




Limited-type completely continuous operators on Banach lattices

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Abstract

This paper is devoted to the study of *disjoint p -limited completely continuous* (dp -lcc) operators, a new class of operators naturally associated with the notion of p -limited sets in Banach lattices. We establish connections between this new class of operators with other limited-type completely continuous classes of operators on Banach lattices. A new Gelfand–Phillips-type property related to the dp -lcc operators is defined. As an application of our results, we provide necessary and sufficient conditions under which the adjoint of every positive dp -lcc operator between two given Banach lattices is also dp -lcc. We also obtain the duality results for others limited-type completely continuous operators.

Keywords p -Limited set · Almost p -limited set · p -Limited completely continuous operator · Almost p -limited completely continuous operator · Gelfand–Phillips property

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1 Introduction

Recall that a subset A of a Banach space X is said to be *limited* if every weak* null sequence in the dual space X^* converges uniformly to 0 on A . Or equivalently, if for every bounded linear operator $T : X \rightarrow c_0$, $T(A)$ is a relatively compact set. Notions related to limitedness have been considered in the mathematical literature. For instance, in the context of summability properties, the so-called *p-limited* ($1 \leq p < \infty$) sets were defined by Karn and Sinha [22] and further studied by Delgado and Piñeiro [12], Galindo and Miranda [18, 19], and Ghenciu [20, 21]. We recall the definition of *p-limited* sets: a subset A of a Banach space X is a *p-limited* set ($1 \leq p < \infty$) if for every weak* *p*-summable sequence $(x_n^*)_n \subset X^*$, there is $a = (a_n)_n \in \ell_p$, such that $|x_n^*(x)| \leq a_n$ for every $x \in A$ and $n \in \mathbb{N}$. Or equivalently, if for every bounded linear operator $T : X \rightarrow \ell_p$, $T(A)$ is an order bounded subset of ℓ_p . The lattice-theoretic variation of limited and *p-limited* sets which were introduced in [11] and [19], respectively, are defined as follows: a subset A of a Banach lattice E is *almost limited* (resp. *almost p-limited*) if for each disjoint weak* null (resp. disjoint weak* *p*-summable) sequence $(x_n^*)_n \subset E^*$, $(\sup_{x \in A} |x_n^*(x)|)_n \in c_0$ (resp. $(\sup_{x \in A} |x_n^*(x)|)_n \in \ell_p$). By a disjoint operator $T : E \rightarrow c_0$ (resp. $T : E \rightarrow \ell_p$), we mean that the weak* null (resp. weak* *p*-summable) sequence $(x_n^*)_n \subset E^*$ that defines T , i.e., $T(x) = (x_n^*(x))_n$ for every $x \in E$, is disjoint.

These classes of sets play an important role in the study of certain properties of Banach spaces and Banach lattices, as well as in the theory of bounded linear operators. Notably, the *Gelfand–Phillips* property, along with its variants—the *strong Gelfand–Phillips property*, *p-Gelfand–Phillips*, *weak p-Gelfand–Phillips* and *strong p-Gelfand–Phillips* properties—establish links between limitedness and compactness in Banach lattices (see [6, 8, 12, 16]). In the scope of operators, we have the following refinements of the well-known class of completely continuous operators: a bounded linear operator $T : X \rightarrow Y$ between two Banach spaces is said to be:

- *limited completely continuous* (lcc) if it maps limited weakly null sequences into norm null sequences (see [24]).
- *p-limited completely continuous* (*p*-lcc) if it maps *p*-limited weakly null sequences into norm null sequences (see [21]). Also, a bounded linear operator $T : E \rightarrow Y$ from a Banach lattice E into a Banach space Y is:
- *almost limited completely continuous* (alcc) if it maps almost limited weakly null sequences into norm null sequences (see [8]).
- *almost p-limited completely continuous* (*ap*-lcc) if it maps almost *p*-limited weakly null sequences into norm null sequences (see [7]).

The Gelfand–Phillips-type properties mentioned above can be characterized in terms of the classes of operators defined above. Indeed, a Banach space X has the GP property (resp. the weak *p*-GP property) if and only if the identity operator Id_X is lcc (resp. *p*-lcc). Also, a Banach lattice E has the strong *p*-GP property if and only if Id_E is *ap*-lcc. Finally, a Banach lattice E with order continuous norm

has the strong GP property if and only if Id_E is alcc. Thus, we have the following list of examples that will be used through this manuscript without further mentioning:

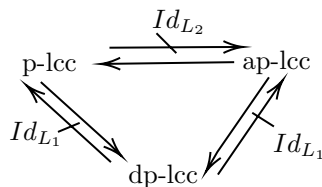
Example 1.1

- (1) If X is reflexive or separable, then Id_X is lcc. On the other hand, Id_{ℓ_∞} is not lcc, see [24].
- (2) If X is reflexive or a separable dual space, then Id_X is p -lcc. However, $Id_{c_0}, Id_{L_1((0,1))}$, and Id_{ℓ_∞} fail to be p -lcc, see [6].
- (3) If E is a discrete Banach lattice with order continuous norm, then Id_E is alcc. Nevertheless, $Id_{L_1((0,1))}$, and Id_{ℓ_∞} are not alcc, see [8].
- (4) If E has the Schur property, then Id_E is ap -lcc. Also, if E is a discrete KB-space, then Id_E is an ap -lcc operator whenever $p \geq 2$. However, $Id_{c_0}, Id_{C(K)}$ for any Hausdorff compact topological space, $Id_{L_p((0,1))}$ with $p \geq 1$ all fail to be ap -lcc, see [7].

In Banach lattices, disjoint sequences often provide the appropriate framework to study classical phenomena. This is reflected, for instance, in the notions of almost limited, almost Dunford–Pettis, and almost p -limited sets, as well as in lattice versions of classical topological and geometric properties of Banach spaces, where disjointness plays a central role. Therefore, when looking for a disjoint analog of p -completely continuous operators, it is natural to restrict the class of testing sequences accordingly. Motivated by this perspective, a sequence $(x_n)_n$ in a Banach lattice E is said to be *disjoint p -limited* if it is disjoint and $\{x_n : n \in \mathbb{N}\}$ is a p -limited subset of E . This leads to the following definition.

Definition 1.2 A bounded operator $T : E \rightarrow X$ is disjoint p -limited completely continuous (*abbr.* dp -lcc) if for every disjoint p -limited sequence $(x_n) \subset E, \|Tx_n\| \rightarrow 0$.

The main purpose of this paper is to initiate a systematic study of the dp -lcc operators, establishing its connections with the other completely continuous-type classes mentioned above. In particular, the diagram below follows from Example 2.1:



In the above diagram, the cut arrows mean that the implication does not hold and the counterexample is the set within the arrow. Moreover, it will follow from Corollary 2.6 that every dp -lcc operator is lcc, whenever $p \geq 2$.

Naturally, a new Gelfand–Phillips-type property related to the class of dp -lcc operators is defined and studied in Sect. 2. In particular, we prove that in a discrete Banach lattice E , this new property is equivalent to the strong p -GP property, the p -GP property and the weak p -GP property whenever $p \geq 2$. Section 3 of this manuscript is devoted to the duality theory of the limited-type completely continuous operators mentioned above. For instance, we proved that the adjoint of every positive dp -lcc (resp. lcc) operator $T : E \rightarrow F$ is dp -lcc (resp. lcc) if and only if E^* has order continuous norm or F^* has the disjoint p -GP property (resp. E^* or F^* has order continuous norm).

We refer the reader to [2, 23] for background on Banach lattices and to [1, 16] for Banach space theory. Throughout this paper, X, Y will denote Banach spaces and E, F will denote Banach lattices. For a Banach space X , B_X denotes its closed unit ball, S_X denotes its unit sphere and X^* denotes its topological dual. For a subset $A \subset E$, we denote by $\text{sol}(A)$ its solid hull. We denote by E^+ the set of all positive elements in E and by A^+ the set of all positive elements in $A \subset E$. Throughout this article, we assume that $1 \leq p < \infty$, unless otherwise stated.

2 Disjoint p -GP property and dp -lcc operators

We begin this section providing examples of dp -lcc operators:

Example 2.1

- (1) It is obvious that every p -lcc operator defined on a Banach lattice is dp -lcc.
- (2) If $T : E \rightarrow X$ is an ap -lcc operator, then T is dp -lcc. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence in E , then $(x_n)_n$ is clearly an almost p -limited sequence. Moreover, since p -limited sets are relatively weakly compact (see [12, Proposition 2.1(2)]), it follows from [2, Theorem 4.34] that $(x_n)_n$ is weakly null. By the assumption, we get that $\|Tx_n\| \rightarrow 0$, proving that T is dp -lcc.
- (3) We recall that a Banach lattice E is said to have the positive Schur property if every positive/disjoint weakly null sequence in E is norm null (see, e.g., [25]). As disjoint p -limited sequences are weakly null, every bounded linear operator defined on a Banach lattice with the positive Schur property is dp -lcc. Thus, the identity operator $Id_{L_p((0,1])}$ is dp -lcc that fails to be ap -lcc and p -lcc too.
- (4) We recall that a bounded linear operator $T : E \rightarrow X$ is said to be disjoint p -convergent if it maps disjoint weakly p -summable sequences to norm null sequences. If E^* has cotype $2 \leq p < \infty$ (see [13, p. 217] or [1, Definition 6.2.10] for definition), then every disjoint p -convergent operator $T : E \rightarrow X$ is dp -lcc. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence in E , then $\{x_n : n \in \mathbb{N}\}$ is a relatively weakly compact set (see [12, Proposition 2.1(2)]), and so $(x_n)_n$ is

- weakly p -summable by [9, Lemma 2.4]. Since T is disjoint p -convergent, we conclude that $\|Tx_n\| \rightarrow 0$ which proves that T is dp -lcc.
- (5) The converse of item (4) is false. For instance, for each $p > 1$, the identity operator $Id_{\mathcal{L}_{p^*}}$ is not disjoint p -summing, because $(e_n)_n$ is a normalized disjoint weakly p -summing sequence in \mathcal{L}_{p^*} . However, $Id_{\mathcal{L}_{p^*}}$ is a dp -lcc operator, because reflexive spaces have the strong p -GP property and disjoint p -limited sequences are weakly null and almost p -limited, thus norm null by [6, Theorem 3.3].
 - (6) We recall that a norm-bounded subset A of a Banach space X (resp. Banach lattice E) is said to be *Dunford-Pettis* (resp. *almost Dunford-Pettis*) if every weakly null sequence in X^* (resp. disjoint weakly null sequence in E^*) converges uniformly to zero on A . In a Banach lattice with the weak Grothendieck property (see, e.g., [17]), every almost Dunford–Pettis set is almost limited. Thus, if E is a Banach lattice with the weak Grothendieck property, then every alcc operator $T : E \rightarrow X$ is dp -lcc. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence, then $(x_n)_n$ is weakly null and Dunford–Pettis by [6, Lemma 3.4]. Hence, $(x_n)_n$ is an almost Dunford–Pettis sequence, and by the assumption $(x_n)_n$ is almost limited. Therefore $\|Tx_n\| \rightarrow 0$, proving that T is dp -lcc.
 - (7) The conclusion in item (6) is false in general, as the identity operator Id_{c_0} is alcc, but it is not dp -lcc, since $(e_n)_n$ is a disjoint normalized p -limited sequence in c_0 . Moreover, $Id_{L_1([0,1])}$ is a dp -lcc operator that is not alcc.

Our next result proves that the dp -lcc operators are precisely those bounded linear operators that take p -limited sets whose elements are mutually disjoint into relatively compact sets.

Theorem 2.2 *A bounded linear operator $T : E \rightarrow X$ is dp -lcc if and only if it maps p -limited sets whose elements are mutually disjoint into relatively compact sets.*

Proof Let $T : E \rightarrow X$ be a dp -lcc operator and let $A \subset E$ be a p -limited set whose elements are mutually disjoint. To prove that $T(A)$ is a relatively compact subset of X , take $(y_n)_n \subset T(A)$. Thus, for each $n \in \mathbb{N}$, there $x_n \in A$, such that $y_n = T(x_n)$. By the assumption on A , we have that $(x_n)_n$ is a disjoint p -limited sequence, and hence, $\|y_n\| = \|Tx_n\| \rightarrow 0$. Therefore, $T(A)$ is relatively compact set in X .

Conversely, assume that T maps p -limited sets whose elements are mutually disjoint into relatively compact subsets. If $(x_n)_n$ is a disjoint p -limited sequence in E , then $A = \{x_n : n \in \mathbb{N}\}$ is a p -limited set whose elements are mutually disjoint, and by the assumption, $T(A)$ is a relatively compact set in X . On the other hand, since disjoint p -limited sequences are weakly null, we have that $Tx_n \xrightarrow{\omega} 0$ in X . Finally, since $\{Tx_n : n \in \mathbb{N}\}$ is a relatively compact set such that $Tx_n \xrightarrow{\omega} 0$, we conclude that $\|Tx_n\| \rightarrow 0$, proving that T is dp -lcc. □

Before proving our next result, we recall that a bounded linear operator $T : E \rightarrow X$ from a Banach lattice E into a Banach space X is order weakly compact if $T([-x, x])$ is a relatively weakly compact subset of X for every $x \in E^+$. Or, equivalently if T maps order bounded disjoint sequences into norm null sequences

([2, Theorem 5.57]). Also, for each bounded linear operator, there exists a lattice seminorm q_T on E defined by

$$q_T(x) = \sup \{ \|T(y)\| : |y| \leq |x| \}, x \in E,$$

such that $\|T(x)\| \leq q_T(x) \leq \|T\| \cdot \|x\|$ for every $x \in E$ (see [23, p. 192] and [3, Section 2]).

Theorem 2.3 *Let $2 \leq p < \infty$. For a bounded linear operator $T : E \rightarrow X$, the following are equivalent:*

- (a) T is a dp -lcc operator.
- (b) T is order weakly compact and each relatively weakly compact almost p -limited subset W of E is approximately order bounded with respect to the lattice seminorm q_T , i.e., for each $\varepsilon > 0$, there exists some $u \in E^+$ such that $q_T((|x| - u)^+) \leq \varepsilon$ holds for every $x \in W$.
- (c) For every weakly null almost p -limited sequence $(x_n) \subset E^+$, $\|Tx_n\| \rightarrow 0$.
- (d) For every weakly null p -limited sequence $(x_n) \subset E^+$, $\|Tx_n\| \rightarrow 0$.
- (e) For every disjoint p -limited sequence $(x_n) \subset E^+$, $\|Tx_n\| \rightarrow 0$.

Proof (a) \Rightarrow (b) We begin checking that T is order weakly compact. Indeed, if $(x_n)_n$ is an order bounded disjoint sequence in E , then $(x_n)_n$ is p -limited by [19, Remark 3.3(4) and Proposition 3.8], and so $\|Tx_n\| \rightarrow 0$ which implies that T is order weakly compact. Let W be a relatively weakly compact almost p -limited subset of E and $\varepsilon > 0$. By combining [2, Theorem 4.34] and [19, Propositions 3.5 and 3.8], we have that every disjoint sequence in $\text{sol}(W)$ is weakly null and p -limited, and hence, $\|Tx_n\| \rightarrow 0$ for every disjoint sequence $(y_n)_n \subset \text{sol}(W)$. Now, we show that $q_T(x_n) \rightarrow 0$ for each disjoint sequence (x_n) in $\text{sol}(W)$. Indeed, by the definition of q_T , there exists a sequence $(y_n)_n \subset E$, such that $|y_n| \leq |x_n|$ and $q_T(x_n) \leq 2\|Ty_n\|$. As $(y_n)_n$ is a disjoint sequence contained in $\text{sol}(W)$, we obtain that $q_T(x_n) \leq \|Ty_n\| \rightarrow 0$. Thus, [2, Theorem 4.36], there exists some $u \in E^+$, such that $q_T((|x| - u)^+) \leq \varepsilon$ for every $x \in \text{sol}(W)$, proving that W is approximately order bounded with respect to the lattice seminorm q_T .

(b) \Rightarrow (c) Let $(x_n)_n$ be a positive weakly null almost p -limited sequence in E and $\varepsilon > 0$. It is clear that $W = \{x_n : n \in \mathbb{N}\}$ is a relatively weakly compact almost p -limited subset of E . By the assumption, there exists some $u \in E^+$, such that $\|T(x_n - u)^+\| \leq q_T((x_n - u)^+) < \frac{\varepsilon}{2}$ for every $n \in \mathbb{N}$. On the other hand, since $0 \leq x_n \wedge u \leq x_n$ holds for every n , we have that $(x_n \wedge u)_n$ is a positive weakly null order bounded sequence in E . Since $T : E \rightarrow X$ is order weakly compact, $\|T(x_n \wedge u)\| \rightarrow 0$ (see [23, Corollary 3.4.9]), and so, there exists $n_0 \in \mathbb{N}$, such that $\|T(x_n \wedge u)\| < \frac{\varepsilon}{2}$ for every $n \geq n_0$. By [2, Theorem 1.7(1)] and the above inequalities,

$$\|Tx_n\| = \|T(x_n \wedge u + (x_n - u)^+)\| \leq \|T(x_n - u)^+\| + \|T(x_n \wedge u)\| < \varepsilon$$

holds for every $n \geq n_0$, proving that $\|Tx_n\| \rightarrow 0$.

The implications (c) \Rightarrow (d) \Rightarrow (e) are obvious.

(e) \Rightarrow (a) If $(x_n)_n$ is a disjoint p -limited sequence in E , then $(|x_n|)_n$ is p -limited by [19, Propositions 3.6 and 3.8] and weakly null by [27, Proposition 1.3]. Thus, the assumption yields that $\|Tx_n\| \rightarrow 0$. \square

Theorem 2.4 *Let $p \geq 2$ and let E and F be two Banach lattices, such that the lattice operations on E are weakly sequentially continuous.*

- (1) *A positive linear operator $T : E \rightarrow F$ is dp -lcc if and only if it is p -lcc.*
- (2) *A bounded linear operator $T : E \rightarrow F$ is dp -lcc if and only if it is ap -lcc.*

Proof

- (1) It was noticed in Examples 2.1(1) that every p -lcc operator is dp -lcc. For the converse, we assume that $T : E \rightarrow F$ is a positive dp -lcc operator and let $(x_n)_n \subset E$ be a p -limited weakly null sequence. We shall prove that $\|T(x_n)\| \rightarrow 0$. To this end, let $\varepsilon > 0$. On the one hand, as $\{x_n : n \in \mathbb{N}\}$ is a relatively weakly compact almost p -limited subset of E and T is a dp -lcc operator there exists, by Theorem 2.3, some $u \in E^+$, such that

$$\|T(|x_n| - u)^+\| \leq q_T((|x_n| - u)^+) \leq \varepsilon/2$$

for every $n \in \mathbb{N}$. On the other hand, since the lattice operations on E are weakly sequentially continuous, $|x_n| \xrightarrow{\omega} 0$, and hence, $\|T(|x_n| \wedge u)\| \leq \|T|x_n|\| \rightarrow 0$, because T is order weakly compact (see [23, Corollary 3.4.9]). Therefore, there exists $n_0 \in \mathbb{N}$, such that $\|Tx_n\| < \varepsilon/2$ for every $n \geq n_0$. Finally, using [2, Theorem 1.7(1)], the positivity of T , and the above inequalities,

$$\begin{aligned} \|Tx_n\| &\leq \|T(|x_n|)\| = \|T(|x_n| \wedge u + (|x_n| - u)^+)\| \\ &\leq \|(|Tx_n| - u)^+\| + \|T(x_n) \wedge u\| < \varepsilon, \end{aligned}$$

proving that T is p -lcc.

- (2) It was noticed in Examples 2.1(2) that each ap -lcc operator is dp -lcc. For the converse, let $T : E \rightarrow F$ be a dp -lcc operator. If $(x_n)_n$ is an almost p -limited weakly null sequence in E , then $(|x_n|)_n \subset E^+$ is also an almost p -limited weakly null sequence, and by Theorem 2.3, we get that $\|T(x_n)\| \rightarrow 0$. \square

By Examples 2.1(3), Theorem 2.4 does not hold without the assumption on E having weakly sequentially continuous lattice operations. The following is an immediate consequence of Theorem 2.3.

Corollary 2.5 *Let $p \geq 2$. If a positive operator $S : E \rightarrow F$ between Banach lattices is dominated by a dp -lcc operator, then S itself is dp -lcc.*

As a final application of Theorem 2.3, we prove that every dp -lcc operator is lcc, whenever $p \geq 2$. Hence, it follows from [4, Theorem 2.5 and Theorem 2.6] that every dp -lcc operator on a Banach lattice, such that its dual has the weak* sequentially continuous lattice operations, is alcc, whenever $p \geq 2$.

Corollary 2.6 *Every order weakly compact operator $T : E \rightarrow X$ is lcc. Consequently, every dp -lcc operator is lcc, whenever $p \geq 2$.*

Proof If $T : E \rightarrow X$ is an order weakly compact operator, there are a Banach lattice F with order continuous norm, a lattice homomorphism $Q : E \rightarrow F$, and a bounded linear operator $S : F \rightarrow X$, such that $T = S \circ Q$ (see [2, Theorem 5.58]). Since every Banach lattice with order continuous norm has the GP property (see [26, Theorem 4.5]), for each limited weakly null sequence $(z_n)_n$ in E , $(Q(z_n))_n$ is norm null in F , and so, $\|T(z_n)\| = \|S(Q(z_n))\| \rightarrow 0$, proving that T is lcc. The final commentary follows from Theorem 2.3. \square

The converse of Corollary 2.6 is false, since the identity operator Id_{c_0} is an lcc operator that fails to be dp -lcc. However, as we will see in the next proposition, every lcc operator on a Banach lattice with the Grothendieck property is dp -lcc. First, we recall that a Banach space X has the Grothendieck property if every weak* null sequence in X^* is weakly null. In particular, every Dunford–Pettis set is a limited set in a Banach space with the Grothendieck property.

Proposition 2.7 *Let E be a Banach lattice with the Grothendieck property and X be a Banach space. If $T : E \rightarrow X$ is an lcc operator, then T is dp -lcc.*

Proof Suppose that T is lcc. If $(x_n)_n$ is a disjoint p -limited sequence, then $(x_n)_n$ is a Dunford–Pettis sequence in E (see [6, Lemma 3.4(1)]), and so, $(x_n)_n$ is a limited sequence in E by the Grothendieck property of E . Also, since disjoint p -limited sequences are weakly null, we obtain that $x_n \xrightarrow{\omega} 0$. Thus, $\|Tx_n\| \rightarrow 0$, proving that T is a dp -lcc operator. \square

Now, we introduce the Gelfand–Phillips-type property related to the class of dp -lcc operators.

Definition 2.8 A Banach lattice E is said to have the *disjoint p -GP property* if the identity operator Id_E is dp -lcc, that is every disjoint p -limited sequence in E is norm null.

Before providing examples, we notice that this property appeared, without this name, in [7, Theorem 2.10, Corollary 3.5 and Corollary 3.10].

Example 2.9

- (1) Every Banach lattice with the positive Schur property has the disjoint p -GP property. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence in a Banach lattice E , then, as already noticed in Examples 2.1(2), $x_n \xrightarrow{\omega} 0$. Therefore, assuming that E has the positive Schur property, we have that $\|x_n\| \rightarrow 0$.
- (2) Since p -lcc operators are dp -lcc, every Banach lattice with the weak p -GP property has the disjoint p -GP property. On the other hand, $L_1([0, 1])$ has the disjoint p -GP property that fails the weak p -GP property.
- (3) We recall from [18] that a bounded subset A of a Banach space X is said to be coarse p -limited if $T(A)$ is a relatively compact subset of ℓ_p for every bounded linear operator $T : X \rightarrow \ell_p$. A Banach space X is said to have the coarse p -GP property whenever every coarse p -limited subset of X is relatively compact. This property was introduced by P. Galindo and the second author in [18]. Every Banach lattice E with the coarse p -GP property has the disjoint p -GP property. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence in E , then $(x_n)_n$ is weakly null (see Examples 2.1(2)) and coarse p -limited by [18]. By [18, Theorem 3], we conclude that $\|x_n\| \rightarrow 0$, proving that E has the disjoint p -GP property.
- (4) Since there are reflexive Banach lattices failing the coarse p -GP property (see [18, Proposition 9]), and every reflexive Banach lattice has the weak p -GP property, hence the disjoint p -GP property, there are Banach lattices with the disjoint p -GP property, failing the coarse p -GP.
- (5) If E has the disjoint p -GP property, then every closed sublattice F of E has the disjoint p -GP property. Indeed, if $i : F \rightarrow E$ is the inclusion operator and $(x_n)_n \subset F$ is a disjoint p -limited sequence, then $(i(x_n))_n$ is a disjoint p -limited sequence in E , and so, $\|x_n\| = \|i(x_n)\| \rightarrow 0$.
- (6) Since the canonical basis $(e_n)_n$ of c_0 is a normalized disjoint p -limited sequence, c_0 does not have the disjoint p -GP property. Consequently, every Banach lattice with the disjoint p -GP property is a KB-space.
- (7) If E^{**} has order continuous norm, then E has the disjoint p -GP property. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence, we already know that $|x_n| \xrightarrow{\omega} 0$. Moreover, it follows from [6, Lemma 3.4] that $(x_n)_n$ is a Dunford–Pettis sequence. Therefore, if $(x_n^*)_n$ is a norm-bounded disjoint sequence in $(E^*)^+$, $(x_n^*)_n$ is weakly null by [23, Theorem 2.4.14], and so, $x_n^*(x_n) \xrightarrow{\omega} 0$. By [14, Corollary 2.6], we conclude that $\|x_n\| \rightarrow 0$.
- (8) If E^* has the cotype $2 \leq p < \infty$, then E has the disjoint p -GP property. Indeed, if $(x_n)_n$ is a disjoint p -limited sequence, $|x_n| \xrightarrow{\omega} 0$. Also, it follows from [7, Lemma 3.2] that each norm-bounded disjoint sequence (f_n) in E^* is weakly p -summable. Since (x_n) is a p -limited sequence, $(f_n(x_n)) \in \ell_p$. Hence from [14, Corollary 2.6], $\|x_n\| \rightarrow 0$.
- (9) If E is a dual Banach lattice with order continuous norm, then it has the disjoint p -GP property. First, let $E = G^*$, for some Banach lattice G . We will use [14, Corollary 2.7] to prove that $\|x_n\| \rightarrow 0$ for every disjoint p -limited sequence $(x_n)_n \subset E$. On the one hand, as noticed already, every disjoint p -limited sequence is weakly null, and so, $|x_n| \xrightarrow{\omega} 0$ in E (see [27, Proposition 1.3]). In particular, $|x_n| \xrightarrow{\omega^*} 0$ in $E = G^*$. On the other hand, if $(z_n)_n$ is a norm-bounded disjoint

sequence in G^+ , then $z_n \xrightarrow{\omega} 0$ in G by [23, Theorem 2.4.14], and hence, $(J(z_n))_n$ is weakly null sequence in $G^{**} = E^*$, where $J : G \rightarrow G^{**}$ denotes the canonical embedding. Finally, since p -limited sets are Dunford-Pettis (see [6, Lemma 3.4]), we have that $x_n(z_n) = J(z_n)(x_n) \rightarrow 0$. Therefore, $\|x_n\| \rightarrow 0$ as desired.

Our following result is similar to [6, Proposition 2.4], and hence, we will omit its proof.

Theorem 2.10 *For a Banach lattice E the following statements are equivalent:*

- (a) *E has the disjoint p -GP property.*
- (b) *Every bounded linear operator from E into an arbitrary Banach space Y is dp -lcc.*
- (c) *Every bounded linear operator from E into ℓ_∞ is dp -lcc.*

Before proving an interesting characterization of the disjoint p -GP property for $p \geq 2$, we recall that a norm-bounded subset A of a Banach lattice E is L -weakly compact if every disjoint sequence contained in $\text{sol}(A)$ is norm null (see [23, Definition 3.6.1]).

Proposition 2.11 *Let $p \geq 2$. For a Banach lattice E , the following statements are equivalent:*

- (a) *E has the disjoint p -GP property.*
- (b) *For every weakly null almost p -limited sequence $(x_n) \subset E^+$, $\|x_n\| \rightarrow 0$.*
- (c) *For every weakly null p -limited sequence $(x_n) \subset E^+$, $\|x_n\| \rightarrow 0$.*
- (d) *For every disjoint p -limited sequence $(x_n) \subset E^+$, $\|x_n\| \rightarrow 0$.*
- (e) *E is a KB-space and every almost p -limited subset of E is approximately order bounded.*
- (f) *Every almost p -limited set in E is L -weakly compact.*
- (g) *For each arbitrary Banach space X , every almost p -limited operator $S : X \rightarrow E$ is L -weakly compact.*

Proof The equivalences (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) follows from Theorem 2.3.

(a) \Rightarrow (e) Suppose that E has the disjoint p -GP. We have from Examples 2.9(6) that E is a KB-space. Now, if $A \subset E$ is an almost p -limited set, then A is also relatively weakly compact (see [6, Theorem 3.10]), and by applying Theorem 2.3 for Id_E , we obtain that A is approximately order bounded.

(e) \Rightarrow (a) Follows from [6, Theorem 3.10] and Theorem 2.3 for Id_E .

(a) \Rightarrow (f) Let $A \subset E$ be an almost p -limited set and let $(x_n)_n$ be a disjoint sequence contained in $\text{sol}(A)$. By applying [19, Propositions 3.5 and 3.8], $(x_n)_n$ is a p -limited sequence, and hence, $\|x_n\| \rightarrow 0$. Therefore, A is L -weakly compact.

(f) \Rightarrow (g) Immediate.

(g) \Rightarrow (b) Let $(x_n)_n \subset E^+$ be a weakly null almost p -limited sequence. We show that $\|x_n\| \rightarrow 0$. Since $|x_n| \xrightarrow{w} 0$, it is enough to show that for each norm-bounded positive disjoint sequence (f_n) in E^* , $f_n(x_n) \rightarrow 0$. For this, consider the operator $S : \ell_1 \rightarrow E$ defined by

$$S(\lambda) = \sum_{n=1}^{\infty} \lambda_n x_n, \text{ lambda} = (\lambda_n) \in \ell_1.$$

Since (x_n) is almost p -limited, we have the operator S is almost p -limited (see the proof of [6, Theorem 4.6(1)]). By our hypothesis, S is L -weakly compact and so $f_n(x_n) = f_n(Se_n) \rightarrow 0$ as desired. Thus, E has the disjoint p -GP property. \square

For a discrete Banach lattice, we have the following equivalence:

Corollary 2.12 *For a discrete Banach lattice E and $p \geq 2$, the following statements are equivalent:*

- (a) E has the disjoint p -GP property.
- (b) E has the strong p -GP property.
- (c) E has the p -GP property.
- (d) E has the weak p -GP property.

Proof The equivalences (a) \Leftrightarrow (b) \Leftrightarrow (c) follow from [5, Corollary 3.8], and the implication (d) \Rightarrow (a) was noticed in Examples 2.9(2). To prove (a) \Rightarrow (b), assume that E has the disjoint p -GP and take a weakly null almost p -limited sequence $(x_n)_n \subset E$. Since E has order continuous norm (because it is a KB-space), $|x_n| \xrightarrow{w} 0$ by [23, Proposition 2.5.23]. Moreover, $(|x_n|)_n$ is almost p -limited by [19, Proposition 3.5]. It follows from Proposition 2.11 that $\|x_n\| \rightarrow 0$, proving that E has the strong p -GP property (see [6, Theorem 3.3]). \square

3 Duality theory of limited-type completely continuous operators

Recall that in [28, Theorem 125.6], Zaanen studied the duality problem of the class of AM -compact operators. He established that if E and F are two Banach lattices such that F has order continuous norm, then every operator $T : E \rightarrow F$ with an AM -compact adjoint is AM -compact. For recent developments on this problem, we refer [10, 15]. In this section, we will address this problem for the classes of dp -lcc operators, lcc operators, alcc operators, and p -lcc operators. We notice that the duality problem for the ap -lcc operators were obtained in [7].

We begin with the dp -lcc operators:

Theorem 3.1 *The adjoint of every positive dp-lcc operator $T : E \rightarrow F$ is dp-lcc if and only if E^* has order continuous norm or F^* has the disjoint p-GP property.*

Proof Let $T : E \rightarrow F$ be a positive dp-lcc operator. If F^* has the disjoint p-GP property, then $T^* : F^* \rightarrow E^*$ is clearly a dp-lcc operator. Suppose now that E^* has order continuous norm and let $(y_n^*)_n$ be a disjoint p-limited sequence in F^* . Since $\|y_n^*\| \xrightarrow{\omega} 0$, we get that $\|T^*y_n^*\| \leq T^*\|y_n^*\| \xrightarrow{\omega} 0$ in E^* . Now, if $(x_n)_n$ is a norm-bounded disjoint sequence in E , then the assumption yields that $x_n \xrightarrow{\omega} 0$ in E (see [23, Theorem 2.4.14]), and hence $J(x_n) \xrightarrow{\omega} 0$ in E^{**} . Moreover, since p-limited sets are Dunford–Pettis sets (see [6, Lemma 3.4]), $(T^*y_n^*)_n$ is a Dunford–Pettis sequence in E^* , which yields that $T^*y_n^*(x_n) = J(x_n)(T^*y_n^*) \rightarrow 0$. We conclude by [14, Corollary 2.7] that $\|T^*y_n^*\| \rightarrow 0$.

To the converse, we suppose that the norm of E^* is not order continuous norm and that F^* fails the disjoint p-GP property. On the one hand, as E^* does not have order continuous norm, there exists a non-norm null positive disjoint order bounded sequence $(x_n^*)_n$ in E^* (see [23, Theorem 2.4.2]). Without loss of generality, we may assume that $\|x_n^*\| = 1$ for every $n \in \mathbb{N}$; thus, we may find a sequence $(x_n)_n \subset S_E^+$, such that $x_n^*(x_n) \geq 1/2$ for every $n \in \mathbb{N}$.

Letting $x^* \in E^*$ be such that $0 \leq x_n^* \leq x^*$ for every n , we have

$$\sum_{n=1}^m |x_n^*(x)| \leq \sum_{n=1}^m x_n^*(|x|) = \left(\bigvee_{n=1}^m x_n^* \right) (|x|) \leq x^*(|x|)$$

for every $m \in \mathbb{N}$ and every $x \in E$. Thus, $S : E \rightarrow \ell_1$ given by $S(x) = (x_n^*(x))_n$ is a well-defined (bounded) linear operator. On the other hand, by the failure of the disjoint p-GP property by F^* , there exists a disjoint p-limited $(y_n^*)_n$ in F^* , such that $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, there exists $y_n \in F^+$, such that $\|y_n\| \leq 1$ and $y_n^*(y_n) \geq \frac{1}{4}$ (see [10, Lemma 2.5]). Defining $R : \ell_1 \rightarrow F$ by $R((a_n)_n) = \sum_{n=1}^{\infty} a_n y_n$ and $T := R \circ S$, we have that T is a positive operator from E to F that is completely continuous, because it factors through ℓ_1 ; hence, T is dp-lcc. For any $k \in \mathbb{N}$, we have $x_k \geq 0, \|x_k\| = 1$ and

$$T^*(y_k^*)(x_k) = \sum_{n=1}^{\infty} x_n^*(x_k) y_k^*(y_n) \geq x_k^*(x_k) y_k^*(y_k) \geq \frac{1}{8},$$

so $\|T^*(y_k^*)\| \geq \frac{1}{8}$ for every $k \in \mathbb{N}$, which implies that T^* fails to be dp-lcc. □

Theorem 3.2 *Let E and F be two Banach lattices, such that F is a dual Banach lattice. Consider the following statements:*

- (a) *Every positive operator $T : E \rightarrow F$ is dp-lcc.*
- (b) *Every positive operator $T : E \rightarrow F$ with a dp-lcc adjoint is dp-lcc.*
- (c) *E has the disjoint p-GP property or F has order continuous norm.*

Thus, (a) \Rightarrow (b) \Rightarrow (c). Moreover, if $p \geq 2$, then (c) \Rightarrow (a).

Proof (a) \Rightarrow (b) is immediate.

(b) \Rightarrow (c) For the sake of contradiction, we assume that (c) fails. On the one hand, there exists a disjoint p -limited sequence (x_n) in E satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$. If $J : E \rightarrow E^{**}$ denotes the canonical embedding, by [10, Lemma 2.5], there exists a sequence a positive sequence $(x_n^*)_n \subset E^*$, such that $\|x_n^*\| = 1$ and $|x_n^*(x_n)| = |J(x_n)(x_n^*)| \geq \frac{1}{4}$ for every $n \in \mathbb{N}$. Thus, $Q(x) = (x_n^*(x))_n, x \in E$, defines a positive operator from E into ℓ_∞ . By the assumptions on F , there exists a lattice embedding $S : \ell_\infty \rightarrow F$ (see [23, Theorem 2.4.14]). Therefore, there exists a $C > 0$ such that $C\|(\lambda_n)\|_\infty \leq \|S(\lambda_n)\|$ for every $(\lambda_n) \in \ell_\infty$. Define $T = S \circ Q$. Since $T^* = Q^* \circ S^*$ factors through the Banach lattice ℓ_∞^* that has the positive Schur property, we have that T^* is an almost Dunford–Pettis operator, and hence dp -lcc, because disjoint p -limited sequences are weakly null. However, since $(x_n)_n$ is a disjoint p -limited sequence in E , such that

$$\|Tx_n\| = \|S(Qx_n)\| \geq C\|(x_k^*(x_n))_k\|_\infty \geq C|x_n^*(x_n)| \geq \frac{C}{4},$$

we obtain that T cannot be dp -lcc.

(c) \Rightarrow (a) Since every operator defined on a Banach lattice with the disjoint p -GP property is itself dp -lcc, we only need to prove the case where F is a dual space with order continuous norm. Indeed, notice that whenever this holds, F has the disjoint p -GP property by Example 2.9(9) it has the disjoint p -GP property. Therefore, if $T : E \rightarrow F$ is a positive operator and $(x_n)_n$ is a positive weakly null p -limited sequence in E , then $(Tx_n)_n$ is a positive weakly null p -limited sequence in F , and by Proposition 2.11 $\|Tx_n\| \rightarrow 0$. We conclude from Theorem 2.3 that T is a dp -lcc operator. \square

The assumption that F is a dual Banach lattice is essential in Theorem 3.2, since every bounded linear operator from ℓ_∞ into c is dp -lcc (see, e.g., [7]). However, ℓ_∞ does not have the disjoint p -GP property and c does not have order continuous norm (it is not a dual space).

We may obtain the implication (c) \Rightarrow (b) of Theorem 3.2 under different assumptions on F :

Proposition 3.3 *Let $T : E \rightarrow F$ be a positive operator between two Banach lattices, such that $T^* : F^* \rightarrow E^*$ is a dp -lcc operator. If F has the weak Grothendieck property or that F^* has cotype $2 \leq p < \infty$, then T is dp -lcc.*

Proof Let $T : E \rightarrow F$ be a positive operator whose adjoint $T^* : F^* \rightarrow E^*$ is dp -lcc. If $(x_n)_n$ is a disjoint p -limited sequence in E , we already known that $|x_n| \xrightarrow{\omega} 0$ in E , and by the inequality $|Tx_n| \leq T|x_n|$, we obtain that $|Tx_n| \xrightarrow{\omega} 0$ in F . Now, if $(y_n^*)_n$ is a norm-bounded disjoint sequence in F^* , then $y_n^* \xrightarrow{\omega^*} 0$ in F^* . We will prove that $y_n^*(Tx_n) \rightarrow 0$ under the assumptions on F , and by [14, Corollary 2.6], we will conclude that $\|Tx_n\| \rightarrow 0$. First, assuming that F has the weak Grothendieck property, we

get that $y_n^* \xrightarrow{\omega} 0$ in F^* , and since p -limited sets are Dunford–Pettis sets by [6, Lemma 3.4], we conclude that $y_n^*(Tx_n) \rightarrow 0$. Finally, assuming that F^* has cotype $2 \leq p < \infty$, $(y_n^*)_n$ is weakly p -summable (see [7, Lemma 3.2]), and so, $(y_n^*(Tx_n))_n \in \ell_p$, because $(Tx_n)_n$ is a p -limited sequence, and we are done. \square

Now, we consider the class of lcc operators.

Theorem 3.4 *Let E and F be two Banach lattices. The adjoint of every positive lcc operator $T : E \rightarrow F$ is lcc if and only if E^* or F^* has order continuous norm.*

Proof Suppose that neither of the two spaces E^* and F^* has order continuous. On the one hand, as E^* does not have order continuous norm, there exists a non-norm null positive disjoint order bounded sequence $(x_n^*)_n$ in E^* (see [23, Theorem 2.4.2]). Without loss of generality, we may assume that $\|x_n^*\| = 1$ for every $n \in \mathbb{N}$; thus, we may find a sequence $(x_n)_n \subset S_E^+$, such that $x_n^*(x_n) \geq 1/2$ for every $n \in \mathbb{N}$. Letting $x^* \in E^*$ be such that $0 \leq x_n^* \leq x^*$ for every n , we have

$$\sum_{n=1}^m |x_n^*(x)| \leq \sum_{n=1}^m x_n^*(|x|) = \left(\bigvee_{n=1}^m x_n^* \right) (|x|) \leq x^*(|x|)$$

for every $m \in \mathbb{N}$ and every $x \in E$. Thus, $S : E \rightarrow \ell_1$ given by $S(x) = (x_n^*(x))_n$ is a well-defined (bounded) linear operator. On the other hand, by [26, Theorem 4.5] and the failure of the GP property by F^* , there exists a weakly null limited sequence $(y_n^*)_n$ in F^* , such that $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, there exists $y_n \in F^+$, such that $\|y_n\| \leq 1$ and $y_n^*(y_n) \geq \frac{1}{4}$ (see [10, Lemma 2.5]). Defining $R : \ell_1 \rightarrow F$ by $R((a_n)_n) = \sum_{n=1}^{\infty} a_n y_n$ and $T := R \circ S$, we have that T is a positive operator from E to F that is completely continuous, because it factors through ℓ_1 ; hence, T is lcc. For any $k \in \mathbb{N}$, we have $x_k \geq 0$, $\|x_k\| = 1$ and

$$T^*(y_k^*)(x_k) = \sum_{n=1}^{\infty} x_n^*(x_k) y_k^*(y_n) \geq x_k^*(x_k) y_k^*(y_k) \geq \frac{1}{8},$$

so $\|T^*(y_k^*)\| \geq \frac{1}{8}$ for every $k \in \mathbb{N}$, which implies that T^* fails to be lcc.

If E^* or F^* has order continuous norm, then E^* or F^* has the GP property [26, Theorem 4.5]. Thus, every operator from F^* into E^* is lcc. \square

Theorem 3.5 *Let E and F be two Banach lattices, such that F is σ -Dedekind complete. The following are equivalent:*

- (a) Every operator $T : E \rightarrow F$ is lcc.
- (b) Every positive operator $T : E \rightarrow F$ is lcc.
- (c) Every positive operator $T : E \rightarrow F$ with an lcc adjoint is lcc.
- (d) E has the GP property or F has order continuous norm.

Proof The implications (a) \Rightarrow (b) \Rightarrow (c) are immediate.

(c) \Rightarrow (d) We assume for the sake of contradiction that (d) fails. On the one hand, since E does not have the GP property, there exists a limited weakly null sequence (x_n) in E satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$. By [10, Lemma 2.5], there exists a positive $(x_n^*)_n \subset E^*$, such that $\|x_n^*\| \leq 1$ and $x_n^*(x_n) = J(x_n)(x_n^*) \geq \frac{1}{4}$ holds for every $n \in \mathbb{N}$, where $J : E \rightarrow E^{**}$ denotes the canonical embedding. Therefore, $Q(x) = (x_n^*(x))_n, x \in E$, defines a positive operator from E into ℓ_∞ . On the other hand, by the assumptions on F , there exists a lattice embedding $S : \ell_\infty \rightarrow F$ (see [23, Corollary 2.4.3]). Thus, there exists $C > 0$, such that $C\|(\lambda_n)\|_\infty \leq \|S(\lambda_n)\|$, for all $(\lambda_n)_n \in \ell_\infty$. The positive operator $T = S \circ Q$ is not lcc, because $(x_n)_n$ is a limited weakly null sequence in E , such that

$$\|Tx_n\| = \|S(Qx_n)\| \geq C\|Qx_n\|_\infty \geq C|x_n^*(x_n)| \geq \frac{C}{4}$$

for every $n \in \mathbb{N}$. However, T^* is lcc, since it factors thorough ℓ_∞ that has the GP property.

(d) \Rightarrow (a) If E has the GP property, then it is immediate that every operator from E into F is lcc. On the other hand, if F has order continuous norm, then F has the GP property (see [26, Theorem 4.5]), and we are done. \square

The assumption that F is σ -Dedekind complete is essential in Theorem 3.5, since every bounded linear operator from ℓ_∞ into c is Dunford–Pettis (see, e.g., [7]), hence lcc. However ℓ_∞ does not have the GP property and c does not have order continuous norm (it is not σ -Dedekind complete).

Now, we provide necessary conditions, so that the adjoint of every alcc (resp. p -lcc) operator is also alcc (resp. p -lcc).

Theorem 3.6 *Let E and F be two Banach lattices.*

- (1) *Suppose that F^* has order continuous norm. If the adjoint of every positive alcc operator $T : E \rightarrow F$ is also alcc, then E^* has order continuous norm or F^* has the strong GP property.*
- (3) *If the adjoint of every positive p -lcc operator $T : E \rightarrow F$ is also p -lcc, then E^* has order continuous norm or F^* has the strong p -GP property.*

Proof (1) Suppose that the norm of E^* is not order continuous and that F^* fails the strong GP property. On the one hand, as E^* does not have order continuous norm, E contains a sublattice isomorphic to ℓ_1 and there exists a positive projection $P : E \rightarrow \ell_1$ (see [23, Theorem 2.4.14 and Proposition 2.3.11]). On the other hand, since F^* does not have the strong GP property, there exists an almost limited weakly null sequence $(y_n^*)_n$ in F^* , such that $\|y_n^*\| = 1$ for every $n \in \mathbb{N}$ (see [8, Theorem 2.15]). Moreover, it follows from [10, Lemma 2.5] that there exists $(y_n)_n \subset B_{F^+}$, such that $y_n^*(y_n) \geq \frac{1}{4}$ for every $n \in \mathbb{N}$. Consider the positive operator $R : \ell_1 \rightarrow F$ defined by

$R((a_n)_n) = \sum_{n=1}^{\infty} a_n y_n$. Thus, the positive operator $T := R \circ P$ is completely continuous, because it factors through ℓ_1 ; hence, T is alcc. However, T^* is not alcc. Indeed, as $P : E \rightarrow \ell_1$ is surjective, there exists $\delta > 0$, such that $\delta B_{\ell_1} \subset P(B_E)$, and hence

$$\begin{aligned} \|T^*(y_n^*)\| &= \sup_{x \in B_E} |y_n^*(Tx)| = \sup_{x \in B_E} |y_n^*(R(P(x)))| = \sup_{z \in P(B_E)} |y_n^*(R(z))| \\ &\geq \sup_{z \in \delta B_{\ell_1}} |y_n^*(R(z))| \geq |y_n^*(R(\delta e_n))| = \delta |y_n^*(y_n)| \geq \frac{\delta}{4} \end{aligned}$$

for every $n \in \mathbb{N}$. This proves that T^* is not an alcc operator, and we are done.

The proof of item (2) follows by reproducing the argument above and making the necessary adjustments. \square

The following example show that the converse of Theorem 3.6 is false, in general.

Example 3.7 Let $T : \ell_{\infty} \rightarrow L_2([0, 1])$ be the completely continuous positive operator from [7, Examples 2.12(2)]. Thus, T is both alcc and p -lcc. However, T^* takes $(r_n)_n$ into a normalized sequence in ℓ_{∞}^* , and since $(r_n)_n$ is both almost limited and p -limited, we obtain that T^* is neither an alcc operator nor a p -lcc operator.

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