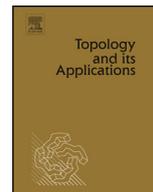




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Twisted conjugacy in fundamental groups of geometric 3-manifolds [☆]

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ABSTRACT

A group G has the R_∞ -property if for every $\varphi \in \text{Aut}(G)$, there are an infinite number of φ -twisted conjugacy classes of elements in G . In this note, we determine the R_∞ -property for $G = \pi_1(M)$ for all geometric 3-manifolds M .

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1. Introduction

Let M be a closed connected n -manifold and $f : M \rightarrow M$ a selfmap. Classical Nielsen fixed point theory is concerned with the minimal number of fixed points among all maps homotopic to f , i.e., the number $MF[f] := \min_{g \sim f} \{\#(\text{Fix } g = \{x \in M \mid g(x) = x\})\}$. If $n \geq 3$, a classical theorem of Wecken asserts that $MF[f] = N(f)$, the Nielsen number of f . For $n = 2$, the difference $MF[f] - N(f)$ can be arbitrarily large. For $n \geq 3$, the computation of $N(f)$ is a central issue but is very difficult in general. When M is a Jiang-type

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space, for instance a generalized lens space, an orientable coset space of a compact connected Lie group, a spherical space form, or a nilmanifold, either $N(f) = 0$ or $N(f) = R(f)$, the Reidemeister number of f . If φ is the induced homomorphism of f on $\pi_1(M)$, $R(f) = R(\varphi)$, the cardinality of the set of φ -twisted conjugacy classes of elements in $\pi_1(M)$. In such a situation, if $R(f) = \infty$ then $N(f) = 0$ which implies that f is homotopic to a fixed point free map. For example, for any $n \geq 5$, there exists an n -dimensional nilmanifold M such that every homeomorphism $f : M \rightarrow M$ is isotopic to a fixed point free map [9]. This result is a consequence of the R_∞ -property of $\pi_1(M)$ for certain nilmanifolds M . Recall that a group G has the R_∞ -property if for every $\varphi \in \text{Aut}(G)$, the set of orbits of the left action $\sigma \cdot \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}$ is infinite. It is therefore natural to ask: for what families of n -manifolds M does $\pi_1(M)$ have the R_∞ -property. For $n = 3$, the Thurston-Perelman Geometrization Theorem asserts that every closed 3-manifold is made up of finite pieces of 3-manifolds equipped with geometries of the following eight types: (I) S^3 (Spherical); (II) $S^2 \times \mathbb{R}$; (III) \mathbb{E}^3 (Euclidean); (IV) Nil; (V) $\widetilde{SL(2, \mathbb{R})}$; (VI) $\mathbb{H}^2 \times \mathbb{R}$; (VII) Sol; (VIII) \mathbb{H}^3 (Hyperbolic). By a *geometric* 3-manifold, we mean a connected 3-manifold equipped with a geometry from (I) - (VIII) with finite volume (see [22]). It turns out that a geometric 3-manifold is compact except in case of the following geometries where the manifold can be either compact or non-compact: $\widetilde{SL(2, \mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$, \mathbb{H}^3 .

The main objective of this note is to determine whether the fundamental group of a geometric 3-manifold has the R_∞ -property. Leaving out the case of spherical geometry where the fundamental group is finite, our main result is the following:

Main Theorem. *Let M be a geometric 3-manifold with infinite fundamental group. Then $\pi_1(M)$ has the R_∞ -property when M has any of the following geometries: $\widetilde{SL(2, \mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$, \mathbb{H}^3 . In the remaining cases, $\pi_1(M)$ has the R_∞ -property with the following exceptions:*

- (a) $S^2 \times S^1$ -geometry: $M \cong S^2 \times S^1, S^2 \tilde{\times} S^1, \mathbb{R}P^2 \times S^1$,
- (b) \mathbb{E}^3 -geometry: *The orientable manifolds with holonomy group $\{1\}, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$,*
- (c) Nil-geometry: *The circle bundles over the torus $S^1 \times S^1$ with non-zero Euler class k , and, Seifert fibre spaces with base the sphere S^2 having four singular points of type $(2, 1)$ and holonomy group \mathbb{Z}_2 .*
- (d) Sol-geometry: *The manifolds having this geometry are of two kinds: $M_0 = T \times I / \sim$, where the boundary tori are glued via a Anosov diffeomorphism; $M_1 = E_0 \sqcup E_1 / \sim$, where E_0, E_1 are twisted I -bundles over the Klein bottle and their boundary tori are glued via an Anosov diffeomorphism.*
 - (1) *The group $\pi_1(M_0) = G = \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}$ where the \mathbb{Z} -action θ on \mathbb{Z}^2 is given by an Anosov matrix $A \in SL(2, \mathbb{Z})$. Then G has the R_∞ -property if and only if any of the following holds: (i) $\det(A) = -1$, (ii) $A, A^{-1} \in SL(2, \mathbb{Z})$ are not conjugates in $GL(2, \mathbb{Z})$, (iii) A, A^{-1} are conjugates in $GL(2, \mathbb{Z})$ but are not conjugate to a matrix of the form $\begin{pmatrix} r & s \\ s & u \end{pmatrix}$, and, furthermore, neither A nor $-A$ equals X^p for some $p \in \mathbb{Z}$ and $X \in GL(2, \mathbb{Z})$ with $\det(X) = -1$.*
 - (2) *The group $\pi_1(M_1)$ has the R_∞ -property.*

We have a related notion of a manifold possessing the R_∞ -property.

Definition 1. We say that a manifold M has the R_∞ -property if, for every self-homotopy equivalence $f : M \rightarrow M$, the Reidemeister number of the automorphism $f_\# : \pi_1(M) \rightarrow \pi_1(M)$ is infinite.

Note that when M is an aspherical space $K(\pi, 1)$, the topological and the algebraic notions of the R_∞ -properties coincide, that is, an aspherical manifold M has the R_∞ -property if and only if the group $\pi_1(M) = \pi$ has the R_∞ -property. Other than spherical- and $S^2 \times \mathbb{R}$ -geometries, the universal covers of the remaining geometric 3-manifolds are diffeomorphic to \mathbb{R}^3 and so they are aspherical. Since the fundamental groups of manifolds admitting spherical geometry are finite, the two notions trivially coincide. We will show that in the remaining case of $S^2 \times \mathbb{R}$ -geometry also, the two notions agree. In general, however, examples of smooth compact manifolds are known which have the R_∞ -property but their fundamental groups do not.

(See the Appendix.) We remark here that, in the case of 3-manifolds, the Borel conjecture is known to be valid: if M, M' are closed aspherical 3-manifolds, any isomorphism $\phi : \pi_1(M) \rightarrow \pi_1(M')$ is induced by a *homeomorphism* $f : M \rightarrow M'$ (see [1, §2.1]).

The proof of the Main Theorem will be spread over several sections, depending on the type of the geometry under consideration. In many cases, the proof can be found or can be derived from results available in the literature. But they are scattered in various papers and often do not specifically address the case of fundamental groups of geometric 3-manifolds. Specifically, the case of hyperbolic geometry follows from the work of Levitt and Lustig (for compact manifolds) and that of Fel'shtyn (for non-compact ones). In the cases of \mathbb{E}^3 -, $S^2 \times S^1$ -, and *Sol*-geometries, the proof (for the most part) follows from the work of Gonçalves, Wong, and Zhao. The result for \mathbb{E}^3 -geometry was also obtained by Dekimpe and Penninckx, who considered the more general case of three-dimensional crystallographic groups. The complete result in the case of Nil-geometry is due to Dekimpe [4, Theorem 4.4]. However, the results on $\mathbb{H}^2 \times \mathbb{R}$ - and $SL(2, \mathbb{R})$ -geometries and the *complete* classification of manifolds admitting *Sol*-geometry whose fundamental groups do *not* have the R_∞ -property, could not be found in the literature.

Our aim here is to present a coherent discussion of all the eight geometries, considering the importance of the role of the fundamental group in the study of 3-manifolds. For the R_∞ -property of fundamental groups of non-prime 3-manifolds see [7].

For the rest of the paper, we leave out the case of spherical geometry.

2. Geometries \mathbb{H}^3 and \mathbb{E}^3

2.1. \mathbb{H}^3 , the hyperbolic geometry

The fundamental group of a compact hyperbolic 3-manifold is known to be a (torsion-free) non-elementary word hyperbolic group. It follows from the main result of [15] that these groups all have the R_∞ -property. The fundamental group of a non-compact, finite volume hyperbolic 3-manifold is relatively hyperbolic (with respect to the finite collection of fundamental groups at the cusps). In this case, Fel'shtyn [5] has shown that such a group has the R_∞ -property (see also [17] and [18]).

2.2. \mathbb{E}^3 , Euclidean or flat geometry

In this case the R_∞ -property has been studied in a more general context. Here we shall confine ourselves to the case of 3-manifolds.

Any such manifold M is a quotient \mathbb{R}^3/π where π is a torsion-free lattice in the group $Iso(\mathbb{R}^3)$ of isometries of \mathbb{R}^3 and is finitely covered by the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$. Thus M is compact and the fundamental group π of M therefore admits a finite index subgroup isomorphic to \mathbb{Z}^3 . It turns out that π has a *unique* maximal normal abelian group $\Gamma \cong \mathbb{Z}^3$. In particular, Γ is characteristic in π . It is the translation part of π . The group $\Phi := \pi/\Gamma$, which is finite, is the *holonomy group* of M . Φ acts on Γ as automorphisms. This is the same as the action of the deck transformation group of the covering $\mathbb{R}^3/\Gamma \rightarrow M$. Thus we have an exact sequence in which $\Gamma \cong \mathbb{Z}^3$ is characteristic and Φ , finite:

$$1 \rightarrow \Gamma \rightarrow \pi \rightarrow \Phi \rightarrow 1. \quad (1)$$

It is known that the fixed subgroup Γ^Φ equals the centre $Z(\pi)$ of π . When $Z(\pi) \cong \mathbb{Z}$, the quotient $\pi/Z(\pi)$ is a planar crystallographic group Λ . Irrespective of the rank of $Z(\Gamma)$, one has a projection of π onto a planar crystallographic group Λ . Thus one has an exact sequence

$$1 \rightarrow Z \rightarrow \pi \rightarrow \Lambda \rightarrow 1. \quad (2)$$

However, only the case when $Z = Z(\pi) \cong \mathbb{Z}$ will be relevant for our purposes.

When M is non-orientable, it turns out that M fibres over a circle with fibre the Klein bottle. This results in an exact sequence

$$1 \rightarrow \pi_1(K) \rightarrow \pi \rightarrow \mathbb{Z} \rightarrow 1. \tag{3}$$

Using the notation of [13], up to diffeomorphism, there are a total of ten flat 3-manifolds whose fundamental groups π are listed below, where the first six are orientable and the remaining four are non-orientable. We also indicate the holonomy group Φ and the centre $Z(\pi)$. Whenever it is relevant for our purposes, we shall indicate the planar crystallographic group $\Lambda = \pi/Z(\pi)$ with $Z(\pi) \cong \mathbb{Z}$. We will use the notation of Lyndon [16] for planar crystallographic groups.

We denote the image of an element $\gamma \in \pi$ under the projection $\pi \rightarrow \Phi$ by $\bar{\gamma}$.

1. $\langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \{1\}$.
2. $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^2, t \alpha_2 t^{-1} = \alpha_2^{-1}, t \alpha_3 t^{-1} = \alpha_3^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_2$. $Z(\pi) = \langle \alpha_1 \rangle$. $\Lambda = \pi/Z(\pi) \cong G_2$.
3. $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^3, t \alpha_2 t^{-1} = \alpha_3, t \alpha_3 t^{-1} = \alpha_2^{-1} \alpha_3^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_3$. $Z(\pi) = \langle \alpha_1 \rangle$. $\Lambda = \pi/Z(\pi) \cong G_3$.
4. $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^4, t \alpha_2 t^{-1} = \alpha_3, t \alpha_3 t^{-1} = \alpha_2^{-1}, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_4$. $Z(\pi) = \langle \alpha_1 \rangle$. $\Lambda = \pi/Z(\pi) \cong G_4$. $\Lambda = \pi/Z(\pi) \cong G_4$.
5. $\langle \alpha_1, \alpha_2, \alpha_3, t \mid \alpha_1 = t^6, t \alpha_2 t^{-1} = \alpha_3, t \alpha_3 t^{-1} = \alpha_2^{-1} \alpha_3, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \langle \bar{t} \rangle \cong \mathbb{Z}_6$. $Z(\pi) = \langle \alpha_1 \rangle$. $\Lambda = \pi/Z(\pi) \cong G_6$.
6. $\langle \alpha_1, \alpha_2, \alpha_3, t_1, t_2, t_3 \mid \alpha_1 \alpha_3 = t_3 t_2 t_1, \alpha_i = t_i^2, t_i \alpha_j t_i^{-1} = \alpha_j^{-1}$ for $i \neq j, \alpha_i \alpha_j = \alpha_j \alpha_i, 1 \leq i, j \leq 3 \rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by \bar{t}_1, \bar{t}_2 . $Z(\pi) = \{1\}$.
7. $\pi_1(K) \times \mathbb{Z} = \langle \alpha, \beta \mid \beta \alpha \beta^{-1} = \alpha^{-1} \rangle \times \langle t \rangle$ where K is the Klein bottle, with holonomy $\Phi = \mathbb{Z}_2$, generated by $\bar{\beta}$. $Z(\pi) = gp\langle \beta^2, t \rangle \cong \mathbb{Z}^2$.
8. $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \alpha \beta \rangle$ with holonomy $\Phi = \mathbb{Z}_2$ generated by $\bar{\beta}$. $Z(\pi) = \langle \beta^2 \rangle \cong \mathbb{Z}$. We have $gp\langle \alpha, \beta \rangle \cong \pi_1(K)$ and so $\pi = \pi_1(K) \rtimes \langle t \rangle$.
9. $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \beta^{-1} \rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by $\bar{t}, \bar{\beta}$. $Z(\pi) = \langle t^2 \rangle \cong \mathbb{Z}$. We have $gp\langle \alpha, \beta \rangle = \pi_1(K)$ and so $\pi \cong \pi_1(K) \rtimes \langle t \rangle$. $\pi_1(K)$ is characteristic.
10. $\langle \alpha, \beta, t \mid \beta \alpha \beta^{-1} = \alpha^{-1}, t \alpha t^{-1} = \alpha, t \beta t^{-1} = \alpha \beta^{-1} \rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by $\bar{t}, \bar{\beta}$. $Z(\pi) = \langle t^2 \rangle \cong \mathbb{Z}$. We have $gp\langle \alpha, \beta \rangle = \pi_1(K)$ and $\pi = \pi_1(K) \rtimes \langle t \rangle$. $\pi_1(K)$ is characteristic.

We now state the result concerning the R_∞ -property of these groups.

Theorem 2. *Let $\pi = \pi_1(M)$ where M is a compact flat 3-manifold. Then π has the R_∞ -property if π is isomorphic to one of the groups (3), (4), (5), (7), (8), (9) or (10). In the case when π is isomorphic to the groups (1), (2), or (6), M admits self-homeomorphisms with finite Reidemeister numbers.*

We merely outline the method of proof here, referring the reader to relevant papers for detailed proofs. In case (1), the manifold M is a torus and the assertion is well-known.

For Cases (4) and (5), we use the exact sequence (2). In these cases, $Z = Z(\pi)$ is characteristic. Since π projects onto a two dimensional crystallographic group $\Lambda = \pi/Z(\pi)$, which is isomorphic to G_4 and G_6 respectively, the R_∞ -property of π follows from the R_∞ -property of Λ by [10].

For Case (7), $\pi = \pi_1(K) \times \mathbb{Z}$ where K is the Klein bottle. This group is known to have the R_∞ -property (see [9, Theorem 2.4]). For Cases (9) and (10), $\pi \cong \pi_1(K) \rtimes \mathbb{Z}$ where $\pi_1(K)$ is characteristic. For any $\varphi \in \text{Aut}(\pi)$, the induced automorphism $\bar{\varphi}$ is either $id_{\mathbb{Z}}$ or $-id_{\mathbb{Z}}$. The former case yields $R(\varphi) = \infty$. In the later case, the set of twisted conjugacy classes of φ' injects into the set of twisted conjugacy classes of φ , i.e., $\mathcal{R}(\varphi') \hookrightarrow \mathcal{R}(\varphi)$. Since $\pi_1(K)$ has the R_∞ -property, it follows that $R(\varphi) = \infty$.

For Cases (2), (3), (8) we consider $\bar{\pi} = \Phi$ and $\Gamma = \mathbb{Z}^3$. For Case (8), $\Phi = \mathbb{Z}_2$ and for Case (3), $\Phi = \mathbb{Z}_3$. In each of these two cases, we use the exact sequence (1). One can find a representative $g \in \pi$ of a suitable element of Φ such that $R(\iota_g \circ \varphi') = \infty$ (see [13, §3.3 and §4.3]). Since Γ is characteristic and since Φ is finite, it follows that $R(\varphi) = R(\iota_g \circ \varphi) = \infty$. For Case (2), an explicit automorphism φ was constructed with $R(\varphi) < \infty$ (see §7 of [13]).

For the Case (6), consider an automorphism φ with restriction φ' on Γ of type II and IV' from [13, Table 4.1]. One can write down all the other three automorphisms using $\theta_1, \theta_2, \theta_3$ given few lines below the table. All such lifts have Reidemeister number finite for suitable values of ϵ, r, s, b . Choosing $\epsilon = 1, r = 1, s = a = 0$ in the notation of [13, Table 4.1, §4.2] we obtain that $\phi' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$ so that $R(\phi') = |\det(I - \phi')| = |\det(I - \theta_i \phi')| = R(\theta_i \phi') = 2$. Now by the addition formula, namely, [8, Lemma 2.1], it follows that $R(\varphi) < \infty$. Moreover, this group in Case (6) is the classical Hantzsche-Wendt group which is known not to have the R_∞ -property (see [3]). In [12], all crystallographic groups of rank 2 are classified in terms of the R_∞ -property. Similarly, the result in [4] includes the full classification of all crystallographic groups of rank 3 in terms of the R_∞ -property, which certainly include the ten 3-dimensional flat manifolds. So the result can be obtained from [4] after identifying explicitly the flat 3-manifolds as described in [13] with the ones as described in [4].

3. Geometries $S^2 \times \mathbb{R}$ and Sol

3.1. $S^2 \times \mathbb{R}$ -geometry

In this case let us first analyze which of the fundamental groups have the R_∞ -property and which of the spaces have the R_∞ -property. For the first question, the groups involved are:

- the fundamental group of $S^2 \times S^1$, which is \mathbb{Z} , and does not have the R_∞ -property;
- the fundamental group of $\mathbb{R}P^2 \times S^1$, which is $\mathbb{Z}_2 \times \mathbb{Z}$, and does not have the R_∞ -property;
- the fundamental group of $S^2 \tilde{\times} S^1$, which is \mathbb{Z} , and does not have the R_∞ -property;
- the fundamental group of $\mathbb{R}P^3 \# \mathbb{R}P^3$, which is $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, and it has the R_∞ -property.

Now we consider the question at the level of spaces. Certainly, since $\pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3)$ has the R_∞ -property, the manifold $\mathbb{R}P^3 \# \mathbb{R}P^3$ has the R_∞ -property.

For the other three manifolds, $M = S^2 \times S^1, \mathbb{R}P^2 \times S^1$, and $S^2 \tilde{\times} S^1$, observe that in all these cases, they are total spaces of fibre bundles over S^1 . In each case, it is easy to construct a fibre preserving map $f : M \rightarrow M$ which induces the reflection map on the base space S^1 . This implies that the induced map $f_\#$ has finite Reidemeister number.

For more details and further results about the Nielsen theory of selfmaps on such manifolds see [14].

3.2. Sol-geometry

Let M be a Sol 3-manifold. Then M is one of the two types:

- a mapping torus of a self-homeomorphism $f : T \rightarrow T$ of the torus $T = S^1 \times S^1$ which induces in $\pi_1(T) = \mathbb{Z}^2$ an automorphism given by an Anosov matrix $A \in GL(2, \mathbb{Z})$,
- the union of two twisted I -bundles over the Klein bottle glued along their common boundaries, which are tori, via an Anosov diffeomorphism. Such a manifold is also known as a sapphire manifold.

The case of a torus bundle. The homeomorphism type of M is determined by the conjugacy class of A in $GL(2, \mathbb{Z})$. Thus M fibres over a circle with fibre T and we have an isomorphism $G := \pi_1(M) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. It turns out that the normal subgroup $N := \mathbb{Z}^2 \subset G$ is characteristic (see [8, Lemma 2.1]). There are several cases to consider depending on the conjugacy class of A . Note that since A is Anosov, (i.e., $|Tr(A)| >$

2, $\det A = \pm 1$), the eigenvalues of A are real and neither of them equals ± 1 . In particular A has infinite order.

Case (a): If $\det A = -1$, it was shown in [11] that any automorphism of G induces the identity map of the quotient $G/N = \mathbb{Z}$. Therefore G has the R_∞ -property in this case.

Case (b): We now assume that $\det A = 1$. Examples of A such that $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ does not have the R_∞ -property were given in [8, Example 4.3]. It was shown that when

$$A = \begin{pmatrix} k^2 + 1 & k \\ k & 1 \end{pmatrix}, \tag{4}$$

G does not have the R_∞ -property.

A necessary condition for an automorphism $\phi : G \rightarrow G$ to have finite Reidemeister number is that the induced automorphism $\bar{\phi}$ on $G/N \cong \mathbb{Z}$ equals $-id$. Let S be the matrix of $\phi|_N$. Then $\bar{\phi} = -id$ if and only if $SA = A^{-1}S$. Conversely, if $S \in GL(2, \mathbb{Z})$ is such that $SAS^{-1} = A^{-1}$, then we obtain an automorphism ϕ of G such that $\phi|_N$ is given by S and $\bar{\phi} = -id$. In particular, if A and A^{-1} are not conjugates in $GL(2, \mathbb{Z})$, then G has the R_∞ -property.

From [6, Proposition 5.8, Theorem 5.9] we obtain that, if $A, A^{-1} \in SL(2, \mathbb{Z})$ are conjugates in $GL(2, \mathbb{Z})$, then A must be conjugate to a matrix of the form A_0, B_0 , or C_0 where

$$A_0 = \begin{pmatrix} r & s \\ s & u \end{pmatrix}, B_0 = \begin{pmatrix} r & s \\ t & r \end{pmatrix}, \text{ and } C_0 = \begin{pmatrix} r & s \\ u - r & u \end{pmatrix}. \tag{5}$$

We remark that an $A \in SL(2, \mathbb{Z})$ may be conjugate to more than one of the above three types.

Type A_0 : If A is conjugate to a matrix of the form A_0 in Equation (5), then there is an automorphism of the group $G = \pi_1(M)$ which has finite Reidemeister number.

Proof. We may (and do) assume that $A = A_0$. Consider the automorphism of $N = \mathbb{Z}^2$ given by the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $JA_0J^{-1} = A_0^{-1}$, J extends to an automorphism ϕ of the group $G = N \rtimes_{A_0} \mathbb{Z}$. Then ϕ induces $-id$ on the quotient $G/N = \mathbb{Z}$. The Reidemeister number $R(\phi)$ can then be calculated using a certain addition formula (see [8, Lemma 2.1]). In our context, we obtain $R(\phi) = R(J) + R(JA)$. Since for any $S \in GL(2, \mathbb{Z})$ we have $R(S) = |\det(I - S)|$ when $\det(S)$ is non-zero, we obtain that $R(\phi) = |\det(I - J)| + |\det(I - A_0J)| = 4$. \square

It was shown in [11, Theorem 2.2] that the Nielsen number of any homeomorphism of M (where the gluing torus homeomorphism corresponds to A_0) is equal to either 0 or 4 and that both possibilities do occur.

We shall now treat simultaneously the remaining cases when A is of type B_0 or C_0 . Recall that $\det(A) = 1$. We say that an element $A_1 \in GL(2, \mathbb{Z})$ is a primitive root of A if $\delta A = A_1^m$ with $m \geq 1$ maximum and $\delta \in \{1, -1\}$.

Types B_0 and C_0 : Let A be conjugate to an Anosov matrix of the form B_0 or C_0 in Equation (5). Then: the group $G = \pi_1(M) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ has the R_∞ -property if and only if any primitive root of A has determinant $+1$.

Proof. We assume, as we may, that $A = \begin{pmatrix} r & s \\ t & r \end{pmatrix}$ when it is of type B_0 and $A = \begin{pmatrix} r & s \\ u - r & r \end{pmatrix}$ when it is of type C_0 . Also, since the case when A is of type A_0 had already been considered, we assume that $t \neq s$.

Let $\phi \in \text{Aut}(G)$. Let $S \in GL(2, \mathbb{Z})$ be the matrix of the automorphism $\phi|_N$, where $N = \mathbb{Z}^2$. Recall that N is characteristic in G . If $\bar{\phi}$ induces the identity on $G/N \cong \mathbb{Z}$, then $R(\phi) = \infty$. So assume that $\bar{\phi} = -id$. Then S satisfies the equation $SAS^{-1} = A^{-1}$.

Consider the group $K(A) = \{X \in SL(2, \mathbb{Z}) \mid XAX^{-1} = \delta A, \delta \in \{1, -1\}\}$. Note that the centralizer $Z(A) \subset SL(2, \mathbb{Z})$ is subgroup of $K(A)$ of index at most 2. Since $A \in SL(2, \mathbb{Z})$ is Anosov, it follows

that its centralizer $Z(A) \subset GL(2, \mathbb{Z})$ is virtually infinite cyclic. In fact, the image of $K(A)$ in $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I_2, -I_2\}$ under the natural projection equals the centralizer $Z(\bar{A})$ where $\bar{A} := \{\pm A\} \in PSL(2, \mathbb{Z})$. Using the length function associated to the free product $PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, we see that \bar{A} has a *unique* primitive root \bar{A}_1 in $PSL(2, \mathbb{Z})$. That is, there exists a unique $\bar{A}_1 \in PSL(2, \mathbb{Z})$ such that $\bar{A} = \bar{A}_1^k$ with $k \geq 1$ largest. Moreover $Z(\bar{A})$ equals $\langle \bar{A}_1 \rangle \cong \mathbb{Z}$. Then $A_1 \in K(A)$ is such that $A_1^k = \delta_0 A$, $\delta_0 \in \{1, -1\}$. The only other element in $SL(2, \mathbb{Z})$ to have this property in $K(A)$ is $-A_1$. Further, $K(A)$ equals $gp\langle A_1, -I_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$. “If” part: Suppose that every primitive root A' of A in $GL(2, \mathbb{Z})$ has determinant $+1$. Then $A' = A_1$ or $-A_1$. We set $S_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ when A is of type B_0 and $S_0 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ when A is of type C_0 . Note that $S_0^2 = I_2$ and $S_0 A S_0^{-1} = A^{-1}$ in each type. We shall show that the same holds for the primitive root A_1 .

Suppose that $YAY^{-1} = \delta A^\epsilon$, $\delta, \epsilon \in \{1, -1\}$. Then $S_0 Y A Y^{-1} S_0^{-1} = \delta S_0 A^\epsilon S_0^{-1} = \delta A^{-\epsilon}$. Hence the group $N(A) := \{S \in GL(2, \mathbb{Z}) \mid SAS^{-1} = \delta A^\epsilon, \delta, \epsilon \in \{1, -1\}\}$ contains $K(A)$ as an index-2 subgroup since A_1 is a primitive root of A in $GL(2, \mathbb{Z})$. Moreover, since $\pm A_1$ are the only primitive roots of A in $GL(2, \mathbb{Z})$, we have $N(A) = K(A) \rtimes \langle S_0 \rangle$ and so, any element of $N(A)$ can be expressed as $\epsilon A_1^p S_0^j$, $\epsilon \in \{1, -1\}, p \in \mathbb{Z}, j = 0, 1$. Thus $N(A)$ acts on $K(A)$ via conjugation and also on $Z(\bar{A})$. In view of the uniqueness of the primitive root of \bar{A} , we see that, $SA_1 S^{-1} = \delta_1 A_1^\epsilon$ for all $S \in N(A)$, where $\delta_1, \epsilon \in \{1, -1\}$. We have $A^{-1} = S_0 A S_0^{-1} = S_0 A_1^k S_0^{-1} = \delta_1^k A^\epsilon$. This implies that $\epsilon = -1$ and $\delta_1^k = 1$. If k is odd, we have $\delta_1 = 1$ and $S_0 A_1 S_0^{-1} = A_1^{-1}$ and so A_1 is of type B_0 (resp. C_0) depending on the matrix S_0 . Since $A_1^k = \delta_0 A$, if k is odd, we replace A_1 by $-A_1$ so that $(-A_1)^k = A$ resulting in $\delta_0 = 1$. Again $-A$ is of type B_0 (resp. type C_0) and the same holds for A_1 as well. It can be shown (by induction) that if X is of type B_0 (resp. C_0), the same is true of X^p for all non-zero integers p . Therefore if $SAS^{-1} = A^{-1}$, then $S, AS \in N(A)$ are of the form $\eta A_1^q S_0$ for some $\eta \in \{1, -1\}, q \in \mathbb{Z}$.

To complete the proof that $R(\phi) = \infty$, we now apply the addition formula $R(\phi) = R(S) + R(AS)$ where $SAS^{-1} = A^{-1}$. First consider the case when A is of type B_0 . We have $S = \eta A_1^p S_0, \eta \in \{1, -1\}$ where $S_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $p = 0$, then $S = \eta S_0$ and $I - \eta S_0$ is singular. So $R(S) = \infty$. Suppose that $p \neq 0$. Write $A_1 = \begin{pmatrix} x & y \\ z & x \end{pmatrix} \in SL(2, \mathbb{Z})$. Then $S = \eta \begin{pmatrix} x & -y \\ z & -x \end{pmatrix}$. So $\det(I_2 - S) = 1 - (x^2 - yz) = 0$ since $A_1 \in SL(2, \mathbb{Z})$. Therefore $R(\phi) = \infty$. The same argument applies in the case of Type C_0 to yield $R(S) = \infty$ and so $R(\phi) = \infty$.

“Only if” part: Suppose that $\delta A = X_0^m$ for some $X_0 \in GL(2, \mathbb{Z}), m \geq 1$, with $\det(X_0) = -1, \delta \in \{1, -1\}$. Then m is even; write $m = 2n$. Let $X_1 = X_0^2$ so that $X_1^n = \delta A$. By what has been shown already, X_1 is of type B_0 or C_0 . We claim that X_0 is of the same type as A — B_0 or C_0 . To see this, write $X_0 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then $X_1^2 = \begin{pmatrix} p^2+qr & q(p+s) \\ r(p+s) & s^2+qr \end{pmatrix}$. If A is of type B_0 , then we must have $p^2 = q^2$. If $p = -q$, then X_1 —and hence A —would be diagonal. So $p = q$ and X_0 is of type B_0 . The proof that X_0 is of type C_0 when A is, is similar and omitted.

Consider the automorphism ϕ of G whose restriction to $N = \mathbb{Z}^2$ is given by $S = S_0 X_0$. Then $SAS^{-1} = S_0 A S_0^{-1} = A^{-1}$. So $\bar{\phi} = -id$. Using the addition formula we obtain that $R(\phi) = R(S) + R(AS)$. Proceeding as before, we see that $R(S) = 2 = R(AS)$.

In summary, we have shown that: *The group $\pi_1(M) = G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ has the R_∞ -property if (i) $\det A = -1$, (ii) $A \in SL(2, \mathbb{Z})$ is not conjugate to A^{-1} , and, (iii) $A \in SL(2, \mathbb{Z})$ is of the type B_0 or C_0 , and, neither A nor $-A$ is in the cyclic group generated by an element of $GL(2, \mathbb{Z})$ having determinant equal to -1 . If $A \in SL(2, \mathbb{Z})$ is of type A_0 , then $\pi_1(M)$ does not have the R_∞ -property.*

This completes the proof of part (d) of Main Theorem for torus bundles case. \square

The case of a sapphire manifold. Recall that a *sapphire* is a 3-manifold obtained from two orientable 3-manifolds which are twisted I -bundles over a Klein bottle glued along their boundary tori. A sapphire which is not a torus bundle over the circle admits Sol-geometry when the gluing map is an Anosov homeomorphism. If M is a sapphire which is not a torus bundle, it is double covered by a torus bundle \tilde{M} with fundamental group $L := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where A is a hyperbolic matrix in $SL(2, \mathbb{Z})$. Moreover the index 2 subgroup L is characteristic in $G := \pi_1(M)$ (see [11, Lemma 3.1]). So, any automorphism ϕ of G restricts to an automorphism

ϕ' of L . It follows that $R(\phi) = \infty$ if $R(\phi') = \infty$. If ϕ' induces id on the quotient $L/N \cong \mathbb{Z}$ where $N \cong \mathbb{Z}^2$ is the characteristic subgroup of L corresponding to the fundamental group of the torus fibre in \tilde{M} , then $R(\phi') = \infty$. In case ϕ' induces $-id$, choose an element $\alpha \in G \setminus L$. Denote by ι_α the inner conjugation by α . Then the automorphism $\iota_\alpha \circ \phi =: \psi$ has the same Reidemeister number as ϕ . Moreover, using the fact that M does not admit an orientation reversing homeomorphism, it can be shown that, when ϕ' induces $-id$ on L/N , then $\iota_\alpha \circ \phi'$ induces id on L/N . Again we are led to the conclusion that $R(\phi) = \infty$. See [11, Theorems 3.4 and 4.2] for a more geometric proof.

4. Geometries $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R})$

In this section, we focus on those geometric 3-manifolds that are finitely covered by 3-manifolds that are S^1 -bundles over hyperbolic surfaces.

Let $\Gamma = \pi_1(M)$ where M admits either a $\mathbb{H}^2 \times \mathbb{R}$ -geometry or an $\widetilde{SL}(2, \mathbb{R})$ -geometry. Then M admits a finite cover $\hat{M} \rightarrow M$ such that \hat{M} fibres over an orientable finite volume hyperbolic surface Σ with fibre S^1 . Thus $\chi(\Sigma) < 0$ and Σ is compact if and only if M is. In the case of $\mathbb{H}^2 \times \mathbb{R}$ -geometry, the S^1 -bundle may be assumed to be the product bundle $\Sigma \times S^1$.

Thus we have an exact sequence

$$1 \rightarrow Z \rightarrow \Lambda \xrightarrow{\eta} \pi_1(\Sigma) \rightarrow 1 \tag{2}$$

where $Z = \pi_1(S^1) \cong \mathbb{Z}$. In the case of $\mathbb{H}^2 \times \mathbb{R}$ -geometry, since $\hat{M} \cong \Sigma \times S^1$, Z equals the centre of Λ .

Suppose that M has the $\widetilde{SL}(2, \mathbb{R})$ -geometry. Let \mathcal{Z} denote the centre of Λ . Since $\pi_1(\Sigma)$ is a nonabelian free group or a higher genus surface group, its centre is trivial. It follows that $\mathcal{Z} \subset Z$. We claim that \mathcal{Z} is non-trivial and hence infinite cyclic. To get a contradiction, suppose that \mathcal{Z} is trivial. Then Λ maps isomorphically onto its image under the projection $p : \widetilde{SL}(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$ in view of the fact that $\ker(p) \cong \mathbb{Z}$ is the centre of $\widetilde{SL}(2, \mathbb{R})$. So $\hat{M} = \widetilde{SL}(2, \mathbb{R})/\Lambda \rightarrow PSL(2, \mathbb{R})/p(\Lambda)$ is an infinite covering projection. This contradicts the finiteness of the volume of \hat{M} . Hence our claim.

Theorem 3. *Let M be a 3-manifold which admits a $\mathbb{H}^2 \times \mathbb{R}$ -geometry or $\widetilde{SL}(2, \mathbb{R})$ -geometry. Then $\pi_1(M)$ has the R_∞ -property.*

Proof. By the above discussion, the group $\Gamma = \pi_1(M)$ has a finite index subgroup Γ_0 whose centre \mathcal{Z} is an infinite cyclic group and Γ_0/\mathcal{Z} is either a nonabelian free group of finite rank or the fundamental group of a closed surface Σ_0 of genus $g \geq 2$. Thus Γ_0/\mathcal{Z} is a non-elementary hyperbolic group and so has the R_∞ -property. Hence, it follows from [9, Lemma 1.1] that Γ_0 has the R_∞ -property. The same argument shows that any finite index subgroup of Γ_0 also has the R_∞ -property.

Since Γ is finitely generated and since Γ_0 has finite index in Γ , it follows that there is a finite index subgroup $K \subset \Gamma_0$ that is characteristic in Γ . (For example, we may take K to be the intersection of all subgroups of Γ having index equal to the index of Γ_0 in Γ .) Then K has the R_∞ -property and so, by [9, Lemma 1.1], Γ also has the R_∞ -property. \square

5. Nil-geometry

The closed 3-manifolds which admit Nil-geometry are listed as the infranilmanifolds M of dimension 3, following Dekimpe [2, Theorem 6.5.5, Chapter 6, p. 154]. The last column denotes the holonomy group F . The group F may be described as the $\pi_1(M)/\Gamma$ where Γ is the unique maximal nilpotent normal subgroup of $\pi_1(M)$. The manifold M is then covered by the compact nilmanifold \tilde{M} with covering group F . Since M has Nil-geometry, it is understood that Γ is not abelian; equivalently \tilde{M} is not the torus.

Type	Set of Seifert invariants	F
i	$M_1(k) = \{k, (o_1, 1); \}$	1
ii	$M_2(k) = \{k - 2, (o_1, 0); (2, 1), (2, 1), (2, 1), (2, 1)\}$	\mathbb{Z}_2
iii	$M_3(k) = \{k, (n_2, 2); \}$	\mathbb{Z}_2
iv	$M_4(k) = \{k - 1, (n_2, 1); (2, 1), (2, 1)\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
v	$M_5(k) = \{k - 2, (o_1, 0); (4, 3), (4, 3), (2, 1)\}$ $M_6(k) = \{k - 1, (o_1, 0); (4, 1), (4, 1), (2, 1)\}$ $M_7(k) = \{k - 2, (o_1, 0); (4, 3), (4, 1), (2, 1)\}$	\mathbb{Z}_4
vi	$M_8(k) = \{k - 2, (o_1, 0); (3, 2), (3, 2), (3, 2)\}$ $M_9(k) = \{k - 1, (o_1, 0); (3, 1), (3, 1), (3, 1)\}$ $M_{10}(k) = \{k - 2, (o_1, 0); (3, 2), (3, 1), (3, 1)\}$ $M_{11}(k) = \{k - 2, (o_1, 0); (3, 2), (3, 2), (3, 1)\}$	\mathbb{Z}_3
vii	$M_{12}(k) = \{k - 2, (o_1, 0); (6, 5), (3, 2), (2, 1)\}$ $M_{13}(k) = \{k - 1, (o_1, 0); (6, 1), (3, 1), (2, 1)\}$ $M_{14}(k) = \{k - 2, (o_1, 0); (6, 1), (3, 2), (2, 1)\}$ $M_{15}(k) = \{k - 2, (o_1, 0); (6, 5), (3, 1), (2, 1)\}$	\mathbb{Z}_6

In this table the integer k is assumed to be strictly bigger than 0. The case of manifolds with the same Seifert invariants but with $k \leq 0$ are either flat manifolds or they represent manifolds which are already homeomorphic to one in the Table for $k > 0$. When the manifold is not flat, one passes from one family of invariants with $k < 0$ to another with $k > 0$ by changing the orientation (see [20]).

The complete answer to the question about R_∞ -property of the manifolds above is given by [4, Theorem 4.4] which in terms of the table above says: A closed infranil-manifold has the R_∞ -property, if and only if it does not belong to the first two lines. Furthermore, for the cases of the manifolds of types (i) and (ii), namely the ones which do not have the R_∞ -property, the Reidemeister spectrum is described in [21, section 5].

Remark 4. The fundamental groups of the Seifert manifolds have a presentation given by [19, Chapter 5 section 5.3]. It follows from such presentation that we have a short exact sequence $1 \rightarrow \langle h \rangle \rightarrow \pi_1(M) \rightarrow \pi_1(B) \rightarrow 1$, where h is a generator which corresponds to the regular fibre. From [19, Chapter 5, §5.3, Lemma 1], it follows that this short exact sequence is characteristic with respect to automorphisms of $\pi_1(M)$. The results obtained for the 3-manifolds may help to study the R_∞ -property for the groups $\pi_1(B)$. The groups $\pi_1(B)$ are most often Fuchsian groups [23].

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Appendix A

The purpose of the appendix is to show that the notions of a space X having the R_∞ -property and the group $\pi_1(X)$ having the R_∞ -property are not equivalent.

Certainly if $\pi_1(X)$ has the R_∞ -property, then the space X has the R_∞ -property. We now provide an example where X has the R_∞ -property but its fundamental group $\pi_1(X)$ does not have.

Let $Y = S^3 \times S^3$ and $h : Y \rightarrow Y$ a homeomorphism which induces on the homology group $H_3(S^3 \times S^3; \mathbb{Z})$ a homomorphism given by a matrix

$$A = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \quad (6)$$

in $GL(2, \mathbb{Z})$.

Let X be the mapping torus of the homeomorphism h of $S^3 \times S^3$, so X fibres over the circle S^1 with fibre $S^3 \times S^3$. Let $f : X \rightarrow X$ be a homotopy equivalence. Then the map f can be deformed to a fibre-preserving map. We assume that f itself is fibre preserving. The induced map \bar{f} on the base S^1 is either of degree 1 or -1 . The induced isomorphism on $H_3(X) \cong \mathbb{Z}^2$ is a matrix B such that: If the degree of \bar{f} is -1 then we must have $BAB^{-1} = A^{-1}$. Let A be a matrix where such B does not exist (see §3.2 above). Then any homotopy equivalence will induce a map of degree 1 on S^1 and hence identity on $\pi_1(X)$. So $R(f) = \infty$, but certainly $\mathbb{Z} = \pi_1(X) = \pi_1(S^1)$ does not have the R_∞ -property. So it suffices to choose a matrix A where B does not exist and determine a homeomorphism h_A . We can take A any matrix such that $\det A = -1$ or one of the matrices from [8, Example 4.4]. In details, for the former case let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad (7)$$

so that

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}. \quad (8)$$

Define $h_A(q_1, q_2) = (q_1q_2, q_1^2q_2)$ and $h_{A^{-1}}(q_1, q_2) = (q_2q_1^{-1}, q_1q_2^{-1}q_1)$.

It is straightforward to verify that $h_A \circ h_{A^{-1}} = h_{A^{-1}} \circ h_A = Id_{S^3 \times S^3}$. So we can apply our construction for h_A and we obtain the result.

For a matrix in the latter case (as in [8, Example 4.4]), let

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \quad (9)$$

so that

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}. \quad (10)$$

Define $h_A(q_1, q_2) = (q_1^4q_2, q_1^3q_2)$ and $h_{A^{-1}}(q_1, q_2) = (q_1q_2^{-1}, q_2q_1^{-1}q_2q_1^{-1}q_2q_1^{-1}q_2)$.

It is straightforward to verify that $h_A \circ h_{A^{-1}} = h_{A^{-1}} \circ h_A = Id_{S^3 \times S^3}$. So we can apply our construction for h_A and we obtain the result.

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