Editors' Suggestion

Legislative rebellions and impeachments in a neural network society

Juan Neirotti *

Department of Mathematics, Aston University, The Aston Triangle, Birmingham B4 7ET, United Kingdom

Nestor Caticha †

Instituto de Fisica, Universidade de Sao Paulo, CEP 05315-970, Sao Paulo, SP, Brazil

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Inspired by studies of legislative-executive conflict in modern presidential democracies in South America, we present an agent-based statistical mechanics exploration of the collective, coordinated action of strategic political actors in the legislative chamber and the conditions that may result in premature changes in the executive officeholder, such as a president's impeachment or a motion of no confidence in a prime minister. The legislative actors are represented by information processing agents equipped with a neural network, and emit opinions about issues in the presidential agenda. We construct a Hamiltonian which is the sum of the costs for the agents to hold a specific set of political positions. We use replica methods for two types of disorder, in the space of weights and in the network of agents' interactions. We obtain the phase diagram of the model, where the control parameters may be loosely described as indices measuring the strategic legislative support, the presidential polling popularity and the volume of the presidential agenda under discussion. It shows an intermediate phase of coexistence of pro and con strategic behavior. This region is surrounded by pure phases where the strategic vote falls completely in the pro or con camps. Driven by external forces, the change of these indices may lead the system out of the coexistence region into the purely anti-executive phase, triggering a phase transition into a state opposing the executive leader and supporting their removal from office by constitutional means. We use data from Brazil, and show the presidential trajectories that led to impeachment or not during the democratic period which began in 1989. These trajectories ended in the region of the phase diagram in accordance to the president being removed or not from office.

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I. INTRODUCTION

The mechanism for deposition of institutional power in several areas of the world has changed in the last half century. A new pattern of government overthrow took over the traditional military coup, specially in Latin America as extensively documented in Ref. [1]. These abrupt changes of collective behavior are typically seen in a society which exerts pressure on the parliament to promote changes outside the election model, by parliamentary impeachment. External pressures which include elements such as the state of the economy, perception of corruption or their combination may be distal reasons, but at a closer look the correlated behavior of the parliament follows the emboldenment that derives from the collective perception that there is sufficient strength in the opposition camp to overthrow the executive. Technically still within the realm of constitutional order, despite being associated to affective rather than ideological affinity [2,3], this transition mechanism seems to bring new theoretical challenges to comparative studies of presidentialism [1].

To illustrate the problem and to better understand the expected characteristics of the model, we will briefly discuss different instances were the executive was either impeached or not by the legislative in Brazil. The reason for this is that we have access to the relevant data [4–6]. The first case corresponds to the presidency of Collor de Mello, from Brazil. Collor won the 1989 elections in the second round with 54% of the votes, but his party only had 8% of the seats in the Chamber of Deputies and 4% in the Senate. By March 1990, when Collor was sworn into office and his approval ratings were at +60%, the consumer price index rose by 84%; at this point Collor launched his first economic plan. But, in spite of the extreme measures imposed, government control of inflation proved elusive and popular discontent increased. The application of a second (unsuccessful) economic plan (Collor II), and a number of corruption cases revealed by the press provoked a plummet on the approval ratings and triggered a number of streets demonstrations. With very few allies in the Legislative, the impeachment was approved by 441 votes against 38 in the chamber of deputies and by 76 votes against 2 in the Federal Senate.

After a short transition government, Cardoso and Lula da Silva led administrations that completed the full term, without successful parliamentary challenges. In more recent times, we have the presidency of Rousseff that also ended in impeachment. Rousseff's presidency began on January 2011, with a healthy approval rating of 47%, a balanced senate and majority in the deputies' chamber. Rousseff was reelected for a second term; however, the composition of the Legislative

*Contact author: j.p.neirotti@aston.ac.uk

†Contact author: ncaticha@usp.br

had changed against Rousseff. Her public image was damaged by the corruption in the management of the state-owned oil company (Petrobras) and the subsequent anticorruption investigation (operation *Car Wash*). The chamber of deputies approved the initiation of the impeachment process (367 in favor, 137 against) and the federal senate finally removed her from office (61 in favor, 20 against).

The presidency of Jair Bolsonaro in Brazil is another example of presidential tenure that did not end in impeachment. His support remained moderate, in the mid 40%. In this case, a large number of impeachment attempts ($>10^2$) were shielded from initiating legislative proceedings by the legislative chamber leader, due to the fact that Mr. Bolsonaro's polling was much higher than those of Collor or Rousseff and had considerable legislative support.

Certain characteristics are shared by the different examples, among them we cite:

(i) Presidential tenures start with high approval ratings and at least some moderate legislative support.

(ii) The members of the legislative chamber discuss the president's proposals under the influence of their internal political alliances and the effective pressure exerted by the presidential approval ratings.

(iii) Presidential approval ratings and legislative chamber's alliances evolve over time due to the emergence of items in the presidential agenda (i.e., proposed policies) or pieces of information about the president (i.e., scandals, state of the economy).

(iv) There are two types of members of Congress. Those in the first group are quenched either in favor or opposite to the president, independently of any external influence. Their voting patterns are trivial to predict. They may be collectively called the expressive members. The second group, which we call the strategic voters, vote in one or another direction.

The separation into expressive and strategic voters has been widely used in Political Science with respect to voters [7–9]. We borrow it to describe members of congress. In this paper we only discuss the behavior of the strategic members of parliament.

Furthermore, intuition permits the expectation that chances of impeachment increase under

(i) an increase of difference in opinions between strategic legislative agents and the president,

(ii) a decrease of the presidential approval rating.

But intuition may not be sufficient. There is a need to understand mechanisms for opinion formation and alliance formation to provide insight for the understanding of modern presidential democracies. Empirical evidence coming from research in psychology supports there is a cost of dissent, with humans trying to attain social conformity modulated by peer pressure [10–13] and that conformity is learned from interactions within a social network, e.g., Ref. [14]. The cost of dissension on some set of issues can be modeled using techniques of statistical mechanics of disordered systems and there is an extensive literature on polarization [15–21] and echo chambers [22–25] in opinion dynamics models. In this paper we address the collective behavior behind constitutional impeachments and legislative rebellions. Our aim is to contribute to a twofold discussion to this area. One is a mathematical approach based on the analysis of complex systems [26], using replica methods and agent-based modeling. This leads to the identification of the pertinent effective control parameters and the order parameters relevant to construct a phase diagram. The second is to encourage political scientists to devise empirical methods to measure these parameters which may better instrumentalize their approach to the study of real legislative rebellions. Before we delve into the problem, we discuss the type of result that emerges from our analysis and give a summary of what might be obtained from this approach. The discussion of how this picture emerges starts in Sec. II.

The two control parameters can be succinctly interpreted as the public approval rating and the strategic legislative chamber ideological alignment with the president. Figure 1 shows the phase diagram, with three regions of different collective behavior of the legislative body, as indicated by an order parameter, with the following meaning: in light gray (yellow online) the coexistence region where the strategic legislative body has a distribution of alignment with two peaks, associated to the groups for and against the executive. The dark gray (orange online) region shows a unified legislative in favor of the president, and in the white region, the president has lost support from the strategic members of congress. The lines were constructed from the data on presidential polling and from legislative voting records. The mean support of the government in legislative is measured between polling dates and only considers what are called issues of substance, where a qualified quorum is needed, as opposed to simple procedural issues, where a win of the government position carries no significant meaning. These trajectories, which are driven by externalities, show for four Brazilian presidents, cases that led to constitutional legislative rebellions and impeachments and others that remained in the stereotypical democratic regime, where dialogue is still possible. In Sec. VII we come back to how this figure was constructed.

II. ORGANIZATION

The model is introduced in Sec. III. The analytical approach derives from the use of statistical mechanics in the space of interactions in the Gardner style [27], here applied not only to a single perceptron agent, but to a population of such agents, that interact on a random graph. The relevant timescales of the different changes that can occur are discussed and this leads to the methodology appropriate for the analysis. In Sec. IV we present the structure of the agents, the relevant order parameters that characterize the macroscopic state of the society and the analytical framework. The two types of quenched disorder, from the issues under discussion and the graph of interactions, lead as shown in Sec. V, to functional mean field equations that determine the thermodynamics of the model. In Sec. VI we present analytical results obtained from the study of the saddle-point equations. Readers interested in the lessons that can be gleaned from this toy model should go to Sec. IX where the interpretation in terms of less mathematical terms can be found. A short version of the extensive calculations is shown in Appendix A.

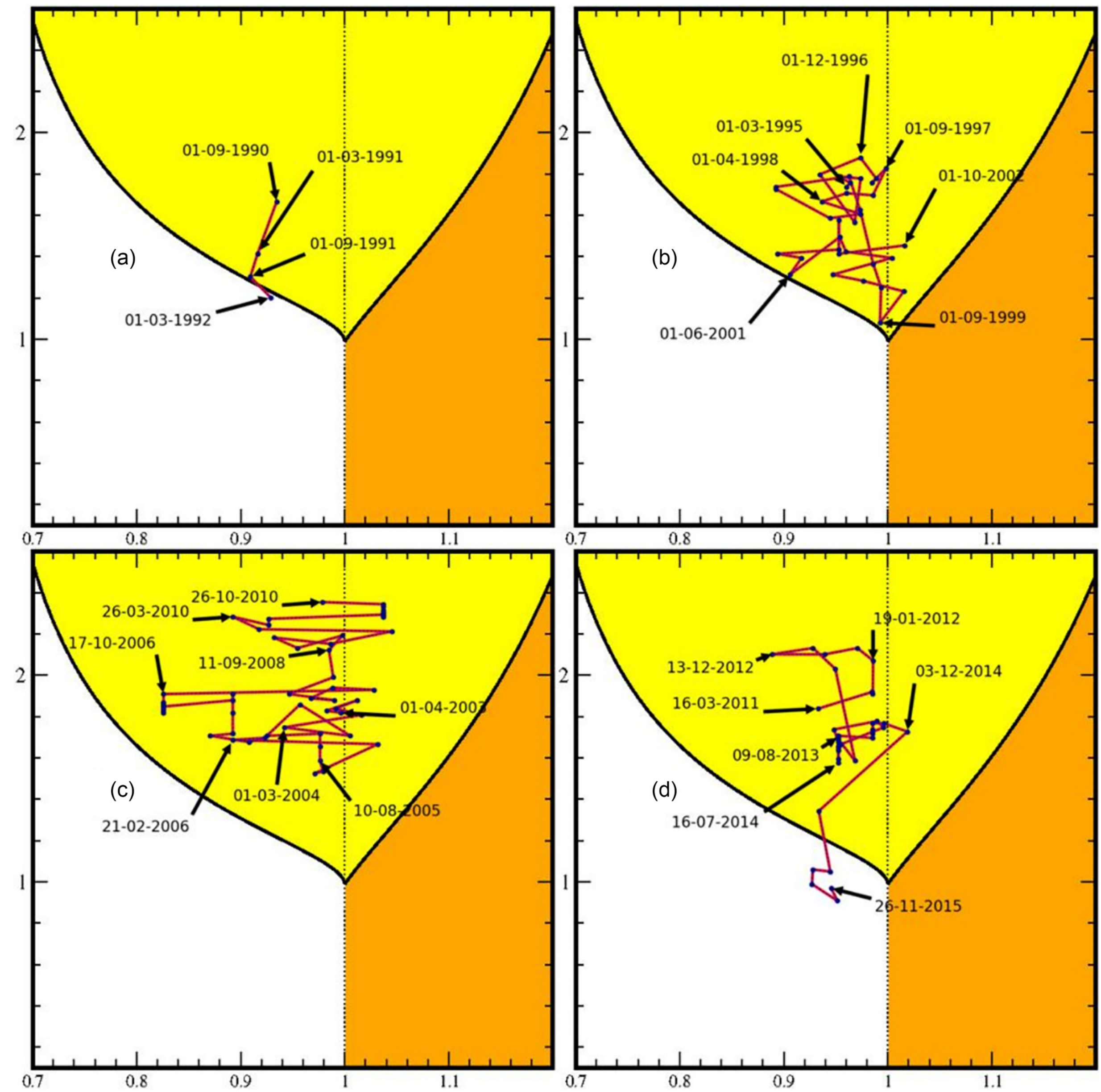

FIG. 1. (a) Collor de Mello was impeached, (b) Cardoso and (c) Lula da Silva were not impeached, and (d) Rousseff was impeached. The x axis is related to parliamentary support to the President, and the y axis to the popular support as discussed in Sec. VII.

III. MODEL

A. Hamiltonian cost

Our strategic members-of-congress agents are simple neural networks that discuss and emit for or against opinions about multidimensional issues. In addition there is a special agent, playing the role of the current executive leader, called the president, also modeled by a neural network.

The agenda, composed of the set of P substantial issues under discussion, is modeled by representing them with N-dimensional binary vectors $\boldsymbol{\xi}_\mu \in \{-1,+1\}^N$. A usual way to characterize its size in units of N is given by $\alpha = P/N$, which is a simple measure of the agenda's complexity.

The ath agent's for or against opinion is $\sigma_{a\mu}$, arising from the issue and its internal state $\mathbf{J}_a \in \mathbb{R}^N$, $\sigma_{a\mu} = \varphi(\mathbf{J}_a, \boldsymbol{\xi}_\mu) \in \{-1,+1\}$, where a runs over the members of congress. Note the internal space of an agent, parameterized by $\mathbf{J}_a$ means this is far from being a simple Ising spin. The president's opinion on issue $\boldsymbol{\xi}_\mu \in \{-1,+1\}^N$ is $\sigma_{B\mu} = \varphi(\mathbf{B}, \boldsymbol{\xi}_\mu)$, where $\mathbf{B} \in \mathbb{R}^N$ is its internal state. A specific choice of the neural networks architecture φ will be postponed for now, but its output is binary, i.e., $\varphi \in \{-1,+1\}$.

The inner circle or clique of a particular congress-agent is represented by an adjacency matrix $\boldsymbol{G}$ with entries $g_{ac} \neq 0$ if agents a cares about the opinion of agent c and zero if not. The weighed opinion on issue $\boldsymbol{\xi}_\mu$ of a's peers is $\Sigma_{a\mu} = \sum_c g_{ac}\sigma_{c\mu}$.

The model follows from simply adding costs due to different interactions that the agents have. We consider the cost for agent a to hold an opinion on the μth issue to arise from two contributions:

$$C_{a,\mu} = -\frac{1+\sigma_{B\mu}\sigma_{a\mu}}{2} - \frac{1-\sigma_{B\mu}\sigma_{a\mu}}{2}\sigma_{a\mu}\Sigma_{a\mu}. \qquad (1)$$

Equation (1) implements a mechanism of *corroboration* as follows. If agent a agrees with **B**, i.e., $\sigma_{B\mu}\sigma_{a\mu}=1$, then only the first term contributes to the cost, which gets reduced in one unit. If a and **B** disagree, $\sigma_{B\mu}\sigma_{a\mu}=-1$, then the second term is different from zero. If the weighed opinion of a's peers is in agreement with **B**, i.e., $\sigma_{B\mu}\Sigma_{a\mu}=-\sigma_{a\mu}\Sigma_{a\mu}>0$, the cost increases, and if $\sigma_{a\mu}\Sigma_{a\mu}>0$ the cost decreases. If agreeing with its peers is less costly than agreeing with **B**, then a can form a local consensus against **B**, through corroboration.

A simple rearrangement of terms allows us to write the cost as

$$2C_{a,\mu}=-\sigma_{B\mu}\sigma_{a\mu}-\Sigma_{a\mu}\sigma_{a\mu}+\Sigma_{a\mu}\sigma_{B\mu}-2. \tag{2}$$

The first term describes the advantage of having the same opinion as the president. The second, of concurring with its peers. The third one can be attributed to the disadvantage that other members of its peer group are in alliance with the president. The last is just an additive constant.

The overall cost for the entire congress, defined by the microscopic states $\{\mathbf{J}_a\}$ of the agents, the topological structure of alliances $\boldsymbol{G}$ and the complete presidential agenda $\mathcal{A}=\{\boldsymbol{\xi}_\mu,\sigma_{B\mu}\}_{\mu=1,\dots P}$, gives the full Hamiltonian cost of the system, up to a constant:

$$\begin{aligned}E(\{\mathbf{J}_a\},\mathcal{A},\boldsymbol{G})&=\sum_{a,\mu}C_{a,\mu}=E_1+E_2+E_3,\\ E_1&=-\sum_{a,\mu}\sigma_{B\mu}\sigma_{a\mu},\\ E_2&=-\sum_{a,\mu}\Sigma_{a\mu}\sigma_{a\mu},\\ E_3&=\sum_{a,\mu}\Sigma_{a\mu}\sigma_{B\mu},\end{aligned} \tag{3}$$

with the three terms, respectively, describing for every agent, the interaction of the agent and the president (E_1), of the agent and its peers (E_2), and of its peers and the president (E_3). The cost for agent a, expression (2), depends on the g_{ac} in two places. In the first, which we leave as shown, it describes the interaction of the peers c with agent a. The second describes the overall influence of the president on the group of peers of a. To simplify matters we disregard fluctuations in the second term, and substitute $\sum_a g_{ca}\approx\langle\sum_a g_{ca}\rangle\approx\nu\eta_0$, where η_0 is the average intensity of the influence exerted by another agent and ν the average size of the group. The parameter $\nu\eta_0$ represents the excess effective number of agents with an opinion against the president's position and is a central effective parameter. Note that we don't disregard fluctuations of $\sum_a g_{ca}\sigma_{a\mu}$, expected to be larger. Then the overall cost, up to an additive constant, simplifies to

$$\begin{aligned}&E_0(\{\mathbf{J}_a\};\mathcal{A},\boldsymbol{G})\\&\quad=-\frac{1}{2}\left[(1-\nu\eta_0)\sum_{a,\mu}\sigma_{B\mu}\sigma_{a\mu}+\sum_{\mu ac}g_{ac}\sigma_{a\mu}\sigma_{c\mu}\right].\end{aligned} \tag{4}$$

B. Timescales

The choice of techniques used to analyze the system follows from a discussion of the relevant "physical" timescales of the problem. While there is no conservation law that applies to the global cost in any strict sense, there are several relevant timescales associated to this discussion. The agenda under discussion and the political alliances are supposed to remain valid on a long timescale τ_q of around one year, certainly less than τ_P, the 4 or 5 years of the presidential cycle. For times of the order of τ_q we expect ν, the dissident clique size; η_0, the dissident interaction strength; and α, a measure of the volume of the agenda covered, to remain constant. Agents interact and may change opinions about the issues on a faster scale τ_{op} of a few days. τ_{op} is the typical time elapsed between the treatment of subsequent agenda items $\boldsymbol{\xi}_\mu$ and $\boldsymbol{\xi}_{\mu+1}$. The expected value of the cost is sufficiently stable on an intermediate timescale τ_C, which is larger than the timescale associated to the dynamics of the agents but much shorter than changes of the issues of national interest, $\tau_{op}\ll\tau_C\ll\tau_q<\tau_P$. τ_C is of the order of weeks, similar to the time validity of presidential polls data. This separation of the timescales determines the methodology of analysis of the problem and leads to a description of the system with a Boltzmann distribution with a β, conjugated to the expected value of the cost, that controls the size of the fluctuations above the ground state. It can be interpreted as the pressure that society at large exerts on congress. As an example, [28] choose β as a measure of the president's polling or presidential approval rating. Since the timescale in which the agenda and the political alliances changes is still larger, their random effect can be averaged over as quenched disorder. This is reasonable, since at least during τ_q the prominent issues of the agenda are to some extent fixed, as are the intraparty alliances.

The macroscopic state of the system is characterized by order parameters to be described below. Still within the presidential cycle, changes due to externalities may lead to changes in the intensive parameters. Phase transitions may occur for a congress divided into a situation and opposition parties, to a majoritarian congress that either supports a state of almost unanimous support for the president or it is in almost total opposition. These transitions are to a constitutional dictatorship regime or to a state where the conditions for impeachment are ripe. They signal presidential crises driven by the collective properties of congress and not by external or internal military forces that act by simple dissolution of congress.

IV. METHODS AND ORDER PARAMETERS

We have not yet made explicit the *a priori* measure of the σ variables. If it were just a product of independent uniform measures, e.g., Ising-like variables, then several interesting features of the problem would remain untouched. Thus we decided for more structured agents, which we model by a neural network classifier with a binary for/against, ±1 output. To keep it analytically manageable, we choose the simplest architecture, the single layer perceptron. Linearly separable models, in some manner similar to the Rescorla-Wagner model [29] from psychology have been shown to be useful in describing human performance in several cases. Therefore the dynamical variables of agent a are N-dimensional vectors $\mathbf{J}_a$ and its opinion on an issue is $\sigma_{a\mu}=\mathrm{sgn}(\mathbf{J}_a\cdot\boldsymbol{\xi}_\mu)$, where for any $\mathbf{V}$ and $\mathbf{U}\in\mathbb{R}^N$ we have that $\mathbf{V}\cdot\mathbf{U}=\sum_{j=1}^N V_jU_j$. The issues from the agenda are constructed by choosing independently $P=\alpha N$ vertices of the N-dimensional hypercube with coordinates of absolute value equal to one. M is the number of congressional agents, which for the Brazilian National Congress is 513.

Under the assumption that the average value of E_0 in Eq. (4), is approximately constant over a cycle of discussions of order τ_C and the random agenda and alliances quenched on the τ_q scale, standard arguments yield the probability distribution of the states of the congress-agents, given by

$$\mathcal{P}(\{\mathbf{J}_a\}|\beta, \mathcal{A}, \mathbf{G}) = \frac{1}{Z}\mathcal{P}_0(\{\mathbf{J}_a\})\exp\{-\beta E_0(\{\mathbf{J}_a\}; \mathcal{A}, \mathbf{G})\}, \quad (5)$$

where $\mathcal{P}_0(\{\mathbf{J}_a\}) = \prod_a (2\pi e)^{-N/2}\,\delta(\mathbf{J}_a\cdot\mathbf{J}_a - N)$ is the *a priori* measure of the agents weights, taken to be independent and uniform over the spherical shell of radius $\sqrt{N}$ in N dimensions. The discussion about the separation of timescales require the use of quenched disorder. The macroscopic properties of the system are obtained from the free energy $f = -\beta^{-1}\overline{\ln Z}$, averaged of the possible agendas and alliances, taken to be fixed on the relevant timescale.

The interaction g between a pair of agents, an element of the matrix $\boldsymbol{G}$, are assumed independent of each other and identically distributed. They are constructed in two steps. First, the random variable $x \in \{0, 1\}$ which has a Bernoulli distribution with parameter p, is used to decide if there is a connection present between two peers. Then, the strength of their interaction η is drawn from a Normal distribution centered at η_0 with variance Δ^2. With $\nu = 2p(M-1)$ [see Eq. (A16)],

$$\begin{aligned}\mathcal{P}(g, \eta, x|\nu, \eta_0, \Delta^2) &= \mathcal{P}(g|\eta, x)\mathcal{P}(x|p)\mathcal{P}(\eta|\eta_0, \Delta^2)\\ &= \delta(g - x\eta)[(1-p)\delta(x) + p\delta(x-1)]\\ &\quad\times\mathcal{N}(\eta|\eta_0, \Delta^2).\end{aligned} \quad (6)$$

The problem is complicated by the existence of two sources of quenched disorder, the agenda $\mathcal{A}$ and the alliances, encoded in the matrix $\boldsymbol{G}$. An adaptation of ideas introduced in Refs. [30–32] to treat this problems associated to coding theory are needed here. Due to the technical impossibility of computing the average of a logarithm we proceed by applying the replica formalism [26], i.e.,

$$f = -\beta^{-1}\lim_{n\to 0}\mathbb{E}_{\xi,\mathbf{G}}\frac{Z^n - 1}{n}, \quad (7)$$

where $Z^n = \prod_{\gamma=1}^n Z^{(\gamma)}$ is the partition function of the replicated system, each of the n systems linked to a replica index γ.

Taking expectations over the alliances brings forward the following population averages order parameters:

$$\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} \equiv \mathbb{E}_{\xi,\mathbf{G}}\left[\frac{1}{M}\sum_a\left(\sigma^{\gamma_1}_{a\mu_1}\sigma_{B\mu_1}\ldots\sigma^{\gamma_\ell}_{a\mu_\ell}\sigma_{B\mu_\ell}\right)\right], \quad (8)$$

where ϱ^γ_μ is the average agreement of the population with $\mathbf{B}$ on the μth issue on the γ replica, and $\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$ are population averages of the agreement of individual a with $\mathbf{B}$ across systems γ_1 (on issue $\boldsymbol{\xi}_{\mu_1}$) to γ_ℓ (on issue $\boldsymbol{\xi}_{\mu_\ell}$). Their expectation values are ℓ-point correlation functions for the opinions. The introduction of these parameters also requires the introduction of conjugate parameters $\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$. Observe that $\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$ is the average of local properties, thus the conjugate variable $\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$ must represent the average effect of the local neighborhood on the local agent. By imposing the replica-symmetric ansatz [30–32], the order parameters should not present any dependency on either replica or agenda item indexes, they should only depend on their number ℓ. By observing that the definition of the order parameters Eq. (8) satisfy $-1 \leqslant \varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} \leqslant 1$, we suppose the existence of a field $\tanh(\beta z)$ which is drawn from two normalized distributions $\pi(z)$ and $\hat{\pi}(z)$ such that

$$\begin{aligned}\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} &= \int \mathrm{d}z\,\pi(z)\tanh^\ell(\beta z),\\ \hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} &= \nu\int \mathrm{d}z\,\hat{\pi}(z)\tanh^\ell(\beta z).\end{aligned} \quad (9)$$

Graph disorder introduces these two probability densities $\pi(z)$ and $\hat{\pi}(s)$, that are functional order parameters that describe the level of consensus at the local and neighborhood levels, respectively. It is their behavior that signals the transitions from a two parties equilibrium to a consensus that can be either for or against the presidential agent.

The introduction of the order parameters in Eq. (9) will be carefully discussed in Eqs. (A22) and (A23). The usual order parameters associated with the agents overlaps and with the president are also introduced:

$$R^\gamma_a = \mathbb{E}\big(\mathbf{J}^\gamma_a\cdot\mathbf{B}/N\big),\ \ q^{\gamma\rho}_a = \mathbb{E}\big(\mathbf{J}^\gamma_a\cdot\mathbf{J}^\rho_a/N\big), \quad (10)$$

$$W^\gamma_{ab} = \mathbb{E}\big(\mathbf{J}^\gamma_a\cdot\mathbf{J}^\gamma_b/N\big),\ \ t^{\gamma\rho}_{ab} = \mathbb{E}\big(\mathbf{J}^\gamma_a\cdot\mathbf{J}^\rho_b/N\big). \quad (11)$$

Under the assumption of replica symmetric saddle points $R^\gamma_a = R, q^{\gamma\rho}_a = q$, and, by Ref. [33], $W^\gamma_{ab} = t^{\gamma\rho}_{ab} = W$, the properties of the system follow from the extrema of the free-energy functional (see Appendix A):

$$\begin{aligned}\beta f[q, R, \pi, \hat{\pi}] = \underset{q,R,\pi,\hat{\pi}}{\mathrm{extr}}\Bigg\{&-\frac{1}{2}\left(\ln(1-q) + \frac{q-R^2}{1-q}\right) + \alpha\nu\int dz\,ds\,\pi(z)\hat{\pi}(s)\,\ln\left(\frac{1+\tanh(\beta s)\tanh(\beta z)}{1-\tanh(\beta s)}\right)\\ &-\alpha\frac{\nu}{2}\int dz_1\,dz_2\,\pi(z_1)\pi(z_2)\,\langle\ln[1+\tanh(\beta\eta)\tanh(\beta z_1)\,\tanh(\beta z_2)]\rangle_\eta\\ &-\alpha\left\langle\left\langle\ln\left[1+\epsilon^{-1}\mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)\right]\right\rangle_x\right\rangle_y\Bigg\},\end{aligned} \quad (12)$$

where the averages are taken over the following random variables $\eta \sim \mathcal{N}(\eta|\eta_0, \Delta^2)$, $y \sim \mathsf{P}(y|\hat{\pi})$, and $x \sim 2\mathcal{N}(x|0,1)\mathcal{H}(-Rx/\sqrt{q-R^2})$, where $\mathcal{H}(t)$ is the Gardner error function $\mathcal{H}(t) = \int_t^\infty \mathrm{d}x\mathcal{N}(x|0,1)$. We used the shorthand $\epsilon = \epsilon(\beta, \nu\eta, y) = [\exp(2\beta(1-\nu\eta_0 + y)) - 1]^{-1}$. The free energy is a functional of the normalized distributions π and $\hat{\pi}$. The new variable y's distribution is

$$\mathsf{P}(y|\hat{\pi}) \equiv \int \frac{d\hat{y}}{2\pi} e^{-iy\hat{y}} \exp\left[\nu\left(\int ds\hat{\pi}(s)e^{i\hat{y}s} - 1\right)\right]. \quad (13)$$

The characteristic function $\phi_s(\hat{y})$ of $\hat{\pi}(s)$, and the generator function of the cumulants of s, $K_s(\hat{y})$ are

$$\phi_s(\hat{y}) = \int ds\hat{\pi}(s)e^{i\hat{y}s}, \quad (14)$$

$$K_s(\hat{y}) = \log \phi_s(\hat{y}). \quad (15)$$

We observe that there exists a random variable u, with a generator of cumulants function given by $K_u(\hat{y}) \equiv \phi_s(\hat{y}) - 1$. Add ν independent copies of u to define $y = \sum_{i=1}^{\nu} u_i$, since

$$\mathsf{P}(y|\hat{\pi}) = \int \frac{d\hat{y}}{2\pi} e^{-iy\hat{y}} \exp\left[\nu K_u(\hat{y})\right], \quad (16)$$

$$= \int \frac{d\hat{y}}{2\pi} e^{-iy\hat{y}} [\phi_u(\hat{y})]^{\nu}, \quad (17)$$

where the u_i are random variables with the property that the rth cumulant of u is equal to $\mathbb{E}(s^r|\hat{\pi})$, the rth moment of s. Hence, as ν grows, y becomes normal. The rth order cumulant $\kappa_r^{(y)}$ of y satisfies

$$\kappa_r^{(y)} = \nu\kappa_r^{(u)} = \nu\mathbb{E}(s^r|\hat{\pi}), \quad (18)$$

so the cumulants of y are constructed by accumulation of the cumulants of u or of the moments of s. It automatically follows that

$$\mathbb{E}(y|\mathsf{P}) = \nu\mathbb{E}(s|\hat{\pi}), \quad (19)$$

$$\mathbb{E}(y^2|\mathsf{P}) - \mathbb{E}(y|\mathsf{P})^2 = \nu\mathbb{E}(s^2|\hat{\pi}). \quad (20)$$

Since $\mathbb{E}(s^2|\hat{\pi})$ turns out to be proportional to η_0^2, y's variance is proportional to $1/\nu$ in the relevant region where $\nu\eta_0$ is of order 1.

V. SADDLE-POINT EQUATIONS

The extreme of the free energy (12) is determined by the saddle-point equations, which determine the order parameters in a self-consistent way. The distribution $\hat{\pi}(s)$ satisfies

$$\hat{\pi}(s) = \int dz \int dy\, \mathsf{P}(z, s, y|\hat{\pi})$$
$$= \int dz \int dy\, \mathsf{P}(y|\hat{\pi})\mathsf{P}(z|y)\mathsf{P}(s|z), \quad (21)$$

where

$$\mathsf{P}(z|y) = \langle\delta[z - \beta^{-1}g(x; \epsilon, q)]\rangle_x, \quad (22)$$

$$\mathsf{P}(s|z) = \langle\delta(s - \beta^{-1}\text{arctanh}[\tanh(\beta\eta)\tanh(\beta z)])\rangle_\eta, \quad (23)$$

and

$$g(x; \epsilon, q) = \frac{1}{2}\ln\frac{1+\epsilon}{\epsilon} + \frac{1}{2}\ln\frac{1-\mathcal{H}_+}{\mathcal{H}_+};$$
$$\text{with } \mathcal{H}_+ = \mathcal{H}\left(\sqrt{\frac{q}{1-q}}x\right). \quad (24)$$

The equations for $\pi(z)$ and $\hat{\pi}(s)$ are

$$\pi(z) = \int dy\, \mathsf{P}(y|\hat{\pi})\langle\delta[z - \beta^{-1}g(x; \epsilon, q)]\rangle_x, \quad (25)$$

$$\hat{\pi}(s) = \int dz\, \pi(z)\langle\delta(s - \beta^{-1}\,\text{arctanh}[\tanh(\beta\eta)\tanh(\beta z)])\rangle_\eta. \quad (26)$$

This shows that $\pi(z)$ is the distribution of the local field z associated to the agent, that is constructed over the influence of its neighborhood through $\mathsf{P}(y|\hat{\pi})$, the distribution of consensus in the neighborhood of the agent, and the influence of the agenda through the average over x. These two contributions represent the sources the agent uses to form its opinion. The distribution of the neighborhood effective field $\hat{\pi}(s)$ acting on the local agent is obtained by averaging over the distribution of the local agent field through $\pi(z)$ and through the distribution of influences through the average over η. Observe that if there is agreement between agent and president and if the influence between peers is strong (large η), the neighborhood field s becomes large and positive. If the agent does not give any importance to its peers ($\eta = 0$), then the distribution $\hat{\pi}(s)$ becomes a δ function centered at zero, and the system decouples.

In addition, these functional saddle-point equations also depend on the usual parameters q and R, which satisfy

$$\frac{q - R^2}{1 - q} = \frac{\alpha}{\pi}\int dy\, \mathsf{P}(y|\hat{\pi}) \times \int \mathcal{D}x \frac{\exp\left(-\frac{qx^2}{1-q}\right)\mathcal{H}\left(-\frac{Rx}{\sqrt{q-R^2}}\right)}{\left[\epsilon + \mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)\right]^2}, \quad (27)$$

$$\frac{R}{\sqrt{1-q}} = \frac{\alpha}{\pi}\sqrt{\frac{q}{q-R^2}}\int dy\, \mathsf{P}(y|\hat{\pi}) \times \int \mathcal{D}x \frac{\exp\left\{-\left(\frac{q}{1-q} + \frac{R^2}{q-R^2}\right)\frac{x^2}{2}\right\}}{\epsilon + \mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)}, \quad (28)$$

The numerical solution of this set of equations is discussed in Appendix B.

VI. MACROSCOPIC CHARACTERIZATION OF THE MODEL

In Appendix B we demonstrate that for sufficiently large neighborhoods ($\nu > O(1)$), for sufficiently high presidential approval β, and for a very narrow distribution of social strengths, i.e., $\Delta \ll \eta_0$ and $\mathcal{P}(\eta) = \mathcal{N}(\eta|\eta_0, \Delta^2)$, there are three possible solutions for Eqs. (25) and (26). Two of them are the pure *supportive* state, in the sense that supports the presidential *status quo*, obtained if $\nu\eta_0 < 1$ and the other is the *opposition* pure state if $\nu\eta_0 > 1$. There is a possibility of a third solution which is a mixture of the two pure states, that appears in the region of the phase space where dialogue between opposite positions may exist. Defining a parameter Λ as

$$\Lambda(R) \equiv \frac{\mathrm{sgn}(R)}{2\beta}\frac{q}{1-q}, \tag{29}$$

permits plotting a partial phase diagram, presented in Fig. 5. $|\Lambda(R)|^{-1}$ plays the role of an effective presidential approval rating. In the region $(|\Lambda|, \eta_0) \in \mathbb{A}$ a convex combination of both pure states is found. For $(|\Lambda|, \eta_0) \notin \mathbb{A}$ the distributions can be expressed as

$$\hat{\pi}_0(z) \equiv \mathcal{N}\big(z\big|\mathcal{I}_0^\star(\Lambda,\eta_0), \eta_0^2 - [\mathcal{I}_0^\star(\Lambda,\eta_0)]^2 + \Delta^2\big), \tag{30}$$

$$\pi_0(s) \equiv \mathcal{N}\bigg(s|1+\nu[\mathcal{I}_0^\star(\Lambda,\eta_0)-\eta_0] + \frac{1}{2}\Lambda, \nu\big(\eta_0^2+\Delta^2\big)+\frac{3}{4}\Lambda^2\bigg), \tag{31}$$

$$\mathsf{P}_0(y|\hat{\pi}) \equiv \mathcal{N}\big(y\big|\nu\mathcal{I}_0^\star(\Lambda,\eta_0), \nu\big(\eta_0^2+\Delta^2\big)\big), \tag{32}$$

where $\mathcal{I}_0^\star$ is the only solution to

$$\mathcal{I}_0^\star = \eta_0\mathrm{erf}\left(\frac{1-\nu\eta_0+\nu\mathcal{I}_0^\star+\frac{1}{2}\Lambda(R)}{\sqrt{2\big[\nu\big(\eta_0^2+\Delta^2\big)+\frac{3}{4}\Lambda(R)^2\big]}}\right) \tag{33}$$

outside region $\mathbb{A}$ [this equation is developed in Appendix B, Eq. (B14)]. We have observed that in the region of interest $\nu\eta_0 \sim O(1)$, the variance of $\mathsf{P}_0(y|\hat{\pi})$ is of order $O(\nu^{-1})$; therefore, we can approximate this distribution by

$$\mathsf{P}_0(y|\hat{\pi}) \approx \delta(y-\nu\mathcal{I}_0^\star). \tag{34}$$

Inside the region $\mathbb{A}$ we have mixed states described by

$$\hat{\pi}_{\mathrm{m}}(z) \equiv h_+\mathcal{N}(s|\mathcal{I}_+^\star, \Delta^2) + h_-\mathcal{N}(s|\mathcal{I}_-^\star, \Delta^2), \tag{35}$$

$$\pi_{\mathrm{m}}(s) \equiv \mathcal{N}\bigg(s\bigg|1+\nu[\mathcal{I}^\star-\eta_0]+\frac{1}{2}\Lambda, \nu\big(\eta_0^2+\Delta^2\big)+\frac{3}{4}\Lambda^2\bigg), \tag{36}$$

$$\mathsf{P}_{\mathrm{m}}(y|\hat{\pi}) \equiv \mathcal{N}(y|\nu\mathcal{I}^\star, \nu\{\langle(\mathcal{I}^\star)^2\rangle+\Delta^2\}), \tag{37}$$

where $\mathcal{I}_\pm^\star$ are the stable solutions to Eq. (33) in $\mathbb{A}$, $h_\pm$ are suitable weights (B24) satisfying $0 \leqslant h_\pm \leqslant 1$ and $h_+ + h_- = 1$, $\mathcal{I}^\star$ is the mixed solution:

$$\mathcal{I}^\star := h_+\mathcal{I}_+^\star + h_-\mathcal{I}_-^\star, \tag{38}$$

$$\langle(\mathcal{I}^\star)^2\rangle := h_+(\mathcal{I}_+^\star)^2 + h_-(\mathcal{I}_-^\star)^2, \tag{39}$$

and, given that for all $(\lambda, \eta_0) \in \mathbb{A}$, $\nu\eta_0 \sim O(1)$, we have that

$$\mathsf{P}_{\mathrm{m}}(y|\hat{\pi}) \approx \delta(y-\nu\mathcal{I}^\star). \tag{40}$$

The application of Eqs. (34) and (40) into Eqs. (27) and (28) produce the following expressions:

$$\frac{q-R^2}{1-q} = \frac{\alpha}{\pi}(e^\kappa-1)^2 \times \int \mathcal{D}x \frac{\exp\left(-\frac{qx^2}{1-q}\right)\mathcal{H}\left(-\frac{Rx}{\sqrt{q-R^2}}\right)}{\left[1+(e^\kappa-1)\mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)\right]^2}, \tag{41}$$

$$\frac{R}{\sqrt{1-q}} = \frac{\alpha}{\pi}(e^\kappa-1)\sqrt{\frac{q}{q-R^2}} \times \int \mathcal{D}x \frac{\exp\left\{-\left(\frac{q}{1-q}+\frac{R^2}{q-R^2}\right)\frac{x^2}{2}\right\}}{1+(e^\kappa-1)\mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)}, \tag{42}$$

where $\kappa \equiv 2\beta(1-\nu\eta_0+\nu\mathcal{I}^\star)$. Observe that these equations are invariant under the following transformation: $(\kappa, q, R) \to (-\kappa, q, -R)$.

At very high presidential approval β, Eqs. (27) and (28) can be expressed as

$$\frac{q_\pm-R_\pm^2}{1-q_\pm} = \frac{\alpha}{\pi}\int \mathcal{D}x \frac{\exp\left(-\frac{q_\pm x^2}{1-q_\pm}\right)\mathcal{H}\left(-\frac{R_\pm x}{\sqrt{q_\pm-R_\pm^2}}\right)}{\left[\mathcal{H}\left(-\sqrt{\frac{q_\pm}{1-q_\pm}}x\right)\right]^2}, \tag{43}$$

$$\frac{R_\pm}{\sqrt{1-q_\pm}} = \pm\frac{\alpha}{\pi}\sqrt{\frac{q_\pm}{q_\pm-R_\pm^2}} \times \int \mathcal{D}x \frac{\exp\left\{-\left(\frac{q_\pm}{1-q_\pm}+\frac{R_\pm^2}{q_\pm-R_\pm^2}\right)\frac{x^2}{2}\right\}}{\mathcal{H}\left(-\sqrt{\frac{q_\pm}{1-q_\pm}}x\right)}, \tag{44}$$

where the subindex $+(-)$ is valid for $\nu\eta_0 < (>)1$. The $\beta \to \infty$ solutions satisfy $q_\pm = \pm R_\pm$. These results justify naming the solution with subindex $+$ as *in favor* and the solution with subindex $-$ as *in opposition*. Similar behavior is observed for finite but large values of the presidential approval.

For sufficiently large presidential approval ratings, sufficiently large ν and a volume of information $\alpha \gg \beta$, we can also demonstrate that

$$q = 1 - \frac{Q_0^2}{\alpha^2} + o(\alpha^{-2}), \tag{45}$$

$$Q_0 = \frac{(2\pi)^{3/2}}{2+\sqrt{\pi}}, \tag{46}$$

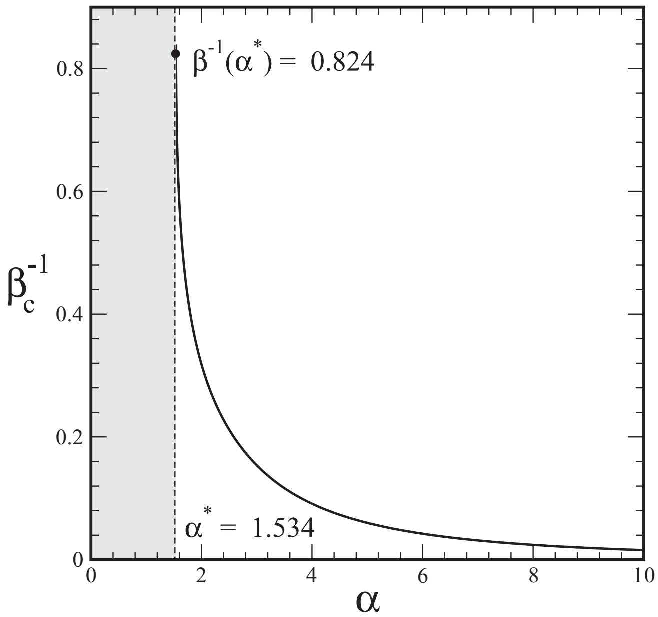

FIG. 2. Critical presidential approval against size of agenda. When the number of substantial issues ($P = \alpha N$) in the legislature is not sufficiently large, there is always a phase around the opinion boundary $\nu\eta = 1$ where opposition and supportive states coexist. This area represents the collection of points (α, β^{-1}) where a discussion between members of the chamber with different positions may occur. There is a critical value of the agenda's size $\alpha^\star = 1.534(1)$ bellow which there is always room for discussion, no matter how low the presidential approval β is. Above this threshold there is always a minimum presidential approval $\beta(\alpha)$ such that above it there is no more discussion and positions are definitely set.

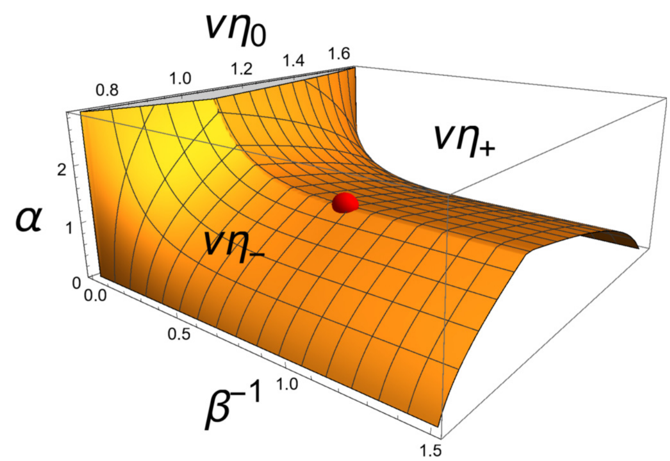

FIG. 3. Phase diagram of the system in terms of the parameters $\nu\eta_0$, β^{-1} and α. There are two phases separated by the plane $\nu\eta_0 = 1$. For $\nu\eta_0 < 1$ we have that $R > 0$ and the average consensus is in favor of **B**. For $\nu\eta_0 > 1$, $R < 0$ and the average position of the agents is to form local alliances against the president **B**. In all the points of the space above the surfaces $\nu\eta_+$ and $\nu\eta_-$, the distribution describing the position of the neighborhood, given by $\hat{\pi}(z)$, is sharply picked at $+\eta_0$, for the supportive position, i.e., $\nu\eta_0 < 1$, or at $-\eta_0$, for the opposition position, i.e., $\nu\eta_0 > 1$. In the region bellow the surfaces $\nu\eta_+$ and $\nu\eta_-$ we have the same phase separation at $\nu\eta_0 = 1$ but the contribution from the neighborhood is a mixture of a opposition component plus a supportive component. The circle at coordinates $\nu\eta_0 = 1$, $\beta^{-1} = 0.824$ and $\alpha = 1.534$ is the critical point presented in Fig. 2. The phase diagram presented in Fig. 4 has been obtained by cutting sections at constant α from this three-dimensional plot, and the the red sphere corresponds to the first value of α $(= \alpha^\star)$ for which the behavior presented in Fig. 4(c) is observed.

and

$$R = \begin{cases} q + 2\pi\sqrt{3}Q_0^3\alpha^{-3}e^{-2\beta} & \nu\eta_0 < 1 \\ -q - 2\pi\sqrt{3}Q_0^3\alpha^{-3}e^{-2\beta(\nu\eta_0-1)} & 1 < \nu\eta_0 \end{cases}. \quad (47)$$

Due to the odd parity of $R(\kappa)$, we can conclude that the plane $(\nu\eta_0, \beta^{-1})$ is divided into two phases, the in-favor phase for which $R > 0$ and $\nu\eta_0 < 1$ and a opposition phase with $R < 0$ and $\nu\eta_0 > 1$.

Consider the set of points with coordinates $(\beta^{-1}, \nu\eta = 1, \alpha)$ such that the parameter defined in Eq. (29) becomes $|\Lambda^\star| = 0.411$ (see Fig. 5). In consequence, the solution of Eq. (33) over this line is $\nu\mathcal{I}^\star = 0.988$ and the correspondent index κ becomes a function of the overlap q, i.e.,

$$\kappa^\star(q) = \frac{\nu\mathcal{I}^\star}{|\Lambda^\star|}\frac{q}{1-q}. \quad (48)$$

By solving the Eqs. (41) and (42) with κ given by Eq. (48), we obtain the curve presented in Fig. 2. We solve for the properties of the equilibrium state valid in the τ_q timescale in the intensive parameters of the system: β the presidential approval, α a measure of the complexity of the agenda and $\nu\eta_0$ a measure of the peer pressure by other agents in congress which arises from the mean number of interlocutors ν and the mean intensity of their interaction η_0. These results are presented as the phase diagrams shown in Fig. 3.

By fixing the value of α, we can study the behavior of the system for a given volume of information. We constructed Fig. 4 by solving Eq. (33) for different values of β^{-1} and $\nu\eta_0$ at fixed values of α with $\nu = 10$ and $\Delta = 0.01$. The full lines separate pure-state areas [in white, for $R < 0$ and in dark gray (orange online) for $R > 0$, given by Eqs. (30)–(32)] from mixed-state areas [in gray (yellow online), given by Eqs. (35)–(37)]. We also found that for values of $\alpha < \alpha^\star = 1.534(1)$ the mixed states are contained into a mixed-triangular-shaped area, with vertexes at $(\beta^{-1} = 0, \nu\eta_0 = 1.651(1))$, $(\beta^{-1} = 0, \nu\eta_0 = 0.717(1))$, and $(\beta = \beta(\alpha), \nu\eta_0 = 1)$. In particular, we observe that $\beta^\star \equiv \beta(\alpha^\star) \approx 1.214(1)$ and for all $\alpha^\star < \alpha' < \alpha$, $\beta(\alpha) > \beta(\alpha') > \beta(\alpha^\star)$. The lightly shaded (yellow online) region close to the boundary $(\nu\eta_0 = 1)$ is characterized by a mixture of states that represents a state of dialogue, where the influence on the agents from their neighborhoods come from both sides of the argument. The larger the complexity of the agenda (α) the smaller the size of this region. To complete our analysis and for very low values of β we obtain the following values for the parameters R and q:

$$q \approx \frac{2\alpha}{\pi}\beta^2(1-\nu\eta_0)^2\left(\frac{2\alpha}{\pi}+1\right), \quad (49)$$

$$R \approx \frac{2\alpha}{\pi}\beta(1-\nu\eta_0)[1-2\beta(1-\nu\eta_0)]. \quad (50)$$

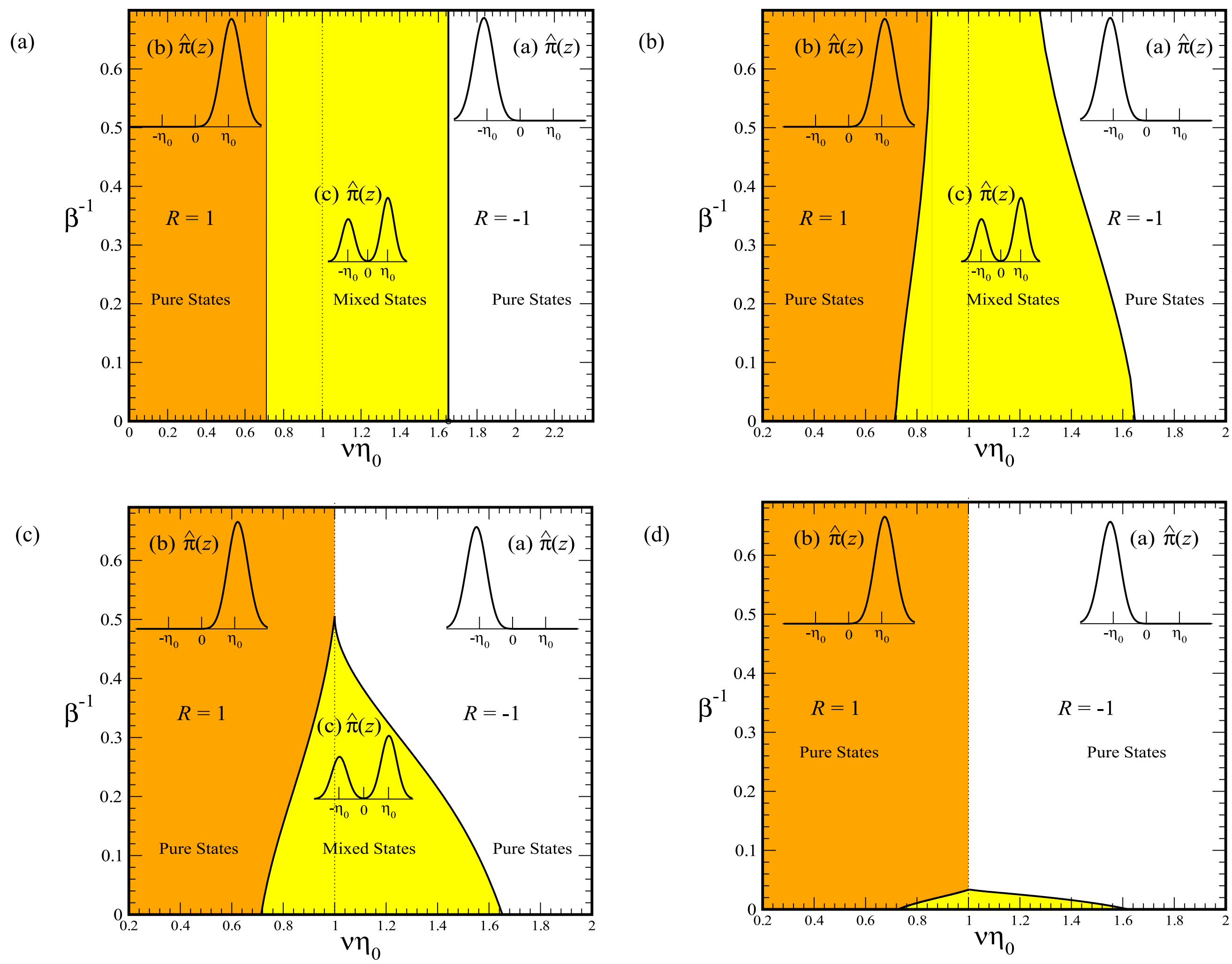

FIG. 4. Phase diagram of the system for different volumes of information α. For $\nu\eta_0 < 1$ and all values of β and α, the average agreement with the executive R is positive and the state of the system is mostly supportive, i.e., the local consensus, described by the distribution $\hat{\pi}$ is centered at $\eta_0 > 0$. For $\nu\eta_0 > 1$ and all values of β and α, the level of agreement with **B** is negative and the local consensus is against the president. For values of $\alpha < \alpha^\star \approx 1.534$ there is always an area around the boundary $\nu\eta_0 = 1$, represented in light gray (yellow online), where mixed states [Eqs. (35)–(37)] develop, whereas outside this area we have pure states only [given by Eqs. (30)–(32)]. For values of $\alpha \geqslant \alpha^\star$ the area where the mixed states develop gets a mixed-triangular shape, with vertexes at $(\beta^{-1} = 0, \nu\eta_0 = 1.651(1))$, $(\beta^{-1} = 0, \nu\eta_0 = 0.717(1))$, and $(\beta = \beta(\alpha), \nu\eta_0 = 1)$. In particular we observe that $\beta^\star \equiv \beta(\alpha^\star) \approx 1.214$ and for all $\alpha^\star < \alpha' < \alpha$, $\beta(\alpha) > \beta(\alpha') > \beta(\alpha^\star)$. These results correspond to fixed values of $\nu = 10$ and $\Delta = 0.01$. The area in white corresponds to pure opposition states and the area in dark gray (orange online) correspond to supportive pure states. Panels (a), (b), (c), and (d) correspond to $\alpha = 0$, $\alpha \lesssim \alpha^\star$, $\alpha \gtrsim \alpha^\star$ and $\alpha \gg \alpha^\star$, respectively.

VII. CONSTRUCTING THE TRAJECTORIES

We now turn to the confrontation of the model and empirical data obtained from legislative voting records [4] and presidential polling data [5]. There are several details [6] of the voting data that have to be taken into account to illustrate the history of events during a presidential tenure. These will be the subject of a forthcoming paper written from the perspective of political science analysis. For the current purposes, of describing collective behavior in a complex system, we do a simple analysis and proceed as follows. There are 25 types of votes in the legislative chamber of Brazil, which can be grouped roughly into substantial and procedural. Some require simple majority to pass, while others require some type of qualified quorum, such as half plus one, or 3/5 of the total number of members of parliament. We selected votes only of the substantial and qualified quorum types.

The frequency of Presidential polls changes, but they roughly occurs every few months. Call the times of occurrence T_n. For each time interval $(T_{n-1}, T_n]$ between polls we measured the mean of votes for and against the government positions. Its difference defines a variable M_n used to represent government support.

Popular support is measured by the approval rate. The results are given as the percentage in three categories: “optimal/good,” “regular,” and “bad/terrible.” The approval A_n index is constructed by adding the “optimal/good” plus half of “regular” at time T_n.

The empirical trajectories are obtained by joining the set of points $\{M_n, A_n\}$. The problem now is to place these paths on the phase diagram, i.e., making them trajectories in the thermodynamic space. Two issues have to be considered. First, which phase diagram? These amounts to choosing the statistical mechanics variables adequate to discuss the thermodynamics state.

What is the analog in the statistical mechanics formulation of the presidential approval? A natural choice is to use the temperature β^{-1}. This seems natural since the Lagrange multiplier controls the magnitude of the fluctuations above the ground state of the cost function. For large temperatures, large fluctuations about the minima are possible and for large popular approval, dissent from peers will not carry a large political weight. But the diagram then depends on q, which itself depends on α; see Fig. 4. However, in terms of the scaling $\tilde{\beta} = \frac{\Lambda^*}{|\Lambda|} \propto \beta(1-q)/q$, the diagram collapses and can be represented in two-dimensional space. These simplifies matters, since no empirical determination of q is needed. Of course, this points into possible future directions for a more detailed analysis. For the other coordinate we use the inverse of $\nu\eta_0$, which describes the effective coupling between agents.

The second issue deals with how to relate the empirical and theoretical pairs. We use linear transformations between empirical and theoretical variables. The parameters of the transformations are chosen such that Cardoso's trajectory remains in the mixed (dialogue) phase. Despite two very serious crises, when the government stability was in peril, he remained in office. The other three trajectories are then fixed and shown in Fig. 1. Collor and Rousseff end up in the region with no support form the strategic representatives, and Lula da Silva is easily within the mixed phase, never being in danger of an impeachment. The transformations are

$$A = a_A\tilde{\beta} + b_A,$$
$$M = a_M\frac{1}{\nu\eta_0} + b_M. \tag{51}$$

Figure 1 uses $[a_A, b_A, a_M, b_M] = [50, 27, 2000, -1790]$. These are not fundamental constants, since changes to the model or to the selection of voting procedures would change their value. The problem of determining the relevant transformations, i.e., the adequate "Boltzmann constants," remains open.

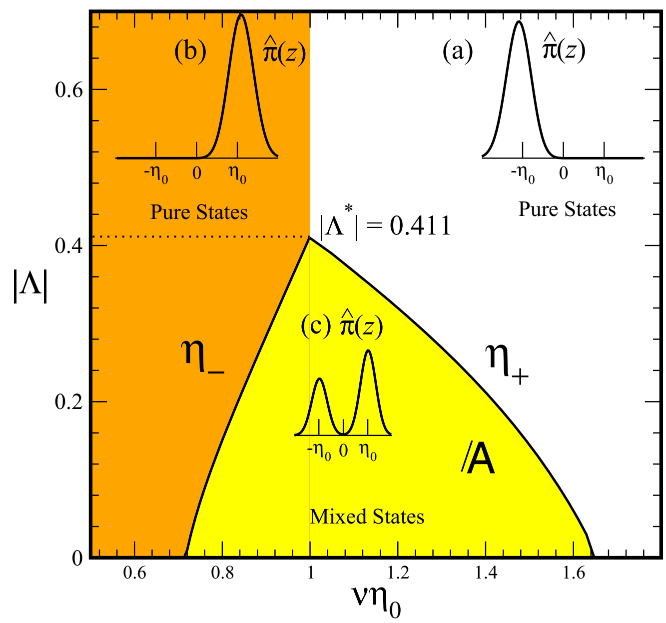

FIG. 5. Phase diagram of the system in terms of the parameters $\nu\eta_0$ and $|\Lambda|$. There are two phases separated by the line $\nu\eta_0 = 1$. For $\nu\eta_0 < 1$ we have that $R > 0$ and the average consensus is in favor of **B**. For $\nu\eta_0 > 1$ $R < 0$ and the average position of the agents is to form local alliances against the president **B**. In all the points of the plain outside the region $\mathbb{A}$, the distribution describing the position of the neighborhood, given by $\hat{\pi}(z)$, is sharply picked at $+\eta_0$, for the supportive position, i.e., $\nu\eta_0 < 1$, or at $-\eta_0$, for the opposition position, i.e., $\nu\eta_0 > 1$. In region $\mathbb{A}$ we have the same phase separation at $\nu\eta_0 = 1$ but the contribution from the neighborhood is a mixture of a opposition component plus a supportive component. We also observe that the vertexes of $\mathbb{A}$ are (1.651,0), (1,0.411), and (0.717,0).

VIII. DISCUSSION

During the past few decades [34,35] the application of statistical mechanics techniques to model social problems have produced a number of interesting results, not only providing new insights about social phenomena but also showing predictive capabilities [36]. Inspired by these ideas, we introduced a model for the phenomenon of impeachment in presidential democracies. The political agents, who have a simple neural network that enables making decisions on issues, interact with an external meta-agent **B**, which represents the executive, and with peers in the legislative chamber. The model balances the need to include psychological characteristics of observed behavior [10,37], the complexity of the social interactions [21,38], and the analytical tractability of the mathematical expressions constructed.

For sufficiently large number of alliances $\nu > O(1)$, the saddle-point Eq. (25)–(28) can be solved in pairs. The first two, Eqs. (25) and (26), involving the distributions π and $\hat{\pi}$ connected with the distribution of alliances and the pair (27) and (28), connected to the parameters associated with the discussion of the presidential agenda. The solution to Eqs. (25) and (26) has been expressed using the parameter Λ defined in Eq. (29), which brings an input from the disorder-from-learning part of the problem into the disorder-from-graph part of the problem. In a similar manner, the parameter κ that helps to express the solution of Eqs. (27) and (28), introduces effects from the disorder-from-graph part of the problem into the disorder-from-learning part of the problem. The constraints that emerged from expressing the solution in terms of these parameters have helped constructing the phase diagram presented in Fig. 5.

We obtained a set of sensible results for a graph with an average of ν links per vertex (the number of dissidents). In this setting and considering a steady president, i.e., **B** constant, we found that there exist two possible pure positions. One characterized by an overall average attitude in favor of the president characterized by a positive and increasing (with the volume of information) average agreement R, that we dubbed the presidential supportive state, and other with a negative and decreasing value of R, the opposition state. These states are also characterized by a sharp picked distribution of neighbors' influences $\hat{\pi}$, centered at $\mathcal{I}_+^\star$ ($\mathcal{I}_-^\star$) for the supportive (opposition) state. From Fig. 4 we also showed that for volumes of information bellow a critical value $\alpha^\star = 1.534(1)$, there is a region in the plane $(\nu\eta_0, \beta^{-1})$, in a form of a band around $\nu\eta_0 = 1$, where mixed states, defined by the Eqs. (35)–(37), exist. The mixture is explicit in Eq. (35) that presents the influence on an agent by its neighborhood ($\hat{\pi}$) as a combination of the two sides of the argument. This band of mixed states collapses into a triangle with vertexes at $(\beta^{-1} = 0, \nu\eta_0 = 1.651(1))$, $(\beta^-1 = 0, \nu\eta_0 = 0.71845(1))$, and $(\beta = \beta(\alpha), \nu\eta_0 = 1)$, for values of $\alpha > \alpha^\star$. We also observed that the larger the volume of information the smaller the triangle area, i.e., $\alpha > \alpha'$ implies that $\beta(\alpha) > \beta(\alpha')$. The interpretation of this behavior is as follows: when information is limited (low α) and for values of effective dissidents $\nu\eta_0 \simeq 1$ the influence from the neighborhood to the agent is formed by a combination of positions in favor and against the executive. In this region the overlap R, which represents the average agreement with $\mathbf{B}$ still has a well defined sign given by $\text{sgn}(R) = \text{sgn}(1 - \nu\eta_0)$, but is the result from two pure-state contributions. In this region coexist the two positions, in pro and against the executive $\mathbf{B}$. Definite positions are not set yet, thus propitiating a state of dialogue. The more information is fed to the system the smaller this region becomes. There is a critical value of information $\alpha^\star = 1.534(1)$, beyond which this behavior is only observed for presidential approval ratings lower than $\beta^\star = 1.214(1)$. In other words, the more information is provided the purer the contribution to the agents opinion from their neighborhoods and the lower the chances for a dialogue between opposed positions. For very large values of α and $\nu\eta_0 \approx 1$ coexistence exists only if the presidential approval β is sufficiently high. Thus, only a president with high index of popularity can guarantee a discussion of the topics in the agenda between opposite positions of the legislative.

Under the light of the cases used as motivation for our model we observe that there are events, represented by particular items of the executive's agenda, that are so momentous in the formation of opinions that can be considered critical (e.g., Collor de Mello's economic plans, Rousseff's Petrobras scandal, Cardoso' s September 1999 crisis, 2001 energy crisis), to the point that, immediately after they occur, opposite positions in pro or against the executive's proposals become more consolidated, and the influence of the neighborhood on the agents becomes more polarized (on either position) and the dialogue-prone region gets reduced. If the public rejects the proposals, then β diminishes and the executive may find itself in front of a polarized legislative chamber that either supports it or not. If the negative information instances persist and neither the public nor the chamber supports the president, then the executive may find itself facing an impeachment procedure. Also observe that to be impeached a president must find itself facing an adverse legislative chamber ($\nu\eta_0 > 1$) outside the dialogue-prone phase (coexistence phase).

Rousseff started her tenure with high values of approval (high β) and a favorable composition of the legislative ($\nu\eta_0 < 1$). Over time and for a sufficiently large value of α the dialogue-prone coexistence phase in the legislative got reduced, the approval rate (β) decreased and the composition of the chamber became adverse ($\nu\eta_0 > 1$), leaving the state of Rousseff's presidency in the side of the adverse pure states of the chamber and open for an impeachment process.

This type of minimal model is capable of capturing some of the complex relationships that link the political powers and the dynamics of social opinions because it takes into account the double random quenched disorder of agenda and alliances. The analytical treatment is possible by assuming that its general properties are robust to a number of approximations which have been used in other contexts dealing with quenched disorder. The replica symmetry remains unbroken. Moreover, the interagent interactions considered are such that only mutual reciprocity between agents is allowed, $\eta_{ab} = \eta_{ba}$, no equilibrium solution could be reached otherwise.

Several theoretical natural extensions to this work can be foreseen. Other connectivity graphs could be considered at the expense of making the calculations much harder. The evolution of opinions in the presence of an adaptive social rule that slowly changes following the average position of the population was studied in Ref. [39]. In the present context, this would translate into allowing a changing $\mathbf{B}$. As a consequence, the contribution from socially neutral issues [40] could become a factor, as it can be observed by the presence of the parameter W, see Eq. (11), which represents the overlap between the representation of different agents (and it is a measure of the level of agreement between them). We expect that, if a similar setting is imposed in the present framework, the free-energy functional should be dependent also on a parameter W, revealing the contribution from the socially neutral issues to the system.

Also and more importantly, the control of the agenda allows strategies that can change the political scenario. Issues at the borders of doubt (i.e., $\boldsymbol{\xi}_0$ such that $\mathbf{B} \cdot \boldsymbol{\xi}_0 = 0$ or $\mathbf{J} \cdot \boldsymbol{\xi}_0 = 0$) are interesting since they lead to exponentially fast learning [41] in the online teacher-student scenario and may provide a recipe to avoid the dynamical traps that lead to quenched or extreme polarization.

IX. CONCLUSION

The characterization of the macroscopic state of the system is a first step in the construction of a framework to understand parliamentary rebellions that might lead or not to impeachments. In Fig. 1 we show the evolution in the space of compound variables $\tilde{\beta}$ and $(\nu\eta_0)^{-1}$. The phase diagrams can be loosely collapsed using $\tilde{\beta} = \beta/\beta_c(\alpha)$, obtained by scaling β with the critical value $\beta_c(\alpha)$ presented in Fig. 2. The exact scaling is obtained from the effective presidential approval $|\Lambda|^{-1}$, defined in Eq. (29). The strategic agents effective value $\nu\eta_0$ arises from the loyal base and opposition relative numbers and a coupling constant describing the peer pressure influence agents exert on each other. As externalities work to produce an external pressure and influence the size of the agenda under discussion, the system evolves in the phase diagram and it can cross phase boundaries into the impeachment regime.

An important characteristic of statistical mechanics is that it can identify aggregate variables, which if known, would characterize macroscopic behavior. Hence, their identification and recognition as relevant, may induce the development of empirical methodology to produce estimates of the parameters, that would in future models, allow predictions and not only description of what has occurred.

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION OF THE AVERAGES OVER $\mathcal{A}$, B and G

By observing that $\mathcal{P}(g_{ac}) = \int d\eta_{ac}\mathcal{N}(\eta_{ac}|\eta_0, \Delta^2)\sum_{x_{ac}=0,1}[p\delta_{x_{ac},1} + (1-p)\delta_{x_{ac},0}]\,\delta(g_{ac} - x_{ac}\eta_{ac})$, where the Kronecker's δ is $\delta_{X,Y} = 1$ if $X = Y$ and 0 otherwise and Dirac's δ is $\int_\Omega \mathrm{d}x\delta(x - x_0) = 1$ if $x_0 \in \Omega$ and 0 otherwise, the replicated partition function is

$$\overline{Z^n}(\beta) \equiv \int d\mathbf{B}\mathcal{P}(\mathbf{B}) \int \prod_\mu d\boldsymbol{\xi}_\mu \mathcal{P}(\boldsymbol{\xi}_\mu) \prod_a \prod_c \int dg_{ac}\mathcal{P}(g_{ac}) \int \prod_{\gamma=1}^{n} \prod_a d\mathbf{J}_a^\gamma \mathcal{P}(\mathbf{J}_a^\gamma)$$
$$\times \prod_{\gamma\mu a} \exp\left\{\beta \sum_{c\in\mathbb{N}_a} x_{ac}\eta_{ac}\mathrm{sgn}\left(\frac{\mathbf{J}_a^\gamma \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}\right)\mathrm{sgn}\left(\frac{\mathbf{J}_c^\gamma \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}\right)\right\} \prod_{\gamma\mu a} \exp\left\{(1 - \nu\eta_0)\beta\mathrm{sgn}\left(\frac{\mathbf{J}_a^\gamma \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}\right)\mathrm{sgn}\left(\frac{\mathbf{B} \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}\right)\right\}, \quad \text{(A1)}$$

and by defining the variables:

$$\lambda_{a,\mu}^\gamma \equiv \frac{\mathbf{J}_a^\gamma \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}, \qquad u_\mu \equiv \frac{\mathbf{B} \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}, \quad \text{(A2)}$$

and by defining the overlaps:

$$R_a^\gamma \equiv \frac{\mathbf{J}_a^\gamma \cdot \mathbf{B}}{N}, \qquad W_{ab}^\gamma \equiv \frac{\mathbf{J}_a^\gamma \cdot \mathbf{J}_b^\gamma}{N},$$
$$q_a^{\gamma\rho} \equiv \frac{\mathbf{J}_a^\gamma \cdot \mathbf{J}_a^\rho}{N}, \qquad t_{ab}^{\gamma\rho} \equiv \frac{\mathbf{J}_a^\gamma \cdot \mathbf{J}_b^\rho}{N}, \quad \text{(A3)}$$

we have that the expectation over patterns is

$$\langle\cdot\rangle_{\mathcal{A}} \equiv \int \prod_\mu d\boldsymbol{\xi}_\mu \mathcal{P}(\boldsymbol{\xi}_\mu) \exp\left(i\sum_{\gamma\mu a} \hat{\lambda}_{a\mu}^\gamma \frac{\mathbf{J}_a^\gamma \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}} + i\sum_\mu \hat{u}_\mu \frac{\mathbf{B} \cdot \boldsymbol{\xi}_\mu}{\sqrt{N}}\right)$$
$$= \int \prod_{\gamma a} \frac{dR_a^\gamma d\hat{R}_a^\gamma}{2\pi/N} \exp\left(i\sum_{\gamma a} \hat{R}_a^\gamma (NR_a^\gamma - \mathbf{J}_a^\gamma \cdot \mathbf{B})\right)$$
$$\times \int \prod_\gamma \prod_{a<b} \frac{dW_{ab}^\gamma d\hat{W}_{ab}^\gamma}{2\pi/N} \exp\left(i\sum_\gamma \sum_{a<b} \hat{W}_{ab}^\gamma (NW_{ab}^\gamma - \mathbf{J}_a^\gamma \cdot \mathbf{J}_b^\gamma)\right)$$
$$\times \int \prod_a \prod_{\gamma<\rho} \frac{dq_a^{\gamma\rho} d\hat{q}_a^{\gamma\rho}}{2\pi/N} \exp\left(i\sum_a \sum_{\gamma<\rho} \hat{q}_a^{\gamma\rho} (Nq_a^{\gamma\rho} - \mathbf{J}_a^\gamma \cdot \mathbf{J}_a^\rho)\right)$$
$$\times \int \prod_{\gamma<\rho} \prod_{a<b} \frac{dt_{ab}^{\gamma\rho} d\hat{t}_{ab}^{\gamma\rho}}{2\pi/N} \exp\left(i\sum_{a<b} \sum_{\gamma<\rho} \hat{t}_{ab}^{\gamma\rho} (Nt_{ab}^{\gamma\rho} - \mathbf{J}_a^\gamma \cdot \mathbf{J}_b^\rho)\right)$$
$$\times \exp\left\{-\frac{1}{2}\sum_\mu \left[\sum_{\gamma a} (\hat{\lambda}_{a\mu}^\gamma)^2 + 2\sum_{\gamma a}\sum_{\gamma<\rho} \hat{\lambda}_{a\mu}^\gamma \hat{\lambda}_{a\mu}^\rho q_a^{\gamma\rho} + 2\sum_{\gamma a}\sum_{a<b} \hat{\lambda}_{a\mu}^\gamma \hat{\lambda}_{b\mu}^\gamma W_{ab}^\gamma\right.\right.$$
$$\left.\left. + 2\sum_{\gamma a}\sum_{\gamma<\rho}\sum_{a<b} \hat{\lambda}_{a\mu}^\gamma \hat{\lambda}_{b\mu}^\rho t_{ab}^{\gamma\rho} + 2\sum_{\gamma a} \hat{u}_\mu \hat{\lambda}_{a\mu}^\gamma R_a^\gamma + \hat{u}_\mu^2\right]\right\} + O(N^{-1}). \quad \text{(A4)}$$

By considering the distribution of the synaptic vector $\mathbf{B}$ as $\mathcal{P}(\boldsymbol{B}) = \prod_k \delta(B_k - 1)$ and by defining the matrices:

$$[\hat{\boldsymbol{Q}}]_{a,b}^{\gamma,\rho} \equiv i\left\{\delta^{\gamma,\rho}\left(\delta_{a,b}\hat{\ell}_a^\gamma + (1 - \delta_{a,b})\hat{W}_{a,b}^\gamma\right) + (1 - \delta^{\gamma,\rho})\left(\delta_{a,b}\hat{q}_a^{\gamma,\rho} + (1 - \delta_{a,b})\hat{t}_{a,b}^{\gamma,\rho}\right)\right\}, \quad \text{(A5)}$$

$$[\boldsymbol{Q}]_{a,b}^{\gamma,\rho} \equiv \delta^{\gamma,\rho}\left(\delta_{a,b} + (1 - \delta_{a,b})W_{a,b}^\gamma\right) + (1 - \delta^{\gamma,\rho})\left(\delta_{a,b}q_a^{\gamma,\rho} + (1 - \delta_{a,b})t_{a,b}^{\gamma,\rho}\right), \quad \text{(A6)}$$

we have that the average over synaptic vectors become

$$\langle\cdot\rangle_{\mathbf{B},\{\mathbf{J}_a^\gamma\}} = \int \prod_{\gamma,a} \frac{d\hat{\ell}_a^\gamma}{4\pi} \exp\left(i\frac{N}{2}\sum_{\gamma,a}\hat{\ell}_a^\gamma - N\ln|\hat{\boldsymbol{Q}}| - \frac{1}{2}\sum_{a,b}\sum_{\gamma,\rho}\hat{R}_a^\gamma[\hat{\boldsymbol{Q}}^{-1}]_{a,b}^{\gamma,\rho}\hat{R}_b^\rho - \frac{nNM}{2}\right), \tag{A7}$$

which renders the following expression for the partition function:

$$\begin{aligned}\overline{Z^n}(\beta) = {}& \int \prod_{\gamma a}\frac{d\hat{\ell}_a^\gamma}{4\pi}\int\prod_{\gamma a}\frac{dR_a^\gamma d\hat{R}_a^\gamma}{2\pi/N}\int\prod_{\gamma}\prod_{a<b}\frac{dW_{ab}^\gamma d\hat{W}_{ab}^\gamma}{2\pi/N}\int\prod_{a}\prod_{\gamma<\rho}\frac{dq_a^{\gamma\rho}d\hat{q}_a^{\gamma\rho}}{2\pi/N}\int\prod_{\gamma<\rho}\prod_{ab}\frac{dt_{ab}^{\gamma\rho}d\hat{t}_{ab}^{\gamma\rho}}{2\pi/N}\\
&\times \exp\left(\frac{N}{2}\mathrm{tr}\boldsymbol{Q}\hat{\boldsymbol{Q}} - \frac{N}{2}\ln|\hat{\boldsymbol{Q}}| - \frac{N}{2}\sum_{ab}\sum_{\gamma,\rho}\hat{R}_a^\gamma[\hat{\boldsymbol{Q}}^{-1}]_{ab}^{\gamma\rho}\hat{R}_b^\rho + iN\sum_{\gamma a}\hat{R}_a^\gamma R_a^\gamma - \frac{nNM}{2}\right)\\
&\times\int\prod_{\gamma\mu a}\frac{d\lambda_{a\mu}^\gamma d\hat{\lambda}_{a\mu}^\gamma}{2\pi}\exp\left(-i\sum_{\gamma\mu a}\hat{\lambda}_{a\mu}^\gamma\lambda_{a\mu}^\gamma\right)\\
&\times\int\prod_{\mu}\mathcal{D}u_\mu\exp\left(i\sum_{\gamma\mu a}\hat{\lambda}_{a\mu}^\gamma R_a^\gamma u_\mu + (1-\nu\eta_0)\beta\sum_{\gamma\mu a}\mathrm{sgn}(\lambda_{a\mu}^\gamma u_\mu)\right)\\
&\times\exp\left\{-\frac{1}{2}\sum_{\mu}\left[\sum_{\gamma a}\left[1-\left(R_a^\gamma\right)^2\right]\left(\hat{\lambda}_{a\mu}^\gamma\right)^2 + 2\sum_{\gamma a}\sum_{\gamma<\rho}\left[q_a^{\gamma\rho}-R_a^\gamma R_a^\rho\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{a\mu}^\rho\right.\right.\\
&\left.\left.+2\sum_{\gamma a}\sum_{a<b}\left[W_{ab}^\gamma - R_a^\gamma R_b^\gamma\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{b\mu}^\gamma + 2\sum_{\gamma a}\sum_{\gamma<\rho}\sum_{b}\left[t_{ab}^{\gamma\rho}-R_a^\gamma R_b^\rho\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{b\mu}^\rho\right]\right\}\\
&\times\left\langle\exp\left\{\beta\sum_{\gamma\mu a}\sum_{a\neq c}x_{ac}\eta_{ac}\mathrm{sgn}\left(\lambda_{a\mu}^\gamma\lambda_{c\mu}^\gamma\right)\right\}\right\rangle_{\boldsymbol{G}} + O(N^{-1}).\end{aligned} \tag{A8}$$

In the limit of large N we find that

$$\hat{R}_a^\gamma = i\sum_{\rho,b}[\hat{\boldsymbol{Q}}]_{a,b}^{\gamma,\rho}R_b^\rho, \tag{A9}$$

$$[\hat{\boldsymbol{Q}}^{-1}]_{a,b}^{\gamma,\rho} = [\boldsymbol{K}]_{a,b}^{\gamma,\rho} \equiv [\boldsymbol{Q}]_{a,b}^{\gamma,\rho} - R_a^\gamma R_b^\rho, \tag{A10}$$

and to express the extraction of the asymptotic behavior of integrals of the form $I_N \equiv \int_{x_1}^{x_2} dx\, e^{-Ng(x)}$ in the limit $N\to\infty$ through Laplace's method, we denote: $\mathrm{extr}_x\, I_N \equiv e^{-Ng(x_0)+O(\log N)}$, where x_0 is such that $g(x_0)\leqslant g(x)$ for all $x\in[x_1,x_2]$, so we can write

$$\begin{aligned}\overline{Z^n}(\beta) = \underset{\boldsymbol{K}}{\mathrm{extr}}&\left\{\exp\left(\frac{N}{2}\ln|\boldsymbol{K}|\right)\int\prod_{\gamma\mu a}\frac{d\lambda_{a\mu}^\gamma d\hat{\lambda}_{a\mu}^\gamma}{2\pi}\exp\left(-i\sum_{\gamma\mu a}\hat{\lambda}_{a\mu}^\gamma\lambda_{a\mu}^\gamma\right)\right.\\
&\times\int\prod_{\mu}\mathcal{D}u_\mu\,\exp\left(i\sum_{\gamma a}\hat{\lambda}_{a\mu}^\gamma R_a^\gamma u_\mu + (1-\nu\eta_0)\beta\sum_{\gamma\mu a}\mathrm{sgn}\left(\lambda_{a\mu}^\gamma u_\mu\right)\right)\\
&\times\exp\left[-\frac{1}{2}\sum_{\gamma\mu a}\left[1-\left(R_a^\gamma\right)^2\right]\left(\hat{\lambda}_{a\mu}^\gamma\right)^2 - \sum_{\gamma\mu a}\sum_{\gamma<\rho}\left[q_a^{\gamma\rho}-R_a^\gamma R_a^\rho\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{a\mu}^\rho\right.\\
&\left.-\sum_{\gamma\mu a}\sum_{a<b}\left[W_{ab}^\gamma-R_a^\gamma R_b^\gamma\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{b\mu}^\gamma - \sum_{\gamma\mu a}\sum_{\gamma<\rho}\sum_{b}\left[t_{ab}^{\gamma\rho}-R_a^\gamma R_b^\rho\right]\hat{\lambda}_{a\mu}^\gamma\hat{\lambda}_{b\mu}^\rho\right]\\
&\left.\times\left\langle\exp\left\{\beta\sum_{\gamma\mu a}\sum_{a\neq c}x_{ac}\eta_{ac}\mathrm{sgn}\left(\lambda_{a\mu}^\gamma\right)\mathrm{sgn}\left(\lambda_{c\mu}^\gamma\right)\right\}\right\rangle_{\boldsymbol{G}}\right\},\end{aligned} \tag{A11}$$

where $\mathcal{D}x \equiv (2\pi)^{-1/2}dx\, \exp(-x^2/2)$. Also, by imposing the replica symmetric (RS) ansatz: $R_a^\gamma = R$, $q_a^{\gamma,\rho} = q$, and by following Ref. [33] we can assume that $t_{ab}^{\gamma\rho} = W_{ab}^\gamma = W$, then by defining

$$\mathcal{C}_{a\mu} \equiv \frac{\sqrt{W-R^2}y_\mu + Ru_\mu + \sqrt{q-W}y_{a\mu}}{\sqrt{1-q}} \tag{A12}$$

and by observing that the logarithm of the matrix $\boldsymbol{K}$ in the RS approach is

$$\ln|\boldsymbol{K}| = nM\left(\ln(1-q) + \frac{q-R^2}{1-q}\right) + O(n^2), \tag{A13}$$

we have that, after the integration over the variables $\{\hat{\lambda}_{a\mu}^\gamma\}$, the partition function becomes

$$\begin{aligned}\overline{Z^n}(\beta) = \operatorname*{extr}_{R,q,W}\Bigg\{&\exp\left[\frac{nNM}{2}\left(\ln(1-q)+\frac{q-R^2}{1-q}\right)\right]\int\prod_\mu \mathcal{D}u_\mu \prod_\mu \mathcal{D}y_\mu \prod_{\mu a}\mathcal{D}y_{a\mu}\prod_{\gamma\mu a}\frac{\mathrm{d}\lambda_{a\mu}^\gamma}{\sqrt{2\pi}}\\ &\times\exp\left[-\frac{1}{2}\sum_{\gamma\mu a}\left(\lambda_{a\mu}^\gamma - \mathcal{C}_{a\mu}\right)^2 + (1-\nu\eta_0)\beta\sum_{\gamma\mu a}\operatorname{sgn}\left(\lambda_{a\mu}^\gamma u_\mu\right)\right]\\ &\times\left\langle \exp\left\{\beta\sum_{\gamma\mu a}\sum_{a\neq c}x_{ac}\eta_{ac}\operatorname{sgn}\left(\lambda_{a\mu}^\gamma\right)\operatorname{sgn}\left(\lambda_{c\mu}^\gamma\right)\right\}\right\rangle_{\boldsymbol{G}}\Bigg\}.\end{aligned} \tag{A14}$$

The average over the graph variables is

$$\begin{aligned}\Upsilon &\equiv \left\langle \exp\left\{\beta\sum_{\gamma\mu a}\sum_{a\neq c}x_{ac}\eta_{ac}\operatorname{sgn}\left(\lambda_{a\mu}^\gamma\right)\operatorname{sgn}\left(\lambda_{c\mu}^\gamma\right)\right\}\right\rangle_{\boldsymbol{G}}\\ &= \int\prod_{ac}\frac{d\eta_{ac}}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(\eta_{ac}-\eta_0)^2}{2\Delta^2}\right]\prod_{ac}\left\{1-p+p\prod_{\gamma\mu}\exp\left[\beta\operatorname{sgn}\left(\lambda_{a\mu}^\gamma\right)\operatorname{sgn}\left(\lambda_{c\mu}^\gamma\right)\right]\right\}\\ &= (1-p)^{M(M-1)}\int\prod_{ac}\frac{d\eta_{ac}}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(\eta_{ac}-\eta_0)^2}{2\Delta^2}\right]\\ &\quad\times\prod_{ac}\left\{1+\frac{p}{1-p}\prod_{\gamma\mu}\left[\cosh(\beta\eta_{ac}) + \operatorname{sgn}\left(\lambda_{a\mu}^\gamma\lambda_{c\mu}^\gamma\right)\sinh(\beta\eta_{ac})\right]\right\}\\ &\quad\times\prod_{ac}\left\{1+\frac{p}{1-p}\cosh(\beta\eta_{ac})^{nP}\prod_{\gamma\mu}\left[1+\tanh(\beta\eta_{ac})\operatorname{sgn}\left(\lambda_{a\mu}^\gamma\lambda_{c\mu}^\gamma\right)\right]\right\}.\end{aligned} \tag{A15}$$

Observe that we are assuming that the number of neighbors, in average, must be $\nu \ll M$:

$$\sum_a x_{ac}\mathcal{P}(x_{ac}) = p(M-1) = \frac{\nu}{2}, \tag{A16}$$

$$\begin{aligned}\Upsilon &= \left(1-\frac{\nu}{2(M-1)}\right)^{M(M-1)}\int\prod_{ac}\frac{d\eta_{ac}}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(\eta_{ac}-\eta_0)^2}{2\Delta^2}\right]\prod_{ac}\Bigg\{1+\frac{p}{1-p}\cosh(\beta\eta_{ac})^{nP}\\ &\quad\times\left[1+\tanh(\beta\eta_{ac})\sum_{\gamma\mu}\operatorname{sgn}\left(\lambda_{a\mu}^\gamma\lambda_{c\mu}^\gamma\right)+\tanh(\beta\eta_{ac})^2\sum_{\langle\gamma_1\mu_1;\gamma_2\mu_2\rangle}\operatorname{sgn}\left(\lambda_{a\mu_1}^{\gamma_1}\lambda_{c\mu_1}^{\gamma_1}\right)\operatorname{sgn}\left(\lambda_{a\mu_2}^{\gamma_2}\lambda_{c\mu_2}^{\gamma_2}\right)+\dots\right]\Bigg\}\\ &= \exp\left\{-\frac{\nu}{2}M+\frac{\nu}{2}M\sum_{\ell=0}^{nP}\langle\cosh(\beta\eta)^{nP}\tanh(\beta\eta)^\ell\rangle_\eta\sum_{\langle\gamma_1\mu_1;\dots;\gamma_\ell\mu_\ell\rangle}\left[\frac{1}{M}\sum_a\operatorname{sgn}\left(\lambda_{a\mu_1}^{\gamma_1}u_{\mu_1}\right)\dots\operatorname{sgn}\left(\lambda_{a\mu_\ell}^{\gamma_\ell}u_{\mu_\ell}\right)\right]^2\right\}.\end{aligned} \tag{A17}$$

Observe that Υ is the part of the replicated partition function that accounts for the interaction between peers and the interaction between peers and graph. If $\eta_0 = \Delta = 0$, then $\Upsilon = 1$. We define

$$\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} \equiv \frac{1}{M}\sum_a \mathrm{sgn}\big(\lambda^{\gamma_1}_{a\mu_1}u_{\mu_1}\big)\ldots\mathrm{sgn}\big(\lambda^{\gamma_\ell}_{a\mu_\ell}u_{\mu_\ell}\big), \tag{A18}$$

$$\varrho_0 \equiv \frac{1}{M}\sum_a 1, \tag{A19}$$

where $\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$ is the average level of agreement per individual, across issues and replicas and ϱ_0, which is fancy way to write 1, will be left as a free parameter for the time being until we apply a variational technique (which will confirm its value, see below).

Applying Laplace's method to the integrals involving the parameters $\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$ and their conjugates $\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}$, we can express the replicated partition function as

$$\begin{aligned}
\overline{Z^n}(\beta) = \underset{q,W,R,\{\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell},\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}\}}{\mathrm{extr}} &\Bigg\{\exp\left[\frac{nNM}{2}\left(\ln(1-q)+\frac{q-R^2}{1-q}\right)\right]\\
&\times\exp\Bigg[-M\sum_{\ell=0}\sum_{\langle\gamma_1\mu_1;\ldots;\gamma_\ell\mu_\ell\rangle}\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}-\frac{\nu}{2}M\\
&+\frac{\nu}{2}M\sum_{\ell=0}\langle\cosh(\beta\eta)^{nP}\tanh(\beta\eta)^\ell\rangle_\eta\sum_{\langle\gamma_1\mu_1;\ldots;\gamma_\ell\mu_\ell\rangle}\left(\varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}\right)^2\Bigg]\\
&\times\int\prod_\mu\mathcal{D}u_\mu\prod_\mu\mathcal{D}y_\mu\Bigg(\prod_\mu\mathcal{D}t_\mu\prod_{\gamma\mu a}\frac{d\lambda^\gamma_\mu}{\sqrt{2\pi}}\exp\Bigg[-\frac{1}{2}\sum_{\gamma\mu}(\lambda^\gamma_\mu-\mathcal{C}_\mu)^2\\
&+\sum_{\ell=0}\sum_{\langle\gamma_1\mu_1;\ldots;\gamma_\ell\mu_\ell\rangle}\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell}\mathrm{sgn}(\lambda^{\gamma_1}_{\mu_1}u_{\mu_1})\ldots\mathrm{sgn}(\lambda^{\gamma_\ell}_{\mu_\ell}u_{\mu_\ell})+(1-\nu\eta_0)\beta\sum_{\gamma\mu}\mathrm{sgn}(\lambda^{\gamma_1}_{\mu_1}u_{\mu_1})\Bigg]\Bigg)^M\Bigg\},
\end{aligned}\tag{A20}$$

where we have disregarded terms of $O(n^2)$, $O(N^{-1})$, and $O(M^{-1})$ in the argument of the exponential and now:

$$\mathcal{C}_\mu \equiv \frac{\sqrt{W-R^2}y_\mu+Ru_\mu+\sqrt{q-W}t_\mu}{\sqrt{1-q}}. \tag{A21}$$

Once more we consider the RS approach by introducing the distribution $\pi(z)$ and its conjugate $\hat{\pi}(z)$:

$$\hat{\varrho}^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} = \mathcal{C}_{\hat{\pi}}\int ds\,\hat{\pi}(s)\tanh^\ell(\beta s),\;\; \varrho^{\gamma_1\ldots\gamma_\ell}_{\mu_1\ldots\mu_\ell} = \int dz\,\pi(z)\tanh^\ell(\beta z), \tag{A22}$$

$$\hat{\varrho}_0 = \mathcal{C}_{\hat{\pi}}\int ds\,\hat{\pi}(s),\;\; \varrho_0 = \int dz\,\pi(z), \tag{A23}$$

where Eqs. (A22) are the definitions of the fields π and $\hat{\pi}$, and Eqs. (A23) are the normalization conditions they must satisfy. By using the symmetry of the RS supposition and by integrating over the variables $\{\lambda^\gamma_\mu\}$, the Hubbard-Stratanovich variables u, t, and y and by applying the scaling condition $P = \alpha N$, we have that the replicated partition function takes the form of

$$\begin{aligned}
\overline{Z^n}(\beta) = \underset{q,R,\varrho_0,\hat{\varrho}_0,\pi,\hat{\pi}}{\mathrm{extr}}&\left\{\exp\left[-M\left(\frac{\nu}{2}-\hat{\varrho}_0+\varrho_0\hat{\varrho}_0-\frac{\nu}{2}\varrho_0^2\right)\right]\left(1+nMN\frac{\nu}{2}\varrho_0^2\alpha\langle\ln\cosh(\beta\eta)\rangle_\eta+\frac{nMN}{2}\left(\ln(1-q)+\frac{q-R^2}{1-q}\right)\right.\right.\\
&-nMN\alpha\mathcal{C}_{\hat{\pi}}\int dz\,ds\,\pi(z)\hat{\pi}(s)\ln\left[1+\tanh(\beta s)\tanh(\beta z)\right]\\
&+\frac{\nu}{2}nMN\alpha\int dz_1\,dz_2\,\pi(z_1)\pi(z_2)\,\langle\ln\left(1+\tanh(\beta\eta)\tanh(\beta z_1)\tanh(\beta z_2)\right)\rangle_\eta\\
&+nNM\alpha e^{-\hat{\varrho}_0}\sum_{C=0}^\infty\frac{\mathcal{C}^\ell_{\hat{\pi}}}{C!}\int_{-\infty}^\infty\prod_{\ell=1}^C ds_\ell\,\hat{\pi}(s_\ell)\,2\int_{-\infty}^\infty\mathcal{D}x\mathcal{H}\left(-\frac{Rx}{\sqrt{q-R^2}}\right)\\
&\left.\left.\times\ln\left[\mathcal{H}\left(\sqrt{\frac{q}{1-q}}x\right)e^{-\beta(1-\nu\eta_0)}\prod_{\ell=1}^C[1-\tanh(\beta s_\ell)]+\mathcal{H}\left(-\sqrt{\frac{q}{1-q}}x\right)e^{\beta(1-\nu\eta_0)}\prod_{\ell=1}^C[1+\tanh(\beta s_\ell)]\right]\right)\right\}.
\end{aligned}\tag{A24}$$

Observe that during the integration process the dependency with respect to W disappears. By adding up the series, can be re-express as

$$\begin{aligned}\overline{Z^n}(\beta) = \underset{q,R,\varrho_0,\hat{\varrho}_0,\pi,\hat{\pi}}{\text{extr}} \Big\{ \exp\Big[-M\Big(\frac{\nu}{2} - \hat{\varrho}_0 + \varrho_0\hat{\varrho}_0 - \frac{\nu}{2}\varrho_0^2\Big)\Big]\Big(1 + nMN\frac{\nu}{2}\varrho_0^2\alpha\langle\ln\cosh(\beta\eta)\rangle_\eta + \frac{nMN}{2}\Big(\ln(1-q) + \frac{q-R^2}{1-q}\Big)\\ - nMN\alpha\mathcal{C}_{\hat{\pi}}\int dz\,ds\,\pi(z)\hat{\pi}(s)\ln\left[1+\tanh(\beta s)\tanh(\beta z)\right]\\ + \frac{\nu}{2}nMN\alpha\int dz_1\,dz_2\,\pi(z_1)\pi(z_2)\,\langle\ln\left(1+\tanh(\beta\eta)\tanh(\beta z_1)\tanh(\beta z_2)\right)\rangle_\eta\\ + nNM\alpha\Big(-\beta(1-\nu\eta_0) + e^{-\hat{\varrho}_0} + \mathcal{C}_{\hat{\pi}}\int ds\,\hat{\pi}(s)\ln[1+\tanh(-\beta s)]\Big)\\ + 2nNM\alpha\int_{-\infty}^{\infty}\mathcal{D}x\mathcal{H}\Big(-\frac{Rx}{\sqrt{q-R^2}}\Big)\\ \times\int\frac{dy}{2\pi}\int d\hat{y}e^{-iy\hat{y}}\exp\Big[\mathcal{C}_{\hat{\pi}}\int ds\,\hat{\pi}(s)e^{i\hat{y}s} - \hat{\varrho}_0\Big]\ln\Big[1 + \big(e^{2\beta(1-\nu\eta_0+y)}-1\big)\mathcal{H}\Big(-\sqrt{\frac{q}{1-q}}x\Big)\Big]\Big)\Big\}.\end{aligned} \tag{A25}$$

Observe that

$$\partial_{\varrho_0}\overline{Z^n} = (-M\hat{\varrho}_0 + \nu M\varrho_0 + O(n))\overline{Z^n}, \tag{A26}$$

$$\partial_{\hat{\varrho}_0}\overline{Z^n} = (M - M\varrho_0 + O(n))\overline{Z^n}, \tag{A27}$$

which implies in the extreme that $\varrho_0 = 1$ [as it was expected from Eq. (A19)] and $\hat{\varrho}_0 = \nu$. From this last equation we have that $\mathcal{C}_{\hat{\pi}} = \nu$. By defining the weight function:

$$\mathsf{P}(y|\hat{\pi}) \equiv \int\frac{d\hat{y}}{2\pi}e^{-iy\hat{y}}\exp\Big[\nu\Big(\int ds\,\hat{\pi}(s)e^{i\hat{y}s} - 1\Big)\Big], \tag{A28}$$

and the distribution

$$\mathcal{P}(x|q,R) \equiv 2\mathcal{N}(x)\mathcal{H}\Big(-\frac{Rx}{\sqrt{q-R^2}}\Big), \tag{A29}$$

the free-energy functional F can be defined as

$$\begin{aligned}\beta F \equiv \frac{1-\overline{Z^n}(\beta)}{nNM} = -\alpha\Big(-\beta(1-\nu\eta_0) + \mathrm{e}^{-\nu} + \frac{\nu}{2}\langle\ln\cosh(\beta\eta)\rangle_\eta\Big) - \frac{1}{2}\Big(\ln(1-q) + \frac{q-R^2}{1-q}\Big)\\ - \frac{\nu}{2}\alpha\int dz_1\,dz_2\,\pi(z_1)\pi(z_2)\,\langle\ln\left(1+\tanh(\beta\eta)\tanh(\beta z_1)\tanh(\beta z_2)\right)\rangle_\eta\\ + \nu\alpha\int dz\,ds\,\pi(z)\hat{\pi}(s)\ln\Big[\frac{1+\tanh(\beta s)\tanh(\beta z)}{1-\tanh(\beta s)}\Big] - \alpha\beta\int dy\mathsf{P}(y|\hat{\pi})(1-\nu\eta_0+y)\\ - \alpha\int dx\mathcal{P}_{Rq}(x)\int dy\mathsf{P}(y|\hat{\pi})\ln\Big[e^{-\beta(1-\nu\eta_0+y)}\mathcal{H}\Big(\sqrt{\frac{q}{1-q}}x\Big) + e^{\beta(1-\nu\eta_0+y)}\mathcal{H}\Big(-\sqrt{\frac{q}{1-q}}x\Big)\Big].\end{aligned} \tag{A30}$$

Density of the system can be expressed as $f = \text{extr}_{q,R,\pi,\hat{\pi}}\,F$.

APPENDIX B: CHARACTERISTIC FUNCTION OF THE DISTRIBUTIONS

By using the Fourier transform in the saddle-point Eqs. (25) and (26) we define the functions

$$\hat{\phi}(\omega) \equiv \int ds\,\hat{\pi}(s)e^{is\omega} = 1 + i\mathcal{I}_0\omega - \frac{\mathcal{R}_0}{2}\omega^2 + O(\omega^3), \tag{B1}$$

$$\phi(\omega) \equiv \int ds\,\pi(s)e^{is\omega}. \tag{B2}$$

Thus,

$$\mathsf{P}(y|\hat{\pi}) \equiv \int\frac{d\omega}{2\pi}\,e^{-iy\omega+\hat{\phi}(\omega)-\nu} \approx \int\frac{d\omega}{2\pi}\,\exp\Big[-\frac{\nu\mathcal{R}_0}{2}\omega^2 + i(\nu\mathcal{I}_0 - y)\omega\Big] = \mathcal{N}(y|\nu\mathcal{I}_0,\nu\mathcal{R}_0), \tag{B3}$$

which is consistent with Eqs. (19) and (20). Consider the definition (29), then the Fourier Transform of $\pi(s)$ can be expressed as

$$\phi(\omega) = \int ds \int dy\, \mathsf{P}(y|\hat{\pi})\, e^{is\omega}\left\langle \delta\left[s - 1 + \nu\eta_0 - y - \frac{\Lambda(x)}{2}x^2\right]\right\rangle_x$$
$$\approx \int dy \mathcal{N}(y|\nu\mathcal{I}_0, \nu\mathcal{R}_0)e^{i\omega y}\left\langle \exp\left[i\omega\left(1 - \nu\eta_0 + \frac{\Lambda(x)}{2}x^2\right)\right]\right\rangle_x, \tag{B4}$$

where the expectation over x is approximated by

$$\left\langle \exp\left\{i\omega\left[1 - \nu\eta_0 + \frac{\Lambda(x)}{2}\right]\right\}\right\rangle_x \approx 2\int_{-\infty}^{\infty} \mathcal{D}x\, \Theta(xR) \exp\left[i\omega\left(1 - \nu\eta_0 + \frac{\Lambda(x)}{2}x^2\right)\right]$$
$$\approx 2\int_0^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2} + i\omega\left(1 - \nu\eta_0 + \frac{\Lambda(R)}{2}x^2\right)\right]$$
$$\approx \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left[-[1 - i\omega\Lambda(R)]\frac{x^2}{2} + is(1 - \nu\eta_0)\right]$$
$$\approx \frac{1}{\sqrt{1 - i\omega\Lambda}} \exp\left[i(1 - \nu\eta_0)\omega\right]$$
$$\approx \exp\left[-\frac{3}{8}\Lambda^2\omega^2 + i\left(1 - \nu\eta_0 + \frac{\Lambda(R)}{2}\right)\omega\right], \tag{B5}$$

where Eq. (B5) is reached by assuming that Λ sufficiently small. From this point onwards we will refer to the parameter $\Lambda(R)$ as the amplified thermal noise. In such a case the function ϕ is Gaussian:

$$\phi(s) \approx \exp\left[-\frac{\nu\mathcal{R}_0 + \frac{3}{4}\Lambda(R)^2}{2}s^2 + i\left(1 - \nu\eta_0 + \nu\mathcal{I}_0 + \frac{\Lambda(R)}{2}\right)s\right], \tag{B6}$$

and so

$$\pi(z) \approx \frac{1}{\sqrt{2\pi\left(\nu\mathcal{R}_0 + \frac{3}{4}\Lambda(R)^2\right)}} \exp\left\{-\frac{\left(z - 1 + \nu\eta_0 - \nu\mathcal{I}_0 - \frac{1}{2}\Lambda(R)\right)^2}{2\left(\nu\mathcal{R}_0 + \frac{3}{4}\Lambda(R)^2\right)}\right\}. \tag{B7}$$

The argument in the Dirac's δ function in Eq. (26) is mildly sensitive to changes in the presidential approval; therefore, we can approximate it by

$$\beta^{-1}\text{arctanh}[\tanh(\beta\eta)\tanh(\beta z)] \approx \frac{|\eta + z| - |\eta - z|}{2} = \text{sgn}(z\eta)\min\{|\eta|, |z|\}, \tag{B8}$$

which allows us to write the expression of the Fourier transform of the distribution $\hat{\pi}$ as

$$\hat{\phi}(\omega) = \int ds\phi(s)\left\langle \int \frac{dz}{2\pi} \exp\left(-isz + i\omega\text{sgn}(z\eta)\min\{|\eta|, |z|\}\right)\right\rangle_\eta, \tag{B9}$$

where the expectation over the social strengths can be demonstrated to be, disregarding terms of $O(\eta_0\Delta^2)$:

$$\mathcal{Q} \equiv \left\langle \int \frac{dz}{2\pi} \exp\left(-isz + i\omega\beta^{-1}\text{arctanh}[\tanh(\beta\eta)\tanh(\beta z)]\right)\right\rangle_\eta$$
$$\approx \delta(s) + \frac{\omega}{\pi}\exp\left(-\frac{\Delta^2 s^2}{2}\right)\frac{\eta_0}{s} - \omega^2\frac{\eta_0^2 + \Delta^2}{2}\delta(s). \tag{B10}$$

Expression (B10), together with Eqs. (B9) and (B1) produce the following expressions for the moments of $\nu^{-1}\hat{\pi}$:

$$1 = \phi(0), \tag{B11}$$

$$\mathcal{I}_0 = -\frac{i\eta_0}{\pi}\int ds\frac{\phi(s)}{s}\mathrm{e}^{-\frac{\Delta^2 s^2}{2}}, \tag{B12}$$

$$\mathcal{R}_0 = \eta_0^2 + \Delta^2, \tag{B13}$$

which implies that

$$\mathcal{I}_0^\star = \eta_0 \mathrm{erf}\left(\frac{1-\nu\eta_0+\nu\mathcal{I}_0^\star+\frac{1}{2}\Lambda(R)}{\sqrt{2\left[\nu\left(\eta_0^2+\Delta^2\right)+\frac{3}{4}\Lambda(R)^2\right]}}\right). \tag{B14}$$

Equation (B14) has one, two, or three solutions depending on the value of the parameters ν, η_0, Δ and the function $\Lambda(R)$. It is easy to see that for the line

$$\eta_0 = \frac{2+\Lambda(R)}{2\nu}, \tag{B15}$$

$\mathcal{I}_0^\star = 0$ is a solution. Thus, we expect that for $\nu\eta_0 > 1+\Lambda(R)/2$, $\mathcal{I}_0^\star \approx -(\eta_0-\epsilon)$ is a solution, and for $\nu\eta_0 < 1+\Lambda(R)/2$, $\mathcal{I}_0 \approx \eta_0-\epsilon$ is a solution (in both cases $0<\epsilon<\eta_0$ is a suitable positive number).

If the derivative with respect to $\mathcal{I}_0$ of the right-hand side of Eq. (B14) evaluated at $\mathcal{I}_0^\star$ is equal to 1, and $\mathcal{I}_0^\star$ is also a solution of Eq. (B14), then Eq. (B14) has two solutions, $\mathcal{I}_0^\star$ and $\eta_0-\epsilon$ or $-(\eta_0-\epsilon)$, depending on whether $\nu\eta_0 < 1+\Lambda(R)/2$ or $\nu\eta_0 > 1+\Lambda(R)/2$, respectively.

If $\eta_-(\Lambda) < \eta_0 < \eta_+(\Lambda)$, then

$$\eta_\pm = \frac{2\mp|\Lambda|}{2\nu} \mp \left[\frac{1}{\nu}\sqrt{\left[\nu(\eta_\pm^2+\Delta^2)+\frac{3}{4}\Lambda(R)^2\right]\log\left(\frac{2}{\pi}\frac{\nu^2\eta_\pm^2}{\nu(\eta_\pm^2+\Delta^2)+\frac{3}{4}\Lambda(R)^2}\right)} - \eta_\pm \mathrm{erf}\left(\sqrt{\log\left(\sqrt{\frac{2}{\pi}\frac{\nu^2\eta_\pm^2}{\nu(\eta_\pm^2+\Delta^2)+\frac{3}{4}\Lambda(R)^2}}\right)}\right)\right] \tag{B16}$$

are the superior (η_+) and inferior (η_-) limits to the area in the plane $(|\Lambda|,\eta_0)$ where three solutions to the Eq. (B14) can be found. Let as define the set $\mathbb{A} := \{(|\Lambda|,\eta_0) : \eta_-(\Lambda) < \eta_0 < \eta_+(\Lambda)\}$. (The use of $|\Lambda|$ instead of Λ is due to the fact that realizable solutions must satisfy $\mathrm{sgn}(R) = \mathrm{sgn}(1-\nu\eta_0)$. This point will be clarified when the equations involving R and q are contemplated (see below). It is important to note that for very high values of the presidential approval β we have that

$$\nu\eta_\pm \approx \frac{1}{1\pm\sqrt{\frac{1}{\nu}\log\left(\frac{2\nu}{\pi}\right)}\mp \mathrm{erf}\left(\sqrt{\frac{1}{2}\log\left(\frac{2\nu}{\pi}\right)}\right)}, \tag{B17}$$

which implies that the segment (at very high presidential approval) of the coexistence of states grows with $\sqrt{\nu}$. From Eq. (B14) we conclude that, to satisfy the saddle-point Eqs. (25) and (26), there must be a set of values of η such that, for a very high presidential approval β there coexist states with different attitudes.

For all the points $(\Lambda,\eta_0)\notin\mathbb{A}$, we have that both conditions (B12) and (B13) are satisfied for a distribution:

$$\hat{\pi}(s) = \mathcal{N}(s|\mathcal{I}_0^\star, \eta_0^2-(\mathcal{I}_0^\star)^2+\Delta^2), \tag{B18}$$

where $\mathcal{I}_0^\star$ is the only solution of Eq. (B14). In this case we also have that

$$\mathsf{P}(y|\hat{\pi}) \approx \mathcal{N}\left(y\middle|\nu\mathcal{I}_0^\star, \nu\left(\eta_0^2+\Delta^2\right)\right). \tag{B19}$$

For the points $(\Lambda,\eta_0)\in\mathbb{A}$ we propose the following form for $\hat{\pi}(s) = \mathcal{Z}^{-1}\exp[-\Phi(s)]$, where $\mathcal{Z}$ is a normalization constant and the function $\Phi(s)$ is defined as

$$\Phi(s) := \frac{s^2}{2\Delta^2} - \frac{\eta_0}{\nu}\frac{1-\nu\eta_0+\nu s+\frac{1}{2}\Lambda(R)}{\Delta^2}\mathrm{erf}\left(\frac{1-\nu\eta_0+\nu s+\frac{1}{2}\Lambda(R)}{\sqrt{2\left[\nu\left(\eta_0^2+\Delta^2\right)+\frac{3}{4}\Lambda(R)^2\right]}}\right) - 2\frac{\eta_0}{\nu}\frac{\nu\left(\eta_0^2+\Delta^2\right)+\frac{3}{4}\Lambda(R)^2}{\Delta^2}\mathcal{N}\left(\nu s\middle|\nu\eta_0-1-\frac{1}{2}\Lambda(R), \nu\left(\eta_0^2+\Delta^2\right)+\frac{3}{4}\Lambda(R)^2\right). \tag{B20}$$

Observe that $\Phi'(\mathcal{I}_0^\star)=0$ is identical to Eq. (B14) and, for all points in $\mathbb{A}$, this equation has three roots, $\mathcal{I}_-^\star < \mathcal{I}_0^\star < \mathcal{I}_+^\star$. The asymptotic behavior of $\lim_{s\to\pm\infty} s^{-2}\Phi(s) > 0$ indicates that $\mathcal{I}_\pm^\star$ are minima with $\mathcal{I}_0^\star$ an intermediate maximum. Let us compute the second derivative of Φ at the minima:

$$\Phi''(\mathcal{I}_\pm^\star) = \frac{1}{\Delta^2}\left(1-\eta_0\frac{d}{ds}\mathrm{erf}\left(\frac{1-\nu\eta_0+\nu s+\frac{1}{2}\Lambda(R)}{\sqrt{2\left[\nu\left(\eta_0^2+\Delta^2\right)+\frac{3}{4}\Lambda(R)^2\right]}}\right)\Bigg|_{s=\mathcal{I}_\pm^\star}\right); \tag{B21}$$

therefore, $\Phi''(\mathcal{I}^\star_\pm) = \mathcal{L}^2_\pm \Delta^{-2}$, with

$$\mathcal{L}_\pm := \sqrt{1 - 2\nu\eta_0\mathcal{N}\left(\nu\mathcal{I}^\star_\pm \middle| \nu\eta_0 - 1 - \frac{1}{2}\Lambda(R), \nu\left(\eta_0^2 + \Delta^2\right) + \frac{3}{4}\Lambda(R)^2\right)}, \tag{B22}$$

which is larger than zero for all $(\Lambda, \eta_0) \in \mathbb{A}$. Thus,

$$\hat{\pi}(s) \approx h_+\mathcal{N}(s|\mathcal{I}^\star_+, \Delta^2) + h_-\mathcal{N}(s|\mathcal{I}^\star_-, \Delta^2), \tag{B23}$$

$$h_\pm \equiv \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{\Phi(\mathcal{I}^\star_+) - \Phi(\mathcal{I}^\star_-)}{4} - \frac{1}{2}\ln\frac{\mathcal{L}_+}{\mathcal{L}_-}\right) \tag{B24}$$

$$\int ds\hat{\pi}(s)s \approx h_+\mathcal{I}^\star_+ + h_-\mathcal{I}^\star_-, \tag{B25}$$

$$\int ds\hat{\pi}(s)s^2 \approx h_+(\mathcal{I}^\star_+)^2 + h_-(\mathcal{I}^\star_-)^2 + \Delta^2. \tag{B26}$$

Observe that the expectation (B23) represents a convex combination between the solutions to Eq. (B14), and $\hat{\pi}'(\mathcal{I}^\star_\pm) \approx 0$. To compute the distribution $\mathsf{P}(y|\hat{\pi})$ we first need to compute the Fourier transform of $\hat{\pi}$:

$$\hat{\phi}(\omega) \approx h_+\exp\left[-\frac{\Delta^2}{2}\omega^2 + i\omega\mathcal{I}^\star_+\right] + h_-\exp\left[-\frac{\Delta^2}{2}\omega^2 + i\omega\mathcal{I}^\star_-\right], \tag{B27}$$

and if we define

$$\begin{aligned}\Upsilon(\omega, y) &:= -iy\omega - \nu + \nu\int ds\frac{\exp[-\Phi(s) + is\omega]}{\int ds'\exp[-\Phi(s')]}\\ &\approx -iy\omega - \nu + \nu\left\{h_+\exp\left[-\frac{\Delta^2}{2}\omega^2 + i\omega\mathcal{I}^\star_+\right] + h_-\exp\left[-\frac{\Delta^2}{2}\omega^2 + i\omega\mathcal{I}^\star_-\right]\right\}\\ &\approx -iy\omega + \nu\left\{i\omega(h_+\mathcal{I}^\star_+ + h_-\mathcal{I}^\star_-) - \frac{h_+(\mathcal{I}^\star_+)^2 + h_-(\mathcal{I}^\star_-)^2 + \Delta^2}{2}\omega^2\right\} + O(\omega^3),\end{aligned} \tag{B28}$$

$$\mathsf{P}(y|\hat{\pi}) = \int\frac{d\omega}{2\pi}e^{\Upsilon(\omega,y)} \approx \mathcal{N}(y|\nu(h_+\mathcal{I}^\star_+ + h_-\mathcal{I}^\star_-), \nu[h_+(\mathcal{I}^\star_+)^2 + h_-(\mathcal{I}^\star_-)^2 + \Delta^2]). \tag{B29}$$

These distributions should be applied depending on the value of η_0 and $\Lambda(R)$, following the diagram presented in Fig. 5.

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