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On the generating function of the decimal expansion of an irracional real number

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Abstract

In this paper we prove that the generating power series associated with the decimal expansion of an irrational real number is a transcendental function. Then we give a condition on the mod p reduced series which ensures that the given real number is transcendental.As an application we give another proof of the transcendence of the Morse number. 1991 Maths. Subject Classif.: 11J91, 30B10.

1 Introduction

Let α be a rational number with a decimal expansion given by $a_0, a_1a_2a_3...$ and let us construct the power series :

$$f_\alpha(z) = \sum_{n=1}^{\infty} a_n z^n$$

It is clear that this series has 1 as radius of convergence or is a polynomial. And since the decimal expansion of a rational number is eventually periodic, it is also clear that $f_\alpha(z)$ is the power series expansion of a rational function of z. The objective os this note is to analyze what happens if α is an irrational number. At first glance the above result could suggest that if α is an algebraic real number then the corresponding $f_\alpha(z)$ is the power series expansion of an algebraic function of z and if α is transcendental then $f_\alpha(z)$ is a transcendental function.

We prove the following:

Theorem 1 *Let α be an irrational real number. Then $f_\alpha(z)$ is a transcendental function.*

So, the power series $f_\alpha(z)$ does not capture the nature of the irrational number α, as regards to algebraicity and transcendency. But we can reduce $f_\alpha(z)$ module a prime p (greater then the highest digit appearing in the decimal expansion of α) and obtain a formal power series over the finite field F_p. We can also ask what happens to the reduced series $\widetilde{f_\alpha}(z)$, whether it is algebraic or not over the rational function field $F_p(z)$. We have:

Theorem 2 *If α is an irrational number and $\widetilde{f_\alpha}(z)$ is an algebraic function over $F_p(z)$ for some prime p as above, then α is transcendental.*

As an application of theorem 2 we give another proof of the transcendence of the Morse number, which can be defined from the following recursive process:

$$a_0 = 1 \qquad\qquad b_0 = 0$$

$$a_1 = a_0 b_0 \qquad b_1 = b_0 a_0$$

$$\ldots\ldots \qquad \ldots\ldots$$

$$a_{n+1} = a_n b_n \qquad b_{n+1} = b_n a_n$$

where $a_n b_n$ and $b_n a_n$ are simply the concatenation of the blocs a_n and b_n in the given order. The Morse number is defined by

$$\alpha = 0, d_0 d_1 d_2 \ldots$$

where $d_0, d_1, \ldots, d_{2^n}$ are defined respectively as the 2^n digits of a_n. So,

$$\alpha = 0, 1001011001101001\ldots$$

2 Proof of theorem 1

To prove theorem 1 we need some simple lemmas:

lemma 1 *If α is an irrational number then $f_\alpha(z)$ is not the power series expansion of a rational function.*

Proof: Let $P(z)$ and $Q(z)$ be two complex polynomials such that

$$\frac{P(z)}{Q(z)} = \sum_{n=0}^{\infty} a_n z^n \qquad (a_n \in \mathbb{C})$$

in some neighborhood of $z = 0$ and let σ be any authomorphism of $\mathbb{C}$. If we denote by $P^\sigma(z)$ the polynomial obtained by applying σ to all coefficients of $P(z)$ then it is easy to see that

$$\frac{P^\sigma(z)}{Q^\sigma(z)} = \sum_{n=0}^{\infty} a_n^\sigma z^n$$

in some neighborhood of $z = 0$. If all a_n are rational numbers then

$$\frac{P^\sigma(z)}{Q^\sigma(z)} = \frac{P(z)}{Q(z)}$$

for all $\sigma \in Aut(\mathbb{C})$. So, the zeros of $P(z)$ and of $Q(z)$ are algebraic numbers and they appear with all their conjugates. We can write:

$$\frac{P(z)}{Q(z)} = \frac{c\prod(z-\zeta_i)}{d\prod(z-\delta_i)}$$

where ζ_i, δ_i are algebraic numbers and the polynomials under the product sign have rational coefficients.

So, we see that $(c/d)^\sigma = (c/d)$ for all $\sigma \in Aut(\mathbb{C})$, and it follows that $(c/d) \in \mathbb{Q}$. We have thus proved that if a power series $\sum a_n z^n$ is the power series expansion of a rational complex function of z, then it is also the expansion of a rational function $P_1(z)/Q_1(z)$ with rational coefficients. But this proves lemma 1 because $\alpha = P_1(1/10)/Q_1(1/10)$ would be rational.

lemma 2 *If a power series with integer coefficients represents an algebraic function which is not rational then it's radius of convergence is lesser then* 1.

Proof: Let $F(z,w)$ be a polynomial of $\mathbb{C}[Z,W]$,

$$F(z,w) = a_0(z)w^m + a_1(z)w^{m-1} + \ldots + a_m(z) \tag{1}$$

and let $f(z)$ be an algebraic function of z such that

$$F(z, f(z)) = 0$$

Let $\beta_1, ..., \beta_k$ be a maximal linearly independent (over $\mathbb{Q}$) subset of the complex coefficients appearing in $F(z,w)$. We can write:

$$F(z,w) = \beta_1 F_1(z,w) + ... + \beta_k F_k(z,w)$$

where $F_i(z,w)$, $1 \leq i \leq k$, are polynomials with rational coefficients. So,

$$\sum_{i=1}^{k} \beta_i F_i(z, f(z)) = 0$$

and if we put

$$F_i(z, f(z)) = \sum_{n=0}^{\infty} \alpha^i{}_n z^n$$

we obtain

$$\sum_{n=0}^{\infty} (\sum_{i=1}^{k} \beta \alpha^i{}_n) z^n$$

Since $f(z)$ has a power series with integral coefficients, $\alpha^i{}_n$ are all rational and so, by the linear independence of $\beta_1, ..., \beta_k$ over $\mathbb{Q}$, we have:

$$F_i(z, f(z)) = 0 \qquad\qquad 1 \leq i \leq k$$

So without loss of generality ,we can suppose that $f(z)$ satisfies a polynomial (1) with integral coefficients. If we put $g(z) = a_0(z) f(z)$ then :

(2) $$g(z)^m + a_1(z) g(z)^{m-1} + a_2(z) a_0(z) g(z)^{m-2} + ... + a_m(z) a_0(z)^{m-1} = 0$$

and the power series for $g(z)$ also has integral coefficients. Now, such a g cannot have a pole in the finite plane $\mathbb{C}$ because equation (2) would give us different orders for this pole. Consequently $g(z)$

must be limited in the unit disc. But it is well known that if a powers series $\sum a_n z^n$ represents a limited function in the unit disc then

$$\sum |a_n|^2 < \infty$$

which is impossible if the a_n are integers, unless $\sum a_n z^n$ is a polynomial. This proves lemma 2.

We can now prove theorem 1: let α be an irrational number and $f_\alpha(z)$ the corresponding series. Then the radius of convergence of this series is 1. By lemma 1, $f_\alpha(z)$ is not a rational function and by lemma 2 it is not an algebraic function.

3 Reduction mod p

Let us now consider the formal reduced series

$$\widetilde{f_\alpha}(z) = \sum_{n=1}^{\infty} \bar{a}_n z^n$$

where $\bar{a}_n$ is the reduction of a_n mod p, and p is a prime greater than the highest digit of de decimal expansion of α. To prove theorem 2 we observe that if $\widetilde{f_\alpha}(z)$ is an algebraic function over $F_p(z)$ then it can be seen,[1], that it satisfies an algebraic equation of the form :

$$p_n(z)\widetilde{f_\alpha}(z)^{p^n} + \ldots + p_1(z)\widetilde{f_\alpha}(z)^p + p_0(z) = 0$$

with $p_i(z) \in F_p[Z]$, and so the "lifted" series $f_\alpha(z)$ satisfies a functional equation of the form:

$$q_m(z)f_\alpha(z^{p^m}) + \ldots + q_1(z)f_\alpha(z^p) + q_0(z) = 0$$

(see [2]), where $q_i(z)$ are polynomials in $\mathbf{z}[z]$. So, we readily see that the digits of the decimal expansion of α verifies a system of linear recurrence equations, or, equivalently, α can be generated by a finite automata, [1], which is not possible if α is algebraic, [3]. This proves theorem 2.

Considering the Morse number defined at the beginning of the paper, one sees that the corresponding series $\widetilde{f_\alpha}(z)$ verifies :

$$(1+z)^3\widetilde{f_\alpha}(z)^2 + (1+z)^2\widetilde{f_\alpha}(z) + z = 0$$

if we reduce mod 2, [1], and so, by the above, α is transcendental.

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