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Minimal repair age replacement in a heterogeneous population: an optimal stopping problem

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Abstract In this note we analyse a minimal repair age replacement policy as a solution of an optimal stopping problem under a point process martingale theory.

Keywords: Semi-martingale, optimal stopping rule, minimal repair, age replacement policy.

1 Introduction

The most common, popular and important replacement policy might be the age dependent preventive maintenance policy. Under this policy, a unit is always replaced at its age T or its failure, whichever occurs first and T is a constant. Later, as the concepts of minimal repairs became more and more stablish, a large variety of extensions and modifications of the age-replacement policy were proposed. An overview on the maintenance theory from a theoretical point of view can be found in Nacagawa (2005). More sophisticated models have also been developed in Mi, J.(1994), Cha, J.H. (2001), Cha, J.H. (2003), Ebrahimi, N. (1997), Aven and Jensen (2000) and Badia, F.G. et al. (2001).

Most of the previous research on maintenance models has been focused on those for items from a homogeneous population, however population heterogeneity has been widely recognized in reliability area and a lot of studies on the stochastic properties of heterogeneous populations have been recently performed in reliability context, as in Badia, F.G. et al. (2003), Cha and Finkelstein (2011) and Finkelstein and Cha (2013).

In this paper we consider a mixed population which is composed of ordered subpopulations through its intensities. An item is selected of an unknown subpopulation and during its operation is minimally repaired on each failure and it is replaced at a fixed age. As the subpopulation are stochastically ordered, the age at which the corresponding item is replaced should be different depending on the subpopulation from which the item in operation is selected. When the operational history of the item, which may contain some important information about the corresponding subpopulation, can be obtained during the field operation, it is reasonable to employ it in determining the replacement policy of the item. To be more specific, the items from stronger subpopulations and those from weaker subpopulations would exhibit different failure patterns during field operations. Thus, news replacement policies can be analysed, the age at which the corresponding item will be replaced is not determined before the operation of the item, but it will be determined based on the failure/repair history of the observed item during its initial operation, which is an stopping time problem. This problem was studied in Cha, J. H. (2015) considering a week and a strong sub-populations in a stochastic process framework. However, the natural approach to consider effectively the information of passed events in time is the point process martingale theory which consider the increasing information in time through increasing families of σ-algebras. In this context, in this paper, we resumes to solve and generalizes the basic minimal repair age replacement optimal stopping problem using an infinitesimal-look-ahead stopping rule. In Section 2 we describe the minimal repair process in an heterogeneous populations and in Section 3 we analyse a minimal repair age replacement optimal stopping problem in heterogenious populations.

2 Minimal repair processes in heterogeneous populations

In this section we describe the heterogeneous population and the corresponding minimal repair process as in Cha and Finkelstein (2011). Let failures of repairable items be repaired instantaneously. Then the process of repairs can be described by a stochastic point process or, equivalently, by its stochastic intensity $\lambda_t, \quad t \geq 0$, (see Aven and Jensen (1991) and Bremaud (1981)). The stochastic intensity of a point process $(N_t)_{t\geq 0}$ is defined as

$$\lambda_t = \lim_{\triangle t \to 0} \frac{P(N_{t,t+\triangle t} = 1|\Im_t)}{\triangle t} = \lim_{\triangle t \to 0} \frac{E[N_{t,t+\triangle t}|\Im_t]}{\triangle t}.$$

where $N_{t,t+\triangle t} = N_{t+\triangle t} - N_t$ and $\Im_t = \sigma\{S_i, 1 \leq i \leq N_t\}$.

A classical example is the deterministic intensity $\lambda_t = \lambda(t)$ which defines the nonhomogeneous Poisson process (NHPP) of repairs with intensity $\lambda(t)$. It is well known that the NHPP can be interpreted as the minimal repair process.

To describe an heterogeneous population we let $S \geq 0$ be an item lifetime com cumulative distribution function $F(t)$ and assume that it is indexed by a random variable Z, i.e.

$$F(t,z) = P(S \leq t|Z = z) = P(S \leq t|z)$$

and that the probability density function $f(t,z)$ exists. Then the corresponding failure rate $\lambda(t,z)$ is $\frac{f(t,z)}{\overline{F}(t,z)}$.

Let Z be a non negative random variable with support in $[a,b], \quad 0 \leq a < b \leq \infty$, and probability density function $\pi(z)$. The setting leads naturally to considering mixtures of distribution functions which are useful to describing heterogeneity:

$$F_m(t) = \int_a^b F(t,z)\pi(z)dz.$$

In accordance with this definition, the mixture failure rate is

$$\lambda_m(t) = \frac{\int_a^b f(t,z)\pi(z)dz}{\int_a^b \overline{F}(t,z)\pi(z)dz} = \int_a^b \lambda(t,z)\pi(z|t)dz,$$

where

$$\pi(z|t) = \pi(z|T > t) = \pi(z)\frac{\overline{F}(t,z)}{\int_a^b \overline{F}(t,z)\pi(z)dz}.$$

is the conditional probability function on $\{S > t\}$.

Let return to the stochastic intensity and modify it with respect to the heterogeneous case, when the orderly point process is indexed (conditioned) by the random variable Z:

$$\lambda_t = \lim_{\triangle t \to 0} \frac{P(N_{t,t+\triangle t} = 1|\Im_t, \sigma(Z))}{\triangle t}$$
$$= \lim_{\triangle t \to 0} E[\frac{P(N_{t,t+\triangle t} = 1|\Im_t, \sigma(Z))}{\triangle t}] = E[\lambda_{t,Z}].$$

where the expectation is with respect to the conditional distribution $(Z|\Im_t)$.

We, now, consider the minimal repair process of an item selected, at time $t = 0$, from our heterogeneous population. At each failure, perform a minimal repair. In this case if $Z = z$ at time $t = 0$, the corresponding intensity realization is deterministic, $\lambda_{t,Z} = \lambda(t, Z), \quad t \geq 0$, in view that at $t = 0$, the information is the degenerated σ-field, $\sigma\{\Omega, \emptyset\}$ and the distribution $\pi(z)$. So, if t_1 is the realization of the first failure T_1 in the interval $[0, t_1)$, $\lambda(t, Z) = \lambda_m(t) = \lambda_m^1(t)$ is just the mixture of the failure rate. The corresponding stochastic intensity λ_t is the expectation of $\lambda(t, Z)$ with respect to the distribution $(Z|\Im_t)$. This operations means that although the value of Z is chosen at $t = 0$ and is fixed, its distribution is updated with time as information about the failures emerges.

In the next interval $[t_1, t_2)$, where t_2 is the realization of T_2, given the additional information that an item has failed at $t = t_1$, the probability density function of Z is actualized as

$$\pi^2(z) = \frac{\lambda(t_1, z)\exp\{-\int_0^{t_1} \lambda(s, z)ds\}\pi(z)}{\int_a^b \lambda(t_1, z)\exp\{-\int_0^{t_1} \lambda(s, z)ds\}\pi(z)dz}.$$

The conditional distribution of $(Z|\Im_t)$ is

$$\frac{\lambda(t_1, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)}{\int_a^b \lambda(t_1, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)dz}$$

and the corresponding stochastic intensity is

$$\lambda_m^2(t) = \int_a^b \lambda(t, z)\frac{\lambda(t_1, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)}{\int_a^b \lambda(t_1, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)dz}$$

in $[t_1, t_2)$.

More generally, for $t \in [t_{n-1}, t_n)$, the conditional distribution $(Z|\Im_t)$ is defined by

$$\pi^n(z|t_1, ..., t_{n-1}) = \frac{\lambda(t_1, z)...\lambda(t_{n-1}, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)}{\int_a^b \lambda(t_1, z)....\lambda(t_{n-1}, z)\exp\{-\int_0^{t} \lambda(s, z)ds\}\pi(z)dz}.$$

Therefore

$$\lambda_m^n(t) = \int_a^b \lambda(t,z)\pi^n(z|t_1,...,t_{n-1})dz$$

in $[t_{n-1}, t_n)$ and

$$\lambda_t = \sum_{n=1}^{\infty} \lambda_m^n(t) 1_{\{T_{n-1} \le t < T_n\}}, \quad T_0 = 0.$$

3 Minimal repair age replacement optimal stopping problem in heterogeneous population.

An item is randomly selected of unknown subpopulation and during its operation is minimally repaired in each failure and replaced at a fixed age T. We suppose a cost c_m for each minimal repair and a cost c_r for replacement. If N_t is the number of minimal repairs in $(0,t]$, the total cost up to time t, for the item, is $c_m N_t + c_r$. The total cost for unit time is $C(t) = \frac{c_m N_t + c_r}{t}$. The goal is to find the replacement age that minimizes the long run average cost per unit time

$$C(T) = \frac{c_m E[N_T] + c_r}{E[T]}.$$

The replacement policies can be strongly connected to the following stopping problem: Minimize

$$C_\tau = \frac{E[Z_\tau]}{E[X_\tau]},$$

in a suitable class of stopping time related to a filtration $(\Im_t)_{t\ge 0}$ which represents our observations, in the probability space $(\Omega, \Im, P)$, assumed to fulfill the usual conditions of right continuity and completeness. The stochastic processes Z_t and X_t are observable in $(\Im_t)_{t\ge 0}$, that is, Z_t and X_t are $\Im_t$-measurable.

As before, T represents the age replacement and S the item lifetime.

We let $(Z_t)_{t\ge 0}$, with $Z_t = c_m N_t + c_r$ and $(X_t)_{t\ge 0}$, with $X_t = t$ be real right continuous stochastic processes adapted to $\Im_t$ and such that $E[Z_S] > -\infty$ and $E[|X_S] < \infty$. We intend to minimize the rate

$$C_\tau = \frac{E[Z_\tau]}{E[X_\tau]}$$

over the class of $\Im_t$-stopping time

$$C_S^{\Im_t} = \{\tau : \tau \ \ is \ \ an \ \ \Im_t - stopping \ \ time, \tau \le S, E[Z_\tau] > -\infty, E[|X_\tau|] < \infty\},$$

that is, to find a stopping time $\sigma \in C_S^{\Im_t}$, with

$$C^* = C_\sigma = \inf\{C_\tau : \tau \in C_S^{\Im_t}\}.$$

It is well known that a smooth semi-martingale (see appendix 2.1) representations for the processes $(Z_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$, is an excellent tool to carry out the stopping problem. Under Section 2, the semi-martingale representation for Z_t is :

$$Z_t = c_m N_t + c_r = c_m\{\int_0^t \sum_{n=1}^{\infty} \lambda_m^n(s)ds + M_t\} + c_r$$

$$= c_m \sum_{n=1}^{\infty} \int_0^t \int_a^b 1_{\{t_{n-1}\leq t<t_n\}} \lambda(t,z)\pi^n(z|t_1,...,t_{n-1})dzds + M_t\} + cr,$$

where M_t is an uniformly integral zero mean $\Im_t$-martingale.

Also $X_t = t = \int_0^t ds = \sum_{n=1}^{\infty} \int_0^t \int_a^b 1_{\{t_{n-1}\leq t<t_n\}} \pi^n(z|t_1,...,t_{n-1})dzds.$

To solve the above stopping problem is equivalent to solve the following maximization problem. Observe that the inequality $C_\tau = \frac{E[Z_\tau]}{E[X_\tau]} \geq C^*$ is equivalent to $C^*E[X_\tau]-E[Z_\tau] \leq 0$ for all $\tau \in C_S^{\Im_t}$, where the equality holds for an optimal stopping time. We have the maximization problem: Find $\sigma \in C_S^{\Im_t}$, with

$$E[Y_\sigma] = \sup\{E[Y_\tau] : \tau \in C_S^{\Im_t}\} = 0,$$

where $Y_t = C^*X_t - Z_t$ and $C^* = \inf\{C_\tau : \tau \in C_S^{\Im_t}\}$.

A smooth semi-martingale representation for Y_t for the minimal repair age replacement policy is

$$Y_t = -c_r + \int_0^t (C^* - c_m\lambda_s)ds + M_t$$

where $\lambda_s = \sum_{n=1}^{\infty} 1_{\{t_{n-1}\leq t<t_n\}} \int_a^b \lambda(t,z)\pi^n(z|t_1,...,t_{n-1})dz.$

Therefore

$$E[Y_t] = E[C^*X_t - Z_t] = -c + E[\int_0^t (C^* - c_m\lambda_s)ds]$$

To find an explicit solution of the stopping problem we adopt a condition called the monotone case:

Definition 3.1 *(MON) Let $Y = (f, M)$ be an SSM. Then the following condition*

$$\{f_t \leq 0\} \subset \{f_{t+h} \leq 0\}, \forall t, h \in \Re^+, \bigcup_{t\in\Re^+} \{f_t \leq 0\} = \Omega$$

is said to be the monotone case and the stopping time

$$\sigma = \inf\{t \in \Re^+ : f_t \leq 0\}$$

is called the ILA-stopping rule (infinitesimal-look-ahead).

Obviously in the monotone case the process f driving the smooth semi-martingale Y_t remains non-positive if once crosses zero from above and the ILA-stopping rule σ is a candidate to solve the maximization problem. Aven and Jensen (1991), proves that

Theorem 3.2 *Let $Y = (f, M)$ be an SSM and σ the ILA-stopping rule. Then, in the monotone case,*

$$E[Y_\sigma] = \sup\{E[Y_\tau] : \tau \in C^{\Im_t}\},$$

where

$$C^{\Im_t} = \{\tau : \tau \ \ is \ \ an \ \ \Im_t \ \ stopping \ \ time, \tau < \infty, E[Y_\tau] > -\infty\}.$$

Clearly, the monotone case holds when λ_s is increasing, a.s., $\lambda_0 < \frac{C^*}{c_m}$ and $\lim_{s\to\infty} \lambda_s > \frac{C^*}{c_m}$. However it is seem too restrictive to demand that λ_s is increasing a.s.. We would like the monotone case to cover cases as the bath-tub-shaped functions, which decrease first up to same value, and increase after that value. The definition of (a, b)-increasing function allows such cases, see appendix A.2.2.

The main idea to solve the stopping problem, for the basic replacement policy, using the monotonicity condition is, instead to considering all stopping time in $C_T^{\Im_t}$ we may restrict the search for an optimal stopping time to the class of index stopping times

$$\rho_x = \inf\{s \in \Re^+ : x - c_m\lambda_s \leq 0\} \wedge T, \inf \emptyset = \infty, x \in \Re.$$

The optimal stopping level can be determined from $E[Y_\tau] = 0$ and coincides with C^* as in the following Theorem from Aven and Jensen (1991).

Theorem 3.3 *If the process $(Y_t)_{t\geq 0}$ in its SSM representation has an intensity with (a, b) increasing paths on $(0, T]$, then*

$$\sigma = \rho_{x^*}, \quad with \quad x^* = \inf\{x \in \Re : xE[\rho_x] - E[Z_{\rho_x}] \geq 0\}$$

is an optimal stopping time with $x^* = C^*$.

In our context

$$\lambda_s = \sum_{n=1}^{\infty} \int_0^t \int_a^b \lambda(t,z)\pi^n(z|t_1,...,t_{n-1})dz.$$

and,

$$E[Y_t] = -c_r + E[\int_0^t (C^* - c_m\lambda_s)ds]$$

$$= -c_r + E\{\int_0^t \sum_{n=1}^{\infty} \int_0^b 1_{\{t_{n-1}\leq s<t_n\}}\pi^n(z|t_1,...,t_{n-1})[C^* - c_m\lambda(s,z)]dzds.$$

$$\rho_x = \inf\{s \in \Re^+ : x - c_m\lambda(s,z) \leq 0\} \wedge T, \inf \emptyset = \infty, x \in \Re$$

considering values of x in $[\frac{c_r}{E[T]} + q\lambda_0, \frac{c_m E[\int_0^T \lambda(s,z)ds]+c_r}{E[T]}]$ obtained from Lemma A.2.3.

This means: Replace the items the first time the sum of the intensities reaches a level x^*. This level has to be determined as

$$x^* = \inf\{x \in \Re : xE[\rho_x] - E[c_m N(\rho_x) + c_r] \geq 0\}.$$

The case of deterministic intensities is of special interest and is stated as a corollary under the assumption of the last theorem.

Lemma 3.4 *If $\lambda(t,z)$ is deterministic with inverse $\lambda^{-1} = \inf\{t \in \Re : \lambda(t,z) \geq x\}, x \in \Re$ and $X_0 = 0$, then $\sigma = t^* \wedge T$ is optimal with $t^* = \lambda^{-1}(C^*) \in \Re_+ \cup \{\infty\}$ and*

$$C^* = \inf\{x \in \Re : \int_0^{\lambda^{-1}(x)} [x - c_m\lambda(s,z)]P(S > s)ds \geq c_r\},$$

where S is the item lifetime.

We observe that the above result is also applied for a doubly stochastic Poisson process, also called Cox process, when the intensity is a random variable which is known at the time origin, that is, λ is $\Im_0$ measurable. We can take $\Im_0 = \sigma\{\lambda_{s,z}, (s,z) \in \Re_+^2\}$ and $\Im_t = \Im_0 \vee \sigma\{N_s, 0 \leq s \leq t\}$.

Consider subpopulations of items with Weibull distribution

$$F(t,\theta) = 1 - \exp[-(\frac{t-\eta}{\theta})^\beta], \quad 0 \leq \eta < t < \infty, \quad 0 < \theta < 1, \quad \beta > 0.$$

The index θ is random with probability density $\pi(\theta) = \frac{(\beta-1)}{\theta^\beta}, \quad 0 < \theta < 1$, An item of an unknown subpopulation θ is selected and during its operation is minimally repaired following a ordered a Weibull process, forming a Cox Process. In practical we consider the ordered lifetimes with a conditional survival function given by

$$\overline{F}(t_i|t_1, t_2, ..., t_{i-1}) = \exp[-(\frac{t_i - \eta_i}{\theta})^\beta + (\frac{t_i - \eta_{i-1}}{\theta})^\beta]$$

for $\eta_i \vee t_{i-1} < t_i$ where t_i are the ordered observations. The density function is

$$f(t_1, t_2, ..., t_n) = \pi_{i=1}^n f(t_i|t_1, t_2, ..., t_{i-1}) =$$

$$(\frac{\beta}{\theta})(\frac{t_1 - \eta_1}{\theta})^{\beta-1} \exp[(\frac{t_1 - \eta_1}{\theta})^\beta] \pi_{i=2}^n (\frac{\beta}{\theta})(\frac{t_i - \eta_i}{\theta})^{\beta-1}$$

$$\exp[-(\frac{t_i - \eta_i}{\theta})^\beta + (\frac{t_i - \eta_{i-1}}{\theta})^\beta].$$

Follows that

$$\lambda(t_i, \theta|t_1, t_2, ..., t_{i-1}) = \frac{f(t_1, t_2, ..., t_n)}{\overline{F}(t_i|t_1, t_2, ..., t_{i-1})} = (\frac{\beta}{\theta})(\frac{t - \eta_i}{\theta})^{\beta-1}, \quad t_{i-1} \leq t < t_i,$$

$t_0 = 0$ is deterministic and

$$\lambda^1(t) = \inf\{s \in \Re^+ : c_m \lambda(s, z) \geq x - c_r\} \wedge T, \inf \emptyset = \infty, x \in \Re$$

We assumes $\beta > 1$, which make $\lambda(t, \theta)$ increasing in t, and

$$\lambda^{-1}(t) = (\frac{(t - c_r)\theta^{\beta-1}}{c_m \beta})^{\frac{1}{\beta-1}} + \eta_i.$$

As, in this case, the intensity is deterministic, we can apply Corollary 3.4 and get the age replacement T_θ, indexed by θ.

4 Appendix.

4.1 A.1

An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n \geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n \to \infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a basis of stochastic processes see the book of Bremaud [2].

4.2 A.2

The stopping problem.

Definition A.2.1 A stochastic process $Z = (Z_t)_{t\geq 0}$ is called a smooth semimartingale representation (SSM) if it has a decomposition of the form

$$Z_t = Z_0 + \int_0^t f_s ds + M_t,$$

where $(f_t)_{t\geq 0}$, is a progressively measurable process with $E[\int_0^t |f_s|ds < \infty$ for all $t \in \Re$, $E[|Z_0|] < \infty$ and $(M_t)_{t\geq 0}$ an zero mean uniformly integrable $\Im_t$ martingale. We denote a SSM by $Z = (f, M)$.

Definition A.2.2 Let $a, b \in \Re \cup \{-\infty, \infty\}, a \leq b$. Then a function $f : \Re^+ \mapsto \Re$ is called (a, b)-increasing if for all $t, h \in \Re^+$,

$$f(t) \geq a, \quad implies \quad f(t+h) \geq \min\{f(t), b\}.$$

Roughly spoken, an (a, b)-increasing function $f(t)$ passes with increasing t the levels a, b from bellow and never falls back bellow such a level. The first step to detect the parameters a and b is to establish bounds for C^*:

Lemma A.2.3 Let $X = (g, L)$ and $Z = (f, M)$ be smooth semimartingales under the above assumptions and

$$q = \inf\{\frac{f_t(w)}{g_t(w)} : 0 \leq t < S(w), w \in \Omega\} > -\infty.$$

Then

$$b_l \leq C^* \leq b_u$$

holds true, where the bounds are given by

$$b_u = \frac{E[Z_S]}{E[X_S]},$$

$$b_l = \frac{E[Z_0 - qX_0}{E[X_S]} + q, if$$

E[Z$_0 - qX_0] > 0$;E[Z$\frac{_0]}{E[X_0], otherwise.}$

5 Conclusions

News replacement policies can be analysed, the age at which the corresponding item will be replaced is not determined before the operation of the item, but it will be determined based on the failure/repair history of the observed item during its initial operation, view as an stopping time problem. We conjecture the validation of such a model for more sophisticated policies.

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