POLYNOMIAL SLOW-FAST SYSTEMS ON THE POINCARÉ–LYAPUNOV SPHERE

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Abstract. The main goal of this paper is to study compactifications of polynomial slow-fast systems. More precisely, the aim is to give conditions in order to guarantee normal hyperbolicity at infinity of the Poincaré–Lyapunov sphere for slow-fast systems defined in $\mathbb{R}^n$. For the planar case, we prove a global version of the Fenichel Theorem, which assures the persistence of invariant manifolds in the whole Poincaré–Lyapunov disk. We also discuss the occurrence of non normally hyperbolic points at infinity, namely: fold, transcritical and pitchfork singularities.

1. Introduction

Slow-fast systems are well-known in the literature due to their vast importance in applied sciences. For instance, the *van der Pol system* [25] was introduced in order to study a vacuum tube triode circuit. Applications in biology can be found in [12] and in references therein, and we refer the book [16] for applications in other branches of sciences.

A system of ODEs of the form

$$\varepsilon\dot{x} = P(x, \mathbf{y}, \varepsilon), \qquad \dot{\mathbf{y}} = \mathbf{Q}(x, \mathbf{y}, \varepsilon), \tag{1}$$

is called *slow-fast system*, where $x \in \mathbb{R}$, $\mathbf{y} = (y_2, \dots, y_n) \in \mathbb{R}^{n-1}$, $0 < \varepsilon \ll 1$. For our purposes, it will supposed that

$$P : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad \mathbf{Q} = (Q^2, \dots, Q^n) : \mathbb{R}^{n+1} \to \mathbb{R}^{n-1}$$

are polynomial functions with respect to the $x, \mathbf{y}$ variables and analytic with respect to ε. Throughout this paper, x and $\mathbf{y}$ will be called *fast* and *slow variables*, respectively. Setting $\varepsilon = 0$ in equation (1), we obtain the so called *slow system* given by

$$0 = P(x, \mathbf{y}, 0), \qquad \dot{\mathbf{y}} = \mathbf{Q}(x, \mathbf{y}, 0), \tag{2}$$

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which is not an ODE, but it is an *algebraic differential equation* (ADE). Solutions of (2) are contained in a codimension one affine algebraic variety

$$C_0 = \Big\{(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}, \quad P(x, \mathbf{y}, 0) = 0\Big\},$$

which will be called *critical set* or *critical manifold*.

In equation (1), the dot $\cdot$ represents the derivative of $x(\tau)$ and $\mathbf{y}(\tau)$ with respect to τ. By taking $\varepsilon t = \tau$, system (1) can be written as

$$x' = P(x, \mathbf{y}, \varepsilon), \qquad \mathbf{y}' = \varepsilon \mathbf{Q}(x, \mathbf{y}, \varepsilon). \tag{3}$$

The apostrophe $'$ in (3) denotes the derivative of $x(t)$ and $\mathbf{y}(t)$ with respect to t. Setting $\varepsilon = 0$ in equation (3) we obtain

$$x' = P(x, \mathbf{y}, 0), \qquad \mathbf{y}' = 0. \tag{4}$$

which will be called *fast system*. System (4) can be seen as a system of ordinary differential equations, with $\mathbf{y} \in \mathbb{R}^n$ being a parameter and the critical set C_0 is a set of equilibrium points of (4).

Observe that systems (1) and (3) are equivalent if $\varepsilon > 0$, since they differ by time scale. The main challenge is to study systems (2) and (4) in order to obtain information of the full system (1). For this purpose, the key tool used in this paper is *Geometric Singular Perturbation Theory* (GSPT for short). Neil Fenichel's seminal work [9] assures that, under the hypothesis of *normal hyperbolicity*, compact limit sets persist for small perturbations. See Subsection 2.1 for further details.

This paper is devoted to study conditions in order to assure normal hyperbolicity near infinity. This problem was motivated by [23], in which all possible global phase portraits of quadratic planar slow-fast systems were given. In such reference, the authors conjectured a *global version* of Fenichel Theorem for quadratic planar slow-fast systems. The contribution of our paper is to give an answer of this problem for polynomial slow-fast systems in general.

The Poincaré compactification is a well known approach used in the study of global dynamics of polynomial vector fields. The main ideas were introduced in [21] by Henri Poincaré for the 2-dimensional case. We refer to [1, 5, 11, 20] for details of such technique, including the case in which the polynomial vector field is defined in $\mathbb{R}^n$.

In our study, it is considered *Poincaré–Lyapunov compactification* (PL-compactification for short) of polynomial vector fields in $\mathbb{R}^n$. The PL-compactification can be seen as a generalization of the well known Poincaré compactification technique, whose construction is very similar to the classical Poincaré compactification, in the sense that we make it quasi-homogeneous instead of homogeneous (see [5, 18]).

Such technique was utilized in several papers by Freddy Dumortier and his collaborators, for example studying *Liénard equations* near infinity (see for instance [2, 3, 4, 8]). In [17, 22] it was given all possible phase portraits in the *Poincaré–Lyapunov disk* (PL-disk for short) of polynomial vector fields having isolated singularities with quasi-homogeneous degree 4 and 5, respectively. Structural stability of quasi-homogeneous polynomial vector fields in the PL-disk was studied in [19]. Global dynamics of *Benoît system* (which is three dimensional) in the *Poincaré–Lyapunov ball* (PL-ball for short) was considered in [18]. See [5, Chapters 5 and 9] for an introduction on such method.

Let Y be a polynomial vector field and let $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}^n$ be a vector of positive integers, which will be called *weight vector*. The *Poincaré–Lyapunov compactification* Y^∞ of Y is an analytic vector field defined in a compact n-dimensional manifold called *Poincaré–Lyapunov sphere* (PL-sphere), which is denoted by $\mathbb{S}^n_\omega$ and it is homeomorphic to $\mathbb{S}^n = \{\sum_{i=1}^{n+1} z_i^2 = 1\} \subset \mathbb{R}^{n+1}$. The phase space $\mathbb{R}^n$ is identified with the northern hemisphere of $\mathbb{S}^n_\omega$, and the set $\{z_{n+1} = 0\} \subset \mathbb{S}^n_\omega$ plays the role of infinity. See subsection 2.2 for details.

In the study of global dynamics of polynomial vector fields, the PL-compactification Y^∞ is a vector field defined in $\mathbb{S}^n_\omega$, however, its global phase portrait is often sketched in the PL-ball. Throughout this paper, the n-dimensional PL-ball will be denoted by $\mathbb{B}^n_\omega$. In particular, the PL-disk will be denoted by $\mathbb{D}_\omega = \mathbb{B}^2_\omega$. The interior of $\mathbb{D}_\omega$ plays the role of $\mathbb{R}^2$, while its boundary plays the role of infinity. Analogously, the interior of the 3-dimensional PL-ball $\mathbb{B}^3_\omega$ plays the role of $\mathbb{R}^3$ and its boundary represents infinity. See figure 1.

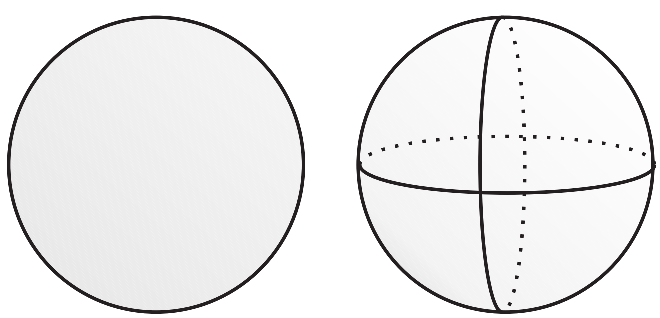

FIGURE 1. PL-disk (left) and 3-dimensional PL-ball (right).

Many interesting phenomena can occur at infinity of the phase space. For instance, consider the slow-fast system

$$x' = y^2 z - \frac{x^2 y}{2} - \frac{x^3}{3}, \quad y' = 0, \quad z' = \varepsilon(ay^3 - xy^2). \tag{5}$$

After Poincaré compactification, in one of the three charts the following system is obtained

$$u' = v - \frac{u^2}{2} - \frac{u^3}{3}, \quad v' = \varepsilon(a - x), \tag{6}$$

which is the van der Pol system studied in [6] (see also figure 2). More generally, system

$$x' = y^{k_1} z - \frac{x^2 y^{k_2}}{2} - \frac{x^3 y^{k_3}}{3}, \quad y' = 0, \quad z' = \varepsilon(a y^{k_4} - x y^{k_5}) \tag{7}$$

presents a van der Pol system at infinity after a PL-compactification with weights $\omega = (\omega_1, \omega_2, \omega_3)$ if, and only if, the positive integers $k_1, \ldots, k_5$ satisfy

$$k_1 = \frac{\delta + \omega_1 - \omega_3}{\omega_2}, \quad k_2 = \frac{\delta - \omega_1}{\omega_2}, \quad k_3 = \frac{\delta - 2\omega_1}{\omega_2}, \quad k_4 = \frac{\delta + \omega_3}{\omega_2}, \quad k_5 = \frac{\delta - \omega_1 + \omega_3}{\omega_2}.$$

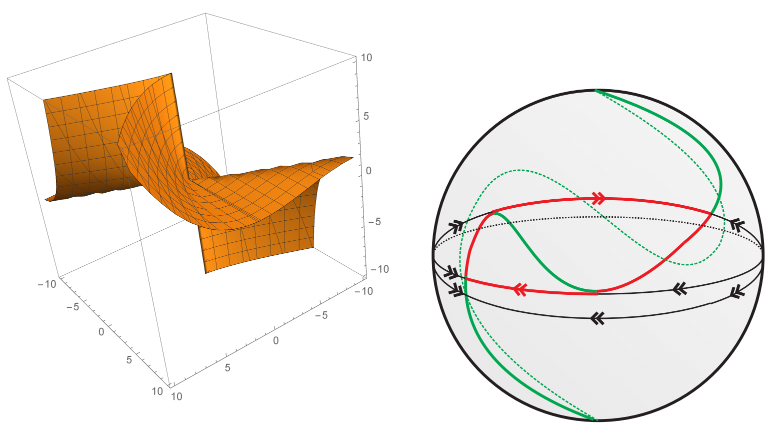

FIGURE 2. Critical manifold of slow-fast system (5) (left) and its phase portrait at infinity (right), which is given by the van der Pol equation (6). The critical manifold is highlithed in green and the canard cycle is highlighted in red (see also [6]).

Let us briefly describe our main results. A preliminary and useful result is given in Proposition 4, which discusses possible dynamics at infinity of a compactified slow-fast system based on the quasi-homogeneous degree of P and $\mathbf{Q}$. Afterwards, in Theorem 7 we state Fenichel Theorem in a suitable way in order to study the perturbed system at infinity (boundary of the PL-ball).

Theorem A establishes conditions that a polynomial slow-fast system in $\mathbb{R}^n$ must satisfy in order to assure normal hyperbolicity at infinity. More precisely, item (a) of Theorem A states an algebraic condition on the polynomial P of the initial slow-fast system so that the origin of each chart of the PL-ball is normally hyperbolic. Such condition implies that, concerning the *Newton polytope* of the slow-fast vector field (see Subsection 4.1 for a precise definition), points associated to higher order monomials are all contained in the same $(n-1)$-dimensional compact face of the polytope. We emphasize that Theorem A item (a) concerns the origin of each chart of the compactification.

On the other hand, in Theorem A item (c) is given a *necessary* condition in order to assure normal hyperbolicity outside the origin, and such condition is based on the transversal intersection of C_0 with infinity. Finally, Theorem A item (b) concerns a degenerate case, in which the *whole* infinity is a component of the critical manifold.

In dimension 2, Theorem B gives sufficient and necessary conditions to assure persistence of invariant manifolds in the *whole* PL-disk. Actually, transversality turns out to be a necessary and sufficient condition in order to assure normal hyperbolicity at infinity. Finally, Theorem C determines conditions that slow-fast systems defined in $\mathbb{R}^3$ must satisfy to generate typical singularities of planar slow-fast systems at infinity, namely fold, transcritical and pitchfork singularities.

This paper is structured as follows. In Section 2 is presented some preliminaries on GSPT and Poincaré–Lyapunov compactification. Section 3 is devoted to discuss some preliminary propositions and examples that will be used in subsequent sections. Theorem A is proven in Section 4, and Section 5 is devoted to proof a global version of Fenichel Theorem in the plane (Theorem B). Finally, in Section 6 is proven Theorem C and some examples are also given.

2. Preliminaries on geometric singular perturbation theory and Poincaré–Lyapunov compactification

2.1. **Geometric singular perturbation theory.** A point $p \in C_0$ is *normally hyperbolic* if $P_x(p) \neq 0$. The set of all normally hyperbolic points of C_0 will be denoted by $\mathcal{NH}(C_0)$. A point $p \in \mathcal{NH}(C_0)$ is called *attracting* or *repelling point* if $P_x(p) < 0$ or $P_x(p) > 0$, respectively.

Fenichel Theorem is a major result in Geometric Singular Perturbation Theory. It assures that, given a j-dimensional compact normally hyperbolic sub-manifold $\mathcal{K} \subset \mathcal{NH}(C_0)$ (possibly with boundary) of system (2), there exists a family of smooth manifolds $\mathcal{K}_\varepsilon$ such that

$\mathcal{K}_\varepsilon \to \mathcal{K}_0 = \mathcal{K}$ according to Hausdorff distance and $\mathcal{K}_\varepsilon$ is a normally hyperbolic locally invariant manifold of (1). Such result was first proved in [9] (see also [24, Theorem 2.2] for a precise statement). In Theorem 7, we stated the Fenichel Theorem in a suitable way in order to assure persistence of invariant manifolds at infinity.

The Fenichel Theorem can be seen as a "generalization" of the theorem of stable and unstable manifolds. The *local invariance* of $\mathcal{K}_\varepsilon$ means that it may exist boundaries through which trajectories can leave. Just as in center manifold theory, in general the locally invariant manifold $\mathcal{K}_\varepsilon$ is not unique. Indeed, it may exist infinitely many invariant manifolds $\mathcal{O}(e^{-\frac{K}{\varepsilon}})$-close to the critical manifold. See Figure 3. The manifold $\mathcal{K}_\varepsilon$ obtained in the Fenichel Theorem is called *slow manifold*.

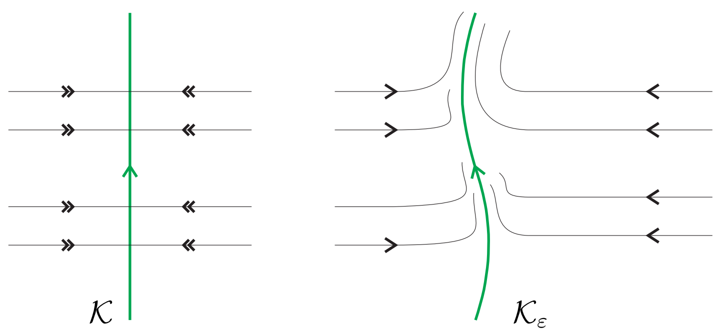

FIGURE 3. Planar slow-fast system for $\varepsilon = 0$ (left) and for $\varepsilon > 0$ sufficiently small (right). The Fenichel Theorem assures the existence of a family of invariant manifolds $\mathcal{K}_\varepsilon$, and the flow on $\mathcal{K}_\varepsilon$ converges to the flow on $\mathcal{K}$. Moreover, Fenichel Theorem also assures the existence of a family of stable manifolds $\mathcal{W}^s_\varepsilon$ of $\mathcal{K}_\varepsilon$.

Concerning the slow system (2), *"any structure in $\mathcal{NH}(C_0)$ which persists under regular perturbations persists under singular perturbation"* [9, pp. 91]. In other words, hyperbolic equilibrium points or limit cycles of (2) in $\mathcal{NH}(C_0)$ persist for ε sufficiently small.

The Fenichel Theorem gives an answer about the dynamics of system (1) near normally hyperbolic manifolds for ε sufficiently small. We refer to [6, 7, 14, 15] for further problems and techniques concerning dynamics of (1) near non-normally hyperbolic manifolds.

2.2. **Poincaré–Lyapunov compactification of polynomial vector fields.** Just as in the homogeneous compactification, the vector field Y^∞ is studied using directional charts U_i and V_i, in which

$$U_i = \{\mathbf{z} \in \mathbb{S}^n_\omega,\ z_i > 0\},\ V_i = \{\mathbf{z} \in \mathbb{S}^n_\omega,\ z_i < 0\},\ \mathbf{z} = (z_1, \dots, z_{n+1}) \in \mathbb{R}^{n+1},$$

for each $i = 1, \dots, n+1$. See Figure 4.

Consider a polynomial vector field $Y(\mathbf{x}) = Y(x_1, \ldots, x_n)$. For every $i = 1, \ldots, n$, the expression of the compactified vector field $Y^\infty(\mathbf{u}) = Y^\infty(u_1, \ldots, u_n)$ in the charts U_i is obtained from the coordinate change

$$x_1 = \frac{u_1}{u_n^{\omega_1}}, \; \ldots, \; x_{i-1} = \frac{u_{i-1}}{u_n^{\omega_{i-1}}}, \; x_i = \frac{1}{u_n^{\omega_i}}, \; x_{i+1} = \frac{u_i}{u_n^{\omega_{i+1}}}, \; \ldots, \; x_n = \frac{u_{n-1}}{u_n^{\omega_n}},$$

and for different charts U_i the coordinate system $(u_1, \ldots, u_n)$ has different meanings. However, for every $i = 1, \ldots, n$ the set $\{u_n = 0\}$ is an invariant set of Y^∞ which plays the role of infinity. On the other hand, the expression of $Y^\infty(\mathbf{u}) = Y^\infty(u_1, \ldots, u_n)$ in V_i is obtained in an analogous way as in U_i, but setting $x_i = -\frac{1}{u_n^{\omega_i}}$ instead of $x_i = \frac{1}{u_n^{\omega_i}}$. The expression of Y^∞ in U_{n+1} coincides with Y, and in V_{n+1} the expression of Y^∞ coincides with Y (up to a multiplication by -1).

3. Poincaré–Lyapunov compactification of slow-fast systems

Consider the polynomial slow-fast system (3). Recall that P and $\mathbf{Q}$ are polynomial with respect to the fast variable x and the slow variables $\mathbf{y}$, but it is analytic with respect to ε. In what follows we present the definitions of quasi-homogeneous polynomial and quasi-homogeneous vector field, which can also be found in [16, Section 7.3].

Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$ be a weight vector. A polynomial $F : \mathbb{R}^n \to \mathbb{R}$ is *quasi-homogeneous of type* ω *and degree* $k \in \mathbb{N}$ if

$$F(\lambda^{\omega_1} x, \lambda^{\omega_2} y_2, \ldots, \lambda^{\omega_n} y_n) = \lambda^k \cdot F(x, y_2, \ldots, y_n), \qquad \forall \lambda \in \mathbb{R}.$$

We say that a polynomial vector field $Y = (Y_1, \ldots, Y_n)$ defined in $\mathbb{R}^n$ is *quasi-homogeneous of type* ω *and degree* $k_\omega \in \mathbb{N}$ if each component $Y_j : \mathbb{R}^n \to \mathbb{R}$ of Y is quasi-homogeneous of type ω and degree $k + \omega_j$. In other words, it satisfies

$$Y_j(\lambda^{\omega_1} x, \lambda^{\omega_2} y_2, \ldots, \lambda^{\omega_n} y_n) = \lambda^{k+\omega_j} \cdot Y_j(x, y_2, \ldots, y_n), \qquad \forall \lambda \in \mathbb{R}.$$

Example 1. *Consider the planar polynomial vector field*

$$Y(x, y) = \big(Y_1(x, y), Y_2(x, y)\big) = (y, x^2)$$

which determines a cusp singularity at the origin. This vector field is quasi-homogeneous of type $\omega = (2, 3)$ *and degree* 1*, because*

$$Y_1(\lambda^2 x, \lambda^3 y) = \lambda^{1+2} Y_1(x, y), \qquad Y_2(\lambda^2 x, \lambda^3 y) = \lambda^{1+3} Y_2(x, y).$$

The vector field associated to the slow-fast system (3) will be denoted by X_ε, whereas its PL-compactification will be denoted by X_ε^∞, which is a vector field defined in $\mathbb{S}^n_\omega \subset \mathbb{R}^{n+1}$. We will also write the polynomial functions P, Q^j as

$$P = \sum_{d=-1}^{\delta_1} P_d, \qquad Q^j = \sum_{d=-1}^{\delta_j} Q^j_d,$$

in which P_d is the quasi-homogeneous component of type ω and degree $d+\omega_1$, and Q^j_d is the quasi-homogeneous component of type ω and degree $d+\omega_j$. The *degree of quasihomogeneity of type* ω *of* P_d *and* Q^j_d will be denoted by $\deg_\omega P_d$ and $\deg_\omega Q^j_d$. The *highest degree of quasihomogeneity of type* ω *of* P *and* Q^j is

$$\deg_\omega P = \max_d\{\deg_\omega P_d\} = \delta_1 + \omega_1, \ \deg_\omega Q^j = \max_d\{\deg_\omega Q^j_d\} = \delta_j + \omega_j.$$

Then, the highest quasi-homogeneous degree component of P and Q^j is, respectively, P_{δ_1} and $Q^j_{\delta_j}$. The degree of quasi homogeneity type ω of the vector field X_ε will be simply denoted by $\deg_\omega X_\varepsilon = \max \delta_l = \delta$.

Example 2. *Let* Y *be the planar polynomial vector field given in the Example 1. If* $\omega = (2,3)$*, then* $\deg_\omega Y = 1$, $\deg_\omega Y_1 = 1+2 = 3$ *and* $\deg_\omega Y_2 = 1+3 = 4$*. Nevertheless, if we consider* $\omega = (1,1)$*, then* $\deg_\omega Y = 1$*,* $\deg_\omega Y_1 = 0+1 = 1$*, and* $\deg_\omega Y_2 = 1+1 = 2$.

In what follows, the expressions of X^∞_ε in each of the $2(n+1)$ local charts of $\mathbb{S}^n_\omega$ are given. The notation $(x, \mathbf{y}, \varepsilon)$ concerns a coordinate system in the finite part of the phase space, whereas $(u, \mathbf{v}, \varepsilon)$ concerns the coordinate system near infinity, in which $\mathbf{v} = (v_2, \ldots, v_n)$. We emphasize that in different open sets U_i of the covering, the coordinates $(u, \mathbf{v}, \varepsilon)$ have different meanings. See Figure 4.

In U_1, X^∞_ε is written as

$$\left\{\begin{array}{rcl} u' &=& \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(\varepsilon Q^2_d - u\frac{\omega_2}{\omega_1}P_d\big),\\ v_2' &=& \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(\varepsilon Q^3_d - v_2\frac{\omega_3}{\omega_1}P_d\big),\\ \vdots &=& \vdots \\ v_{n-1}' &=& \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(\varepsilon Q^n_d - v_{n-1}\frac{\omega_n}{\omega_1}P_d\big),\\ v_n' &=& -\displaystyle\frac{1}{\omega_1}\sum_{d=-1}^{\delta} v_n^{\delta+1-d}P_d, \end{array}\right. \tag{8}$$

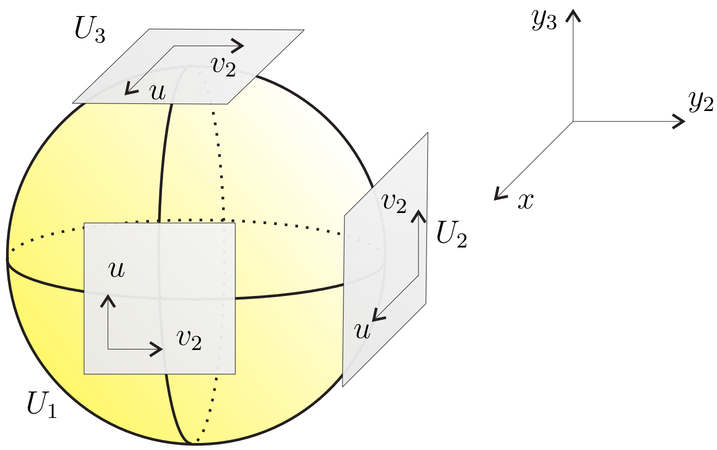

FIGURE 4. Directional charts that cover the PL-ball $\mathbb{B}^3_\omega$. Following the terminology of Definition 3, the slow-fast vector field obtained in the chart U_1 is the compactification in the fast direction, whereas the vector field obtained in a chart U_l is the compactification in the slow direction, for $l = 2, \ldots, n$.

where P_d, Q_d^j are computed in $(1, u, v_2, \ldots, v_{n-1}, \varepsilon)$ for all $j = 2, \ldots, n$. Observe that it was used the quasi homogeneity of P_d, Q_d^j in order to obtain system (8).

For $l = 2, \ldots, n$, in U_l the compactification is written as

$$
(9) \qquad \left\{\begin{array}{rcll}
u' & = & \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(P_d - \varepsilon u\frac{\omega_1}{\omega_l}Q_d^l\big), & \\
v_i' & = & \varepsilon \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(Q_d^i - v_i\frac{\omega_i}{\omega_l}Q_d^l\big), & 1 < i < l \\
v_{i-1}' & = & \varepsilon \displaystyle\sum_{d=-1}^{\delta} v_n^{\delta-d}\big(Q_d^i - v_{i-1}\frac{\omega_i}{\omega_l}Q_d^l\big), & l < i \leq n \\
v_n' & = & -\dfrac{\varepsilon}{\omega_l}\displaystyle\sum_{d=-1}^{\delta} v_n^{\delta+1-d}Q_d^l, &
\end{array}\right.
$$

where $j = 2, \ldots, n$, and the polynomial functions P_d, Q_d^j are computed in $(u, v_2, \ldots, v_{l-1}, 1, v_l, \ldots, v_{n-1}, \varepsilon)$. Once again it was used the quasi homogeneity of P_d, Q_d^j to obtain system (9).

The expression of X_ε^∞ in U_{n+1} is precisely the expression of (3). In the open set V_i, X_ε^∞ is obtained by replacing $\frac{1}{v_n^{\omega_i}}$ by $-\frac{1}{v_n^{\omega_i}}$ in the coordinate change of chart U_i, for all $i = 1, \ldots, n+1$. Furthermore, in any local chart U_i and V_i the set $\{v_n = 0\}$ is an invariant set of X_ε^∞ that plays the role of infinity.

Definition 3. *The vector field obtained in the chart U_1 will be called* compactification of X_ε in the positive fast direction. *The vector field obtained in U_l, for $l = 2, \dots, n$, will be called* compactification of X_ε in the l-th positive slow direction.

From the expression of X_ε^∞ in a chart U_l for $l = 1, \dots, n$, one can conclude the following proposition:

Proposition 4. *Let X_ε be the polynomial vector field associated to the slow-fast system (3) and denote its PL-compactification by X_ε^∞. Then, in a chart U_l for $l = 1, \dots, n$, it follows that:*

- **(a):** *The PL-compactification of X_ε in the fast direction is not a slow-fast system, but it is a singular perturbation problem. In addition, for $\varepsilon = 0$, the set of equilibria is given by $\{(0,0)\} \cup \{P_{\delta_1}(1, u, v_2, \dots, v_{n-1}, 0) = 0\}$.*
- **(b):** *The PL-compactification of X_ε in the l-th slow direction is a slow-fast system, for all $l = 2, \dots, n$.*
- **(c):** *Suppose X_ε is a n-dimensional vector field for $n \geq 3$. Then, for $l = 2, \dots, n$, the vector field X_ε^∞ defines a slow-fast system at infinity $\{v_n = 0\}$ in a chart U_l if, and only if, $\delta = \delta_1 = \delta_{j_0}$, for some j_0. Moreover, such slow-fast system has one fast variable and $n-2$ slow variables.*
- **(d):** *If $\delta = \delta_1 > \delta_j$ for all j, then $\varepsilon\mathbf{Q}$ does not affect the dynamics at infinity $\{v_n = 0\}$. On the other hand, if $\delta_1 < \delta_{j_0} = \delta$ for some j_0, in the limit $\varepsilon = 0$ the infinity is filled with equilibria.*

Proof. Assertions (a) and (b) follow directly from the expressions (8) and (9) of the vector fields in the fast and slow-directions, respectively. In order to prove assertions (c) and (d), assume that X_ε is a n-dimensional vector field for $n \geq 3$. From Equation (9), if $\delta_1 > \delta_j$ for all j, then only terms of P play role at infinity $\{v_n = 0\}$. In other words, if one sets $v_n = 0$ in Equation (9), then only terms of P will remain, thus $\varepsilon\mathbf{Q}$ does not affect the dynamics at infinity. The same reasoning can be used to prove that, if $\delta_1 < \delta_{j_0}$ for some j_0, then only terms of $\varepsilon\mathbf{Q}$ play role at $\{v_n = 0\}$. Setting $\varepsilon = 0$, then $\{v_n = 0\}$ is filled with equilibria. Finally, if $\delta = \delta_1 = \delta_{j_0}$ for some j_0, then terms of both P and Q^{j_0} play role at infinity, and therefore dynamics at infinity of U_l is given by a slow-fast system. □

Example 5. *Consider the planar slow-fast system*

$$x' = P(x, y, \varepsilon) = -x, \qquad y' = \varepsilon Q(x, y, \varepsilon) = \varepsilon(y^2 - x^3). \tag{10}$$

After a PL-compactification with weights $\omega = (2, 3)$, the systems obtained in the fast and slow directions U_1 and U_2 are, respectively,

$$\left\{\begin{array}{rcl} u' & = & \varepsilon(u^2-1)+\dfrac{3uv^3}{2}, \\ v' & = & \dfrac{v^4}{2}, \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{rcl} u' & = & -uv^3+\dfrac{2\varepsilon u(u^3-1)}{3}, \\ v' & = & \dfrac{\varepsilon v(u^3-1)}{3}. \end{array}\right. \tag{12}$$

In this example, we have $\delta_1 < \delta_2$ *because* $\deg_\omega P = \delta_1 + 2 = 2$ *and* $\deg_\omega Q = \delta_2 + 3 = 6$*. As expected from item (a) of Proposition 4, the system (11) defined in* U_1 *is not a slow-fast system. From item (d), the infinity is filled with equilibria when* $\varepsilon = 0$ *in equation (12). Observe that in* U_2 *the critical manifold is given by* $\{uv^3 = 0\}$*. See Figure 5.*

Example 6. *Consider the slow-fast system*

$$x' = P(x, y, z, \varepsilon) = x(y^2 - z^2), \quad y' = \varepsilon, \quad z' = \varepsilon. \tag{13}$$

After a PL-compactification with weights $\omega = (1, 1, 1)$ *(classical Poincaré compactification), one obtains in* U_1*,* U_2 *and* U_3*, respectively,*

$$u' = u(v^2-u^2)+\varepsilon w^3, \quad v' = v(v^2-u^2)+\varepsilon w^3, \quad w' = w(v^2-u^2), \tag{14}$$

$$u' = u(1 - v^2 - \varepsilon w^3), \quad v' = \varepsilon w^3(1 - v), \quad w' = -\varepsilon w^4, \tag{15}$$

$$u' = u(1 - v^2 - \varepsilon w^3), \quad v' = \varepsilon w^3(1 - v), \quad w' = -\varepsilon w^4. \tag{16}$$

As expected from item (d) of Proposition 4, it follows that, at infinity $\{w = 0\}$*, only terms of* P *play role. Moreover, from item (a), the compactification in the fast direction is not a slow-fast system.*

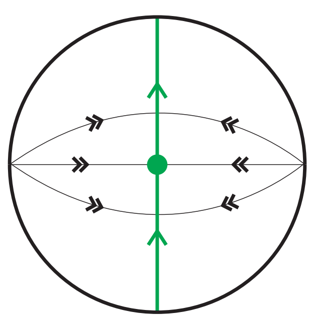

FIGURE 5. Phase portrait of the slow-fast system (10) in the PL-disk. The critical manifold is highlighted in green.

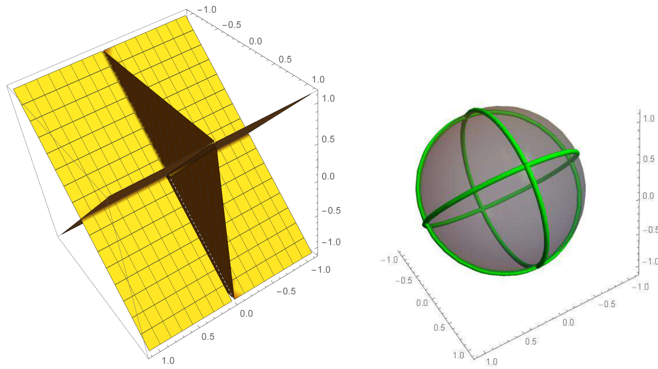

FIGURE 6. Critical manifold of the compactified slow-fast systems (13) and (17) in the Poincaré Ball.

4. Geometric singular perturbation theory at infinity

This section is devoted to study conditions to assure normal hyperbolicity at infinity. We start our analysis stating a suitable version of Fenichel Theorem at infinity, which is given in Theorem 7. Afterwards it is shown that the Newton polytope of a polynomial slow-fast system carries information about the normal hyperbolicity at infinity. Finally, it is given a geometric condition that assures normal hyperbolicity, based on the intersection of the critical manifold with infinity.

In Theorem 7, suppose $n \geq 3$, consider the polynomial slow-fast system (3) and C_0^∞ the critical manifold of system (9) at a generic chart U_l, for $2 \leq l \leq n$. Suppose also $\delta = \delta_1 = \delta_{j_0}$ for some j_0 (see item (c) of Proposition 4).

Theorem 7. *(Fenichel Theorem at infinity) Let $\mathcal{K} \subset \mathcal{NH}(C_0^\infty)$ be a j-dimensional compact normally hyperbolic sub-manifold (possibly with boundary) at infinity $\{v_n = 0\}$ of the slow system associated to (9). Let $\mathcal{W}^s$ be the $(j + j^s)$-dimensional stable manifold of $\mathcal{K}$. Then, at infinity $\{v_n = 0\}$, there is $\tilde{\varepsilon} > 0$ sufficiently small such that for $0 < \varepsilon < \tilde{\varepsilon}$ the following hold:*

- **(F1):** *There exists a family of smooth manifolds $\mathcal{K}_\varepsilon$ such that $\mathcal{K}_\varepsilon \to \mathcal{K}_0 = \mathcal{K}$ according to Hausdorff distance and $\mathcal{K}_\varepsilon$ is a normally hyperbolic locally invariant manifold of (9),*
- **(F2):** *There is a family of $(j+j^s+k^s)$-dimensional manifolds $\mathcal{W}_\varepsilon^s$ such that $\mathcal{W}_\varepsilon^s$ is local stable manifolds of $\mathcal{K}_\varepsilon$.*

Analogous conclusions hold for the $(j + j^u)$-dimensional unstable manifold $\mathcal{W}^u$ at infinity.

Example 8. *Consider the slow-fast system*

$$x' = P(x,y,z,\varepsilon) = x(y^2 - z^2), \quad y' = \varepsilon z^3, \quad z' = \varepsilon y^3. \tag{17}$$

After Poincaré compactification, systems obtained in U_1, U_2 *and* U_3 *are, respectively,*

$$u' = -u^3 + uv^2 + \varepsilon v^3, \quad v' = \varepsilon u^3 - u^2 v + v^3, \quad w' = w(v^2 - u^2), \tag{18}$$

$$u' = u(1 - v^2 - \varepsilon v^3), \quad v' = \varepsilon(1 - v^4), \quad w' = -\varepsilon v^3 w, \tag{19}$$

$$u' = u(v^2 - 1 - \varepsilon v^3), \quad v' = \varepsilon(1 - v^4), \quad w' = -\varepsilon v^3 w. \tag{20}$$

By Theorem 7, at infinity $\{w = 0\}$*, the dynamics near compact normally hyperbolic sets persist for* $\varepsilon > 0$ *sufficiently small. However, for both systems* (19) *and* (20)*, there are two non normally hyperbolic points. See Figure 6.*

4.1. **Newton polytope of a polynomial vector field.** This subsection aims to recall the classical definition of Newton polytope associated to a polynomial vector field (see also [13]). Let $Y = (F_1, \dots, F_n)$ be a n-dimensional polynomial vector field. For each component F_i of Y, we introduce the notation

$$\mathbf{a} = (a_1, \dots, a_n), \quad \mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n},$$

$$F_i(\mathbf{x}) = \sum_{\mathbf{a}_i \in \mathbb{Z}^n} c_{\mathbf{a}_i} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}_i} \in \mathbb{R}, \quad \mathbf{a}_i = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n).$$

Let $Y = (F_1, \dots, F_n)$ be a n-dimensional polynomial vector field. The *support of* Y is the set $\mathcal{S}_Y$ given by $\mathcal{S}_Y = \bigcup_{i=1}^{n} \mathcal{S}_{Y,i}$, in which $\mathcal{S}_{Y,i} = \{\mathbf{a}_i \in \mathbb{Z}^n,\ c_{\mathbf{a}_i} \neq 0\}$. The *Newton polytope* $\mathcal{P}_Y \subset \mathbb{R}^n$ of a n-dimensional polynomial vector field Y is the convex hull of the support $\mathcal{S}_Y$.

Example 9. *Consider the vector field* $Y(x,y) = \big(F_1(x,y), F_2(x,y)\big) = (x + y, x^2)$*. Thus* $\mathcal{S}_{Y,1} = \{(0,0), (-1,1)\}$ *and* $\mathcal{S}_{Y,2} = \{(2,-1)\}$*. Therefore* $\mathcal{S}_Y = \{(0,0), (-1,1), (2,-1)\}$*. See Figure 7(a).*

Example 10. *Consider the* 3*-dimensional polynomial vector field*

$$Y(x,y,z) = \big(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\big) = (-1 + xy, yz^2, xz).$$

Then $\mathcal{S}_{Y,1} = \{(-1,0,0), (0,1,0)\}$, $\mathcal{S}_{Y,2} = \{(0,0,2)\}$ *and* $\mathcal{S}_{Y,3} = \{(1,0,0)\}$*. Therefore* $\mathcal{S}_Y = \{(-1,0,0), (1,0,0), (0,1,0), (0,0,2)\}$*. See Figure 7(b).*

(a)

(b)

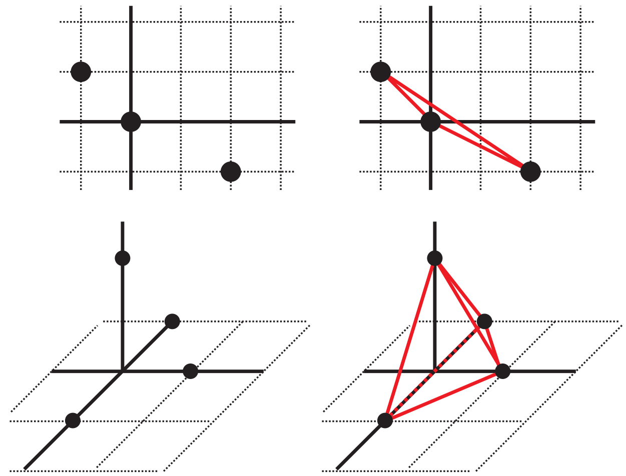

FIGURE 7. Figure (a): Support (left) and Newton polytope (right) of the planar polynomial vector field of the Example 9. Figure (b): Support (left) and Newton polytope (right) of the 3-dimensional polynomial vector field of the Example 10.

4.2. **Normal hyperbolicity at infinity.** We have already stated all preliminary definitions and results needed to prove our main results. In what follows, it will be studied necessary conditions in order to assure normal hyperbolicity at infinity. Denote $P(x,\mathbf{y},\varepsilon) = \sum_{d=-1}^{\delta_1} P_d(x,\mathbf{y},\varepsilon)$, which P_d is a quasi-homogeneous polynomial of type ω of degree $d+\omega_1$ and $\delta_1+\omega_1 = \deg_\omega P$. The degree of quasi homogeneity type ω of the slow-fast system (3) is denoted by $\deg_\omega X_\varepsilon = \delta \geq \delta_1$.

Theorem A. *Let X_ε be a n-dimensional polynomial vector field associated to system (3), whose critical manifold is $C_0 = \{P(x,\mathbf{y},0)=0\}$. Then, near the boundary of the PL ball the following hold:*

(a): *Suppose $\delta = \delta_1$. If the component P of type ω and degree $\delta_1+\omega_1$ has monomials of the form*

$$c_1x^{r_1+1} + \sum_{i=2}^{n} \left(c_i x y_i^{r_i} + d_i y_i^{s_i}\right),$$

satisfying $c_i^2+d_i^2 \neq 0$ for all $i=1,\dots,n$, and $(r_1,r_2,\dots,r_n)$ and $(r_1,s_2,\dots,s_n)$ satisfy the equation of the hyperplane $\{\omega_1a_1+\dots+\omega_na_n = \delta\}$, then the origin of each chart U_l is a normally hyperbolic point of C_0. In particular, the origin of U_1 is a hyperbolic node of the compactified vector field X_ε^∞.

(b): *If $\delta > \delta_1$, then $\{v_n = 0\}$ is a non normally hyperbolic component of C_0.*

(c): *If $p \in U_l$ is a point at infinity (with $2 \leq l \leq n$), a necessary condition to assure normal hyperbolicity is that C_0 intersects the infinity $\{v_n = 0\}$ transversely at p.*

Remark 11. *The hypotheses of Condition (a) of Theorem A implies that the Newton polytope $\mathcal{P}_{X_\varepsilon}$ has a compact face containing the intersection of the hyperplane $\{\omega_1 a_1 + \ldots + \omega_n a_n = \delta\}$ with $(\mathbb{R}_{\geq 0})^n$. Such feature of $\mathcal{P}_{X_\varepsilon}$ turns out to be a necessary condition in order to assure normal hyperbolicity at the origin of each chart at infinity. From a practical way of view, one can use the Newton polytope to detect non normally hyperbolic points at the origin of each chart. Finally, observe that in condition (c) we do not require that p is the origin.*

Proof. From Proposition 4, we know that the compactification X_ε^∞ defines a slow fast system in a chart U_l for $l = 2, \ldots, n$, but it is not in the chart U_1. From Equation (9), the expression of $C_0 = \{P(x, \mathbf{y}, 0) = 0\}$ in a chart U_l is

$$\sum_{d=0}^{\delta} v_n^{\delta-d} P_d\big(u, \ldots, v_{l-1}, 1, v_l, \ldots, v_{n-1}, 0\big) = 0.$$

Therefore, normal hyperbolicity near infinity $\{v_n = 0\}$ means

$$v_n^{\delta-\delta_1} P_{\delta_1} = 0, \qquad v_n^{\delta-\delta_1} \frac{\partial P_{\delta_1}}{\partial u} \neq 0, \tag{21}$$

in which such functions are applied in $(u, \ldots, v_{l-1}, 1, v_l, \ldots, v_{n-1}, 0)$. So we divide our analysis in two cases.

(a) Suppose $\delta = \delta_1$. In this case, a necessary condition to assure that the origin of the chart U_l is normally hyperbolic is to require that the original polynomial P has monomials of the form $c_l x y_l^{r_l} + d_l y_l^{s_l}$, in which $c_l, d_l \in \mathbb{R}$, $r_l, s_l \in \mathbb{N}$ and $c_l^2 + d_l^2 \neq 0$, $r_l = \frac{\delta}{\omega_l}$ and $s_l = \frac{\delta + \omega_1}{\omega_l}$.

Indeed, if $d_l \neq 0$, then C_0^∞ does not intersect the origin of the chart U_l. On the other hand, if $d_l = 0$ then $c_l \neq 0$ so C_0^∞ is normally hyperbolic at the origin of U_l. Recall that C_0^∞ is the critical manifold of the compactified system (9) at a generic chart U_l, for $2 \leq l \leq n$.

Concerning the support $\mathcal{S}_{X_\varepsilon}$, if $c_l \neq 0$ then $\mathcal{S}_{X_\varepsilon}$ contains the point $(0, \ldots, 0, r_l, 0, \ldots, 0)$, in which r_l is positioned in the l-th coordinate. Finally, if $d_l \neq 0$ then $\mathcal{S}_{X_\varepsilon}$ contains the point $(-1, 0, \ldots, 0, s_l, 0, \ldots, 0)$, in which s_l is positioned in the l-th coordinate.

The compactification of the critical set can be normally hyperbolic in one chart and not in another. Therefore, in order to assure that, for all $l = 2, \ldots, n$ the origin of U_l is normally hyperbolic, we must require that the quasi-homogeneous component of P of type ω and degree $\delta_1+\omega_1$ has monomial of the form

$$x\Big(c_1x^{r_1}+c_2y_2^{r_2}+\cdots+c_ny_n^{r_n}\Big)+\Big(d_2y_2^{s_2}+\cdots+d_ny_n^{s_n}\Big),$$

in which $c_i^2+d_i^2\neq 0$ for all $i=1,\ldots,n$.

Observe that points of $\mathcal{S}_{X_\varepsilon}$ related to these monomials are contained in the hyperplane $\{\omega_1a_1+\ldots+\omega_na_n=\delta\}$. Moreover, since all the natural numbers r_i and s_i concerns higher order terms of the vector field X_ε, then all the other points of the support $\mathcal{S}_{X_\varepsilon}$ are either contained in such hyperplane, or they are contained in the half-space $\{\omega_1a_1+\ldots+\omega_na_n<\delta\}$. This implies that the Newton polytope $\mathcal{P}_{X_\varepsilon}$ has a compact face that contains $\{\omega_1a_1+\ldots+\omega_na_n=\delta\}\cap(\mathbb{R}_{\geq 0})^n$. See Figure 8. If this is the case, in U_1 the origin is a hyperbolic node of X_ε^∞.

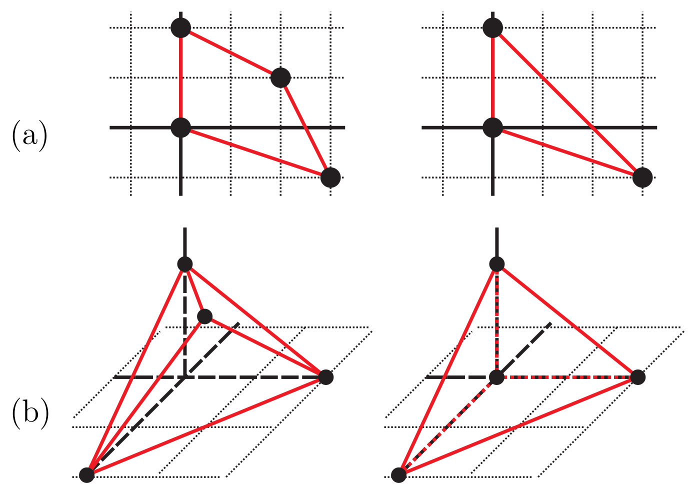

FIGURE 8. Figure (a): On the left, an example of Newton polytope of that gives rise to non normally hyperbolic points at infinity. On the other hand, the slow-fast system associated to the Newton polytope on the right will present normally hyperbolic points at the origin of U_1 and U_2 (under a suitable choice of ω). Figure (b): description analogous to the Figure (a), but in dimension 3.

(b) Suppose $\delta>\delta_1$. By (21), the infinity $\{v_n=0\}$ is a non normally hyperbolic component of the critical manifold, for each U_l.

(c) Denote by C_0^∞ the critical manifold C_0 in a chart U_l, for each $l=2,\ldots,n$. The infinity is represented by the hyperplane $\{v_n=0\}$, and the vector $(0,\ldots,0,1)$ is normal to it, for every point $p\in\{v_n=0\}$. On the other hand, the vector $\nabla P_\delta(p)$ is normal to C_0^∞ at p. If C_0^∞ is normally hyperbolic at p, from equation (21) we know that the first coordinate of $\nabla P_\delta(p)$ is non zero. Therefore, $(0,\ldots,0,1)$ and $\nabla P_\delta(p)$

are linearly independent, which implies that $T_p C_0^\infty \pitchfork T_p\{v_n = 0\}$. We conclude that a necessary condition to assure normal hyperbolicity at infinity is that C_0^∞ intersects $\{v_n = 0\}$ transversely. See Figure 9. □

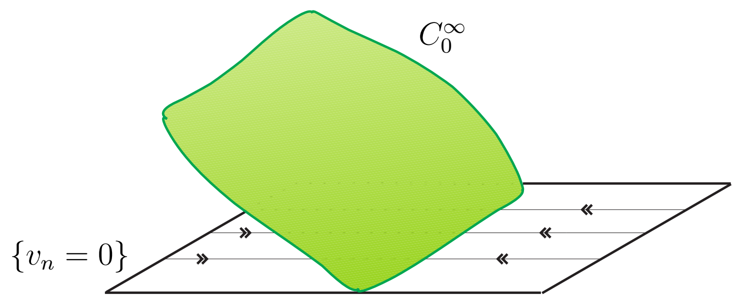

FIGURE 9. Critical manifold C_0^∞ (highlighted in green) intersects the infinity transversally.

Remark 12. *From Propositon 4 item (d), in each chart U_l we know that, if $\delta > \delta_1$, by setting $\varepsilon = 0$, the infinity $\{v_n = 0\}$ is filled with equilibrium points. Due to item (b) of Theorem A, now we know that, in fact, the infinity is a component of the critical manifold C_0.*

In what follows we present an example showing that transversality is not a sufficient condition to assure normal hyperbolicity at infinity.

Example 13. *Consider the polynomial slow-fast system*

$$x' = y + z, \quad y' = \varepsilon(x + z), \quad z' = \varepsilon(x + y). \tag{22}$$

The critical manifold associated to (22) is $C_0 = \{y + z = 0\}$, which intersects the infinty transversaly. After Poincaré compactification, one obtains the following slow-fast system in both charts U_2 and U_3:

$$u' = 1 + v - \varepsilon u(u + v), \quad v' = \varepsilon(1 + u - uv - v^2), \quad w' = -\varepsilon w(u + v).$$

In both charts U_2 and U_3, the infinity and the critical manifold are given by $\{w = 0\}$ and $C_0^\infty = \{v+1 = 0\}$, respectively. This implies that $\mathcal{NH}(C_0^\infty) = \emptyset$. Geometrically, in U_2 and U_3 the set C_0^∞ is a horizontal line. See Figure 10.

It is very difficult to study conditions to assure normal hyperbolicity for the *whole* infinity in arbitrary dimension. However, it can be given an answer for the 2-dimensional case (see Theorem B). It will be clear that, in dimension 2, the transversality condition presented in Theorem A is sufficient and necessary to assure normal hyperbolicity at infinity.

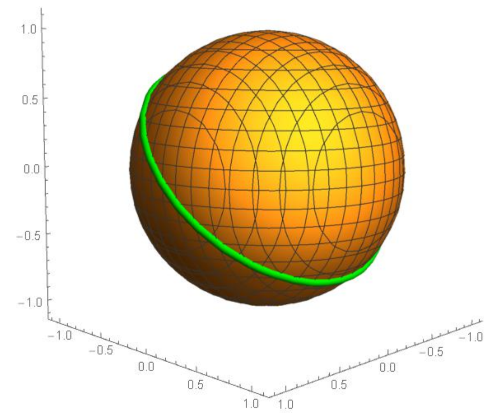

FIGURE 10. Compactification of C_0 related to system (22). The critical manifold C_0 is highlighted in green.

5. Planar polynomial slow-fast systems

Consider the 2-dimensional polynomial slow-fast system

$$x' = P(x, y, \varepsilon), \qquad y' = \varepsilon Q(x, y, \varepsilon). \tag{23}$$

As usual, P_d and Q_d are quasi-homogeneous components of type ω and degree $d + \omega_1$ and $d + \omega_2$, respectively. The highest quasihomogeneous degree component of P and Q is, respectively, P_{δ_1} and Q_{δ_2}. Due to statements (a) and (b) of Proposition 4, for our purposes in this section we will further suppose that $\delta_1 = \delta_2 = \delta$.

The polynomial functions P and Q will be written as

$$P(x, y, \varepsilon) = \sum_{i=-1}^{\delta} P_i(x, y, \varepsilon), \quad Q(x, y, \varepsilon) = \sum_{j=-1}^{\delta} Q_j(x, y, \varepsilon),$$

$$P_i(x, y, \varepsilon) = \sum_{r=0}^{i} c_{\varepsilon,r,i} x^r y^{\frac{i+\omega_1(1-r)}{\omega_2}}, \quad Q_j(x, y, \varepsilon) = \sum_{s=0}^{j} d_{\varepsilon,s,j} x^s y^{\frac{j+\omega_2-\omega_1 s}{\omega_2}},$$

in which $\delta = \deg_\omega X_\varepsilon$ and the notation $c_{\varepsilon,r,i}$, $d_{\varepsilon,s,j}$ indicates that such coefficients depend analytically on ε. Moreover, for each i and j, the powers of the monomials of $P_i(x, y, \varepsilon)$ and $Q_j(x, y, \varepsilon)$ satisfies, respectively, $a\omega_1 + b\omega_2 = i + \omega_1$ and $a\omega_1 + b\omega_2 = j + \omega_2$.

The compactification X_ε^∞ in the fast and slow direction is given by, respectively,

$$
(24) \qquad \begin{cases} u' &=& \displaystyle\sum_{i=-1}^{\delta} v^{\delta-i}\Big(-uP_i(1,u,\varepsilon)+\varepsilon Q_i(1,u,\varepsilon)\Big), \\ v' &=& -\displaystyle\sum_{i=-1}^{\delta} v^{\delta-i+1}P_i(1,u,\varepsilon), \end{cases}
$$

$$
(25) \qquad \begin{cases} u' &=& -\displaystyle\sum_{i=-1}^{\delta} v^{\delta-i}\Big(-u\varepsilon Q_i(u,1,\varepsilon)+P_i(u,1,\varepsilon)\Big) \\ v' &=& -\varepsilon\displaystyle\sum_{i=-1}^{\delta} v^{\delta-i+1}Q_i(u,1,\varepsilon). \end{cases}
$$

Before we start our analysis in the charts U_1 and U_2, let us introduce a useful Lemma, that can be found for instance in [10, pp. 72]. Recall that x_0 is a *simple root* of $P \in \mathbb{R}[x]$ if $P(x) = (x-x_0)\cdot\widetilde{P}(x)$ and x_0 is not root of $\widetilde{P}(x)$.

Lemma 14. *Let P be a polynomial over $\mathbb{R}$. Then x_0 is a simple root of P if, and only if, $P(x_0)=0$ and $P'(x_0)\neq 0$.*

Let us start our study by considering the compactification in the slow direction (25).

Proposition 15. *Let (23) be a planar polynomial slow-fast system and consider its PL-compactification in the slow direction (25). Then $(\tilde{u},0)$ is an equilibrium point at infinity $\{v=0\}$ for $\varepsilon=0$ if, and only if, the critical manifold C_0 intersects the infinity at $(\tilde{u},0)$.*

Proof. Recall that $\delta = \deg_\omega X_\varepsilon$. Therefore, in U_2, C_0 is given by $\{P_\delta(u,1,0)=0\}$, which is a curve of equilibria of system (25) when $\varepsilon=0$. Such curve intersects infinity at points of the form $(\tilde{u},0)$, with $\tilde{u}$ being a root of the polynomial $P_\delta(u,1,0)$. □

Proposition 16. *Let $p=(\tilde{u},0)\in\{v=0\}$ be an equilibrium point of (25) at infinity. Then the following statements are equivalent:*

- **(a):** *p is normally hyperbolic for (25) when $\varepsilon=0$.*
- **(b):** *$\tilde{u}$ is a hyperbolic equilibrium point of the ODE $\dot{u}=P_\delta(u,1,0)$.*
- **(c):** *$\tilde{u}$ is a simple root of the polynomial $P_\delta(u,1,0)$.*
- **(d):** *The critical manifold intersects the infinity transversely.*

Proof. Observe that (a) $\Leftrightarrow$ (b) because $\frac{\partial}{\partial u}P_\delta(\tilde{u},1,0)\neq 0$ if, and only if, $\tilde{u}$ is a hyperbolic equilibrium point of the ODE $\dot{u}=P_\delta(u,1,0)$. In addition, from Lemma 14, it follows that (a) $\Leftrightarrow$ (c) because $\frac{\partial}{\partial u}P_\delta(\tilde{u},1,0)\neq 0$

and $P_\delta(\tilde{u},1,0)=0$ if, and only if, $\tilde{u}$ is a simple root of $P_\delta(u,1,0)$. The equivalence (a) $\Leftrightarrow$ (d) follows because $\frac{\partial}{\partial u}P_\delta(\tilde{u},1,0)\neq 0$ means that $\nabla P_\delta(\tilde{u},1,0)$ and $\vec{n}=(0,1)$ are linearly independent, in which $\vec{n}$ is normal to the line that represents the infinity. $\square$

Due to Propositions 15 and 16, the transversality condition presented in Theorem A is a sufficient and necessary condition to assure normal hyperbolicity at infinity in the 2-dimensional case.

Corollary 17. *A necessary condition for the existence of simple roots of $P_\delta(u,1,0)$ is $c_{0,0,\delta}\neq 0$ or $c_{0,1,\delta}\neq 0$.*

Proof. From the expression of $P_i(x,y,\varepsilon)$, if $c_{0,0,\delta}\neq 0$, then the component P_δ of P has a monomial of the form $c_{0,0,\delta}y^{\frac{\delta+\omega_1}{\omega_2}}$. On the other hand, if, $c_{0,1,\delta}\neq 0$, then the component P_δ of P has a monomial of the form $c_{0,1,\delta}xy^{\frac{\delta}{\omega_2}}$. In both cases, by setting $(u,1,0)$, the origin of U_2 will be either a regular point or a hyperbolic equilibrium point of (25). $\square$

Since we have studied the dynamics of the compactification in the slow direction, it is sufficient to study the dynamics near the origin of the compactification in the fast direction (24). Recall that (24) is not a slow-fast system.

Proposition 18. *Let (23) be a planar polynomial slow-fast system and consider its compactification in the fast direction (24). Then, for $\varepsilon=0$, the following statements are true:*

- **(a):** *The origin of the chart U_1 is an equilibrium point of (24).*
- **(b):** *The critical manifold C_0 intersects the origin of U_1 if, and only if, $c_{0,\frac{\delta}{\omega_1},\delta}=0$. In this case, the origin is a non-hyperbolic equilibrium point of (24).*
- **(c):** *The origin of U_1 is an hyperbolic node of (24) if, and only if, $c_{0,\frac{\delta}{\omega_1},\delta}\neq 0$.*

Proof. Recall the expression of $P_i(x,y,\varepsilon)$. It is straightforward from equation (24) that the origin of U_1 is an equilibrium point for $\varepsilon=0$. Therefore, items (b) and (c) aims to understand features of such equilibrium. Furthermore, in this chart we must study the polynomial P applied in points of the form $(1,u,0)$.

In the chart U_1 the critical manifold C_0 is the zero set of $P_\delta(1,u,0)$, which represents a curve of singularities for $\varepsilon=0$. Therefore, C_0 intersects the origin of U_1 if, and only if, $c_{0,\frac{\delta}{\omega_1},\delta}=0$. Moreover, the origin will be non hyperbolic for (24) if such a point is contained in C_0. This proves item (b) Finally, assuming $\varepsilon=0$, it can be easily checked that $c_{0,\frac{\delta}{\omega_1},\delta} \neq 0$ if, and only if, the origin of U_1 is a hyperbolic node of (24), which proves item (c). □

Now, we are able to state, for the planar case, a *global* version of the Fenichel Theorem, which assures the persistence of invariant manifolds in the *whole* Poincaré–Lyapunov disk. The proof of Theorem B is given by combining Fenichel Theorem (for the finite part) and Propositions 15, 16 and 18 (for the infinite part). In its statement, the compactified critical manifold is denoted by $\mathbf{C}_0$, which is the union of the finite and infinite parts of C_0.

Theorem B. *Consider the planar polynomial slow-fast system* (23)*. Suppose that $\mathcal{NH}(C_0) = C_0$, C_0 intersects the infinity of $\mathbb{D}_\omega$ transversely, and it does not intersect the origin of U_1, V_1. Then there exist $0 < \tilde{\varepsilon} \ll 1$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the following hold in the whole $\mathbb{D}_\omega$:*

- **(G1):** *There exist a family of smooth manifolds $\mathbf{C}_\varepsilon$ such that $\mathbf{C}_\varepsilon \to \mathbf{C}_0$ according to Hausdorff distance and $\mathbf{C}_\varepsilon$ is locally invariant of* (23)*.*
- **(G2):** *If $p_0 \in \mathbf{C}_0$ and $\mathbf{W}^s$ is its stable manifold, then there is a family $\mathbf{W}^s_\varepsilon$ of stable manifolds of $p_\varepsilon \in \mathbf{C}_\varepsilon$, in which $p_\varepsilon \to p_0$. The same conclusion holds if one consider the unstable manifold $\mathbf{W}^u$ of $p_0 \in \mathbf{C}_0$.*

Example 19. *Consider the slow fast system*

$$x' = x^2 + xy - 1, \qquad y' = \varepsilon Q(x, y, \varepsilon), \tag{26}$$

in which $Q(x, y, \varepsilon)$ is a polynomial function of degree less than or equal to 2. We apply the classical Poincaré compactification. Note that $\deg_\omega X_\varepsilon = \delta = 1$ and $\deg_\omega Q = \delta_2 \leq 1$. In U_1 and U_2, the dynamics at infinity are respectively given by

$$u' = u(-1 - u + v^2) + \varepsilon v^{2-\delta_2} Q(1, u), \qquad v' = v(-1 - u + v^2), \tag{27}$$

$$u' = u + u^2 - v^2 - \varepsilon u v^{2-\delta_2} Q(u, 1), \qquad v' = -\varepsilon v^{3-\delta_2} Q(u, 1). \tag{28}$$

All points of $C_0 = \{x^2 + xy = 1\}$ are normally hyperbolic. The origin of U_1 is a hyperbolic node and C_0 is normally hyperbolic at infinity (see Proposition 16). Therefore, as a consequence of Theorem B, global dynamics of system (26) *persist for ε sufficiently small. See figure 11.*

6. Non normally hyperbolic points at infinity

In this section is discussed some examples of 3-dimensional polynomial slow fast systems that present non normally hyperbolic singularities at infinity, namely: fold, transcritical and pitchfork singularities

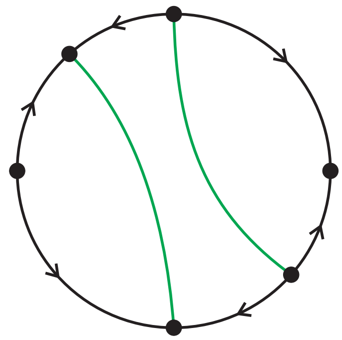

FIGURE 11. Poincaré compactification of system (26). The critical manifold C_0 is highlighted in green and the dots denote equilibria at infinity.

(see [14, 15]). OSince the phase-space is 3-dimensional, the slow-fast system at infinity is 2-dimensional.

Firstly, recall normal forms of such singularities in the literature. According to [14, 15], the non degeneracy conditions that a planar slow-fast system (3) must satisfy in order to present (respectively) a fold, transcritical and pitchfork singularity is given (respectively) by equations (29), (30) and (31) as follows.

$$\begin{aligned} f_x(0,0,0) = 0, \quad f_{xx}(0,0,0) \neq 0, \\ f_y(0,0,0) \neq 0 \ \text{ and } \ g(0,0,0) \neq 0. \end{aligned} \tag{29}$$

$$\begin{aligned} f(0,0,0) = f_x(0,0,0) = f_y(0,0,0) = 0, \\ \det \text{Hes}(f) < 0, \quad f_{xx}(0,0,0) \neq 0 \neq g(0,0,0), \end{aligned} \tag{30}$$

in which Hes(f) denotes the Hessian matrix of f evaluated at $(0,0,0)$.

$$\begin{aligned} f(0,0,0) = f_x(0,0,0) = f_{xx}(0,0,0) = f_y(0,0,0) = 0, \\ f_{xxx}(0,0,0) \neq 0, \quad f_{xy}(0,0,0) \neq 0, \quad g(0,0,0) \neq 0. \end{aligned} \tag{31}$$

Theorem 20 gathers results on normal forms of slow-fast systems, based on the non degeneracy conditions above. The notation $\mathcal{O}$ denotes higher order terms, whereas λ denotes a constant that depends on the non-degeneracy conditions of each singularity (see [14, 15] for details).

Theorem 20. *There exists a smooth coordinate change such that for* (x, y) *sufficiently small a planar slow-fast system is written as*

(a): *If system* (3) *satisfies the non-degeneracy conditions* (29) *of a planar generic fold:*

$$x' = y + x^2 + \mathcal{O}(x^3, xy, y^2, \varepsilon), \quad y' = \varepsilon\Big(\pm 1 + \mathcal{O}(x, y, \varepsilon)\Big), \tag{32}$$

(b): *If system* (3) *satisfies the non-degeneracy conditions* (30) *of a generic transcritical singularity:*

$$x' = x^2-y^2+\lambda\varepsilon+\mathcal{O}(x^3,x^2y,xy^2,y^3,\varepsilon x,\varepsilon y,\varepsilon^2),\quad y'=\varepsilon\Big(1+\mathcal{O}(x,y,\varepsilon)\Big), \tag{33}$$

(c): *If system* (3) *satisfies the non-degeneracy conditions* (31) *of a pitchfork singularity:*

$$x' = x(y-x^2)+\lambda\varepsilon+\mathcal{O}(x^2y,xy^2,y^3,\varepsilon x,\varepsilon y,\varepsilon^2),\quad y'=\varepsilon\Big(\pm1+\mathcal{O}(x,y,\varepsilon)\Big). \tag{34}$$

The main goal of this section is to study conditions that a 3-dimensional polynomial slow-fast system of the form

$$x' = P(x,y,z,\varepsilon),\quad y'=\varepsilon Q(x,y,z,\varepsilon),\quad z'=\varepsilon R(x,y,z,\varepsilon) \tag{35}$$

must satisfy in order to present a fold, transcritical or pitchfork singularity at infinity after Poincaré–Lyapunov compactification with weight $\omega=(\omega_1,\omega_2,\omega_3)$. Without loss of generality, in what follows is studied conditions to assure that the origin of the chart U_2 is one of the non normally hyperbolic points given by Theorem 20. Moreover, if X_ε is the vector field associated to (35), then $\deg_\omega X_\varepsilon=\delta$.

Theorem C. *Consider the 3-dimensional slow fast system* (35) *and its Poincaré–Lyapunov compactification* X_ε^∞ *with weight* $\omega=(\omega_1,\omega_2,\omega_3)$. *If* $\deg_\omega X_\varepsilon=\delta$, *then the following hold for every positive integer* k_1,k_2:

(a): *If* $P_\delta(x,y,z,\varepsilon)=x^2y^{k_1}-y^{k_2}z$, *then the critical manifold of* X_ε^∞ *has a fold singularity at the origin of* U_2 *if, and only if,* $k_1\omega_2=\delta-\omega_1$ *and* $k_2\omega_2=\delta+\omega_1-\omega_3$.
(b): *If* $P_\delta(x,y,z,\varepsilon)=x^2y^{k_1}-y^{k_2}z^2$, *then the critical manifold of* X_ε^∞ *has a transcritical singularity at the origin of* U_2 *if, and only if,* $k_1\omega_2=\delta-\omega_1$ *and* $k_2\omega_2=\delta+\omega_1-2\omega_3$.
(c): *If* $P_\delta(x,y,z,\varepsilon)=xy^{k_1}z-x^3y^{k_2}$, *then the critical manifold of* X_ε^∞ *has a pitchfork singularity at the origin of* U_2 *if, and only if,* $k_1\omega_2=\delta-\omega_3$ *and* $k_2\omega_2=\delta-2\omega_1$.

Proof. The proof is given by straightforward computations. We present the computation of item (a). The other items are completely analogous.

Consider system (35). We recall from Proposition 4 item (c) that the vector field in U_2 is a slow-fast system if, and only if, $\deg_\omega X_\varepsilon=\deg_\omega P=\deg_\omega Q$ or $\deg_\omega X_\varepsilon=\deg_\omega P=\deg_\omega R$.

Given the weight vector $\omega = (\omega_1, \omega_2, \omega_3)$, the expression of the compactified slow-fast system in the chart U_2 is

$$\left\{ \begin{array}{rcl} u' &=& w^{\omega_1} P - \varepsilon u \frac{\omega_1}{\omega_2} w^{\omega_2} Q, \\ v' &=& \varepsilon \big(w^{\omega_3} R - v \frac{\omega_3}{\omega_2} w^{\omega_2} Q \big), \\ w' &=& -\varepsilon \frac{w^{\omega_2 + 1}}{\omega_2} Q, \end{array} \right. \tag{36}$$

in which P, Q and R are applied in $(\frac{u}{w^{\omega_1}}, \frac{1}{w^{\omega_2}}, \frac{v}{w^{\omega_3}}, \varepsilon)$.

The highest quasi-homogeneous component of P is $P_\delta(x, y, z, \varepsilon) = x^2 y^{k_1} - y^{k_2} z$, then, after multiplying the vector field by w^δ, system (36) is rewritten as

$$\left\{ \begin{array}{rcl} u' &=& \big(\dfrac{u^2 w^\delta}{w^{\omega_1 + k_1 \omega_2}} - \dfrac{v w^\delta}{w^{\omega_3 + k_2 \omega_2 - \omega_1}} \big) + \displaystyle\sum_{d=-1}^{\delta-1} w^{\delta - d} P_d - \varepsilon u \frac{\omega_1}{\omega_2} \sum_{d=-1}^{\delta} w^{\delta - d} Q_d, \\ v' &=& \varepsilon \displaystyle\sum_{d=-1}^{\delta} w^{\delta - d} \big(R_d - v \frac{\omega_3}{\omega_2} Q_d \big), \\ w' &=& -\dfrac{\varepsilon}{\omega_2} \displaystyle\sum_{d=-1}^{\delta} w^{\delta + 1 - d} Q_d, \end{array} \right. \tag{37}$$

in which the polynomial functions P, Q and R are applied in $(u, 1, v, \varepsilon)$. Therefore, setting $w = 0$ and $\varepsilon = 0$ in equation (37), it follows that the origin of the chart U_2 is a generic fold singularity if, and only if, $k_1 \omega_2 = \delta - \omega_1$ and $k_2 \omega_2 = \delta + \omega_1 - \omega_3$. □

Theorem C gives conditions on the highest quasi-homogeneous degree of the polynomial P and on the weights $\omega = (\omega_1, \omega_2, \omega_3)$ in order to assure that the origin of the chart U_2 is one of the non normally hyperbolic singularities given by Theorem 20. However, it is important to remark that, depending on the weight vector ω, it is not possible to generate such singularities. This fact will be clear in the next examples.

Example 21. *Under the hypothesis of Theorem C, suppose that $\omega_1 = 1$ and $\omega_2 = \omega_3 = 2$. For any positive integers $k_1 = k_2 = \frac{\delta - 1}{2}$, the origin of U_2 is a fold singularity of X_ε^∞. However, if $\omega_1 = 3, \omega_2 = 2$ and $\omega_1 = 1$, then it does not exist k_1, $k_2 \in \mathbb{Z}$ satisfying conditions (a) of Theorem C. See Figure 12.*

Example 22. *Suppose that $\omega_1 = \omega_3$ and $\omega_2 = 1$. For any positive integers $k_1 = k_2 = \delta - \omega_1$, the origin of U_2 is a transcritical singularity of X_ε^∞. See Figure 12.*

Example 23. *Suppose that $\omega_1 = 2$ and $\omega_2 = \omega_3 = 1$. For any positive integers $k_1 = \delta - 1$ and $k_2 = \delta - 4$, the origin of U_2 is a pitchfork* *singularity of* X_ε^∞. *Nevertheless, if* $\omega_1 = 3, \omega_2 = 2$ *and* $\omega_1 = 1$, *then it does not exist* k_1, $k_2 \in \mathbb{Z}$ *satisfying conditions (c) of Theorem C. See Figure 12.*

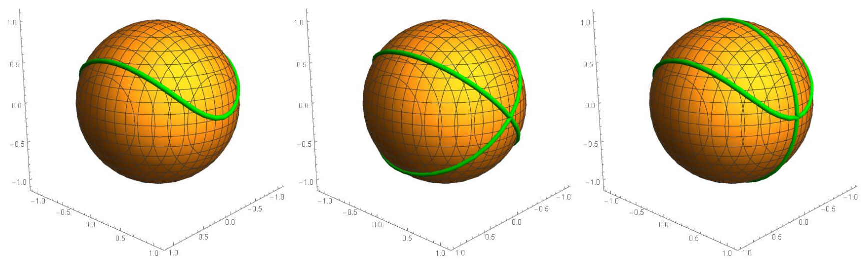

FIGURE 12. Generic non normally hyperbolic singularities at infinity. From the left to the right: fold (Example 21), transcritical (Example 22) and pitchfork (Example 23). The critical manifold is highlighted in green.

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8. Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] A. Cima, J. Llibre. *Bounded polynomial vector fields.* **Trans. of the American Math. Soc.** v. 318, n. 2, 557–579, 1990.

[2] F. Dumortier. *Compactification and desingularization of spaces of polynomial Liénard equations.* **J. Diff. Equations** 224, 296–313, 2006.

[3] F. Dumortier, C. Herssens. *Polynomial Liénard Equations near Infinity.* **J. Diff. Equations** 153, 1–29, 1999.

[4] F. Dumortier, C. Li. *Quadratic Liénard Equations with Quadratic Damping.* **J. Diff. Equations** 139, 41–59, 1997.

[5] F. Dumortier, J. Llibre, J.C. Artés. *Qualitative Theory of planar differential systems*. Universitext, 2006

[6] F. Dumortier, R. Roussarie. *Canard cycles and center manifolds*. **Memoirs Amer. Mat. Soc.** 121, 1996.

[7] F. Dumortier, R. Roussarie. *Geometric Singular Perturbation Theory Beyond Normal Hyperbolicity*. In: Jones C.K.R.T., Khibnik A.I. (eds) Multiple-Time-Scale Dynamical Systems. Springer, New York, NY, pp. 29–63, 2001.

[8] F. Dumortier, C. Rousseau. *Cubic Liénard equations with linear damping*. **Nonlinearity** 3, 1015–1039, 1990.

[9] N. Fenichel. *Geometric singular perturbation theory for ordinary differential equations*. **J. Diff. Eq.** 31, 53–98, 1979.

[10] A. Garcia, Y. Lequain. *Elementos de Álgebra*. Projeto Euclides, IMPA: Rio de Janeiro, 2001.

[11] E. A. González–Velasco. *Generic properties of polynomial vector fields at infinity*. **Trans. Amer. Math. Soc.** 143, 201–222, 1969.

[12] G. Hek. *Geometric singular perturbation theory in biological practice*. **J. Math. Biol.** 60, 347–386, 2010.

[13] E. Kappos. *Dynamics of polynomial systems at infinity*. **Eletronic J. Diff. Equations** 22, 1–15, 2001.

[14] M. Krupa, P. Szmolyan. *Extending geometric singular perturbation theory to nonhyperbolic points-fold and canard points in two dimensions*. **SIAM J. Math. Anal.**, 33(2), pp. 286–314, 2001.

[15] M. Krupa, P. Szmolyan. *Extending slow manifolds near transcritical and pitchfork singularities*. **Nonlinearity** 14, pp. 1473–1491, 2001.

[16] C. Kuehn. *Multiple Time Scale Dynamics. Applied Mathematical Sciences*, vol 191, Springer, 2015.

[17] H. Liang, J. Huang, Y. Zhao. *Classification of global phase portraits of planar quartic quasi-homogeneous polynomial differential systems*. **Nonlinear Dyn.** 78, 1659–1681, 2014.

[18] M.F.S. Lima, J. Llibre. *Global dynamics of the Benoît system*. **Annali di Matematica** 193, 1103–1122, 2014.

[19] R. Oliveira, Y. Zhao. *Structural stability of planar quasihomogeneous vector fields*. **Qual. Theory Dyn. Syst.** 13, 39–72, 2014.

[20] L. Perko. *Differential Equations and Dynamical Systems*. 3rd ed., Texts in Applied Mathematics, Springer, 2001.

[21] H. Poincaré. *Mémoire sur les courbes définies par une équation différentielle I*. **J. de mathématiques pures et appliquées**, 7, pp 375–422, 1881.

[22] B. Qiu, H. Liang. *Classification of global phase portrait of planar quintic quasi-Homogeneous coprime polynomial systems*. **Qual. Theory Dyn. Syst.** 16, 417–451, 2017.

[23] I.S. Meza–Sarmiento, R. Oliveira, P.R. da Silva. *Quadratic slow-fast systems on the plane. Nonlinear Analysis: Real World Applications* 60, 103286, 2021.

[24] P. Szmolyan. *Transversal heteroclinic and homoclinic orbits in singular perturbation problems*. **J. Diff. Eq.** 92, 252–281, 1991.

[25] B. van der Pol. *A theory of the amplitude of free and forced triode vibrations*. **Radio Review** 1, 701–710, 1920

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