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ON A QUASILINEAR SCHRÖDINGER-POISSON SYSTEM

GIOVANY M. FIGUEIREDO[1,†] & GAETANO SICILIANO[2,‡].

[1]Departamento de Matemática - Universidade de Brasilia, Brazil, [2]Instituto de Matemática e Estatística - Universidade São Paulo, Brazil.

[†]giovany@unb.br, [‡]sicilian@ime.usp.br

Abstract

We consider a quasilinear Schrödinger-Poisson system in $\mathbb{R}^3$ under a critical nonlinearity and depending on a parameter $\varepsilon > 0$. We prove existence of solutions and study the behaviour whenever ε tend to zero, recovering a solution of the classical Schrödinger-Poisson system.

1 Introduction

In the recent papers [2, 4] Kavian, Benmlih, Illner and Lange have attracted the attention on a new kind of elliptic system, which was already known in the physical literature: the *quasi-linear Schrödinger-Poisson system*, (see [1] where the authors proposed and discussed this new model from a physical point of view).

The existing literature on this problem is restricted to very few papers, in contrast to the literature concerning the well known and "classical" Schrödinger-Poisson system. The advantage of working with the classical Poisson equation is that the solution is explicitly given by the convolution $\phi^{\text{Poiss}}(u) = |\cdot|^{-1} * u^2$ (up to a multiplicative factor) so that many good properties of the solution are known; in particular the homogeneity $\phi^{\text{Poiss}}(tu) = t^2\phi^{\text{Poiss}}(u), t \in \mathbb{R}$. As a matter of fact, the main difficult dealing with the quasilinear Poisson equation of type

$$-\Delta\phi - \Delta_4\phi = u^2$$

is due exactly to the lack of good properties for the solution ϕ.

Here we consider a system where the Schrödinger equation has a critical nonlinearity and the electrostatic potential satisfy a quasilinear equation. More specifically, we are concerning here with the following system

$$\begin{cases} -\Delta u + u + \phi u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \qquad (P_{\lambda,\varepsilon})$$

where $\lambda > 0$ and $\varepsilon > 0$ are parameters, $2^* = 6$ is the critical Sobolev exponent in dimension 3, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the following assumptions

1. $f(x,t) = 0$ for $t \leq 0$,
2. $\lim_{t\to 0} \dfrac{f(x,t)}{t} = 0$, uniformly on $x \in \mathbb{R}^3$,
3. there exists $q \in (2, 2^*)$ verifying $\lim\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{q-1}} = 0$ uniformly on $x \in \mathbb{R}^3$,
4. there exists $\theta \in (4, 2^*)$ such that

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \leq tf(x,t), \quad \text{for all } x \in \mathbb{R}^3 \text{ and } t > 0.$$

2 Main results

The results we obtain are the following.

Theorem 2.1. *Assume that conditions* (1)-(4) *on* f *hold. Then, there exists* $\lambda^* > 0$, *such that*

$$\forall \lambda \geq \lambda^*, \varepsilon > 0 : \textit{problem } (P_\varepsilon) \textit{ admit a solution } (u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H^1(\mathbb{R}^3) \times \Big(D^{1,2}(\mathbb{R}^3) \cap D^{1,4}(\mathbb{R}^3)\Big).$$

Moreover $\phi_{\lambda,\varepsilon}, u_{\lambda,\varepsilon}$ *are nonnegative and for every fixed* $\varepsilon > 0$*:*

1. $\lim_{\lambda\to+\infty} \|u_{\lambda,\varepsilon}\|_{H^1} = 0$,
2. $\lim_{\lambda\to+\infty} \|\phi_{\lambda,\varepsilon}\|_{D^{1,2}\cap D^{1,4}} = 0$,
3. $\lim_{\lambda\to+\infty} |\phi_{\lambda,\varepsilon}|_\infty = 0$.

We study also the behaviour with respect to ε of the solutions given in Theorem 2.1, indeed we prove they converge to the solution of the Schrödinger-Poisson system.

Theorem 2.2. *Assume that conditions* (1)-(4) *hold. Let* $\lambda^* > 0$ *be the one given in Theorem 2.1 and* $\overline{\lambda} \geq \lambda^*$ *be fixed. Let* $\{(u_{\overline{\lambda},\varepsilon}, \phi_{\overline{\lambda},\varepsilon})\}_{\varepsilon>0}$ *be the solutions given above in correspondence of such fixed* $\overline{\lambda}$*. Then*

1. $\lim_{\varepsilon\to0^+} u_{\overline{\lambda},\varepsilon} = u_{\overline{\lambda},0}$ *in* $H^1(\mathbb{R}^3)$,
2. $\lim_{\varepsilon\to0^+} \phi_{\overline{\lambda},\varepsilon} = \phi_{\overline{\lambda},0}$ *in* $D^{1,2}(\mathbb{R}^3)$,

where $(u_{\overline{\lambda},0}, \phi_{\overline{\lambda},0}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *is a positive solution of the Schrödinger-Poisson system*

$$\begin{cases} -\Delta u + u + \phi u = \overline{\lambda} f(x,u) + |u|^{2^*-2}u & \textit{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \textit{in } \mathbb{R}^3. \end{cases}$$

The important point of Theorem 2.1 is the vanishing of the solutions whenever λ is larger and larger. Moreover, thanks to a Moser iteration scheme, we get $u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon} \in L^\infty(\mathbb{R}^3)$. This allow us to treat also the supercritical case, that is when $p > 2^*$ and indeed we have similar results.

Our approach is variational; indeed a suitable functional can be defined whose critical points are exactly the solutions of (P_ε). Then suitable estimates permits to pass to the limit in ε.

In proving our results, we have to manage with various difficulties. Firstly, the fact that the problem is in the whole $\mathbb{R}^3$ and no symmetry conditions on the solutions and on the datum f are imposed; even more we are in the critical case, then there is a clear lack of compactness. We are able to overcome this difficulty thanks to the Concentration Compactness of Lions and taking advantage of the parameter λ.

Secondly, we have to face with the fact that the solution in the second equation of (P_ε), which is quasilinear, has not an explicit formula, neither has homogeneity properties. To circumvent this last difficulty, a suitable truncation is used in front of the "bad" part of the functional.

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