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# CODIMENSION GROWTH OF SIMPLE JORDAN SUPERALGEBRAS

BY

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ABSTRACT

We study asymptotic behaviour of graded and non-graded codimensions of simple Jordan superalgebras over a field of characteristic zero. It is known that the PI-exponent of any finite-dimensional associative or Jordan or Lie algebra $A$ is a non-negative integer less than or equal to the dimension of algebra $A$. Moreover, the PI-exponent is equal to the dimension if and only if $A$ is simple provided that the base field is algebraically closed. In the present paper we prove that for a Jordan superalgebra $P(t) = H(M_{t|t}, \mathrm{trp})$ its non-graded and $\mathbb{Z}_2$-graded exponents are strictly less than $\dim P(t)$. In particular, $\exp P(2)$ is fractional.

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## 1. Introduction

We consider finite-dimensional Jordan superalgebras over a field of characteristic zero and study their $\mathbb{Z}_2$-graded and non-graded identities. Given an algebra $A$, one can associate the sequence $\{c_n(A), n = 1, 2, \ldots\}$, of codimensions of $A$ depending on the polynomial identities of $A$. The asymptotic behavior of $\{c_n(A)\}$ has been studied for several classes of algebras such as associative algebras (see, for instance, references in [3], [4], [5], [12], [14]), Lie algebras (e.g., [17], [19], [20]) and superalgebras (e.g., [14], [25]), Jordan algebras (e.g., [13], [9]), Novikov algebras [8] and many others.

If $A$ is an associative algebra without additional poynomial identities, then $c_n(A) = n!$, that is the codimension sequence has an overexponential growth. But an unexpected result by Regev [22] has shown that if $A$ satisfies a non-trivial polynomial identity (that is $A$ is PI) then $\{c_n(A)\}$ is exponentially bounded.

At the end of the 1980's Amitsur conjectured that for any PI-algebra $A$ the limit of the sequence $\sqrt[n]{c_n(A)}$ exists and is an integer called the PI-exponent of $A$. This conjecture was confirmed in [10, 11]. It was also shown that $\exp(A) \le \dim A$ for a finite-dimensional algebra $A$. Moreover, $\exp(A) = \dim A$ if and only if $A$ is simple provided that the base field is algebraically closed.

In the non-associative case there are a lot of algebras with non-exponential codimension growth. For example, the relatively free Lie algebra $L$ of countable rank of the variety $\mathbf{AN}_2$ has the growth $c_n(L) \sim \sqrt{n!}$ (see [27]). Moreover, for any integer $t \ge 2$ there exists a Lie algebra $L$ with $c_n(L) \sim n^{\frac{t-1}{t}}$ (see [20]). The class of Jordan algbras also contains examples of overexponential growth as follows from [6, 16]. In [15] it was shown that for any real $0 < \alpha < 1$ there exists a two-step right nilpotent algebra $A_\alpha$ with $c_n(A_\alpha) \sim n^{n^\alpha}$.

Nevertheless, the class of algebras with exponentially bounded codimension sequence is sufficiently wide. It contains, for example, all infinite-dimensional simple Lie algebras of Cartan type [19], all affine Kac–Moody algebras [28], all Novikov algebras [8] and all finite-dimensional algebras [2, 13].

In the light of previous discussion three main questions arise. Given an algebra $A$ with exponentially bounded codimension sequence, can one show that its PI-exponent exists? In the case of a positive answer, is it an integer? Is it true that if the field is algebraically closed and $A$ is finite-dimensional, then the equality $\exp(A) = \dim A$ is equivalent to the simplicity of $A$?

If $A$ is graded by a finite group, say, if $A$ is $\mathbb{Z}_2$-graded, one can consider graded codimensions $c_n^{\rm gr}(A)$ and graded PI-exponent $\exp^{\rm gr}(A)$. Then the same questions look reasonable for graded codimensions and graded PI-exponents.

The second question was answered in the negative in the class of finite-dimensional Lie superalgebras both for an ordinary PI-exponent [14] and for $\mathbb{Z}_2$-graded PI-exponent [24]. Inequalities $\exp(L), \exp^{\rm gr}(L) < \dim L$ were also proved for a series of finite-dimensional simple Lie superalgebras [14, 23].

In the present paper we discuss the codimension growth and graded codimension growth of finite-dimensional Jordan superalgebras. We prove that both $\exp(H(M_{t|t}, {\rm trp}))$ and $\exp^{\rm gr}(H(M_{t|t}, {\rm trp}))$ are strictly less than $\dim H(M_{t|t}, {\rm trp})$ for all $t \geq 2$ (Theorem 1 and Theorem 2 below). Then we compute the precise value $\exp^{\rm gr}(H(M_{2|2}, {\rm trp}) = 4 + 2\sqrt{3}$ of $H(M_{2|2}, {\rm trp})$ (Theorem 3). Finally, we prove that the ordinary PI-exponent of $H(M_{2|2}, {\rm trp})$ exists and is fractional, namely, $7 < \exp(H(M_{2|2}, {\rm trp})) < 8 = \dim H(M_{2|2}, {\rm trp})$ (Theorem 4). All details concerning polynomial identities and their numerical invariants one can find in [12].

## 2. Preliminaries

Throughout the paper $F$ is a field of characteristic zero. Denote by $F\{X\}$ the free non-associative algebra over $F$ with an infinite set $X$ of free generators. Let $A$ be an algebra over $F$ and let ${\rm Id}(A) \subseteq F\{X\}$ be the set of all polynomial identities of $A$. Then ${\rm Id}(A)$ is an ideal of $F\{X\}$. Consider a subspace $P_n = P_n(x_1, \ldots, x_n) \subseteq F\{X\}$ of all multilinear polynomials on $x_1, \ldots, x_n$. Then the intersection ${\rm Id}(A) \cap P_n$ is in fact the set of all multilinear identities of $A$ of degree $n$ in $x_1, \ldots, x_n$. Denote

$$P_n(A) = \frac{P_n}{P_n \cap {\rm Id}(A)}, \quad c_n(A) = \dim P_n(A).$$

The sequence $c_n(A), n = 1, 2 \ldots$, called the codimension sequence of $A$, is exponentially bounded for any finite-dimensional algebra $A$, $c_n(A) \leq d^{n+1}$ where $d = \dim A$ ([2, 13]), and one can define

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)},$$

the lower and upper PI-exponents of $A$, respectively, and also the (ordinary) PI-exponent

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

if the lower and upper exponents coincide. The existence of the PI-exponent of a finite-dimensional algebra is an open problem, but $\exp(A)$ does exist if $A$ is simple. Moreover, $\exp(A) = \dim A$ if $A$ is an associative, alternative, Jordan or Lie algebra provided that $F$ is algebraically closed. For some series of simple Lie superalgebras the same equality also occurs. However, for simple Lie superalgebras of the type $P(t)$, $t \geq 2$, we have the inequality $\exp(P(t)) < 2t^2 - 1 = \dim P(t)$ ([14]).

In the current paper we are interested in numerical characteristics of the identities of Jordan superalgebras, therefore we need to consider also the corresponding graded objects. Let $F\{X, Y\}$ be the free non-associative algebra with infinite sets of even generators $X$ and odd generators $Y$. The notion of parity can be uniquely extended to all monomials on $X \cup Y$ and hence $F\{X, Y\}$ becomes a $\mathbb{Z}_2$-graded algebra.

Given non-negative integers $0 \leq k \leq n$, denote by $P_{k,n-k}$ the subspace of all multilinear polynomials $f(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in F\{X, Y\}$ of degree $k$ on even variables and of degree $n - k$ on odd variables. Given a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$, denote by $\mathrm{Id}^{\mathrm{gr}}(A)$ the ideal of all graded identities of $A$. Then $P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(A)$ is the subspace of all multilinear graded identities of total degree $n$ depending on $k$ even variables and $n - k$ odd variables. Also denote by $P_{k,n-k}(A)$ the quotient

$$P_{k,n-k}(A) = \frac{P_{k,n-k}}{P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(A)}.$$

Then the graded $(k, n-k)$-codimension of $A$ is

$$c_{k,n-k}(A) = \dim P_{k,n-k}(A)$$

and the total graded codimension of $A$ is

$$c_n^{\mathrm{gr}}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(A).$$

It is also known (see [2]) that (as in the non-graded case) if $\dim A = d < \infty$, then $c_n^{\mathrm{gr}}(A) \leq d^{n+1}$ and one can consider the related sequence $\{\sqrt[n]{c_n^{\mathrm{gr}}(A)}\}$. The latter sequence has the lower and upper limits

$$\underline{\exp}^{\mathrm{gr}}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(A)}, \quad \overline{\exp}^{\mathrm{gr}}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(A)},$$

called the lower and upper graded PI-exponents, respectively. If an ordinary limit exists, it is called an (ordinary) graded PI-exponent of $A$,

$$\exp^{\mathrm{gr}}(\mathrm{A}) = \lim_{\mathrm{n}\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(A)}\ .$$

According to [2] we have

$$c_n(A) \le c_n^{\mathrm{gr}}(A) \tag{1}$$

and hence $\overline{\exp^{\mathrm{gr}}(A)} \ge \overline{\exp(A)}$.

## 3. Symmetric group action on multilinear polynomials

Symmetric groups and their representations play an important role in the theory of codimensions. All details concerning the application of representation theory of symmetric groups to the study of polynomial identities can be found in [12], [1], [7].

We start with the non-graded case. Consider $S_n$-action on $P_n$ by setting

$$\sigma \circ f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \sigma \in S_n.$$

The subspace $P_n$ of $F\{X\}$ has the structure of an $S_n$-module and $P_n \cap \mathrm{Id}(A)$ is its submodule for an algebra $A$. Hence $P_n(A)$ is also an $S_n$-module and can be decomposed into the sum of irreducible components,

$$P_n(A) = M_1 \oplus \cdots \oplus M_q. \tag{2}$$

Note that according to Mashke's Theorem one can identify $P_n(A)$ with a submodule of $P_n$.

There exists a one-to-one correspondence between irreducible $S_n$-representations and partitions of $n$ (see, for example, [18]). We recall the main constructions which we use subsequently. Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a partition of $\lambda_1 + \cdots + \lambda_k = n$, $\lambda_1 \ge \cdots \ge \lambda_k > 0$. The Young diagram $D_\lambda$ is a tableau with $n$ boxes, $\lambda_1$ in the 1st row, $\lambda_2$ in the 2nd row, and so on. The Young tableau $T_\lambda$ is the Young diagram $D_\lambda$ filled up by integers $1, 2, \dots, n$. Given a Young tableau of shape $\lambda \vdash n$, let $R_{T_\lambda}$ and $C_{T_\lambda}$ denote the subgroups of $S_n$ of the row and column stabilizers of $T_\lambda$, respectively, and set

$$R_{T_\lambda}^{+} = \sum_{\sigma \in R_{T_\lambda}} \sigma, \quad C_{T_\lambda}^{-} = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\tau)\tau.$$

The element

$$e_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda} \in S_n \tag{3}$$

is an essential idempotent, that is $e^2_{T_\lambda} = \gamma e_{T_\lambda}, 0 \neq \gamma \in \mathbb{Q}$, and $FS_n e_{T_\lambda}$ is the minimal left ideal of the group ring $FS_n$. Its $S_n$-character is denoted by $\chi_\lambda$. Decomposition (2) can be written as

$$\chi_n(A) = \chi_n(P_n(A)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{4}$$

where non-negative integers $m_\lambda$ are multiplicities of $S_n$-modules isomorphic to $FS_n e_{T_\lambda}$ among $M_1, \ldots, M_q$. The character $\chi_n(A)$ is called the $n$th cocharacter of $A$. If $d_\lambda = \deg \chi_\lambda = \dim FS_n e_{T_\lambda}$ then

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda.$$

In particular,

$$c_n(A) \geq d_\lambda \tag{5}$$

for any $\lambda \vdash n$ with $m_\lambda \neq 0$ in (4). We will use relation (5) for getting the lower bound of PI-exponents.

In the case of graded identities, the subspace $P_{k,n-k} \subset F\{X, Y\}$ has a natural structure of an $S_k \times S_{n-k}$-module where $S_k$ acts on even variables $x_1, \ldots, x_k$ whereas $S_{n-k}$ acts on odd variables $y_1, \ldots, y_{n-k}$. Clearly, $P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(A)$ is the submodule under an $S_k \times S_{n-k}$-action and we get an induced $S_k \times S_{n-k}$-action on $P_{k,n-k}(A)$. The character $\chi_{k,n-k}(A) = \chi(P_{k,n-k}(A))$ is called the $(k, n-k)$-cocharacter of $A$. This character can be decomposed into the sum of irreducible characters

$$\chi_{k,n-k}(A) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \chi_{\lambda,\mu} \tag{6}$$

where $\lambda$ and $\mu$ are partitions of $k$ and $n-k$, respectively.

The irreducible $S_k \times S_{n-k}$-module with the character $\chi_{\lambda,\mu}$ is the tensor product of the $S_k$-module with the character $\chi_\lambda$ and the $S_{n-k}$-module with the character $\chi_\mu$. In particular, the dimension $\deg \chi_{\lambda,\mu}$ of this module is the product $d_\lambda d_\mu$. Taking into account multiplicities $m_{\lambda,\mu}$ in (6) we get the relation

$$c_{k,n-k}(A) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu. \tag{7}$$

The number of irreducible components in the decomposition (6), i.e., the sum

$$l_{k,n-k}(A) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu},$$

is called the partial $(k, n-k)$-colength of $A$, and the total sum

$$l_n^{\mathrm{gr}}(A) = \sum_{k=0}^{n} l_{k,n-k}(A)$$

is called the graded colength of $A$. If $\dim A < \infty$ then the sequence $\{l_n^{\mathrm{gr}}(A)\}$ is polynomially bounded.

Proposition 1 (see [29, Theorem 1]): *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra with $\dim A = d < \infty$. Then $l_n^{\mathrm{gr}}(A) \le d(n+1)^{d^2+d+1}$.* ∎

This Proposition shows that the principle part of the exponential growth is defined by dimensions $d_\lambda$ with $m_\lambda \neq 0$ in (7). For this purpose it is convenient to use the following function. Given a partition $\mu = (\mu_1, \ldots, \mu_d) \vdash n$ we define the function

$$\Phi(\mu) = \frac{1}{z_1^{z_1}\cdots z_d^{z_d}}$$

where

$$z_1 = \frac{\mu_1}{n}, \ldots, z_d = \frac{\mu_d}{n}.$$

The value of $\Phi(\mu)$ is closely connected with $d_\mu = \deg \chi_\mu$. By the Young–Frobenius formula for dimensions of irreducible $S_n$-representations we have

$$d_\lambda = \frac{n!}{\lambda_1!\cdots\lambda_d!}\prod_{1\le i<j\le d}(l_i - l_j)$$

where $l_j = \lambda_j + d - j$, $j = 1, \ldots, d$. From the Stirling formula for factorials

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}, \quad 0 < \theta_n < 1,$$

it easily follows that

$$\frac{1}{n^d}\Phi(\lambda)^n \le d_\lambda \le n^{d^2}\Phi(\lambda)^n. \tag{8}$$

One can easily check that

$$\Phi(z_1, \ldots, z_d) = \frac{1}{z_1^{z_1}\cdots z_d^{z_d}}$$

achieves the maximal value $\Phi = d$ only if $z_1 = \cdots = z_d = \frac{1}{d}$.

We will also use the next property of $\Phi(\lambda)$. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ be two partitions of $m$ with corresponding Young diagrams $D_\lambda$ and $D_\mu$, respectively. We say that $D_\mu$ is obtained from $D_\lambda$ by pushing down one box if there exist $1 \le i < j \le d$ such that $\mu_i = \lambda_i - 1, \mu_j = \lambda_j + 1$ and $\mu_k = \lambda_k$ for all remaining $k$.

Lemma 1 (see [14, Lemma 3]): *Let $D\mu$ be obtained from $D_\lambda$ by pushing down one box. Then $\Phi(\mu) \ge \Phi(\lambda)$.* ∎

For a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of $n$ we define the height of $\lambda$ as $\mathrm{ht}(\lambda) = d$.

Lemma 2 (see [12, Lemma 6.2.4]): *Let $\lambda$ and $\mu$ be two partitions of $n$ and $n+1$, respectively, of the same height $d$, such that $\lambda$ is obtained from $\mu$ by erasing one box. Then $d_\lambda \le d_\mu \le (n+1)d_\lambda$.* ∎

Finally, we note that from standard arguments and from the structure of essential idempotent (3) the following statement holds (see, for example, [12, Theorem 4.6.1]).

Lemma 3: *Let $A = A_0 \oplus A_1$ be a finite-dimensional $\mathbb{Z}_2$-graded algebra with $d_0 = \dim A_0$, $d_1 = \dim A_1$. Then $m_{\lambda,\mu} = 0$ in the decomposition (6) for $A$ as soon as $\mathrm{ht}(\lambda) > d_0$ or $\mathrm{ht}(\mu) > d_1$.* ∎

## 4. Upper bound for codimensiion growth

We start with recalling the Jordan superalgebra $H(M_{t|t}, \mathrm{trp})$ (see, for example, [21]) and denote it for brevity by $P(t)$. Then $P(t) = P_0 \oplus P_1$ is a subspace of the $2t \times 2t$ matrix algebra $\mathrm{Mat}_{2t}(F)$ where

$$P_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \mid A \in M_t(F) \right\}, \quad T : X \to X^T \text{ is the transpose involution}$$

and

$$P_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B^T = B,\, C^T = -C,\, B, C \in M_t(F) \right\}.$$

Elements from $P_0 \cup P_1$ are called homogeneous. Any $x \in P_0$ (resp. $x \in P_1$) has the homogeneous degree $|x| = 0$ (resp. $|x| = 1$). Algebra $P(t)$ is a Jordan superalgebra with $\mathbb{Z}_2$-grading if we define the product $\{*, *\}$ of two homogeneous elements $x, y \in P(t)$ as

$$\{x, y\} = xy + (-1)^{|x||y|} yx.$$

It is easily seen that $P(t)$ has also a natural $\mathbb{Z}$-grading

$$P(t) = P^{(-1)} \oplus P^{(0)} \oplus P^{(1)},$$

where $P^{(0)} = P_0$,

$$P^{(-1)} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C^T = -C,\, C \in M_t(F) \right\},$$

$$P^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B^T = B,\, B \in M_t(F) \right\}.$$

In order to get the main result of this section we need first to prove some technical results. Coonsider the decomposition

$$\chi_{k,n-k}(H(M_{t|t}, \mathrm{trp})) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \chi_{\lambda,\mu} \tag{9}$$

of the partial $(k, n-k)$-cocharacter of $H(M_{t|t}, \mathrm{trp}) = P(t)$.

Lemma 4: *Let $m_{\lambda,\mu} \neq 0$ in (9). Then $\Phi(\lambda) \leq t^2$ and $d_\lambda \leq k^{t^4} t^{2k}$.*

*Proof.* By Lemma 3 the height $\mathrm{ht}(\lambda)$ does not exceed $t^2$. If $\mathrm{ht}(\lambda) = d \leq t^2$ then, as mentioned in the previous Section, $\Phi(\lambda) \leq d$. Hence, $d_\lambda \leq k^{t^4} t^{2k}$ by (8), and we are done. ∎

Similarly,

$$\Phi(\mu) \leq t^2 - 1 < t\sqrt{t^2 - 1} \quad \text{and} \quad d_\mu \leq (n-k)^{t^4} (t^2 - 1)^{n-k}$$

as soon as $m_{\lambda,\mu} \neq 0$ in (9) and $\mathrm{ht}(\mu) \leq t^2 - 1$. Hence throughout the rest of the section we consider only the case $\mathrm{ht}(\mu) = t^2$.

Now we define the weight of a partition $\mu = (\mu_1, \ldots, \mu_{t^2}) \vdash m$ as follows. Fill up the Young diagram of $\mu$ with the integers $-1, 1$ by inserting $-1$ into the boxes of the first $\frac{t(t-1)}{2}$ rows and 1 into the boxes of the remaining $\frac{t(t+1)}{2}$ rows. Then $wt(\mu)$ is defined as the sum of all such integers appearing in the diagram of $\mu$. Hence

$$wt\, \mu = -(\mu_1 + \cdots + \mu_{\frac{t(t-1)}{2}}) + (\mu_{\frac{t(t-1)}{2}+1} + \cdots + \mu_{t^2}).$$

Lemma 5: *Let $m_{\lambda,\mu} \neq 0$ in (9). Then $wt\,\mu \leq 1$. If $m = n-k$ is even, then $wt\,\mu$ is an even non-positive integer.*

*Proof.* From the structure of essential idempotent $e_{T_\lambda}$ (see (3)) it follows that any irreducible $S_k \times S_{n-k}$-submodule in $P_{k,n-k}(P(t))$ with the character $\chi_{\lambda,\mu}$ can be generated by a multilinear polynomial $f = f(x_1, \ldots, x_k, y_1, \ldots, y_m)$ such that the set $Y = \{y_1, \ldots, y_m\}$ of odd generators is a disjoint union $Y = Y_1 \cup \cdots \cup Y_p$, $p = \mu_1$, and $f$ is alternating on each subset $Y_1, \ldots, Y_p$.

First fix a basis $b_1, \ldots, b_r, c_1, \ldots, c_s$ of $P_0$ where

$$r = \frac{t(t+1)}{2}, \quad s = \frac{t(t-1)}{2}, \quad b_1, \ldots, b_r \in P^{(1)}, \quad c_1, \ldots, c_s \in P^{(-1)},$$

and a basis $a_1, \ldots, a_{t^2}$ of $P_0$. Since $f$ is multilinear, in order to check an inclusion $f \in \mathrm{Id}^{\mathrm{gr}}(P(t))$ it is sufficient to consider only evaluations

$$\varphi : \{x_1, \ldots, x_k\} \to \{a_1, \ldots, a_{t^2}\} : \ \varphi : \{y_1, \ldots, y_m\} \to \{b_1, \ldots, b_r, c_1, \ldots, c_s\}.$$

Denote by $\deg w$ the degree in $\mathbb{Z}$-grading of a homogeneous element $w \in P(t)$. In particular,

$$\deg a_1 = \cdots = \deg a_{t^2} = 0, \tag{10}$$

$$\deg c_1 = \cdots = \deg c_r = -1, \deg b_1 = \cdots = \deg b_r = 1.$$

Suppose that $wt\,\mu \geq 2$ and consider an evaluation $\varphi$. Fix one of the skew-symmetric sets $Y_i$. If we substitute more than $s = \frac{t(t-1)}{2}$ elements from $\{c_1, \ldots, c_r\}$ instead of $y_\alpha \in Y_i$, then we will get $\varphi(f) = 0$ due to skew symmetry. Hence the maximal number of $y_j \in Y$ such that $\varphi(y_j) \in \{c_1, \ldots, c_r\}$ does not exceed

$$\mu_1 + \cdots + \mu_{\frac{t(t-1)}{2}}.$$

According to (10) we obtain

$$\deg \varphi(f) \geq -(\mu_1 + \cdots + \mu_{\frac{t(t-1)}{2}}) + (\mu_{\frac{t(t-1)}{2}+1} + \cdots + \mu_{t^2}) = wt\,\mu \geq 2.$$

Since $P(t)^{(q)} = 0$ in $\mathbb{Z}$-grading of $P = P(t)$ for any $q \geq 2$ we complete the proof of the first part of our lemma.

Now let $m$ be even. We can split the Young diagram $D_\mu$ into two parts $D_{\mu^{(I)}} \cup D_{\mu^{(II)}}$, where

$$\mu^{(I)} = (\mu_1, \ldots, \mu_{\frac{t(t-1)}{2}}) \vdash p, \quad \mu^{(II)} = (\mu_{\frac{t(t-1)}{2}+1} + \cdots + \mu_{t^2}) \vdash q,$$

and $m = p + q$. Since $wt\,\mu = q - p = m - 2p$ and $m$ is even, then $wt\,\mu$ is also even and $\max wt\,\mu \leq 0$. ∎

We will use the following technical result from [23].

Lemma 6 ([23, Lemma 5]): *Let $m$ be a multiple of $t(t^2-1)$ and let $\nu=(\nu_1,\dots,\nu_{t^2})$ be a partition of $m$ with $wt\,\nu=0$. Then $\Phi(\nu)\le t\sqrt{t^2-1}$.* ∎

Now we will restrict $d_\mu$ in (7) for $H(M_{t|t},\mathrm{trp})$ provided that $m_{\lambda,\mu}\neq 0$.

Lemma 7: *Let $m$ be a multiple of $2t(t^2-1)$. Then there is a partition $\nu=(\nu_1,\dots,\nu_{t^2})$ of $m$ with $wt\,\nu\le 0$ such that $\Phi(\mu)\le\Phi(\nu)\le t\sqrt{t^2-1}$ and $d_\mu\le m^{t^4}(t\sqrt{t^2-1})^m$.*

*Proof.* If $wt\,\mu=0$, then there is nothing to prove by Lemma 6. Otherwise we align the partition $\mu$ and Young diagram $D_\mu$ in the following sense. As in Lemma 5 we split $\mu$ into two parts,

$$\mu^{(I)}=(\mu_1,\dots,\mu_{\frac{t(t-1)}{2}})\vdash p,\quad \mu^{(II)}=(\mu_{\frac{t(t-1)}{2}+1}+\dots+\mu_{t^2})\vdash q,$$

and start to push down boxes in $D_\mu$. If we push down one box inside $\mu^{(I)}$ or $\mu^{(II)}$, then we do not change the weight of the partition. If we move a box from $D_{\mu^{(I)}}$ to $D_{\mu^{(II)}}$ we get $\mu'$ with $wt\,\mu'=wt\,\mu+2$. This process will stop in two cases: either we will get a partition $\nu$ with $wt\,\nu=0$ and $\Phi(\mu)\le\Phi(\nu)$ by Lemma 1, or we will get a partition $\nu\vdash m$ which does not admit pushing down boxes.

Suppose $\nu$ does not admit to push boxes down. Then $\nu$ is a rectangular partition

$$\nu=(\underbrace{q,\dots,q}_{t^2})$$

or almost rectangular, that is

$$\nu=(\underbrace{q+1,\dots,q+1}_{r},\underbrace{q,\dots,q}_{t^2-r})$$

for some $0<r<t^2$. In the first case we have

$$wt\,\nu=q\frac{t(t+1)}{2}-q\frac{t(t-1)}{2}=qt>0.$$

Consider the second option. In this case $m=qt^2+r$ and minimal weight for a fixed $q$ is equal to

$$qt-\frac{t(t-1)}{2}.$$

Since $m \geq 2t(t^2-1)$ then

$$qt^2 = m - r \geq 2t(t^2-1) - \frac{t^2-t}{2} = \frac{4t^3-t^2-3t}{2}.$$

Hence $qt \geq \frac{4t^2-t-3}{2}$ and $wt\,\nu \geq \frac{4t^2-t-3}{2} - \frac{t^2-t}{2} = \frac{3t^2-3}{2} > 0$ for all $t \geq 2$.

This means that we obtain a partition $\nu \vdash m$ such that $\Phi(\mu) \leq \Phi(\nu)$ and $\Phi(\nu) \leq t\sqrt{t^2-1}$ by Lemma 6 and

$$d_\mu \leq m^{t^4}(t\sqrt{t^2-1})^m$$

by (8). ∎

Now we continue with $m$ which is not a multiple of $2t(t^2-1)$.

Lemma 8: *Let $m = 2qt(t^2-1) + r$, $0 < r < 2t(t^2-1)$, and let $\mu$ be a partition of $m$ with $wt\,\mu \leq 1$. Then there exists a polynomial $\psi(m)$ of degree at most $t^4 + 2t^3 - 2t + 1$ such that*

$$d_\mu \leq \psi(m)(t\sqrt{t^2-1})^m.$$

*Proof.* First let $wt\,\mu$ be even. Then $wt\,\mu \leq 0$ and we glue $m-r < 2t(t^2-1)$ boxes to the first row of $D_\mu$. As a result we obtain a partition $\nu$ of $m_0 \leq m+2t(t^2-1)$ with a non-negative weight and $m_0$ is a mutiple of $2t(t^2-1)$. By Lemma 7 we have

$$d_\nu < m_0^{t^4}(t\sqrt{t^2-1})^{m_0} \leq (m+2t(t^2-1))^{m_0-m}(t\sqrt{t^2-1})^m.$$

Appplying Lemma 2 several times we also obtain

$$d_\mu \leq m_0^{m_0-m} d_\nu \leq \varphi(m)(t\sqrt{t^2-1})^m$$

where

$$\varphi(m) = (t\sqrt{t^2-1})^{2t(t^2-1)}(m+2t(t^2-1))^{t^4+2t^3-2t}.$$

Obviously, the same arguments work if $wt\,\mu \leq 0$ while $m$ is odd. Finally, let $wt\,\mu = 1$. We glue one extra box to the first row of $D_\mu$ and get a partition $\mu'$ of $m+1$ with even $m+1$ and $wt\,\mu' = 0$. According to the previous conclusion, $d_{\mu'} \leq \varphi(m+1)(t\sqrt{t^2-1})^{m+1}$ and hence by Lemma 2

$$d_\mu \leq \psi(m)(t\sqrt{t^2-1})^m$$

where $\psi(m) = t\sqrt{t^2-1}(m+1)\varphi(m+1)$. ∎

Now we restrict partial graded codimensions $c_{k,n-k}(H(M_{t|t}, \mathrm{trp}))$.

Lemma 9: *Let $P(t)$ be an algebra $H(M_{t|t},\mathrm{trp})$. Then*

$$c_{k,n-k}(P(t)) \le \alpha(n)t^{2k}(t\sqrt{t^2-1})^{n-k}$$

*for all $k = 0,\dots,n$ and for some polynomial $\alpha(n)$ of degree at most $t^7$.*

*Proof.* The total number of summands in the right-hand side of (7) does not exceed $t^2(n+1)^{t^4+t^2+1}$ by Proposition 1. By Lemma 4, Lemma 7 and Lemma 8,

$$d_\lambda \le n^{t^4}n^{2k}, \quad d_\mu \le \psi(n-k)(t\sqrt{t^2-1})^{n-k}, \quad \deg\psi \le t^4+2t^3-2t+1.$$

Hence

$$c_{k,n-k}(P(t)) \le \alpha(n)t^{2k}(t\sqrt{t^2-1})^{n-k},$$

where $\alpha(n)$ is a polynomial of degree at most

$$t^4+t^2+1+t^4+t^4+2t^3-2t+1 = 3t^4+2t^3+t^2-2t+1 < t^7$$

for all $t \ge 2$. ∎

Theorem 1: *The graded PI-exponent of a simple Jordan superalgebra $H(M_{t|t},\mathrm{trp})$ exists and is less than or equal to $t^2+t\sqrt{t^2-1}$. In particular, $\exp^{\mathrm{gr}}(H(M_{t|t},\mathrm{trp}))$ is strictly less than $2t^2 = \dim H(M_{t|t},\mathrm{trp})$.*

*Proof.* The existence of $\exp(H(M_{t|t},\mathrm{trp}))$ has been proved earlier [26]. The required inequality follows immediately from Lemma 8 since

$$\begin{aligned} c_n^{\mathrm{gr}}(H(M_{t|t},\mathrm{trp})) &= \sum_{k=0}^{n}\binom{n}{k}c_{k,n-k}(H(M_{t|t},\mathrm{trp})) \\ &\le \alpha(n)\sum_{n=0}^{k}\binom{n}{k}t^{2k}(t\sqrt{t^2-1})^{n-k} \\ &= \alpha(n)(t^2+t\sqrt{t^2-1})^n. \end{aligned}$$ ∎

Taking into account the inequality (1) and the results of [14] we get an upper bound for the ordinary PI-exponent.

Theorem 2: *The ordinary PI-exponent of the algebra $H(M_{t|t},\mathrm{trp})$ exists and*

$$\exp(H(M_{t|t},\mathrm{trp})) \le t^2+t\sqrt{t^2-1}.$$

*In particular, $\exp(H(M_{t|t},\mathrm{trp}))$ is strictly less than $2t^2 = \dim H(M_{t|t},\mathrm{trp})$.* ∎

## 5. The graded PI-exponent of algebra $H(M_{2|2}, \mathrm{trp})$

Futhermore we will not use associative multiplication. This allows us to omit the super-Jordan circle, i.e., to write $ab$ instead of $\{a, b\}$. We will also use the notation $ab \cdots c$ for the left-normed product

$$\{\{\cdots\{a, b\}\cdots\}, c\}.$$

We will also use the following agreement for denoting alternating sets of variables. If we apply to a multilinear polynomial

$$f = f(x_1, \dots, x_m, y_1, \dots, y_k)$$

the operator of alternation on variables $x_1, \dots, x_m$, then we shall put the same symbol (bar, double bar, tilde, double tilde, etc.) over variables $x_1, \dots, x_m$, that is

$$f = f(\bar{x}_1, \dots, \bar{x}_m, y_1, \dots, y_k) = \sum_{\sigma \in S_m} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(m)}, y_1, \dots, y_k).$$

For example, $\bar{x}a\bar{y} = xay - yax$, or

$$\begin{aligned} \bar{x}a\bar{y}\bar{\bar{z}}\bar{\bar{t}} &= xay\bar{\bar{z}}\bar{\bar{t}} - yax\bar{\bar{z}}\bar{\bar{t}} \\ &= xayzt - xaytz - yaxzt + yaxtz. \end{aligned}$$

We shall also use the same notation for non-multilinear polynomials with repeating variables as follows:

$$\begin{aligned} \bar{x}_1\bar{x}_2a\bar{\bar{x}}_1\bar{\bar{x}}_2 &= x_1x_2a\bar{\bar{x}}_1\bar{\bar{x}}_2 - x_2x_1a\bar{\bar{x}}_1\bar{\bar{x}}_2 \\ &= x_1x_2ax_1x_2 - x_1x_2ax_2x_1 - x_2x_1ax_1x_2 + x_2x_1ax_2x_1. \end{aligned}$$

Fix the following basis of $P(2) = H(M_{2|2}, \mathrm{trp})$:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

for even part $P(2)_0$, and

$$Y_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

for odd part $P(2)_1$.

Under our agreement the multiplication table has the following form:

$$\begin{aligned} aI &= 2a \quad \text{for all } a \in P(2), \\ Z^2 &= Y_i^2 = X_i Z = X_i Y_i = 0, \quad i = 1, 2, 3, \\ -X_1^2 &= X_2^2 = X_3^2 = 2I, \end{aligned}$$

and

$$\begin{array}{llll} Y_1 Z = X_1, & Y_2 Z = X_2, & Y_3 Z = -X_3, & \\ X_1 Y_2 = -2Y_3, & X_1 Y_3 = 2Y_2, & X_2 Y_1 = 2Y_3, & \\ X_2 Y_3 = 2Y_1, & X_3 Y_1 = 2Y_2, & X_3 Y_2 = 2Y_1, & X_1 X_2 = X_1 X_3 = X_2 X_3 = 0. \end{array}$$

Now we compute some expressions in $H(M_{2|2}, \mathrm{trp})$. First note that

$$[[a, c], b] = a \circ (b \circ c) - (a \circ b) \circ c$$

in any associative algebra, that is any left-normed Lie commutator of odd degree is a Jordan product. If an associative algebra is equipped with $\mathbb{Z}_2$-grading, then the commutator $[a, b]$ of two odd elements is their super-Jordan product in the associated Jordan super-algebra. Hence the element

$$B = [X_1, [Y_1, Z], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3] Y_2 (ZI) \tag{11}$$

of $\mathrm{Mat}_4(F)$ lies in $H(M_{2|2}, \mathrm{trp})$. By the multiplication rule for basis elements of $H(M_{2|2}, \mathrm{trp})$ we have (if we first compute associative products in $\mathrm{Mat}_4(F)$)

$$\begin{aligned} &[X_1, \bar{X}_1, \bar{X}_2, \bar{X}_3] \\ &= [[X_1, X_1], [X_2, X_3]] - [[X_1, X_2], [X_1, X_3]] + [[X_1, X_3], [X_1, X_2]] \\ &= -2[[X_1, X_2], [X_1, X_3]] = 8[X_3, X_2] = 16X_1 \end{aligned} \tag{12}$$

and hence

$$[X_1,[\bar{Y}_1,Z],[\bar{Y}_2,Z],[\bar{Y}_3,Z]]=[X_1,\bar{X}_1,\bar{X}_2,-\bar{X}_3]=-16X_1. \tag{13}$$

Note that the expression on the left hand side of (13) is also alternating on $Y_1,Y_2,Y_3$ and the first $Z$, that is

$$[X_1,[\bar{Y}_1,\bar{Z}],[\bar{Y}_2,Z],[\bar{Y}_3,Z]]=2[X_1,[\bar{Y}_1,Z],[\bar{Y}_2,Z],[\bar{Y}_3,Z]]=-32X_1. \tag{14}$$

It is also clear that

$$[X_1,\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3]Y_2(Z\widetilde{I})=2[X_1,\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3]Y_2Z=32X_1Y_2Z. \tag{15}$$

From (14) and (15) it follows that the alternation of $B$ from (11) separately on $\{Y_1,Y_2,Y_3,Z\}$ and on $\{X_1,X_2,X_3,I\}$ gives us an expression

$$[X_1,[\bar{Y}_1,\bar{Z}],[\bar{Y}_2,Z],[\bar{Y}_3,Z],\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3]Y_2(Z\widetilde{I})=-2^{10}X_1Y_2Z.$$

Generalizing this construction we obtain

$$\begin{aligned} W=[X_1,&\underbrace{[\bar{Y}_1,\bar{Z}],[\bar{Y}_2,Z].[\bar{Y}_3,Z],\dots,[\bar{\bar{Y}}_1,\bar{\bar{Z}}],[\bar{\bar{Y}}_2,Z],[\bar{\bar{Y}}_3,Z]}_{r\text{ times}}, \\ &\underbrace{\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3,\dots,\widetilde{\widetilde{X}}_1,\widetilde{\widetilde{X}}_2,\widetilde{\widetilde{X}}_3}_{s\text{ times}}]Y_2(Z\underbrace{\widetilde{I}\cdots\widetilde{\widetilde{I}}}_{s\text{ times}}) \\ &=(-1)^r\alpha' X_1Y_2Z=\alpha X_3\neq 0, \end{aligned} \tag{16}$$

where $\alpha$ is a non-zero scalar.

The expression $W$ contains $r$ four-alternating sets $\{Y_1,Y_2,Y_3,Z\}$, $s$ four-alternating sets $\{X_1,X_2,X_3,I\}$ plus $2r+1$ factors $Z$ out of alternating sets, plus one $X_1$ and one $Y_2$. Besides, $W$ is a super-Jordan polynomial provided that $r+s$ is even.

Consider a super-Jordan polynomial

$$\begin{aligned} w=[x_0,[\bar{y}_1^{(1)},\bar{z}^{(1)}],[\bar{y}_2^{(1)},z_1],[\bar{y}_3^{(1)},z_2],\dots,[\bar{\bar{y}}_1^{(r)},\bar{\bar{z}}^{(r)}],[\bar{\bar{y}}_2^{(r)},z_{2r-1}],[\bar{\bar{y}}_3^{(r)},z_{2r}], \\ \widetilde{x}_1^{(1)},\widetilde{x}_2^{(1)},\widetilde{x}_3^{(1)},\dots,\widetilde{\widetilde{x}}_1^{(s)},\widetilde{\widetilde{x}}_2^{(s)},\widetilde{\widetilde{x}}_3^{(s)}]y_2(z_0\widetilde{u}_1\cdots\widetilde{\widetilde{u}}_s) \end{aligned}$$

depending on even variables $x_0,\{x_j^{(i)}\},\{u_i\}$ and odd variables $z_0,\{z_i\},\{z^{(i)}\}$.

Let $k = 4s$, $m = 6r$, and let the symmetric groups $S_k$, $S_m$ act on the sets

$$\{x_j^{(i)}, u_i | 1 \leq i \leq s, j = 1,2,3\} \quad \text{and} \quad \{y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, z^{(i)}, z_1, \dots, z_r | 1 \leq i \leq r\},$$

respectively. Relation (16) shows that $\varphi(w) = W \neq 0$ in $H(M_{2|2}, \mathrm{trp})$ under the evaluation

$$\begin{aligned} \varphi : x_0 \to X_1,\ y_1^{(j)} \to Y_1,\ y_2^{(j)} \to Y_2,\ y_3^{(j)} \to Y_3,\ z^{(j)} \to Z, \qquad & j = 1,\dots,r,\\ x_1^{(i)} \to X_1,\ x_2^{(i)} \to X_2,\ x_3^{(i)} \to X_3, \quad i = 1,\dots,s, \quad z_i \to Z, \quad & i = 0,\dots,2r,\\ u_j \to I, \qquad & i = 1,\dots,s. \end{aligned}$$

This means that the corresponding symmetrization $w$ is not a graded identity of $H(M_{2|2}, \mathrm{trp})$ and generates in $P_{4s+1,6r+2}(H(M_{2|2}, \mathrm{trp}))$ an irreducible $S_k \times S_m$-module $M$ with the character $\chi_{\lambda,\mu}$ where $\lambda = (s,s,s,s) \vdash k, \mu = (3r,r,r,r) \vdash m$. In particular,

$$\dim P_{4s+1,6r+2}(H(M_{2|2}, \mathrm{trp})) \geq d_\lambda d_\mu.$$

It is easy to see that

$$\Phi(\lambda) = 4 \quad \text{and} \quad \Phi(\mu) = (3/6)^{-3/6}(1/6)^{-3/6} = 2\sqrt{3}.$$

Then, according to (8), we have

$$d_\lambda \geq \frac{1}{(4s)^4}\Phi(\lambda)^k \geq \frac{1}{n^4}\cdot 4^k, \quad d_\mu \geq \frac{1}{(6r)^4}\Phi(\mu)^m \geq \frac{1}{n^4}(2\sqrt{3})^m.$$

Hence, we have proved the following inequality.

Lemma 10: *Let $k = 4s, m = 6r$ with even $s + r$ and let $n = k + m + 3$. Then*

$$\dim P_{k+1,m+2}(H(M_{2|2}, \mathrm{trp})) \geq \frac{1}{n^4}4^k(2\sqrt{3})^m. \qquad \blacksquare$$

One can generalize Lemma 10 to almost arbitrary $n$ and $k$.

Lemma 11: *Let $P$ be an algebra $H(M_{2|2}, \mathrm{trp})$. Then for any $n$ and $k \geq 5$ with $n - k \geq 8$ one has*

$$c_{k,n-k}(P) \geq \frac{1}{4^{15}n^4}4^k(2\sqrt{s})^{n-k}.$$

*Proof.* It is not difficult to see that for any $n$ and $k \geq 5$ with $n - k \geq 8$, one can choose an even sum $r + s$ such that $k - 1 = 4s + i$, $0 \leq i < r$, $n - k - 2 = 6r + j$, $0 \leq j \leq 5$ and

$$(17) \qquad c_{k-1-i,n-k-2-j}(P) \geq \frac{1}{n^4}4^{k-1-i}(2\sqrt{3})^{n-k-2-j} \geq \frac{1}{4^{15}}4^k(2\sqrt{3})^{n-k}$$

by Lemma 10.

Now we prove that, given $n, n', k, k'$ such that $n' \geq n$, $k' \geq k$, $n' - k' \geq n - k$, and $n-k$ is even, the graded codimension $c_{k,n-k}(P)$ does not exceed $c_{k',n'-k'}(P)$. Let

$$T = c_{k,n-k}$$

and let $f_1, \ldots, f_T$ be multilinear polynomials on $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ linearly independent modulo $\mathrm{Id}^{\mathrm{gr}}(P)$. Since $n-k$ is even, all $f_1, \ldots, f_T$ are even in $\mathbb{Z}_2$-grading. First suppose that $k' - k = m > 0$. Then the polynomials

$$f_i' = f_i x_1' \cdots x_m' \quad 1 \leq i \leq T, \quad \text{with even } x_1' \ldots, x_m'$$

are even and linearly independent modulo $\mathrm{Id}^{\mathrm{gr}}(P)$. Indeed, if

$$\alpha_1 f_1' + \cdots + \alpha_T f_T' \equiv 0,$$

then

$$\varphi(\alpha_1 f_1 + \cdots + \alpha_T f_T) \underbrace{I \cdots I}_{m \text{ times}} = \varphi(\alpha_1 f_1 + \cdots + \alpha_T f_T) = 0$$

for any evaluation $\varphi$ of indeterminates $x_1, \ldots, y_{n-k}$ in $P$. This means that $\alpha_1 f_1 + \cdots + \alpha_T f_T \equiv 0$ and $\alpha_1 = \cdots = \alpha_T = 0$.

The previous remark allows us to reduce the proof to the case where $k' = k$. Let now $(n' - k') - (n - k) = m > 0$. If $m \geq 2$, then consider the polynomials

$$f_i' = f_i y_1' y_2', \quad 1 \leq i \leq T,$$

with odd $y_1', y_2'$. As before, the relation $\alpha_1 f_1' + \cdots + \alpha_Y f_T' \equiv 0$ implies equalities

$$\varphi(\alpha_1 f_1 + \cdots + \alpha_T f_T) Y_i Z = 0, \quad 1 \leq i \leq 3,$$

for any evaluation $\varphi$ where $Y_i, Z$ are basis elements of the odd part of $P(2)$ that is possible only if $\alpha_1 = \cdots = \alpha_T = 0$.

Finally, let $m = 1$. Similar arguments as before show that $f_1 y_1', \ldots, f_T y_1'$ are linearly independent modulo $\mathrm{Id}^{\mathrm{gr}}(P)$. Now we refer to (17) and complete the proof of the lemma. ∎

We are ready to prove the main result of this Section.

THEOREM 3: *Let $P$ be a simple Jordan algebra of the type $H(M_{2|2}, \mathrm{trp})$. Then the graded PI-exponent $\exp^{\mathrm{gr}}(P)$ has a fractional value. Moreover,*

$$\exp^{\mathrm{gr}}(P) = 4 + 2\sqrt{3}.$$

*Proof.* By Lemma 11 we have

$$\begin{aligned} c_n^{\mathrm{gr}}(P) &= \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(P) \geq \frac{1}{4^{15}n^4} \sum_{k=5}^{n-8} \binom{n}{k} 4^k (2\sqrt{3})^{n-k} \\ &= \frac{1}{4^{15}n^4} \sum_{k=0}^{n} \binom{n}{k} 4^k (2\sqrt{3})^{n-k} - B \\ &= \frac{1}{4^{15}n^4} (4+2\sqrt{3})^n - B, \end{aligned}$$

where

$$B = \frac{1}{4^{15}n^4} \sum_{k=0}^{4} \binom{n}{k} c_{k,n-k}(P) + \frac{1}{4^{15}n^4} \sum_{k=n-7}^{n} \binom{n}{k} c_{k,n-k}(P).$$

By Lemma 9,

$$c_{k,n-k}(P) \leq \alpha(n) 4^k (2\sqrt{3})^{n-k} \leq \alpha(n) 4^n$$

for some polynomial $\alpha(n)$ of degree $\deg \alpha \leq 7$. Therefore $B \leq \beta(n) n^4$ where

$$\beta(n) = \frac{2\alpha(n)}{4^{15}n^4} \left( \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{7} \right),$$

a polynomial of degree at most 10. Hence

$$c_n^{\mathrm{gr}}(P) \geq (4+2\sqrt{3})^n - \beta(n) n^4. \tag{18}$$

Relation (18) implies that for any $\varepsilon > 0$,

$$c_n^{\mathrm{gr}}(P) \geq (4+2\sqrt{3}-\varepsilon)^n$$

for $n$ large enough and

$$\liminf_{n\to\infty} \geq 4+2\sqrt{3}.$$

Since

$$\limsup_{n\to\infty} \leq 4+2\sqrt{3}$$

by Theorem 1, we complete the proof. ∎

## 6. Lower bound for non-graded PI-exponent of $H(M_{2|2}, \mathrm{trp})$

For computing the lower bound of non-graded codimensions we need to modify substantially the construction of multialternating polynomials from the previous Section. Denote

$$B_1 = [X_2, [Y_1, Z], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3], \quad C_1 = ZI,$$

and set

$$B_2 = [B_1, [Y_1, Z], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3], \quad D_1 = \mathrm{Alt}_1\{Y_2 C_1 B_2\},$$

where $\mathrm{Alt}_1$ is the alternation of the product $D_1$ on $Y_1, Y_2, Y_3, X_1, X_2, X_3$ from $B_1$, $I$ from $C_1$ and $Z$ from the commutator $[Y_1, Z]$ in $B_2$. That is, $D_1$ is the expression

$$D_1 = Y_2(Z\widetilde{I})[X_2, [\widetilde{Y}_1, Z], [\widetilde{Y}_2, Z], [\widetilde{Y}_3, Z], \widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \\ [Y_1, \widetilde{Z}], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3]. \tag{19}$$

For computing the value of $D_1$ we first note that if $Z$ comes to $B_1$ instead of $Y_1, Y_2$ or $Y_3$, then we get 0. Similarly, we cannot replace $I$ in $C_1$ with $Z$. Second, we cannot take out $Y_1, Y_2, Y_3$ from $B_1$. Hence the right-hand side of (19) is equal to

$$Y_2(Z\widetilde{\widetilde{I}})[X_2, [\widetilde{Y}_1, Z], [\widetilde{Y}_2, Z], [\widetilde{Y}_3, Z], \widetilde{\widetilde{X}}_1, \widetilde{\widetilde{X}}_2, \widetilde{\widetilde{X}}_3, \\ [Y_1, \widetilde{\widetilde{Z}}], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3]. \tag{20}$$

From the multiplication rules for $H(M_{2|2}, \mathrm{trp})$ it follows that the product (20) up to a scalar factor is equal to

$$Y_2(ZI)[X_2, \widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{\widetilde{X}}_1, \widetilde{\widetilde{X}}_2, \widetilde{\widetilde{X}}_3, X_1, X_2, X_3, X_1, X_2, X_3].$$

Like in (12) one can see that

$$[X_2, \widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3] \approx X_2,$$

where we write for brevity $a \approx b$ if $a = \alpha b$ for some non-zero scalar $\alpha$. Since $[X_2, X_1, X_3] \approx X_2$, we have as a result

$$D_1 \approx Y_2 Z X_2 \approx I. \tag{21}$$

Then we generalize the construction of $D_1$ by setting, for $k \geq 2$,

$$C_k = C_{k-1} I, \quad B_{k+1} = [B_k, [Y_1, Z], [Y_2, Z], [Y_3, Z], X_1, X_2, X_3]$$

and

$$D_k = \mathrm{Alt}_k\{Y_2 C_k B_{k+1}\},$$

where $\mathrm{Alt}_k$ is the alternation on $Y_1, Y_2, Y_3, X_1, X_2, X_3$ from $B_k$, $I$ from $C_k$ and $Z$ from $[Y_1, Z]$ in $B_{k+1}$. Applying the same arguments as in the proof of (21) we have

$$D_k \approx I. \tag{22}$$

Now we associate with $D_k$ a multilinear polynomial of degree $10k+11$ as follows:

$$\begin{aligned}
b_1 &= [x_2^{(0)}, [y_1^{(1)}, z_1^{(0)}], [y_2^{(1)}, z_2^{(0)}], [y_3^{(1)}, z_3^{(0)}], x_1^{(1)}, x_2^{(1)}, x_3^{(1)}],\\
c_0 &= z_0^{(0)}, \quad c_1 = c_0 u^{(1)},\\
b_2 &= [b_1, [y_1^{(2)}, z_1^{(1)}], [y_2^{(2)}, z_2^{(1)}], [y_3^{(2)}, z_3^{(1)}], x_1^{(2)}, x_2^{(2)}, x_3^{(2)}],\\
d_1 &= \mathrm{Alt}_1\{y_2^{(0)}(c_0 u^{(1)}) b_2\}
\end{aligned}$$

where all $x_j^{(i)}$ are even variables whereas all $z_j^{(i)}, y_j^{(i)}$ are odd and $\mathrm{Alt}_1$ is the alternation on the set

$$\{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, u^{(1)}, z_1^{(1)}\}.$$

Starting from this expression we set, for all $k \geq 2$,

$$b_{k+1} = [b_k, [y_1^{(k+1)}, z_1^{(k)}], [y_2^{(k+1)}, z_2^{(k)}], [y_3^{(k+1)}, z_3^{(k)}], x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}],$$

and

$$c_k = c_{k-1} u^{(k)}, \quad d_k = \mathrm{Alt}_k\{y_2^{(0)}(c_{k-1} u^{(k)}) b_{k+1}\},$$

where $\mathrm{Alt}_k$ is the alternation on the set

$$\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, y_1^{(k)}, y_2^{(k)}, y_3^{(k)}, u^{(k)}, z_1^{(k)}\}.$$

We split all variables in $d_k$ to two parts. The first part of order $10k$ consists of $k$ eight-alternating sets

$$\{x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, y_1^{(j)}, y_2^{(j)}, y_3^{(j)}, u^{(j)}, z_1^{(j)} | 1 \leq j \leq k\}$$

plus $2k$ extra variables $z_2^{(j)}, z_3^{(j)}, j = 0, \ldots, k-1$. The second one contains the remaining eleven variables

$$x_0^{(0)}, z_0^{(0)}, y_2^{(0)}, y_1^{(k+1)}, y_2^{(k+1)}, y_3^{(k+1)}, x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}, z_2^{(k)}, z_3^{(k)}.$$

We may consider the subspace $P_n$ of multilinear polynomials as an $FS_m$-module where $m = 10k$ and the symmetric group $S_m$ acts on $10k$ indeterminates from the first part. Denote

$$f_k = \mathrm{SYMM}(d_k),$$

where SYMM is the action of eight symmetrizations of variables, namely:

- three on $\{y_i^{(1)}, \ldots, y_i^{(k)}\}$, $i = 1, 2, 3$;
- three on $\{x_i^{(1)}, \ldots, x_i^{(k)}\}$, $i = 1, 2, 3$;
- one on $\{u^{(1)}, \ldots, u^{(k)}\}$;
  and
- one on $\{z_1^{(1)}, z_2^{(1)}, z_3^{(1)}, \ldots, z_1^{(k)}, z_2^{(k)}, z_3^{(k)}\}$.

Computation of the value of $D_k$ (see (22)) shows that the polynomial $f_k$ takes a non-zero value in $H(M_{2|2}, \mathrm{trp})$ under evaluation

$$\varphi(y_j^{(i)}) = Y_j, \quad \varphi(x_j^{(i)}) = X_j, \quad \varphi(z_j^{(i)}) = Z, \quad \varphi(u^{(i)}) = I.$$

On the other hand, $f_k$ generates an irreducible $FS_m$-module with the character $\chi_\lambda$ where

$$\lambda = (3k, \underbrace{k, \ldots, k}_{7 \text{ times}}),$$

as follows from the structure of the essential idempotent $e_{T_\lambda}$ (see (3)). Since $f_k$ is not an identity of $H(M_{2|2}, \mathrm{trp})$, it follows that

(23)
$$c_n(H(M_{2|2}, \mathrm{trp})) \geq d_\lambda \geq \frac{1}{m^2}\Phi(\lambda)^m \geq \frac{1}{(n-11)^8}\Phi(\lambda)^{n-11} \geq 8^{-11} n^{-8} \Phi(\lambda)^n$$

according to (8) since $\Phi(\lambda) \leq 8$. The next lemma is a direct corollary of (23) and the equality

$$\Phi(\lambda) = \frac{1}{(\frac{3}{10})^{\frac{3}{10}} (\frac{7}{10})^{\frac{7}{10}}} = 10 \cdot 3^{-\frac{3}{10}}.$$

Lemma 12: *Let $P$ be the Jordan superalgebra $H(M_{2|2}, \mathrm{trp})$ and let $n = 10k + 11$. Then*

$$c_n(P) \geq 8^{-11} n^{-2} \left(\frac{10}{3^{\frac{3}{10}}}\right)^n. \qquad \blacksquare$$

Combining all previous estimations we get the lower and upper bounds for the ordinary PI-exponent of $H(M_{2|2}, \mathrm{trp})$.

Theorem 4: *The PI-exponent of the simple Jordan superalgebra* $H(M_{2|2}, \mathrm{trp})$ *is strictly less than the dimension of* $H(M_{2|2}, \mathrm{trp})$. *Moreover,* $\exp(H(M_{2|2}, \mathrm{trp}))$ *is not an integer and*

$$7.19 < 10 \cdot 3^{-\frac{3}{10}} \leq \exp(H(M_{2|2}, \mathrm{trp})) \leq 4 + 2\sqrt{3} < 7.47.$$

*Proof.* The upper bound was obtained in Theorem 2. To obtain the lower restriction we apply Lemma 12. If $n - 11$ is a multiple of 10 then

$$(c_n(H(M_{2|2}, \mathrm{trp})))^{\frac{1}{n}} \geq (8^{-11} n^{-8})^{\frac{1}{n}} \cdot 10 \cdot 3^{-\frac{3}{2}},$$

hence

$$\liminf_{k\to\infty} (c_{10k+11}(H(M_{2|2}, \mathrm{trp}))^{\frac{1}{n}}) \geq 10 \cdot 3^{-\frac{3}{10}}.$$

For all the other $n$ we may use the inequality

$$c_{n+1}(H(M_{2|2}, \mathrm{trp})) \geq c_n(H(M_{2|2}, \mathrm{trp}))$$

that holds since $H(M_{2|2}, \mathrm{trp})$ is a unital algebra. Clearly, if $\{\alpha_n\}$ is a non-decreasing sequence and $\alpha_n \geq \gamma a^n$ for some $n$, and for some fixed $a > 1$, then $\alpha_{n+j} \geq \frac{\gamma}{a^{10}} a^{n+j}$ provided that $1 \leq j \leq 10$. ∎

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