathcal{M}^2(a,b)$, $a = (a_0, a_1, a_2)$ and $b = (b_0, b_1, b_2, b_3)$.

Let $F : M \times \mathbb{R}, (0,0) \to V_3$ be the Monge-Taylor map and let $\pi : M \times \mathbb{R}, (0,0) \to \mathbb{R}, 0$ be the natural projection. We have seen that there are two smooth codimension 1 submanifolds of V_3 at $F(0,0)$ which intersect in a D_4 stratum and a $2S_2$ stratum. Let their equations be given by $g_1, g_2 : V_3, 0 \to \mathbb{R}, 0$. We are interested in the projection $\pi : M \times \mathbb{R}, F^{-1}(p), (0,0) \to \mathbb{R}, 0$. We have seen however that the type of this projection is determined by the contact with the smooth surfaces $(g_i \circ F)^{-1}(0)$. This contact is also encoded (see [22]) by the map $g_i \circ F : M \times \{0\}, (0,0) \to \mathbb{R}, 0$. Conditions (2) and (3) in Lemma 3.4 correspond to the assertions that the $g_i \circ F$ have A_1 singularities and in the hyperbolic cases the branches are pairwise transverse.

We have seen above (Lemma 3.6) that the g_i are of the form $a_1 + h_i(a,b)$, $h_i \in \mathcal{M}^2(a,b)$.

Proposition 3.7 *In a generic 1-parameter family of surfaces, the contact $g_i \circ F$ is of type A_1 and in the hyperbolic case the branches are pairwise transverse.*

Proof The proof follows by using Thom's transversality theorem in the Monge-Taylor space V_3. Note that the non-versal umbilic set is a variety of codimension 3, so this singularity occurs generically at isolated points in a family of surfaces. If in addition the contact $g_i \circ F$ is of type worse than A_1 or in the hyperbolic case the branches are not pairwise transverse, then this adds an extra condition that is, defines a variety of codimension 4 in V_3 which is therefore avoided by the map F.

Corollary 3.8 *In a generic 1-parameter family the sub-parabolic line changes as follows at umbilic points.*

I. Lemon to monstar (or vice versa) changes across the hypocycloid $w = -3(2e^{i\theta} + e^{-2i\theta})$. *This is the transition (I) in Theorem 2.3, Figure 2(i).*

II. Birth of two umbilics (a monstar and a star) across the circle $|\beta| = 3$. *See Figure 5.*

Note that the circle $|\beta| = 3$ corresponds to the birth of umbilics. The two newly born umbilics are of opposite index, *i.e.* one is a star and the other a monstar(see [24]). This implies that we must have a crossing of the two sub-parabolic lines (of different colour; otherwise we have in addition a C_3-singularity) before the moment of transition.

Figure 5 here

4 Geometric characterisation of simple singularities of the folding map

In this section, we generalise the duality results in [16] and characterise geometrically the simple singularities of the folding map. These singularities are as follows (see [21]).

Normal form	Name	$\mathcal{A}_e$ − codimension	C
(x, y^2, xy)	*Crosscap*	0	0
$(x, y^2, x^2y \pm y^{2k+1}), k \geq 1$	$B_k^{\pm}$	k	2
$(x, y^2, y^3 \pm x^{k+1}y), k \geq 1$	$S_k^{\pm}$	k	0
$(x, y^2, xy^3 \pm x^k y), k \geq 3$	$C_k^{\pm}$	k	k
$(x, y^2, x^3y + y^5)$	F_4	4	1

Here C denotes the number of (complex) crosscaps that emerge in a small generic deformation of the given germ. We gave in the previous section geometric interpretations of singularities of codimension ≤ 3. We aim to characterise all the simple singularities of the folding map in terms of the geometry of the surface M, its focal set and its symmetry set.

4.1 The S_k series

These singularities are associated with parabolic points on the smooth parts of the focal surface. Their bifurcation sets are discriminants of A_k-singularities [16]. This is not surprising as we know from [3] that the dual of a smooth surface is the discriminant of the family of height functions, and that the bifurcation set of the folding map is the dual of the focal surface.

We have shown in [8] that the A_k-singularities of the height function on a surface X have a simple geometric characterisations. Assume the ridge and the parabolic curve (on M or on its focal surface) are smooth. Then:

Theorem 4.1 *[8] The height function on X has an A_k ($k \geq 3$) singularity if and only if the ridge and the parabolic curve on X have $(k-2)$-point contact.*

Taking X to be the focal set of the surface we obtain the following.

Corollary 4.2 *The folding map on M has generically an S_k singularity if and and only if the ridge and the parabolic curve on the focal surface of M have $(k-2)$-point contact.*

The set of points on the surface M corresponding to parabolic points on the focal surface is the sub-parabolic line of M and has the property of being the locus of points where the lines of curvatures are momentarily geodesics [23]. It would be interesting to find an analogous interpretation for the set of points on M corresponding to the ridge line on the focal surface.

The S_k-series can also be characterized by the contact of the sub-parabolic line and the other line of curvature. A consequence of Lemma 4.5 below is that the folding map has an S_2-singularity at a sub-parabolic point if and only if the sub-parabolic line is transverse to the other line of curvature ($c_1 \neq 0$). It has a singularity of type $S_k, k > 2$ when the two curves are tangential. The degree of tangency is the number of (possibly complex) transverse intersections in a generic perturbation. Thus the arguments used [5] can be used to establish the following:

Corollary 4.3 *Generically the folding map of M has an S_k singularity if and and only if the sub-parabolic line and the other line of curvature have $(k-1)$-point contact.*

The bifurcation sets of multi-local singularities are often also discriminants [8] and a similar result can be made established here, that is

Corollary 4.4 *Generically the folding map has a multi-local singularity of type A_k if and and only if the ridge and the parabolic curve on the symmetry set have $(k-2)$-point contact.*

The same question can also be asked concerning the S_k series, that is to find a geometric interpretation of the ridge and the parabolic curve on the symmetry set of the surface M which ensure a singularity of this type. This question remains to be investigated.

4.2 The B_k-series

We have seen in the previous section that a B_3 singularity is characterised as a cusp point on the focal set in the closure of the parabolic curve of the symmetry set. The conditions on the coefficients of the Monge form for a $B_{\geq 3}$ singularity imply that the distance squared function has an A_3 singularity which is versally unfolded by the family of distance squared functions. So the focal surface remains a cuspidal edge for all the B_k-singularities.

The bifurcation set of a B_k singularity can also be viewed as the discriminant of a function on a surface with boundary [21]. By the duality result described earlier, the singularities of this discriminant are related to the parabolic set on the symmetry set.

4.3 The C_k-series

Let the surface M be given locally at the origin in the Monge form $(x, y, f(x, y))$ where f has no constant or linear terms. As before write $f = f_2 + f_3 + f_4 + O(5)$ with $f_2 = a_0x^2 + a_1xy + a_2y^2$, $f_3 = \sum_{i=0}^{3} b_ix^{3-i}y^i$ and $f_4 = \sum_{i=0}^{4} c_ix^{4-i}y^i$. It follows from above that folding along the principal direction $(0, 1)$ (away from umbilics and parabolic points) produces a singularity of type $C_k (k \geq 3)$ when

$$a_1 = 0, \quad b_1 = 0, \quad b_3 = 0, \quad c_3 \neq 0.$$

The singularity is of type C_3 when furthermore $c_1 \neq 0$. These conditions can be interpreted geometrically as follows:

$a_1 = 0$: the folding plane $y = 0$ contains the principal direction $(0, 1)$

$a_1 = b_3 = 0$: the origin is a ridge point

$a_1 = b_1 = 0$: the origin is a sub-parabolic point

In order to give a geometric interpretation of the C_k series we need to look at the way the ridge and the sub-parabolic line meet at the origin. We have the following.

Lemma 4.5 *Suppose the origin is neither a parabolic nor an umbilic point. Then:*

(i) the leading terms in the equation of the ridge are

$$[(a_0 - a_2)c_3 - b_1b_2]x - [b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3)]y + \ldots = 0.$$

(ii) The leading terms in the equation of the sub-parabolic line are

$$3c_1(a_0 - a_2)x + [2(a_0 - a_2)(c_2 - 2a_0a_2^2) - b_2(3b_0 - 2b_2)]y + \ldots = 0$$

Proof These equations are found using the computer algebra package Maple.

Lemma 4.5 provides a great deal of information. We can, for instance, deduce the conditions for the ridge to be singular. These coincide with the Monge-Taylor map failing to be transverse to the $A_{\geq 3}$ stratum of the distance squared function (see §2.2).

We can also see that the tangent line to the ridge is along the principal direction $(0,1)$ if and only if the distance squared function has an A_4 singularity. This characterisation of an A_4 singularity as a point where a line of curvature and a ridge meet tangentially is already known [24].

We can deduce from Lemma 4.5 that, generically, when the folding map has a C_k singularity the ridge and the sub-parabolic line remain smooth. When $b_1 = 0$ the initial terms in the equation for the ridge become

$$(a_0 - a_2)c_3 x - [b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3)]y + \ldots = 0$$

so that the condition $c_3 \neq 0$ expresses the fact the ridge is transverse to the 'other' line of curvature. Moreover from the equation of the sub-parabolic line we see that $c_1 = 0$ if and only if the sub-parabolic line and the 'other' line of curvature are tangent. A simple calculation shows that the ridge and the sub-parabolic line are tangential when

$$c_3[2(a_0 - a_2)(c_2 - 2a_0a_2^2) - b_2(3b_0 - 2b_2)] - 3c_1[b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3)] = 0$$

This is exactly the condition for the family of folding maps to fail to versally unfold the singularity C_3, see [29].

This condition can also be recovered by looking at the Monge-Taylor map. Using the notation of §2.2, the sub-parabolic stratum is given by $b_1 = 0$ in the transversal to the $SO(2)$-orbit, and the tangent space to this stratum in V_3 is the kernel of the 1-form

$$\xi_1 = (3b_0 - 2b_1)\mathrm{d}a_1 - 2(a_0 - a_2)\mathrm{d}b_1.$$

(Basically the 1-form annihilates elements in the transversal given by $b_1 = 0$ as well as the tangent vector v to the $SO(2)$ orbit of f given in §2.2.)

For the ridge stratum, given by $b_3 = 0$ in the transversal, the tangent space in V_3 is the kernel of the 1-form

$$\xi_2 = b_2\mathrm{d}a_1 - 2(a_0 - a_2)\mathrm{d}b_3.$$

The tangent space to the C_3 stratum (given in the transversal by $b_1 = b_3 = 0$) is the intersection of the kernels of ξ_1 and ξ_2. Now the Monge-Taylor map fails to be transverse to this stratum if and only if there exists a non-zero vector of the form $\lambda u_1 + \mu u_2$ such that $\xi_1(\lambda u_1 + \mu u_2) = \xi_2(\lambda u_1 + \mu u_2) = 0$, that is when

$$\begin{vmatrix} \xi_1(u_1) & \xi_1(u_2) \\ \xi_2(u_1) & \xi_2(u_2) \end{vmatrix} = 0;$$

equivalently, when

$$\begin{vmatrix} (a_0 - a_2)c_3 & b_2^2 - 4(a_0 - a_2)(c_4 - a_2^3) \\ 3(a_0 - a_2)c_1 & 2(a_0 - a_2)(c_2 - 2a_0a_2^2) - b_2(3b_0 - 2b_2) \end{vmatrix} = 0.$$

The rows of the above determinant give the coordinates of the tangent vectors to the ridge and sub-parabolic lines at a C_3. Non-transversality of the Monge-Taylor map to the C_3 stratum is therefore equivalent to the family of folding maps not being versal.

Note that at singularities of type C_k, $k \geq 4$ or F_4, the Monge-Taylor map is tranverse to the C_3-stratum, so on the surface the sub-parabolic line and the ridge still meet transversally at such singularities.

We can also deduce from above the conditions for the sub-parabolic line or the ridge to be singular. These are given by asking that the Monge-Taylor fails to be transverse to the sub-parabolic or the ridge stratum respectively. We thus have

Theorem 4.6 *Suppose that the origin is neither a parabolic nor an umbilic point, but is a ridge point and a sub-parabolic point. Then*

(i) The folding map has a C_3 singularity if and only if the ridge meets the sub-parabolic line of the same colour and both curves are transverse to the other line of curvature. The family of folding maps fails to versally unfold the C_3 singularity if furthermore these two curves are tangential. This is equivalent to the Monge-taylor map not being transverse to the C_3-stratum. See Figure 6.

(ii) The folding map has a $C_k, k > 3$, singularity if and only if the ridge meets the sub-parabolic line of the same colour and is transverse to the other line of curvature while the sub-parabolic line is tangent to it. See Figure 6.

Figure 6 here

We also obtain the following geometrical interpretation of the C_k series.

Theorem 4.7 *Suppose that the hypothesis of Theorem 4.6 (ii) holds. Then the folding map has a C_k singularity if and only if the sub-parabolic line and the other line of curvature have $(k-2)$-point contact.*

Proof The proof is straightforward. Since the hypothesis of Theorem 4.6 (ii) holds, the folding map has a singularity of type $C_k, k > 3$. It is then a matter of determining the integer k.

It follows from Mond's adjacency diagram in [21] that the most degenerate ajacent singularity in the S_k series to a C_k singularity is S_{k-1}. By Corollary 4.3 such a singularity is if and only if the sub-parabolic line and the other line of curvature have $(k-2)-$point contact.

So a C_k singularity can be viewed as an accumulation of an S_{k-1} singularity together with a C_3-singularity. In a generic perturbation of a C_k-singularity, $(k-3)$ singularities of type S_3 and one C_3 should appear on the sub-parabolic line, that is the parabolic set of the focal surface should have $(k-3)$ cusps of Gauss and a point of intersection with the cuspidal edge of the focal set.

4.4 The F_4 singularity

Another application of Lemma 4.5 is the provision of an interpretion the F_4 singularity. With f as before, the folding map has an F_4 singularity when

$$a_1 = 0, \quad b_1 = 0, \quad b_3 = 0, \quad c_3 = 0, \quad c_1 \neq 0, \quad d_5 \neq 0.$$

Sucht conditions will be satisfied in a generic 1-parameter family. Thus,

Proposition 4.8 *Generically the folding map has a singularity of type F_4 if and only if the ridge meets the sub-parabolic line and is tangent to the other line of curvature while the sub-parabolic line is transverse to it. (See Figure 6(iv).)*

The F_4 singularity is of $\mathcal{A}_e$-codimension 4. It can therefore occur at isolated points in generic 1-parameter family. In order to recover the 3-dimensional pictures, we need to consider generic functions on the full bifurcation set of F_4.

A result of Mond [21] shows that the singularities of folding maps, which can always be put in the form $F = (x, y^2, yp(x, y^2))$, are determined by the singularities of $p(x, y)$ as a function on the surface with boundary $\{(x, y) : y \geq 0\}$. The full bifurcation set of F is the discriminant of a versal unfolding of p.

It is easy to generalise the work in [1] on discriminants to discriminants of functions on surfaces with boundary.

Let f be a smooth function on the surface with boundary $\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 \geq 0\}$, with $\mathcal{O}(x)/\langle x_1 \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \rangle$ a vector space of finite dimension. Let $g_1, ..., g_p$ be a basis for this space. It follows that $F(x, a) = f(x) + \sum_{i=0}^{n} a_i g_i$ is a versal unfolding of f and that $g_1, ..., g_p$ generates freely

$$\mathcal{O}(x, a)/\langle x_1 \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, ..., \frac{\partial F}{\partial x_n} \rangle$$

as an $\mathcal{O}(a)$-module. Therefore there exists unique $a_{ij} \in \mathcal{O}(a)$ such that

$$F.g_j = \sum_{i=0}^{n} a_{ij} g_i \ \ mod\ \mathcal{O}(x, a)/\langle x_1 \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, ..., \frac{\partial F}{\partial x_n} \rangle.$$

Theorem 4.9 *The vector fields $\theta_j = \sum_{i=0}^{n} a_{ij} \partial/\partial a_i$ are smooth vector field tangent to the discriminants D of F and form a free $\mathcal{O}(a)$ basis for $Der(log D)$.*

We also have the following result concerning stable functions on the discriminant D of F.

Theorem 4.10 *([13]). A function $h : D \subset \mathbb{R}^p \to \mathbb{R}$ is stable if and only if*

$$\mathcal{M}(a) \ \subset \ \langle \theta_j(h) \rangle + \ \mathcal{M}^2(a)$$

where $\mathcal{M}(a)$ is the maximal ideal in $\mathcal{O}(a)$.

For the F_4 singularity $(x, y^2, y(x^3 + y^4))$ the associated function on a surface with boundary is $p(x, y) = x^3 + y^2$ whose versal unfolding is given by

$$F(x, y, t, u, v, w) = x^3 + y^2 + txy + uy + vx + w.$$

The discriminant of F has two components,

$$\begin{aligned} \Delta_1 &= \{(t, u, v, w) : F = \tfrac{\partial F}{\partial x} = \tfrac{\partial F}{\partial y} = 0\} \\ &= \{(t, -2y - tx, -ty - 3x^2, y^2 + 2x^3 + txy), y \geq 0\}, \end{aligned}$$

$$\begin{aligned} \Delta_2 &= \{(t, u, v, w) : y = F = \tfrac{\partial F}{\partial x} = 0\} \\ &= \{(t, u, -3x^2, 2x^3)\}. \end{aligned}$$

Calculations show that the 1-jets of the vector fields $\theta_j (j = 1...4)$ in the theorem are given by

$$\begin{aligned}
\theta_1 &= w\partial/\partial t \\
\theta_2 &= \tfrac{2}{3}v\partial/\partial t + w\partial/\partial u \\
\theta_3 &= \tfrac{1}{2}u\partial/\partial t + w\partial/\partial v \\
\theta_4 &= \tfrac{1}{6}t\partial/\partial t + \tfrac{1}{2}u\partial/\partial u + \tfrac{2}{3}v\partial/\partial v + w\partial/\partial w
\end{aligned}$$

It is clear then that there is only one stable function on $\Delta = \Delta_1 \cup \Delta_2$ with representative $(t, u, v, w) \mapsto t$.

We shall investigate the geometry of Δ by slicing it by planes t =constant. We are therefore considering in $\mathbb{R}^3$ the sets parametrised as $\Delta_1^t = \{(-2y - tx, -ty - 3x^2, y^2 + 2x^3 + txy), y \geq 0\}$ and $\Delta_2^t = \{(u, -3x^2, 2x^3)\}$.

The surface Δ_2^t does not depend on t (we shall denote it by Δ_2). It is a cuspidal edge with the singular set along the u-axis.

At $t = 0$, Δ_1^0 intersects Δ_2 in two curves one of which is the image of the boundary $y = 0$ and is given by $\Gamma_1^0 = \{(0, -3x^2, 2x^3)\}$. This curve has an ordinary cusp at the origin. The surface Δ_1^0 is in fact a cuspidal edge away from Γ_1. The cuspidal edge curve is given by $\Gamma_2 = \{(-2y, 0, y^2)\}$.

When $t \neq 0$ the intersection curve $\Gamma_1^t = \{(-tx, -3x^2, 2x^3)\}$ (image of the boundary) becomes smooth and meets the cuspidal edge curve $\Gamma_2^t = \{(-2y - \frac{1}{12}t^3, -ty - \frac{1}{48}t^4, y^2 + \frac{1}{12}t^3y + \frac{1}{864}t^6), y \geq 0\}$ of Δ_2^t at $(-\frac{1}{12}t^3, -\frac{1}{48}t^4, \frac{1}{864}t^6)$. This point is a B_3 singularity of the folding map.

The F_4 may thus be characterised as an accumulation of a B_3 and a C_3 singularity. Figure 7 illustrates the generic sections of the full bifurcation set of F_4. One can see, and easily prove, that the C_3 singularity changes from C_3^+ to C_3^- at the moment of transition.

Figure 7 here

Proposition 4.11 *If the folding maps arising from a family of embeddings parametrised by t is versal at an F_4 singularity, then the projection along the t-parameter yields generic sections on the bifurcation set of this family.*

4.5 The non-transverse C_3

We have stated in Theorem 4.6 that the family of folding maps fails to be a versal unfolding of C_3 when the ridge and the sub-parabolic line become tangential. This may occur in a generic 1-parameter family. So we need to consider Morse sections of the product of the full bifurcation set of C_3 with a line. A versal unfolding of the C_3 singularity is

$$(x, y^2, xy^3 + y(\pm x^3 + wx^2 + vx + u))$$

and the full bifurcation set is given by

$$Bif(C_3) = \{(0, v, w), v \leq 0\} \cup \{(\pm 2x^3 + wx^2, -2wx \mp 3x^2, w)\}.$$

A Whitney stratification of $Bif(C_3) \times \mathbb{R}$ is given by the product of the stratification of $Bif(C_3)$ with $\mathbb{R}$. It follows from [6] that Morse functions on the product set are determined by the Morse functions on $Bif(C_3)$ whose base stratum is a single point. There is only one non-general transverse plane to the stratification of $Bif(C_3)$. This is given by $u = 0$. It follows again from [6] that topologically there is only one stable function on $Bif(C_3)$ with representative w. Therefore the Morse functions on $Bif(C_3) \times \mathbb{R}$ are represented by $w + t^2$ and $w - t^2$. These are equivalent when we allow scaling in the target.

In order to obtain the right pictures up to diffeomorphism we need to classify germs of submersions in $\mathbb{R}^3$ up to diffeomorphisms preserving $Bif(C_3)$ in the source. We proceed as for the F_4 case. The associated function on a surface with boundary for the C_3 singularity is given by $p(x, y) = xy \pm x^3$ whose versal unfolding is

$$F(x, y, u, v, w) = xy \pm x^3 + wx^2 + vx + u.$$

The 1-jets of the vector fields $\theta_j (j = 1, 2, 3)$ in Theorem 4.9 are given by

$$\begin{aligned} \theta_1 &= u\partial_w \\ \theta_2 &= \tfrac{2}{3}v\partial_w + u\partial_v \\ \theta_3 &= \tfrac{1}{3}w\partial_w + \tfrac{2}{3}v\partial_v + u\partial_u \end{aligned}$$

It is clear that there is only one stable germ whose representative is given by $(w, v, u) \mapsto w$. The Morse function on the full bifurcation set is thus of the form $w + t^2$, which agrees with the topological pictures. Figure 8(ii) illustrates these generic sections and Figure 8(i) illustrates the situation on the surface.

Figure 8 here

Proposition 4.12 *If a generic 1-parameter family of embeddings yields, at $t = 0$ a family of folding maps which has a non-versal C_3 singularity, then locally the projection along the t-parameter yields generic Morse sections on the bifurcation set of this family as described above.*

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Figure 1. (i) Generic ridges through (left to right) a hyperbolic umbilic, and the two types of elliptic umbilic; (ii) The partition of the space of cubic forms. Types I, II, III mark the boundaries.

Figure 2. Generic evolutions of ridges at umbilics (i) via a D_5 singularity, (ii) with the birth of a pair of hyperbolic umbilics, (iii) corresponding to tangent ridges at an elliptic umbilic.

Figure 3. Morse transitions on ridges. (Two A_4-points on one side of the transition corresponding to the tangency of the ridge and a line of curvature of the same colour.)

Figure 4. A_5 transition.

Figure 5. Generic transitions on subparabolic lines at umbilics.

Figure 6. Ridge (thick line) and subparabolic line (dashed), against the lines of curvature (thin lines). Geometric characterization of (i) C_3, (ii) non-transverse C_3, (iii) $C_k, k > 3$, (iv) F_4.

Figure 7. Evolution of full bifurcation sets: the F_4 transition.

Figure 8. A non-transverse C_3 transition (i) on the surface, (ii) on the bifurcation set.

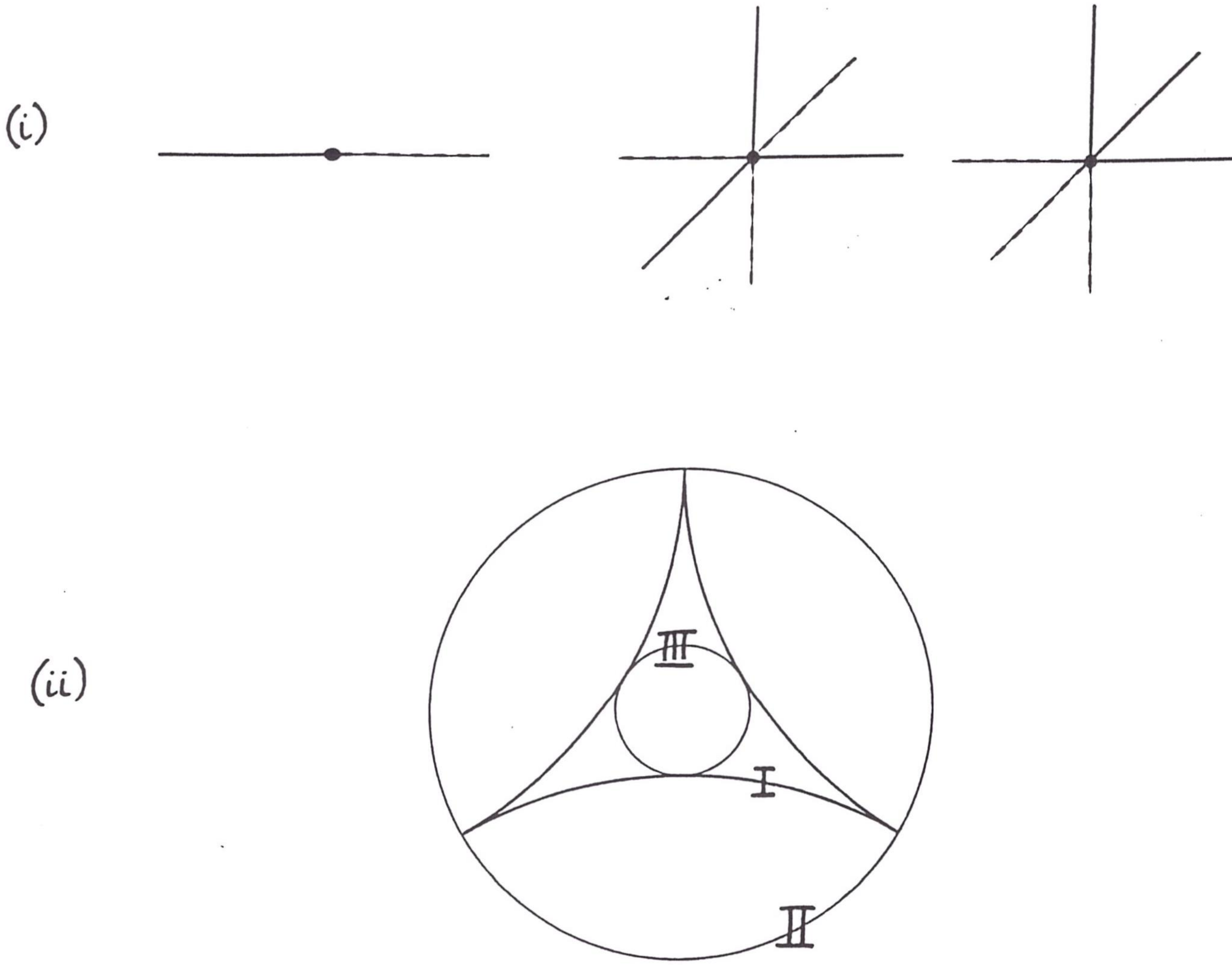

Figure 1

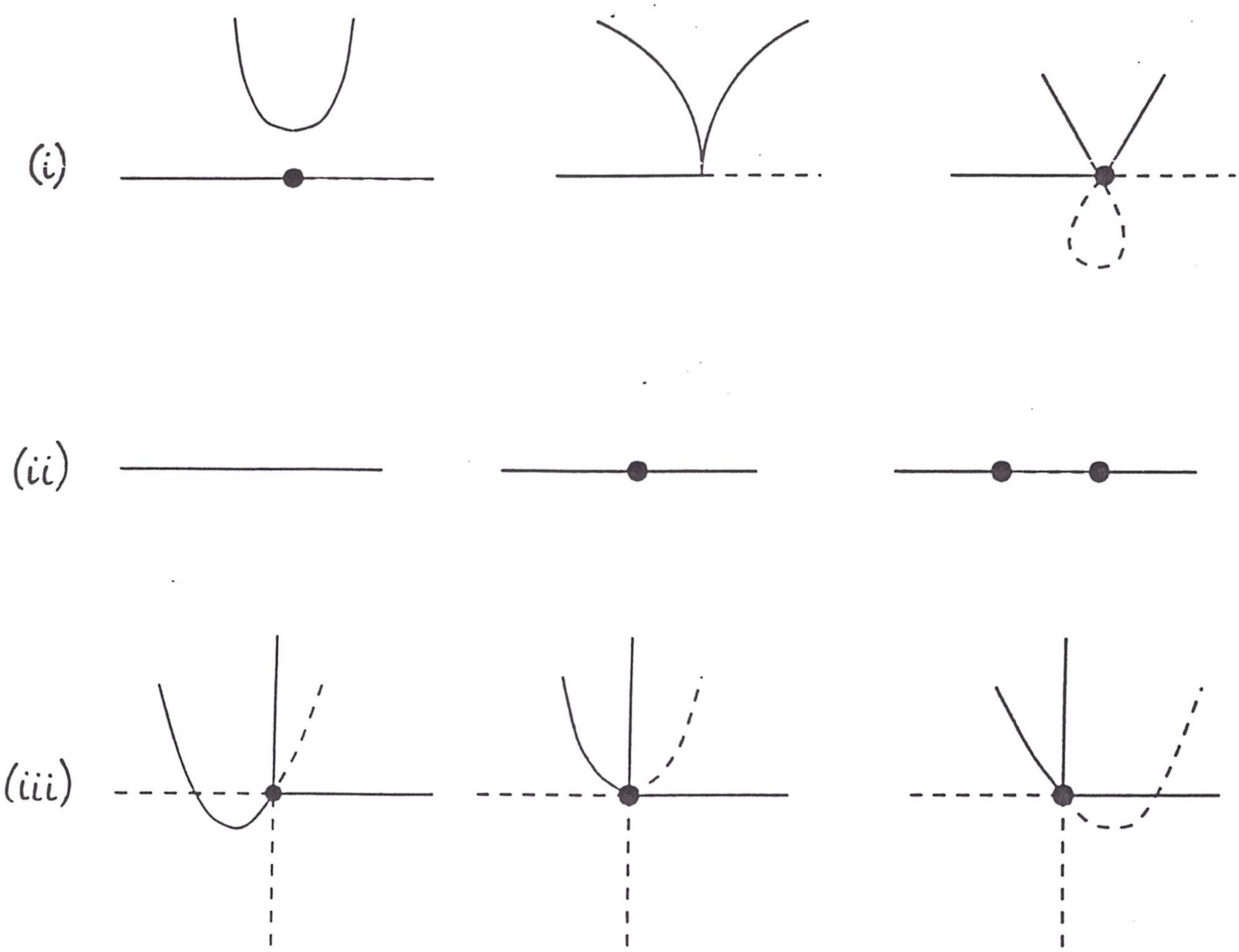

Figure 2

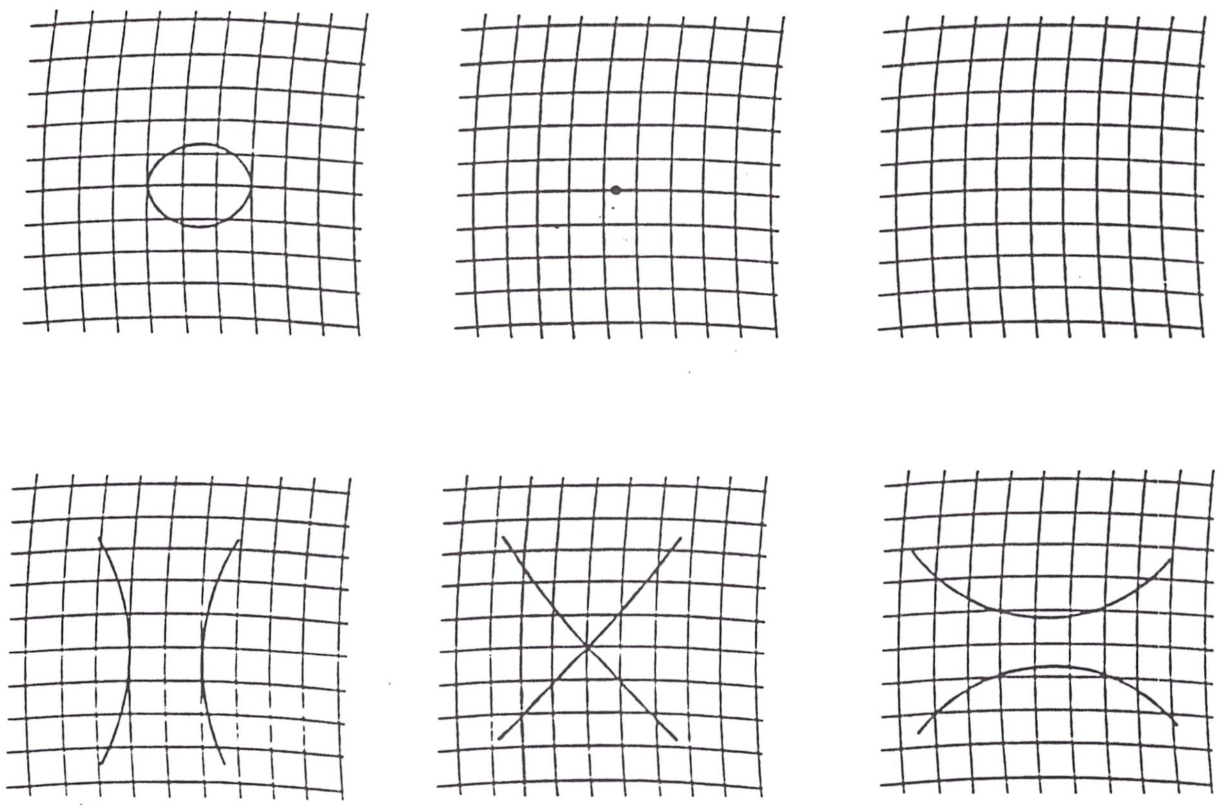

Figure 3

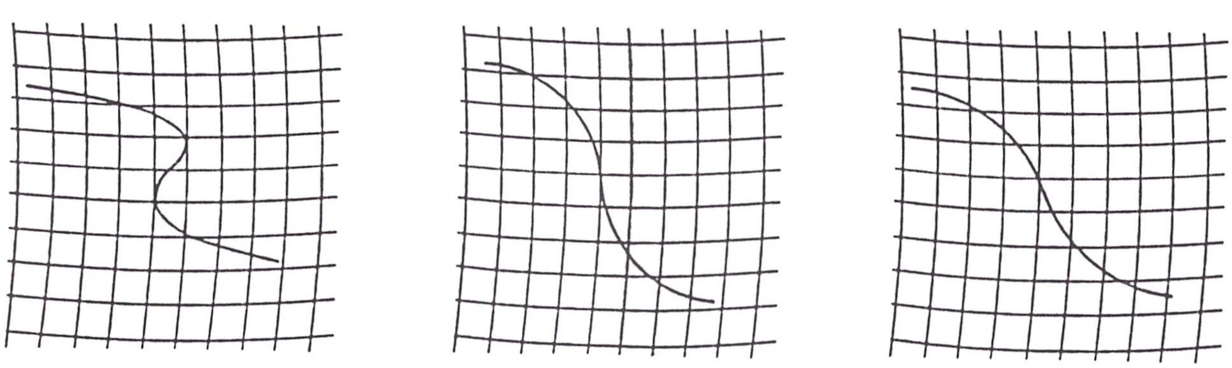

Figure 4

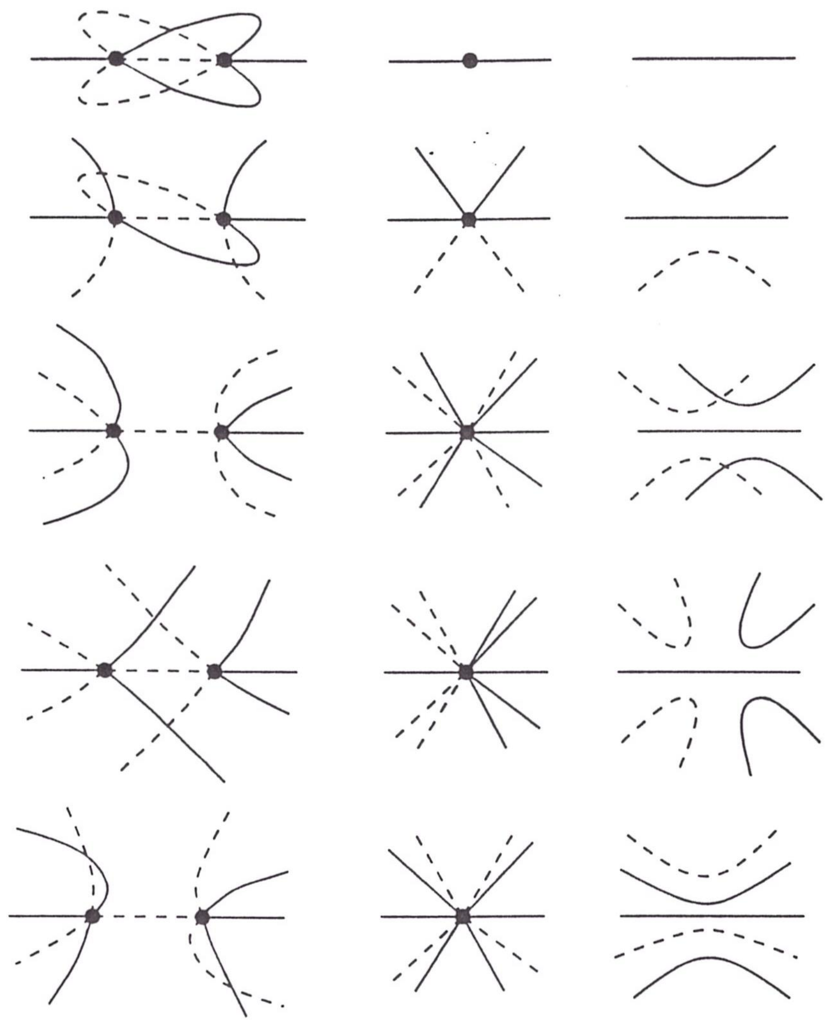

Figure 5

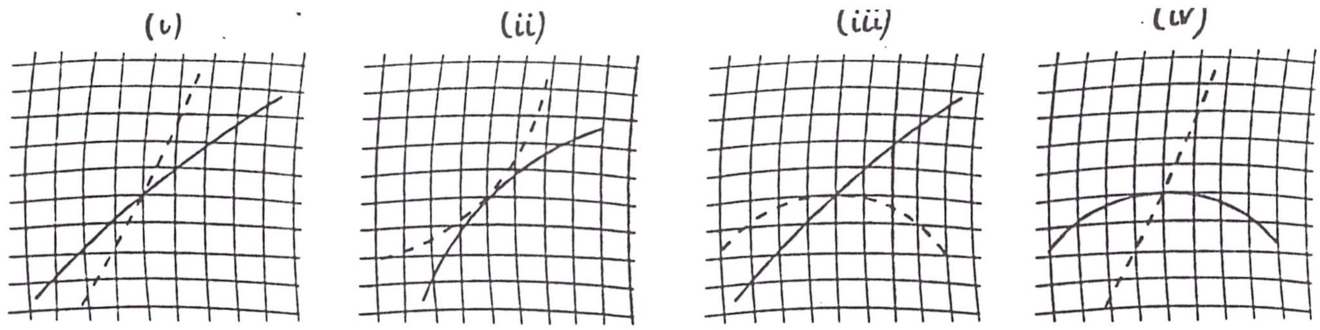

Figure 6

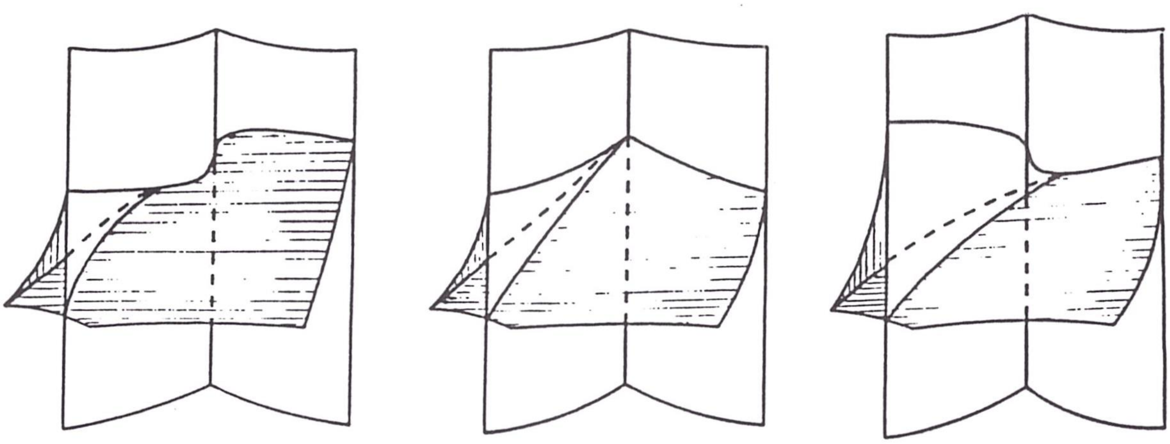

Figure 7

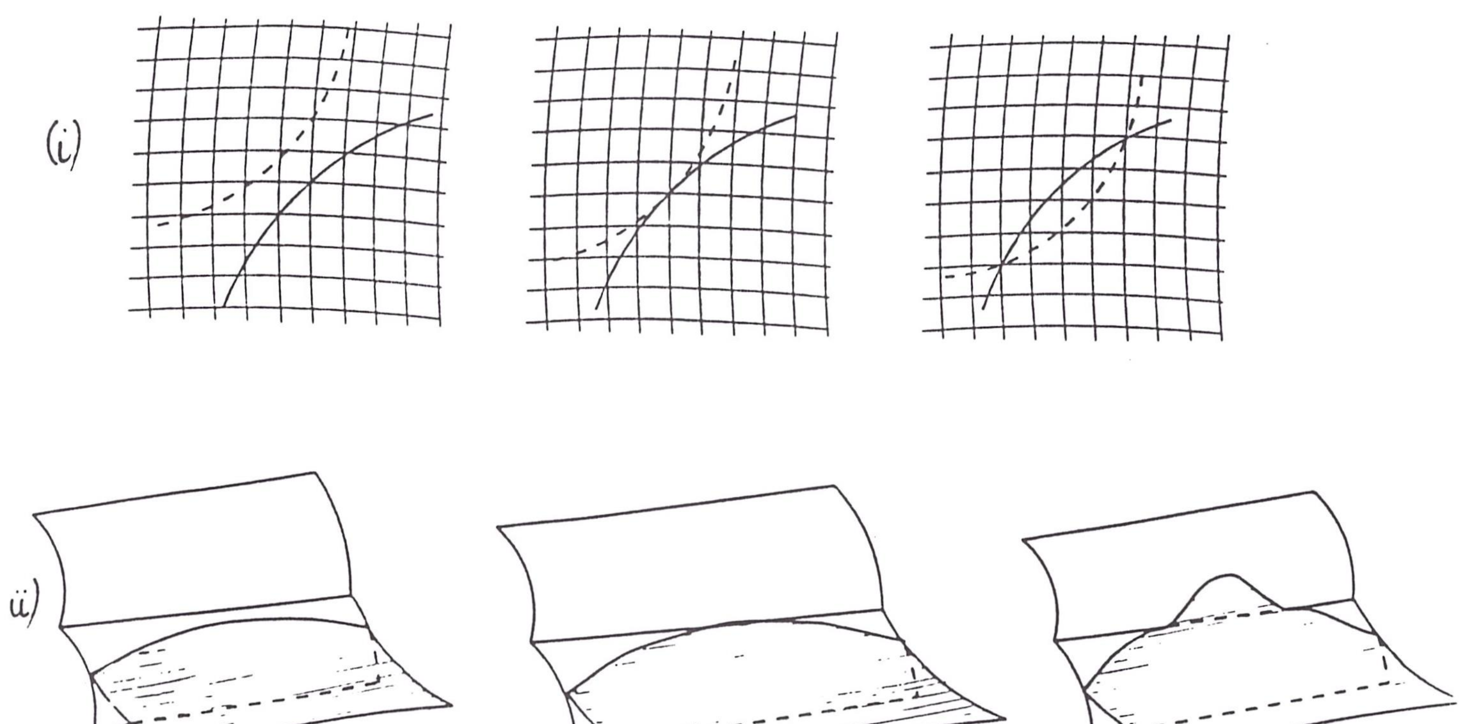

Figure 8

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