Physics Letters B 772 (2017) 621–627

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Exact self-dual skyrmions

L.A. Ferreira [a], Ya. Shnir [b,c,d]

[a] Instituto de Física de São Carlos; IFSC/USP; Universidade de São Paulo, USP; Caixa Postal 369, CEP 13560-970, São Carlos-SP, Brazil
[b] BLTP, JINR, Dubna 141980, Moscow Region, Russia
[c] Department of Theoretical Physics and Astrophysics, BSU, Minsk 220004, Belarus
[d] Department of Theoretical Physics, Tomsk State Pedagogical University, Russia

ARTICLE INFO

Article history:
Received 13 June 2017
Received in revised form 29 June 2017
Accepted 17 July 2017
Available online 21 July 2017
Editor: M. Cvetič

ABSTRACT

We introduce a Skyrme type model with the target space being the sphere S^3 and with an action possessing, as usual, quadratic and quartic terms in field derivatives. The novel character of the model is that the strength of the couplings of those two terms are allowed to depend upon the space–time coordinates. The model should therefore be interpreted as an effective theory, such that those couplings correspond in fact to low energy expectation values of fields belonging to a more fundamental theory at high energies. The theory possesses a self-dual sector that saturates the Bogomolny bound leading to an energy depending linearly on the topological charge. The self-duality equations are conformally invariant in three space dimensions leading to a toroidal ansatz and exact self-dual Skyrmion solutions. Those solutions are labelled by two integers and, despite their toroidal character, the energy density is spherically symmetric when those integers are equal and oblate or prolate otherwise.

1. Introduction

Self-dual field configurations possess very nice physical and mathematical properties, and they are important in the study of non-linear aspects of field theories possessing topological solitons. The best known examples are the instanton solutions of the Yang-Mills theory in four dimensional Euclidean space [1] and the self-dual Bogomol'nyi–Prasad–Sommerfield (BPS) monopoles in the $3+1$ dimensional Yang–Mills–Higgs theory [2,3]. The self-dual solitons satisfy first order differential equations which yields the absolute minimum of the energy, and by construction they are also solutions of the full dynamical system of the field equations. Another feature of the self-dual field configurations is that the corresponding topological solitons always saturate the topological bound, their static energy (or the Euclidean action in the case of the Yang–Mills instantons) depends linearly on the topological charge. Moreover, there are very elegant mathematical methods of construction of various multi-soliton configurations in these models, the Nahm equation [4] and the algebraic Atiyah–Hitchin–Drinfeld–Manin scheme [5].

However the usual Skyrme model [6,7], which can be suggested as an effective low-energy theory of pions, do not support self-dual equations [8], the mass of the soliton solutions for this model, the Skyrmions, is always above the topological lower bound in a given topological sector [9]. As a consequence, there is no exact mathematical scheme of construction of multi-soliton solutions of the Skyrme model, the only way to obtain these solutions in any topological sector, is to implement various numerical methods, some of them are rather sophisticated, they usually need a large amount of computational power.

Recently some modification of the Skyrme model was proposed to construct the soliton solutions which satisfy the first-order Bogomol'nyi-type equation [10–12]. In the first case the conventional Skyrme model was drastically changed via replacement of the usual sigma model term and the quartic Skyrme term with a term sextic in first derivatives and a potential [10,11]. In the second case the usual Skyrme model is coupled to the infinite tower of vector mesons [12]. These self-dual models are directly related to the usual Skyrme model since they can be considered as submodels of a general model of that type. Further, it was shown very recently that the standard Skyrme model without the potential term can be expressed as a sum of two BPS submodels with different solutions [13]. The corresponding submodels, however, are not directly related to the generalized Skyrme model of any type.

Another modification of the Skyrme model, which supports self-dual soltions and has an exact BPS bound, was suggested in [14]. Similar to the usual Skyrme model, or its generalizations,

E-mail address: shnir@maths.tcd.ie (Ya. Shnir).

http://dx.doi.org/10.1016/j.physletb.2017.07.040
0370-2693/

the field of the new model is a map from compactified coordinate space S^3 to the $SU(2)$ group space. The corresponding first order equations are equivalent to the so-called force free equation well known in solar and plasma physics, see e.g. [15]. The drawback of this construction is that, due to an argument by Chandrasekhar [16], those equations do not possess finite energy solutions on $\mathbb{R}^3$, although it supports regular solutions on three sphere S^3 [14].

In this paper we propose generalization of the self-dual Skyrme-type model discussed in [14], which possess the regular solution on $\mathbb{R}^3$. Similar to the usual Skyrme model, we consider a non-linear scalar sigma-model, parameterized by two complex scalar fields Z_a, $a = 1, 2$, satisfying the constraint $Z_a^* Z_a = 1$. The action of the model is given by

$$S = \int d^4x \left(\frac{m^2(x^\rho)}{2} A_\mu^2 - \frac{1}{4e^2(x^\rho)} H_{\mu\nu}^2 \right) \tag{1}$$

with two couplings $m(x^\rho)$ and $e(x^\rho)$, dependent upon the space–time coordinates, which are of dimension of mass and dimensionless, respectively. In addition, μ, $\nu = 0, 1, 2, 3$, and

$$A_\mu = \frac{i}{2} \left(Z_a^* \partial_\mu Z_a - Z_a \partial_\mu Z_a^* \right) \quad \text{and} \quad H_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{2}$$

The model (1) is similar to the one considered in [14], with the main difference being the fact that the coupling constants now are allowed to depend upon the space–time coordinates. That plays a crucial role in the properties of the model. In the first place, it circumvents the famous Chandrasekhar's argument [16] that prevents the existence of finite energy solutions extending over the whole $\mathbb{R}^3$ space. In addition, as we explain below, it renders the self-duality equations conformally invariant in the three dimensional space $\mathbb{R}^3$. As it is usual in many effective field theories, coupling constants that depend upon the space–time coordinates correspond in fact to low energy expectation values of fields belonging to a more fundamental theory at higher energies. At the end of the paper we shall discuss some possibilities for the introduction of a dilation type field that could account for the space–time dependent coupling constants appearing in (1).

The model that we have proposed opens the way for several physical applications, in particular for those ones where the Skyrme model is treated as a low energy effective theory for nuclear physics. Of course, the conformal symmetry has to be broken for those applications to be implemented, and that can be achieved by a potential term in the Lagrangian, or by the introduction of a dilation type field. In addition, the proposed model could also have applications in solar and plasma physics. Indeed, our self-duality equations (see (14)) correspond to the so-called force free equations used in magnetohydrodynamics [15,16], if one interprets the vector $\vec{A}$ in those equations as the physical magnetic field. In fact, our static energy density for self-dual configurations (see (15)) would then correspond to the square of the magnetic field multiplied by the square of the space dependent coupling constant $e\left(x^\mu\right)$ (or equivalently $m\left(x^\mu\right)$), which perhaps could be interpreted as magnetic susceptibility of the medium. The introduction of those non-constant couplings is one of the main ingredients that makes it possible the existence of finite energy solutions in the whole $\mathbb{R}^3$. It would be interesting to verify if the introduction of a space dependent magnetic susceptibility could have a similar role in circumventing the Chandrasekhar argument [16] in solar and plasma physics.

We now discuss the properties of the self-dual sector of the theory (1).

2. The self-duality equations

It will be convenient for our purposes to represent the corresponding static energy functional via the dual of H_{ij}, defined as

$$B_i = \frac{1}{2} \varepsilon_{ijk} H_{jk} , \qquad i, j, k = 1, 2, 3 \tag{3}$$

Then we can write the static energy associated to (1) as

$$E = \frac{1}{2} \int d^3x \left(m^2(\vec{r})\, A_i^2 + \frac{1}{e^2(\vec{r})} B_i^2 \right) \tag{4}$$

In order to have finite energy solutions the fields Z_a, $a = 1, 2$, have to approach fixed constant values at spatial infinity, and so as long as topological arguments are concerned, we can compactify the physical space $\mathbb{R}^3$ to S^3. Thus the field of the model (1) becomes a map $Z_a : S^3 \to S^3$. The mapping is labeled by the topological invariant $Q = \pi_3(S^3)$, which is the winding number of the field configuration, and it can be calculated by the following integral

$$\begin{aligned} Q &= \frac{1}{12\pi^2} \int d^3x\, \varepsilon_{abcd}\, \varepsilon_{ijk}\, \Phi_a\, \partial_i \Phi_b\, \partial_j \Phi_c\, \partial_k \Phi_d \\ &= \frac{1}{4\pi^2} \int d^3x\, A_i\, B_i , \end{aligned} \tag{5}$$

where we have written $Z_1 \equiv \Phi_1 + i\,\Phi_2$, $Z_2 \equiv \Phi_3 + i\,\Phi_4$, and $a, b, c, d = 1, 2, 3, 4$.

Note that on the right hand side of (5) we have written Q in terms of the vectors A_i and B_i defined in (2) and (3), respectively. Evidently, this structure reminds the Hopf invariant used in the theories with the target space being S^2, like in the Skyrme–Faddeev model [17]. However, our target space is still S^3 and we are not projecting the map down to S^2 as is the case of the first Hopf map.

Next we follow the arguments presented in [18]. Let us denote by χ_α, $\alpha = 1, 2, 3$, the independent fields of the target space S^3. The topological charge Q given in (5) is invariant under infinitesimal smooth (homotopic) deformations of the fields $\delta\chi_\alpha$, and so, without the use of the equations of motion, one finds that $\delta Q = 0$. Since the variations are arbitrary one gets from (5) that the vectors A_i and B_i have to satisfy

$$B_i \frac{\delta A_i}{\delta\chi_\alpha} - \partial_j \left(B_i \frac{\delta A_i}{\delta \partial_j \chi_\alpha} \right) + A_i \frac{\delta B_i}{\delta\chi_\alpha} - \partial_j \left(A_i \frac{\delta B_i}{\delta \partial_j \chi_\alpha} \right) = 0. \tag{6}$$

On the other hand, the static Euler–Lagrange equations associated to (1) are given by

$$\begin{aligned} & m^2(\vec{r})\, A_i \frac{\delta A_i}{\delta\chi_\alpha} - \partial_j \left(m^2(\vec{r})\, A_i \frac{\delta A_i}{\delta \partial_j \chi_\alpha} \right) \\ & \quad + \frac{1}{e^2(\vec{r})} B_i \frac{\delta B_i}{\delta\chi_\alpha} - \partial_j \left(\frac{1}{e^2(\vec{r})} B_i \frac{\delta B_i}{\delta \partial_j \chi_\alpha} \right) = 0. \end{aligned} \tag{7}$$

If one now imposes the self-duality equation

$$m(\vec{r})\, A_i = \pm \frac{1}{e(\vec{r})} B_i \tag{8}$$

one gets that (6) becomes

$$\begin{aligned} & \pm m(\vec{r})\, e(\vec{r})\, A_i \frac{\delta A_i}{\delta\chi_\alpha} - \partial_j \left(\pm m(\vec{r})\, e(\vec{r})\, A_i \frac{\delta A_i}{\delta \partial_j \chi_\alpha} \right) \\ & \pm \frac{1}{m(\vec{r})\, e(\vec{r})} B_i \frac{\delta B_i}{\delta\chi_\alpha} - \partial_j \left(\pm \frac{1}{m(\vec{r})\, e(\vec{r})} B_i \frac{\delta B_i}{\delta \partial_j \chi_\alpha} \right) = 0. \end{aligned} \tag{9}$$

If, in addition, we impose that

$$m(\vec{r}) = m_0\, f(\vec{r}) \qquad e(\vec{r}) = e_0\, f(\vec{r}) \tag{10}$$

with m_0 and e_0 being constants, then (9) becomes the same as (7). The conclusion is that, for the choice (10), the self-duality equation (8) implies (7), when the identity (6), coming from the topological charge, is used. Therefore, using (10), the static energy (4) becomes

$$E = \frac{1}{2}\int d^3x \left[\left(m_0^2\, f^2\, A_i^2 + \frac{1}{e_0^2\, f^2}\, B_i^2\right)\right] \tag{11}$$

which can be written as

$$E = \frac{1}{2}\int d^3x \left(m_0 f\, A_i \mp \frac{1}{e_0 f}\, B_i\right)^2 \pm \frac{m_0}{e_0}\int d^3x\, A_i\, B_i \tag{12}$$

Therefore the lower energy bound is

$$E \geq 4\pi^2\, \frac{m_0}{e_0}\; \mid Q \mid \tag{13}$$

which is saturated for the solutions of the self-duality equation

$$m_0 e_0\, f^2\, A_i = \pm B_i \tag{14}$$

For such self-dual field configurations we have

$$E = m_0^2 \int d^3x\, f^2 A_i^2 = \frac{1}{e_0^2}\int d^3x\, \frac{B_i^2}{f^2} \tag{15}$$

Note that if we treat f as an independent field then the static Euler–Lagrange equation coming from (11) is given by

$$m_0^2\, f^2\, A_i^2 = \frac{1}{e_0^2\, f^2}\, B_i^2 \tag{16}$$

which certainly follows from the self-duality equation (14). Therefore, the first order self-duality equations (14) imply all the static second order Euler–Lagrange equations associated to the theory (1), when the coupling constants have the form given in (10). As we shall see, the dilaton function $f(r)$ can regularize solutions of the self-duality equation providing a way to evade the usual arguments [16,14] that there can be no finite energy solutions of the force free equation.

3. Conformal symmetry of the model

Remarkably, the self-dual sector of the model (1) is invariant under conformal transformations in three space dimensions. In order to see it, we will follow the approach of [19] and consider a general infinitesimal space transformations of the form $\delta x_i = \zeta_i$, such that

$$\delta Z_a = 0; \qquad \delta \partial_i Z_a = -\partial_i \zeta_j\, \partial_j Z_a\, . \tag{17}$$

Therefore

$$\delta A_i = -\partial_i \zeta_j\, A_j \qquad \delta H_{ij} = -\partial_i \zeta_k H_{kj} - \partial_j \zeta_k H_{ik}\, , \tag{18}$$

and

$$\delta B_i = -\varepsilon_{ijk}\, \partial_j \zeta_l H_{lk} = -\partial_j \zeta_l \varepsilon_{ijk}\, \varepsilon_{lkm}\, B_m = \partial_j \zeta_i\, B_j - \partial_j \zeta_j\, B_i \tag{19}$$

Let us consider how the self-duality equations (14) change under such transformations. It is convenient to write them in the form

$$\Lambda_i \equiv \lambda\, f^2\, A_i - B_i = 0 \qquad \lambda = \eta\, m_0\, e_0 \qquad \eta = \pm 1\, , \tag{20}$$

and so

$$\begin{aligned} \delta\Lambda_i =& 2\,\frac{\delta f}{f}\,\lambda\, f^2\, A_i - \partial_i \zeta_j\, \lambda\, f^2\, A_j - \partial_j \zeta_i\, B_j + \partial_j \zeta_j\, B_i \\ =& \left[2\,\frac{\delta f}{f}\,\delta_{ij} - (\partial_i \zeta_j + \partial_j \zeta_i) + \partial_l \zeta_l\, \delta_{ij}\right] \lambda\, f^2\, A_j\, . \end{aligned} \tag{21}$$

Hence, in order to remain invariant with respect to the transformations (17), the variations of the space coordinates ζ_i must satisfy

$$\partial_i \zeta_j + \partial_j \zeta_i = 2\, D\, \delta_{ij} \tag{22}$$

for some function D. Therefore,

$$\delta\Lambda_i = \left[2\,\frac{\delta f}{f} + D\right] \lambda\, f^2\, A_i \tag{23}$$

and the self-duality equation (14) remains invariant if

$$\delta f = -\frac{D}{2}\, f \tag{24}$$

As is was shown in [19], the transformations satisfying (22) are actually the conformal transformations. Indeed, we have that the possibilities are

$$\begin{aligned} &\zeta_i^{(P_j)} = \varepsilon^{(P_j)}\, \delta_{ij} \\ &D^{(P_j)} = 0 \qquad \text{(translations)} \\ &\zeta_i^{R_{jk}} = \varepsilon^{(R_{jk})}\, (\delta_{ki}\, x_j - \delta_{ji}\, x_k) \quad j \neq k \\ &D^{(R_{ij})} = 0 \qquad \text{(rotations)} \\ &\zeta_i^{(d)} = \varepsilon^{(d)}\, x_i \\ &D^{(d)} = \varepsilon^{(d)} \qquad \text{(dilatations)} \\ &\zeta_i^{(c_j)} = \varepsilon^{(c_j)} \left(x_i\, x_j - \frac{1}{2}\, x_l^2\, \delta_{ij}\right) \\ &D^{(c_j)} = \varepsilon^{(c_j)}\, x_j \qquad \text{(special conf.)} \end{aligned} \tag{25}$$

Therefore the self-duality equations (14) are invariant under conformal transformations in three dimensional space. Note that f is a scalar field under translations and rotations but not under dilatations and special conformal transformations. Further, one can check that

$$\delta A_i^2 = -2\, D\, A_i^2\, ; \quad \delta B_i^2 = -4\, D\, B_i^2\, ; \quad \delta(A_i\, B_i) = -3\, D\, A_i\, B_i \tag{26}$$

and the volume element transforms as

$$\delta(d^3x) = 3\, D\, d^3x \tag{27}$$

Hence both the static energy functional (11) and the topological charge (5) are conformally invariant.

4. The toroidal ansatz and exact Skyrmion solutions on $\mathbb{R}^3$

Here we again follow the reasonings of [19] to construct an ansatz for our self-duality equations, which is invariant under the diagonal subgroup of two commuting $U(1)$'s in the conformal group and other two commuting $U(1)$'s in the internal symmetry group of the model (1). Note that the model (1) is invariant under the $U(2)$ global transformations

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to U \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \qquad U \in U(2) \tag{28}$$

The Cartan subgroup of the $U(2)$ includes two commuting $U(1)$ elements, namely

$$Z_1 \to e^{i\alpha}\, Z_1 \qquad Z_2 \to Z_2 \tag{29}$$

and

$$Z_1 \to Z_1 \qquad Z_2 \to e^{i\beta}\, Z_2 \tag{30}$$

In addition we also have, in the conformal group in three dimensions, two commuting $U(1)$ elements, which correspond to the vector fields $\mathcal{V}_\zeta = \zeta_i\, \partial_i$ with (see [19])

$$\partial_\varphi \equiv \mathcal{V}_\varphi = x_2\partial_1 - x_1\partial_2 \tag{31}$$

$$\partial_\xi \equiv \mathcal{V}_\xi = \frac{x_3}{a}\left(x_1\partial_1 + x_2\partial_2\right) + \frac{1}{2a}\left(a^2 + x_3^2 - x_1^2 - x_2^2\right)\partial_3 \tag{32}$$

with a being a length scale factor, and where we have introduced two angles, φ and ξ, such that the vectors fields, $\mathcal{V}_\varphi$ and $\mathcal{V}_\xi$, generate rotations along those angular directions. Note, that $\mathcal{V}_\varphi$ is the generator of rotations on the plane $x_1\, x_2$. On the other hand, $\mathcal{V}_\xi$ is a linear combination of the special conformal generator $V^{(C_3)} = x_3\, x_i \partial_i - \frac{1}{2}x_l^2\,\partial_3$, and the translation generator $V^{(P_3)} = \partial_3$ (see (25)). One can easily check that indeed $\left[\partial_\varphi\,,\,\partial_\xi\right] = 0$. The third curvilinear coordinate in $\mathbb{R}^3$ which is orthogonal to φ and ξ is

$$z = \frac{4\,a^2(x_1^2+x_2^2)}{(x_1^2+x_2^2+x_3^2+a^2)^2} \tag{33}$$

One can check that indeed $\partial_\varphi z = \partial_\xi z = 0$. It turns out that $(z\,,\,\xi\,,\,\varphi)$ constitute the toroidal coordinates in $\mathbb{R}^3$ defined as[1]

$$x^1 = \frac{a}{p}\sqrt{z}\cos\varphi\,;\quad x^2 = \frac{a}{p}\sqrt{z}\sin\varphi\,;\quad x^3 = \frac{a}{p}\sqrt{1-z}\sin\xi \tag{34}$$

where

$$p = 1 - \sqrt{1-z}\cos\xi \qquad 0 \le z \le 1 \qquad 0 \le \varphi\,,\,\xi \le 2\pi \tag{35}$$

We now want field configurations that are invariant under the diagonal subgroup of the tensor product of the internal $U(1)$ defined in (29) and the external $U(1)$ generated by ∂_φ given in (31). In addition we want those same field configurations to be invariant under the diagonal subgroup of the tensor product of the internal $U(1)$ defined in (30) and the external $U(1)$ generated by ∂_ξ given in (32). That brings us to the toroidal ansatz defined by

$$Z_1 = \sqrt{F(z)}\,e^{i\,n\,\varphi} \qquad Z_2 = \sqrt{1-F(z)}\,e^{i\,m\,\xi} \tag{36}$$

where m and n are two integers, to keep the configuration single valued in $\mathbb{R}^3$.

In order to proceed it is convenient to write the self-duality equation in terms of vector calculus notation, and so we have that (14) can be written as

$$\vec{\nabla}\wedge\vec{A} = \eta\, m_0\, e_0\, f^2(\vec{r})\,\vec{A}; \qquad \eta = \pm 1\,. \tag{37}$$

Writing $\vec{A}$ in terms of the unit vectors of the toroidal coordinates as $\vec{A} = \frac{V_z}{h_z}\vec{e}_z + \frac{V_\xi}{h_\xi}\vec{e}_\xi + \frac{V_\varphi}{h_\varphi}\vec{e}_\varphi$ (see (66)), we have that $V_\zeta = \frac{i}{2}\left(Z_a^*\partial_\zeta Z_a - Z_a\partial_\zeta Z_a^*\right)$, with $\zeta \equiv z, \xi, \varphi$, and where the scaling factors h_ζ are defined in (63). Therefore, (37) can be written in components as

$$\begin{aligned} \kappa\, f^2\, V_z &= \frac{p}{2\,z\,(1-z)}\,\partial_\xi V_\varphi \\ \kappa\, f^2\, V_\xi &= -2\,(1-z)\,p\,\partial_z V_\varphi \\ \kappa\, f^2\, V_\varphi &= 2\,z\,p\left[\partial_z V_\xi - \partial_\xi V_z\right] \end{aligned} \tag{38}$$

where we have introduced the dimensionless quantity $\kappa \equiv \eta\, m_0\, e_0\, a$, with $\eta = \pm 1$ (see (20)). Substituting the ansatz (36) in the self-duality equations (38) we get

$$\begin{aligned} \partial_\xi F &= 0 \\ \frac{\kappa\, m\, f^2}{p}\,(1-F) &= -2\,(1-z)\,n\,\partial_z F \\ \frac{\kappa\, n\, f^2}{p}\,F &= -2\,z\,m\,\partial_z F \end{aligned} \tag{39}$$

[1] We have replaced the usual toroidal coordinate η by z, these coordinates are related as $z = \tanh^2\eta$, with $\eta > 0$.

Now we can eliminate the derivative $\partial_z F$ from this system, it yields a simple algebraic solution of the self-duality equations (39) for any values of the integers m and n

$$F = \frac{m^2\,z}{m^2\,z + n^2(1-z)} \qquad f^2 = \frac{2\,p}{m_0\,e_0\,a}\,\frac{|\,m\,n\,|}{[m^2\,z + n^2(1-z)]} \tag{40}$$

where, to keep f real, we had to choose the sign of $(m\,n)$ to be related to the sign η of the self-duality as $\eta = -\text{sign}\,(m\,n)$. Thus, the explicit form of the solution for the self-dual model (1) on $\mathbb{R}^3$ is

$$\begin{aligned} Z_1 &= \sqrt{\frac{m^2\,z}{m^2\,z + n^2\,(1-z)}}\; e^{i\,n\,\varphi} \\ Z_2 &= \sqrt{\frac{n^2\,(1-z)}{m^2\,z + n^2\,(1-z)}}\; e^{i\,m\,\xi} \\ f &= \sqrt{\frac{2\,|\,m\,n\,|\left(1-\sqrt{1-z}\,\cos\xi\right)}{m_0\,e_0\,a\left[m^2\,z + n^2(1-z)\right]}} \end{aligned} \tag{41}$$

The vector field $\vec{A}$ takes the following form when evaluated on the solutions (41)

$$\vec{A} = -m\,n\,\frac{p/a}{m^2\,z + n^2\,(1-z)}\left[\vec{e}_\xi\, n\sqrt{1-z} + \vec{e}_\varphi\, m\sqrt{z}\right] \tag{42}$$

and so

$$A_i^2 = m^2\,n^2\,\frac{p^2/a^2}{m^2\,z + n^2\,(1-z)} \tag{43}$$

Note that $\vec{A}$ is tangent to the toroidal surfaces defined by $z =$ constant. On the circle on the $x_1\,x_2$ plane defined by $z = 1$ (see the appendix Appendix A), one has that $\vec{A}_{\text{circle}} = -n\,\vec{e}_\varphi/a$. At spatial infinity, where $z = 0$ and $\xi = 0$, one has that $\vec{A}_{\text{infinity}} = 0$. On the x_3-axis, where $z = 0$, one has $\vec{A}_{x_3\text{-axis}} = -(m/a)(1-\cos\xi)\,\vec{e}_\xi$.

Evaluating the static energy (15) on the solutions (41), we get

$$\begin{aligned} E &= m_0^2 \int d^3x\, f^2\,\vec{A}^2 \\ &= 4\,\pi^2\,\frac{m_0}{e_0}\,|\,m\,n\,|\; m^2\,n^2 \int_0^1 dz\,\frac{1}{\left[m^2\,z + n^2(1-z)\right]^2} \end{aligned} \tag{44}$$

Using the fact that

$$\int_0^1 dz\,\frac{1}{\left[m^2\,z + n^2(1-z)\right]^2} = \frac{1}{m^2\,n^2}$$

one gets

$$E = 4\,\pi^2\,\frac{m_0}{e_0}\,|\,m\,n\,| \tag{45}$$

Further, using (14) into the definition of the topological charge (5), we get that the solutions (41) have topological charges given by

$$Q = \frac{1}{4\,\pi^2}\int d^3x\,A_i\,B_i = \frac{\eta\,m_0\,e_0}{4\,\pi^2}\int d^3x\,f^2\,A_i^2 = -m\,n \tag{46}$$

where we have used that the sign $\eta = \pm 1$, in the self-duality equation (14) is related to $m\,n$ as $\eta = -\text{sign}\,(m\,n)$ (see below (40)). Thus, these field configurations exactly saturate the topological bound (13) for any values of n, m.

Note that the solutions (41) are very similar to those exact solutions constructed in [20], and possessing in fact the same topological charges. The model in [20] however, is defined on target

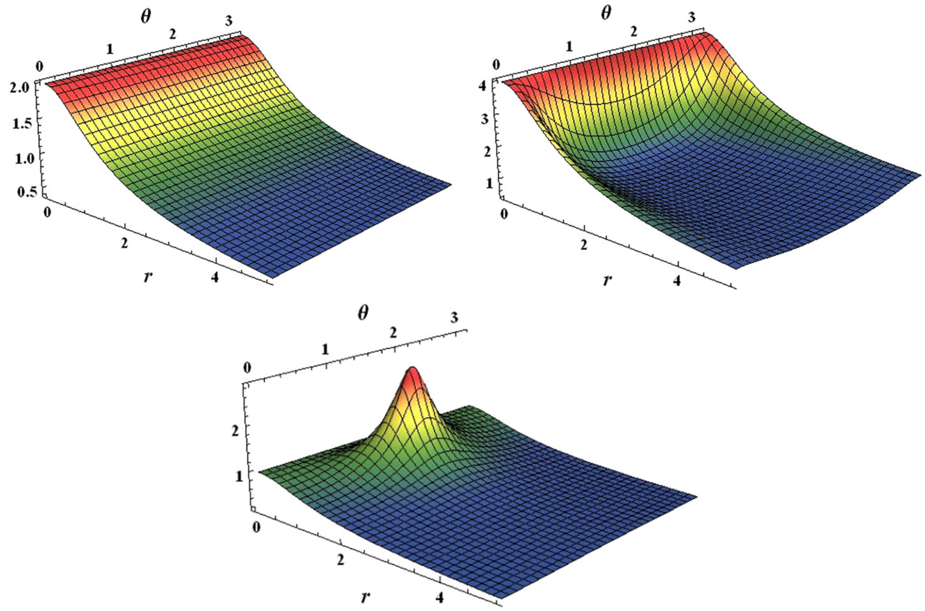

Fig. 1. The function $f(r,\theta)$ for the solutions (41) of the model (1) at $a=1$, $m_0=1$, $e_0=1$, for $n=1$, $m=1$ (left plot), $n=1$, $m=4$ (middle plot), and $n=4$, $m=1$ (right plot).

space S^2 and it does not possess a self-dual sector, even though it presents conformal symmetry in three space dimensions.

If one of the integers, m or n vanishes, the solutions (41) become trivial with $f=0$ everywhere in space. Note that in the particular case where $n=\pm m$ the general solution (41) is reduced to

$$Z_1=\sqrt{z}\,e^{i\,n\,\varphi}, \qquad Z_2=\sqrt{1-z}\,e^{\pm i\,n\,\xi},$$
$$f=\sqrt{\frac{2}{m_0e_0a}\,(1-\sqrt{(1-z)}\cos\xi)} \tag{47}$$

Clearly, the field Z_a in (47) resembles the form of the solution of the model (1) on S^3, constructed in [14], however the angular variables in the latter case are related with the angular coordinates on the three-sphere and the function f, which is a regulator on $\mathbb{R}^3$, does not appear there.

Remind that in the toroidal coordinates (34) the spacial infinity corresponds to $z=0$, $\xi=0$ and the origin corresponds to $z=0$, $\xi=\pi$. Evidently, for the solutions (41) we have the asymptotic behavior

$$Z_1(r\to\infty)=0, \qquad Z_2(r\to\infty)=1, \qquad f(r\to\infty)=0 \tag{48}$$

and

$$Z_1(r\to 0)=0, \qquad Z_2(r\to 0)=-1;$$
$$f(r\to 0)=\frac{2}{\sqrt{m_0e_0a}}\sqrt{\left|\frac{m}{n}\right|} \tag{49}$$

which agrees with the topological boundary conditions imposed on the field Z_a. The general solution is axially symmetric, and on the x^3-axis, corresponding to $z=0$, we have

$$Z_1(0,0,x^3)=0, \qquad Z_2(0,0,x^3)=e^{i\,n\,\xi},$$
$$f(0,0,x^3)=\sqrt{\frac{|\,m\,|}{|\,n\,|}\frac{2(1-\cos\xi)}{m_0e_0a}} \tag{50}$$

thus, the solutions are regular everywhere in space.

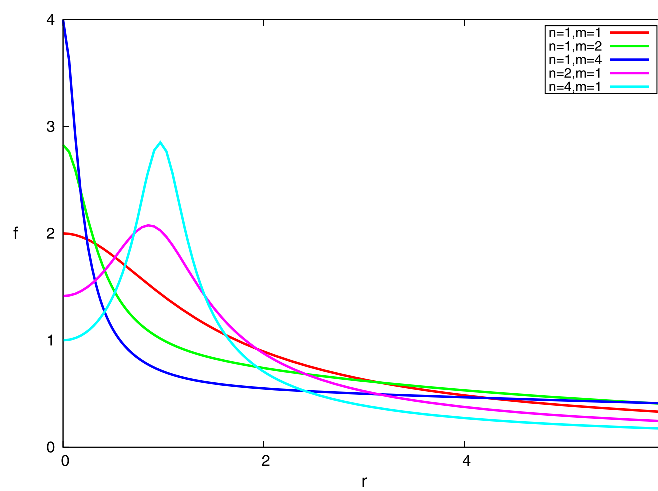

Fig. 2. The function $f(r,\theta=\pi/2)$ of the self-dual solutions (41) for a few values of the integers n,m, at $a=1$, $m_0=1$, $e_0=1$.

In Figs. 1–2 we show the function f in terms of spherical coordinates r,θ. For $n=m$ the solutions possess spherical symmetry. For $n\neq m$, the configuration becomes axially symmetric, it is oblate for $n>m$ and it is prolate for $n<m$.

The solutions (41) can be written in the spherical coordinates in a more transparent form. Indeed, using the expressions (62), we can write the energy density of the general solution (41) as

$$\mathcal{E}=\frac{16m_0}{e_0\,a^3}|mn|^3\frac{((r/a)^2+1)}{[4(\rho/a)^2(m^2-n^2)+n^2((r/a)^2+1)^2]^2}\,. \tag{51}$$

If $m^2=n^2$, the configuration becomes spherically symmetric, then

$$\mathcal{E}=\frac{16m_0}{e_0\,a^3}\frac{n^2}{((r/a)^2+1)^3} \tag{52}$$

Note that in both cases the energy density decays as $1/r^6$ as $r\to\infty$. In addition, it scales as $1/a^3$, and so, the total energy is scale invariant and that is a consequence of the conformal invariance of the model.

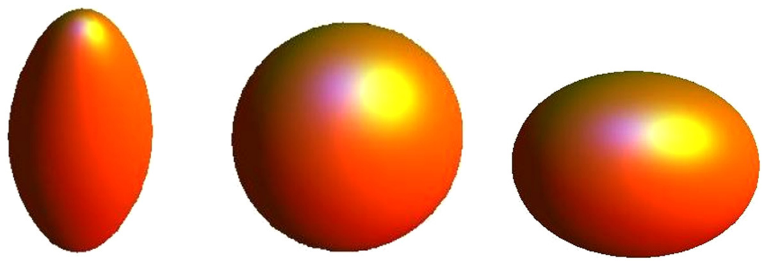

Fig. 3. The isosurfaces of the energy density of the $n = 1$, $m = 4$ (left plot), $n = 2$, $m = 2$ (middle plot), and $n = 4$, $m = 1$ (right plot) solutions of the model (1) at $a = 1$, $m_0 = 1$, $e_0 = 1$.

In Fig. 3 we display the energy density iso-surfaces (see (51)) for the cases ($n = 1$, $m = 4$), ($n = 2$, $m = 2$), and ($n = 4$, $m = 1$). Note that all these configurations have the same total energy.

Finally, let us note that the solutions for the Z_a fields given in (41) do not depend on the arbitrary scale parameter a, as one should expect since they are scalar under conformal transformations (see (17)). The function f however scales as $1/\sqrt{a}$, and that is a consequence of the fact it is not a scalar under dilatations and special conformal transformations, see (24). Thus, similar to the self-dual soliton solution of the non-linear $O(3)$ sigma model in $2+1$ dimensions [21] the instanton solution of the Yang–Mills theory in Euclidian four-dimensional space [1], and the exact Hopfions constructed in [20] those field configurations (except for f) are scale invariant.

5. Solutions in terms of the $SU(2)$ fields

We have presented the solutions in terms of the complex fields Z_a, $a = 1, 2$, that parameterize the 3-sphere S^3. However, we can write those solutions in terms of the $SU(2)$ group elements as follows. We parameterize the $SU(2)$ group elements in the spinor representation as

$$U = \begin{pmatrix} Z_2 & i\,Z_1 \\ i\,Z_1^* & Z_2^* \end{pmatrix} ; \qquad \mid Z_1 \mid^2 + \mid Z_2 \mid^2 = 1 \tag{53}$$

Then the vector A_i, introduced in (2), can be written as

$$A_i = \frac{i}{2}\,\mathrm{Tr}\left(\partial_i U\, U^\dagger \sigma_3\right) \tag{54}$$

where σ_3 is the Pauli matrix, i.e. $\sigma_3 \equiv \mathrm{diag}\,(1, -1)$. Note that the global transformations

$$U \to U\, g \qquad U \to e^{i\,\gamma\,\sigma_3}\, U \tag{55}$$

with $g \in SU(2)$ leave the vector (2) invariant. Therefore, the model (1) has the symmetry $U(2) \equiv U(1)_L \otimes SU(2)_R$, that corresponds to the symmetry discussed in (28). So, the fact that only the σ_3-component of the Maurer–Cartan form $\partial_i U\, U^\dagger$ enters in the action of the model (1), makes the symmetry $SU(2)_L \otimes SU(2)_R$, of the usual Skyrme model, to be broken down to $U(1)_L \otimes SU(2)_R$.

Note that the two commuting $U(1)$'s, introduced in (29) and (30), and used to build the ansatz (36), correspond to

$$U \to e^{i\,\alpha\,\sigma_3/2}\, U\, e^{-i\,\alpha\,\sigma_3/2} ; \quad \text{and} \quad U \to e^{i\,\beta\,\sigma_3/2}\, U\, e^{i\,\beta\,\sigma_3/2} \tag{56}$$

In addition, the boundary conditions (48), (49) and (50) satisfied by the solutions, imply that

$$U\,(r \to \infty) = \mathbb{1}\,; \quad U\,(r \to 0) = -\mathbb{1}\,; \quad U\,(0\,,\,0\,,\,x_3) = e^{i\,m\,\xi\,\sigma_3} \tag{57}$$

Using spherical polar coordinates

$$x_1 = r\,\sin\theta\,\cos\varphi\,; \quad x_2 = r\,\sin\theta\,\sin\varphi\,; \quad x_3 = r\,\cos\theta \tag{58}$$

and the results of Appendix A, one can write (for $m \neq 0$ and $n \neq 0$)

$$F = \frac{\left(m^2/n^2\right) 4\,a^2\,r^2\,\sin^2\theta}{\left(r^2+a^2\right)^2 - \left(1 - m^2/n^2\right) 4\,a^2\,r^2\,\sin^2\theta}\,;$$
$$e^{i\,\xi} = \frac{\left(r^2-a^2\right) + i\,2\,a\,r\,\cos\theta}{\sqrt{\left(r^2+a^2\right)^2 - 4\,a^2\,r^2\,\sin^2\theta}} \tag{59}$$

Therefore, one can express the solutions (41) in terms of the U-fields. In particular for the case where $m = \pm 1$, one gets that

$$U^{(\pm 1\,,\,n)} = \frac{\left(r^2-a^2\right)\,\mathbb{1} \pm i\,2\,a\,r\,T}{\sqrt{\left(r^2+a^2\right)^2 - \left(1-1/n^2\right)\,4\,a^2\,r^2\,\sin^2\theta}} \tag{60}$$

with

$$T = \begin{pmatrix} \cos\theta & \pm\sin\theta\, e^{i\,n\varphi}/\mid n \mid \\ \pm\sin\theta\, e^{-i\,n\varphi}/\mid n \mid & -\cos\theta \end{pmatrix} \tag{61}$$

Note that the multi-Skyrmions solutions we have constructed are all centered at the origin. If some of those configurations can be separated into lower charge Skyrmions, they will certainly be sitting on top of each other.

6. Conclusions

The main purpose of this work was to construct exact analytical and regular self-dual solutions of a Skyrme theory with target space S^3. The crucial ingredient that made that possible was the conformal symmetry of the self-duality equations in three space dimensions. On its turn, such symmetry was possible due to the fact that the strengths of the couplings of the quadratic and quartic terms in the action have a space dependence encoded in a quantity f. The physical nature of such quantity is still to be understood, but it is quite natural to relate it to low energy expectation values of fields of a more fundamental theory in higher energies that would contain our Skyrme model as a low energy effective theory. Note from (24) and (26) that the quantity f transforms under the conformal group, in the same way as $(A_i\,B_i)^{1/6}$, i.e. a fractional power of the topological charge density. In fact, f and $(A_i\,B_i)^{1/6}$ differ by a multiplicative constant, when evaluated on the solutions (41) for the case $m^2 = n^2$, i.e. the solutions with spherically symmetric energy densities. Such a fact could perhaps be a hint on how one could try to extend our model by a scalar dilation type field or even vector fields.

Certainly our results open the way for further investigations on the properties of the proposed Skyrme model, and perhaps on its possible physical applications. Of course, it would be interesting to study how the conformal symmetry could be broken leading to scale dependent solutions and bringing a physical scale to the theory. The introduction of a potential or even of the dilation field mentioned above are some of the possibilities. It would also be important to investigate the rotational modes of the solutions and their semi-classical quantization. Rotating solutions not only would break the conformal symmetry but also would split the energy degeneracies of our self-dual spectrum. We hope to report on those issues elsewhere.

Acknowledgements

The authors are grateful to Profs. Wojtek Zakrzewski and Nobuyuki Sawado for valuable discussions. LAF is partially supported by CNPq-Brazil. YS thanks Ilya Perapechka for relevant discussion. YS is grateful to Fundação de Apoio à Pesquisa do Estado de São Paulo, FAPESP, for the financial support under the grant 2015/25779-6, he also gratefully acknowledges support from the Russian Foundation for Basic Research (Grant No. 16-52-12012), the Ministry of Education and Science of Russian Federation, project No 3.1386.2017, and DFG (Grant LE 838/12-2). YS would like to thank the Instituto de Física de São Carlos for its kind hospitality.

Appendix A. Toroidal coordinates

Here we give some useful formulas related to the toroidal coordinates (34), and that are needed for the explicit calculations leading to the exact solutions (41). Inverting the relations (34) one gets that

$$z = \frac{4a^2\left(x_1^2+x_2^2\right)}{\left(x_1^2+x_2^2+x_3^2+a^2\right)^2}\,;$$
$$\xi = \text{ArcTan}\left[\frac{2\,a\,x_3}{\left(x_1^2+x_2^2+x_3^2-a^2\right)}\right];$$
$$\varphi = \text{ArcTan}\left(\frac{x_2}{x_1}\right) \tag{62}$$

The metric in toroidal coordinates is $ds^2 = h_z^2\,dz^2 + h_\xi^2\,d\xi^2 + h_\varphi^2\,d\varphi^2$, with scaling factors being

$$h_z = \frac{a}{p}\,\frac{1}{2\sqrt{z\,(1-z)}} \qquad h_\xi = \frac{a}{p}\sqrt{1-z} \qquad h_\varphi = \frac{a}{p}\sqrt{z} \tag{63}$$

The volume element is then

$$dx^1\,dx^2\,dx^3 = \frac{1}{2}\,\frac{a^3}{p^3}\,dz\,d\xi\,d\varphi \tag{64}$$

Note that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = a^2\,\frac{\left(1+\sqrt{1-z}\,\cos\xi\right)}{\left(1-\sqrt{1-z}\,\cos\xi\right)} \tag{65}$$

Therefore the spatial infinity corresponds to $z = 0$, and $\xi = 0$ (or 2π). The x^3-axis corresponds to $z = 0$, for $0 < \xi < 2\pi$. The origin corresponds to $z = 0$, and $\xi = \pi$. In addition, $z = 1$ corresponds to the circle $x_1^2 + x_2^2 = a^2$, and $x_3 = 0$.

The unit vectors are defined as $\vec{e}_\zeta = \frac{1}{h_\zeta}\frac{d\vec{r}}{d\zeta}$, for $\zeta \equiv z, \xi, \varphi$, and so we have that

$$\begin{aligned}
\vec{e}_z &= \frac{1}{p}\big[(\sqrt{1-z}-\cos\xi)\cos\varphi\,\vec{e}_1 \\
&\quad + (\sqrt{1-z}-\cos\xi)\sin\varphi\,\vec{e}_2 - \sqrt{z}\sin\xi\,\vec{e}_3\big] \\
\vec{e}_\xi &= -\frac{1}{p}\big[\sqrt{z}\cos\varphi\sin\xi\,\vec{e}_1 \\
&\quad + \sqrt{z}\sin\varphi\sin\xi\,\vec{e}_2 + (\sqrt{1-z}-\cos\xi)\,\vec{e}_3\big] \\
\vec{e}_\varphi &= -\sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2
\end{aligned} \tag{66}$$

where $\vec{e}_i$, $i = 1, 2, 3$, are the unit vectors in Cartesian coordinates.

References

[1] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, Y.S. Tyupkin, Pseudoparticle solutions of the Yang–Mills equations, Phys. Lett. B 59 (1975) 85.
[2] M.K. Prasad, C.M. Sommerfield, An exact classical solution for the 't Hooft monopole and the Julia–Zee dyon, Phys. Rev. Lett. 35 (1975) 760.
[3] E.B. Bogomolny, Stability of classical solutions, Sov. J. Nucl. Phys. 24 (1976) 449, Yad. Fiz. 24 (1976) 861.
[4] W. Nahm, A simple formalism for the BPS monopole, Phys. Lett. B 90 (1980) 413.
[5] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld, Y.I. Manin, Construction of instantons, Phys. Lett. A 65 (1978) 185.
[6] T.H.R. Skyrme, A non-linear field theory, Proc. R. Soc. Lond. 260 (1961) 127.
[7] T.H.R. Skyrme, A unified field theory of mesons and baryons, Nucl. Phys. 31 (1962) 556.
[8] N.S. Manton, P.J. Ruback, Skyrmions in flat space and curved space, Phys. Lett. B 181 (1986) 137.
[9] L.D. Faddeev, Some comments on the many dimensional solitons, Lett. Math. Phys. 1 (1976) 289.
[10] C. Adam, J. Sanchez-Guillen, A. Wereszczynski, A Skyrme-type proposal for baryonic matter, Phys. Lett. B 691 (2010) 105.
[11] C. Adam, J. Sanchez-Guillen, A. Wereszczynski, A BPS Skyrme model and baryons at large N_c, Phys. Rev. D 82 (2010) 085015.
[12] P. Sutcliffe, Skyrmions, instantons and holography, J. High Energy Phys. 1008 (2010) 019.
[13] C. Adam, J. Sanchez-Guillen, A. Wereszczynski, BPS submodels of the Skyrme model, arXiv:1703.05818 [hep-th].
[14] L.A. Ferreira, W.J. Zakrzewski, A Skyrme-like model with an exact BPS bound, J. High Energy Phys. 1309 (2013) 097, http://dx.doi.org/10.1007/JHEP09(2013)097, arXiv:1307.5856 [hep-th].
[15] Gerald E. Marsh, Force-Free Magnetic Fields: Solutions, Topology and Applications, World Scientific, 1996.
[16] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publication, Inc., 1981.
[17] L.D. Faddeev, Quantization of Solitons, Preprint-75-0570, IAS, Princeton, 1975; L.D. Faddeev, A.J. Niemi, Nature 387 (1997) 58.
[18] C. Adam, L.A. Ferreira, E. da Hora, A. Wereszczynski, W.J. Zakrzewski, Some aspects of self-duality and generalised BPS theories, J. High Energy Phys. 1308 (2013) 062, http://dx.doi.org/10.1007/JHEP08(2013)062, arXiv:1305.7239 [hep-th].
[19] O. Babelon, L.A. Ferreira, Integrability and conformal symmetry in higher dimensions: a model with exact Hopfion solutions, J. High Energy Phys. 0211 (2002) 020, http://dx.doi.org/10.1088/1126-6708/2002/11/020, arXiv:hep-th/0210154.
[20] H. Aratyn, L.A. Ferreira, A.H. Zimerman, Exact static soliton solutions of $(3+1)$-dimensional integrable theory with nonzero Hopf numbers, Phys. Rev. Lett. 83 (1999) 1723, http://dx.doi.org/10.1103/PhysRevLett.83.1723, arXiv:hep-th/9905079.
[21] A.M. Polyakov, A.A. Belavin, Metastable states of two-dimensional isotropic ferromagnets, JETP Lett. 22 (1975) 245, Pis'ma Zh. Eksp. Teor. Fiz. 22 (1975) 503.