Integral Representations of Cyclic p-Groups*

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1. INTRODUCTION

The representations of the cyclic group of order p^2, for p prime, by matrices over the integers, were classified by Reiner in [3]. This was done by giving a complete list of all the indecomposable lattices over the group ring ZG, and a set of invariants which characterizes the isomorphism class of every direct sum of indecomposables. The methods used by Reiner are extended here in order to give some of the integral representations of the cyclic group G of order p^n for any n.

For all unexplained terminology, we refer the reader to [2].

If ζ is a primitive root of 1 of order p^n, we write $\zeta_i = \zeta^{p^{n-i}}$ and $R_i = Z[\zeta_i]$ for any $1 \leq i \leq n$. Considering a fixed pair i,j, with $1 \leq i < j \leq n$, we classify the isomorphism classes of the ZG-lattices M such that the simple components of the QG-lattice QM are isomorphic to $Q, Q(\zeta_i)$, or $Q(\zeta_j)$. In particular, for $n = 2$ these lattices furnish all the integral representations of the cyclic group of order p^2. Then we describe the isomorphism classes of the direct sums of the lattices M obtained as above, for different pairs i,j. The proofs of these results are too long to be given in this note; they will appear elsewhere.

Let G_j be the cyclic group of order p^j, and Φ_j the cyclotomic polynomial with root ζ_j as above. There is an epimorphism:

$$ZG_j \simeq Z[X]/(X^{p^j} - 1) \rightarrow Z(X]/(\Phi_j) \simeq R_j.$$

Thus every torsion-free R_j-module is a lattice over ZG_j and hence over ZG, through the epimorphism $G \rightarrow G_j$. On the other hand, for every ZG_j lattice M, if a is the generator of G_j then the submodule L of M annihilated by a^{p^i}-1 is a ZG_i-lattice. If M is a lattice of the type we want to study, then $N = M/L$ is annihilated by $\Phi_j(a)$ and hence is an R_j-module. Further, N is torsion-free over R_j. It is known (see [3] and [4]) that:

$$\mathrm{Ext}^1_{ZG}(N,L) \simeq \bar{L} = L/pL.$$

Because QL and QN have no common composition factors, two extensions of N by L are isomorphic if and only if one is obtained from the other by the action of an automorphism γ of L, on the left, and an automorphism δ of N on the right. The actions of γ and δ on $\mathrm{Ext}^1_{ZG}(N,L)$ are given by applying the functors $\mathrm{Ext}^1_{ZG}(N,\cdot)$ and $\mathrm{Ext}^1_{ZG}(\cdot,L)$ to γ and δ. The ZG_j-lattice M is then determined by the ZG_i-sublattice L, the R_j-lattice N, and an element $u \in \bar{L}$, which represents an orbit of $\mathrm{Ext}^1_{ZG}(N,L)$ under these actions. Such a lattice we indicate, as in [3], with the notation (L,N,u).

If $N = R_j$ and L is one of the ZG_i-lattices:

$$Z, R_i, E_i = Z[X]/(X-1)(\Phi_i),$$

then there is a ring epimorphism $R_j \rightarrow \bar{L}$, and the group of automorphisms of R_j acts on $\bar{L}$ through this epimorphism. Since $U(R_j)$ maps onto $U(\bar{Z})$, there are only two extensions of R_j by Z. The unique non-split extension is $E_j = (Z, R_j, 1)$. The same holds if R_j is replaced by an ideal B of R_j. We then use the notation $E(B) = (Z,B,1)$.

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If ϕ is the Euler function and $\lambda = X - 1$, then letting bars denote lattices modulo p, we have:

$$\bar{R}_i = \bar{Z}[\lambda]/(\lambda)^{\phi(p^i)}, \bar{E}_i = \bar{Z}[\lambda]/(\lambda)^{\phi(p^i)+1}.$$

Every element of any of these rings can then clearly be written as the product of a unit by a power of the residue of λ.

It is well known (see [2]) that a ZG-lattice M is indecomposable if and only if the lattice I_pM is indecomposable over the p-adic group ring I_pG. A lattice M′ is said to be locally isomorphic to M, or to belong to the same genus as M, when $I_pM' \simeq I_pM$ as I_pG-lattices. The ZG-lattices considered in theorems 1 and 2 are determined by first finding all their genera and then determining the isomorphism classes in each genus.

2. INDECOMPOSABLE LATTICES

Consider now a fixed pair i,j where $1 \le i < j \le n$. Using the preceding notations, we list the following types of indecomposable ZG-lattices. With the methods of [3] it can be shown (see also [1]) that this list contains one representative for each genus of the set of all the indecomposable ZG-lattices M, such that the composition factors of QM are isomorphic to Q, $Q(\zeta_i)$ or $Q(\zeta_j)$.

1.º Z; 2.º R_i, R_j; 3.º E_i, E_j; 4.º(R_i, R_j, λ^r), $r = 0, \ldots, \phi(p^i) - 1$; 5.º (E_i, R_j, λ^r), $r = 0, \ldots, \phi(p^i)$; 6.º $(Z \oplus R_i, R_j, 1 + \lambda^r)$, $r = 0, \ldots, \phi(p^i) - 1$; 7.º if $p^i \neq 2$, $(Z \oplus E_i, R_j, 1 + \lambda^r)$, $r = 1, \ldots, \phi(p^i) - 1$.

In order to determine the isomorphism classes of these indecomposable ZG-lattices some further notation is needed.

For each r, $0 \le r < \phi(p^i)$, let $t = \phi(p^i) + - r$ and consider the composite

$$\chi: R_i \to \bar{R}_i \to \bar{Z}[\lambda]/(\lambda)^t.$$

The action of R_j on $\bar{R}_i$ gives another epimorphism θ from R_j to $\bar{Z}[\lambda]/(\lambda)^t$. Denote:

$$U^*(R_i) = \chi(U(R_i)), U^*(R_j) = \theta(U(R_j)).$$

The norm from $Q(\zeta_j)$ to $Q(\zeta_i)$ determines a map $N: R_j \to R_i$. It can be easily shown that $\chi = \theta N$, hence $U^*(R_j \subset U^*(R_i)$. For $t = 1, \ldots, \phi(p^i)$ we write:

$$\bar{U}_t = U(\bar{Z}[\lambda]/(\lambda)^t), \; V_t = \bar{U}_t/U^*(R_i).$$

Let now $\tilde{U}(\bar{R}_i)_r$ be a set of representatives $u \in \bar{R}_i$ of V_t with $t = \phi(p^i) - r$. For each ideal A of R_i and each ideal B of R_j, such a u defines a ZG-lattices (A,B,u) because $\bar{A} = \bar{R}_i$.

Now for $0 \le r \le \phi(p^i)$, let $t = \phi(p^i) + + 1 - r$ and consider:

$$\pi: E_i \to \bar{E}_i \to \bar{Z}[\lambda]/(\lambda)^t.$$

If $j > i$ we also have a map ν from R_j to $\bar{Z}[\lambda]/(\lambda)^t$. Let

$$U^*(E_i) = \pi(U(E_i)), \; U^*(R_j) = \nu(U(R_j)).$$

For $t = 1, \ldots, \phi(p^i) + 1$ consider $\bar{U}_t$ as before and

$$U_t = \bar{U}_t/U^*(E_i)U^*(R_j).$$

We remark that it can be shown that $U_t = V_t$ for $t = 1, \ldots \phi(p^i)$.

Let $\tilde{U}(\bar{E}_i)_r$ be a set of representatives $u \in \bar{E}_i$ of U_t for $t = \phi(p^i) + 1 - r$. Each such u determines the extension (E(A),B,u) because $\overline{E(A)} = \bar{E}_i$. With similar notations we denote extensions of B $Z \oplus A$ and by $Z \oplus E(A)$.

Let q be a fixed integer which is a quadratic non-residue modulo p.

By determining the isomorphism classes in each of the genera of the first list, we obtain the indecomposable ZG-lattices of Theorem 1.

THEOREM 1 – *Every indecomposable ZG-lattice with components* Q, $Q(\zeta_i)$, $Q(\zeta_j)$ *over* QG *is isomorphic to one of the following, where* A *runs through the ideal classes of* R_i *and* B *through the ideal classes of* R_j, *and these indecomposables are non-isomorphic.*

1.º Z; 2.º A,B; 3.º E(A), E(B);
4.º $(A,B,\lambda^r u)$, $r = 0, \ldots, \phi(p^i) - 1$, $u \in \tilde{U}(\bar{R}_i)_r$;
5.º $(E(A),B,\lambda^r u)$, $r = 0, \ldots, \phi(p^i)$, $u \in \tilde{U}(\bar{E}_i)_r$;
6.º $(Z \oplus A,B,1 + \lambda^r u)$, $r = 0, \ldots, \phi(p^i) - 1$, $u \in \tilde{U}(\bar{R}_i)_r$;

7.º *if* $p^i \neq 2$: $(Z \oplus E(A),B,1 + \lambda^r u)$, $r = 1,\ldots,\phi(p^i) - 1$, $u \in \tilde{U}(\bar{E}_i)_{r+1}$;

7*.º *if* $p = 1 (\mathrm{mod}\ 4)$, $(Z \oplus E(A),B,1 + \lambda^r uq)$, $r = 1,\ldots,\phi(p^i) - 1$, $u \in \tilde{U}(\bar{E}_i)_{r+1}$.

3. DIRECT SUMS OF INDECOMPOSABLES

In this section we consider a direct sum S of indecomposable ZG-lattices expressed as in Theorem 1. We denote by $A_i(S)$ the product of all ideals of R_i appearing in the summands of S, and by $A_j(S)$ the product of all ideals of R_j. Let r_1 be the maximum exponent of λ in the summands of type 5 of S, r_2 the maximum exponent of λ coming from summands of the other types, and $r(S) = \max(r_1 - 1, r_2)$. We indicate with u(S) the product of the images in U_t, $t = \phi(p^i) - r(S)$, of all the u's and q's of the summands of types 4 to 7 which form S.

THEOREM 2 – *Let* S *be a direct sum of indecomposable* ZG-*lattices of the types given in Theorem* 1. *The isomorphism class of* S *determines the following invariants.*

I. *The genera of the indecomposables which form* S.

II. *The ideal class of* $A_i(S)$ *in* R_i *and the ideal class of* $A_j(S)$ *in* R_j.

If S *satisfies the following conditions then the isomorphism class of the sum is determined by* I *and* II,

a – S *has a summand of type* 2 *or of type* 3.

b – *Either* $p \neq 1$ (mod 4, *or* $p = 1$ (mod 4) *and among the summands which form* S *there is one of type* 1, 3, 5 *or* 6.

If S *has no summands of type* 2 *nor* 3, *then* u(S) *is invariant under isomorphisms, while if* b *is not satisfied, then* IV *is invariant. In these cases* III *and* IV, *respectively, must be added to* I *and* II, *in order to determine* S.

III. *The element* u(S).

IV. *The parity of the number of summands of type* 7* *which form* S.

We now consider a sum S of modules S_{ij} for various pairs i,j, where each S_{ij} is a sum of indecomposables as described in theorem 2. The characterization of the isomorphism class of S becomes manageable if we reduce the problem to the cases where it is a sum of two such modules S_{ij}.

Let S be a lattice of the form $S = S_{ij} \oplus S_{k\ell}$, with i,j,k,ℓ all different.

If both S_{ij} and $S_{k\ell}$ satisfy a, and if the whole sum S satisfies b of theorem 2, then the invariants which determine the isomorphism class of S are I: the local types of the indecomposable summands and II: the ideal classes of $A_i(S)$, $A_j(S)$, $A_k(S)$ and $A_\ell(S)$. When these conditions are not satisfied, the following invariants must be respectively given: $u(S_{ij})$, $u(S_{k\ell})$ and the invariant IV of S.

Consider now sums of the form $S = S_{ij} \oplus S_{i\ell}$ where $j \neq \ell$. If S satisfies the conditions a and b of Theorem 2, then this sum is determined by I: the local types of the indecomposable summands and II: the ideal classes of $A_i(S)$, $A_j(S)$ and $A_\ell(S)$. If S does not satisfy a then the following invariant must be considered: III: the element $u(S) \in U_t$, with t and u(S) defined as above, but taking into account all the indecomposables of both S_{ij} and $S_{i\ell}$. If S does not satisfy b then the invariant IV of S must be given.

Now let $S = S_{ij} \oplus S_{kj}$ with $i < k$.
Take $U_t = \bar{U}_t/U^*(R_i)$, where

$$t = \min(\phi(p^i) - r(S_{ij}), \phi(p^k) - r(S_{kj})).$$

Let u(S) be the product of the images in U_t of all q's and all u's coming from the indecomposable summands of S. With this u(S) the result can be extended in the same way as in the preceding case.

These methods can still be pushed further and applied to determine invariants for sums of the form $S = S_{ij} \oplus S_{j\ell}$. The isomorphisms between two of these sums are not expressed as simply as in the other cases, because in general this S is an extension of lattices N by L, such that QN and QL have a common composition factor. But the result is simple to state when the sum S is, as in [4], formed by ideals of R_i,

R_j and R_ℓ (that is, indecomposables of type 2) and extensions of ideals by ideals (type 4).

In this case, if the sum includes indecomposables of type 2, then S is determined by I: the local types of indecomposable summands and II: the ideal classes of $A_j(S)$, $A_i(S)$ and $A_\ell(S)$. Otherwise S is determined by I, II and the invariant u(S) defined as follows. Let

$$t = \min(\phi(p^i) - r_2(S_{ij}), \phi(p^j) - r_2(S_{j\ell})).$$

Take u(S) to be the product in U_t of the images of the u's coming from S_{ij} and the inverses of the u's coming from $S_{j\ell}$.

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5. REFERENCES

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