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A CUMULATIVE RESIDUAL INACCURACY MEASURE FOR COHERENT SYSTEMS AT COMPONENT LEVEL AND UNDER NONHOMOGENEOUS POISSON PROCESSES

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Inaccuracy and information measures based on cumulative residual entropy are quite useful and have attracted considerable attention in many fields including reliability theory. Using a point process martingale approach and a compensator version of Kumar and Taneja's generalized inaccuracy measure of two nonnegative continuous random variables, we define here an inaccuracy measure between two coherent systems when the lifetimes of their common components are observed. We then extend the results to the situation when the components in the systems are subject to failure according to a double stochastic Poisson process.

Keywords: coherent system, cumulative residual inaccuracy measure, joint signature point process, minimal repair, nonhomogeneous Poisson process, signature point process

1. INTRODUCTION

An alternate measure of entropy, based on the distribution function instead of the density function of a random variable, called cumulative residual entropy (CRE), was first proposed by Rao *et al.* [29]. It was subsequently extended to cumulative residual inaccuracy (CRI) measure by Kumar and Taneja [20]. When the lifetimes of common components of two coherent systems are observed, we adopt a point process martingale approach to extend here this notion to a symmetric inaccuracy measure. Furthermore, we define an inaccuracy measure between two coherent systems whose components are subject to failure according to a double stochastic Poisson process.

The main inaccuracy measure for the uncertainty of two positive and absolutely continuous random variables, S and T, defined in a complete probability space $(\Omega, \Im, P)$, is

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Kerridge's inaccuracy measure given by (see [18])

$$H(S,T) = E[-\log g(T)] = -\int_0^\infty \log g(x) f(x)\, dx, \tag{1}$$

where f and g are the probability density functions of T and S, respectively. Equation (1) plays a significant role in reliability and survival analysis.

In the case when S and T are identically distributed, Kerridge's inaccuracy measure gives the well-known Shannon's entropy defined as (see [34])

$$H(T) = E[-\log f(T)] = -\int_0^\infty \log f(x) f(x)\, dx.$$

In recent years, several authors have studied various properties of this information measure. For example, Ebrahimi [13] has proposed a measure of uncertainty about the remaining lifetime of a system working at time t as $H(T_t)$ where $T_t = (T - t \,|\, T > t)$. Kayal *et al.* [17] and Kayal and Sunoj [16] have proposed a generalization of (1) and developed various theoretical properties.

One main drawback of Shannon entropy is that for some probability distributions, it may be negative and then it is no longer an uncertainty measure. This drawback is removed in the Varma [36] entropy that is a generalization of both Shannon entropy and Renyi [30] entropy.

Rao *et al.* [29] and Rao [28] have provided an extension of the above measure, the cumulative residual entropy for T, by using survival functions of T in place of their probability density functions in Shannon's entropy. Asadi and Zohrevand [4] have considered the corresponding dynamic measure using the conditional survival function $P(T - t \,|\, T > t)$. An analogous measure based on the distribution function has been introduced by Di Cresccenzo and Longobardi [12], which is known as the cumulative past entropy of T.

Kerridge's measure of inaccuracy has also been extended in a similar way by Kumar and Taneja [20,21] and Kundu *et al.* [22].

Kumar and Taneja's CRI measure between S and T is given by

$$\varepsilon(S,T) = -\int_0^\infty \overline{F}(t) \log \overline{G}(t)\, dt = E\left[\int_0^T \bigwedge_S (s)\, ds\right], \tag{2}$$

where $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ are the reliability functions of T and S, respectively, with F and G being their distribution functions, and $\bigwedge_S(t) = -\log \overline{G}(t)$ being the cumulative hazard function of lifetime S. It is important to note that the expression in (2) makes sense in the set $\{t < S \wedge T\}$, where t is a time constant and $S \wedge T = \min\{S, T\}$ and we set, by convention, $0 \log 0 = 0$. When S and T are identically distributed, Eq. (2) becomes the measure introduced by Rao *et al.* [29].

Indeed, $\varepsilon(S,T)$ represents the information content when using $\overline{G}(t)$, the survival function asserted by the experimenter, due to missing/incorrect information, instead of the true survival function $\overline{F}(t)$. Some transformation of this measure can be seen in the work of Psarrakos and Di Crescenzo [26].

Similarly, Kumar and Taneja [21] have introduced cumulative past inaccuracy measure of S and T by replacing the distributions functions by survival functions in the measure of Di Crescenzo and Longobardi [12]. Kundu *et al.* [22] considered the measures given by Kumar and Taneja [20,21] and obtained several properties when random variables are left, right, and doubly truncated. Quite recently, bivariate extensions of cumulative residual (past) inaccuracy measures have been discussed by Goosh and Kundu [14].

This paper consists of two parts and proceeds as follows. In Section 2, we introduce a symmetric CRI measure between two lifetimes as a metric. Next, we use a point process martingale approach in Section 3 to introduce a signature point process. In Section 4, we then extend the results to coherent systems. We also define a measure to compare the relative performance of different coherent systems. These form the first part of the paper. In the second part of the paper, we begin by defining a joint signature point process for two coherent systems in Section 5, which will be useful to calculate the residual cumulative inaccuracy measure between two coherent systems. We also provide asymptotic reasoning to extend the inaccuracy measure for a double stochastic Poisson process. In Section 6, we define the cumulative inaccuracy measure for coherent systems under a double stochastic Poisson process and focus especially on a nonhomogeneous Poisson process to model a minimal repair coherent system. We present some examples to illustrate the calculation of the inaccuracy measure between minimal repair point processes. Finally, we make some concluding comments in Section 7.

2. CRI MEASURE BETWEEN TWO LIFETIMES

2.1. CRI Measure as a Metric

Suppose we observe twocomponent lifetimes, T and S, which are finite positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(S \neq T) = 1$, through the family of sub σ-algebras $(\Im_t)_{t\geq 0}$ of $\Im$, where

$$\Im_t = \sigma\{1_{\{S>s\}}, 1_{\{T>s\}}, 0 \leq s < t\}$$

satisfies Dellacherie's conditions of right continuity and completeness.

In our general setups, for simplifying the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets always hold with probability 1, which means that the term P-a.s. is suppressed.

We now assume that S and T are totally inaccessible $\Im_t$-stopping times. An extended and positive random variable τ is a $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; a $\Im_t$-stopping time τ is said to be predictable if an increasing sequence $(\tau_n)_{n\geq 0}$ of $\Im_t$-stopping times, $\tau_n < \tau$, exists such that $\lim_{n\to\infty} \tau_n = \tau$; a $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping times σ. In this way, absolutely continuous lifetimes are thought of as totally inaccessible $\Im_t$-stopping times. For a mathematical basis of stochastic processes applied to reliability theory, one may refer to the books by Aven and Jensen [5] and Bremaud [8].

The cumulative hazard functions of T and S are given by $\bigwedge_T(t) = -\log \overline{F}(t)$ and $\bigwedge_S(t) = -\log \overline{G}(t)$. Then, CRI of S and T is given by Eq. (2).

With respect to $(\Im_t)_{t\geq 0}$ and using Doob-Meyer decomposition, we consider the predictable compensator processes $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ such that $1_{\{T\leq t\}} - A_t$ and $1_{\{S\leq t\}} - B_t$ are 0 means $\Im_t$-martingales. From the total inaccessibility of S and T, A_t and B_t are continuous; see Dellacherie [11].

The compensator process is expressed in terms of conditional probabilities, given the available information, and it generalizes the classical notion of hazard. Intuitively, it corresponds to producing whether the failure is going to occur now, on the basis of all observations available up to, but not including the present.

It then follows, by well-known equivalence results between distribution functions and compensator processes, that $A_t = -\log \overline{F}(t \bigwedge T)$ and $B_t = -\log \overline{G}(t \bigwedge S)$; see Arjas and

Yashin [1]. Identifying $\bigwedge_S(t)$ and B_t, in the set $\{S > t\}$, we obtain Eq. (2) as follows:

$$\begin{aligned}\varepsilon(S,T) &= E\left[\int_0^T B_s\,ds\right] = E\left[\int_0^T \left(\int_0^s dB_t\right) ds\right] \\ &= E\left[\int_0^T \left(\int_t^T ds\right) dB_t\right] = E\left[\int_0^T (T-t)\,dB_t\right].\end{aligned}$$

As $\psi(t) = T - t$ is a left continuous function, it is an $\Im_t$-predictable process (see Bremaud [8]), and we can conclude that

$$M_t = \int_0^T (T-t)\,d(1_{\{S\le t\}} - B_t)$$

is a mean 0 $\Im_t$-martingale. We then have

$$\begin{aligned}\varepsilon(S,T) &= E\left[\int_0^T (T-t)\,dB_t\right] = E\left[\int_0^T (T-t)\,d1_{\{S\le t\}}\right] \\ &= E\left[1_{\{S\le T\}}(T-S)\right] = E\left[1_{\{S\le T\}}|T-S|\right].\end{aligned} \tag{3}$$

EXAMPLE 2.1: *An engineering system is a coherent system if its components are relevant and its structure function is monotone increasing; see Barlow and Proschan [7]. A special case of a coherent system is the k-out-of-n system which works when at least k of its components work. In particular, when $k = n$ the system is series, and when $k = 1$ the system is parallel. A mixed system is a stochastic mixture of coherent systems. Let T and $S = \min\{T_1, \ldots, T_n\}$ be lifetimes of coherent (mixed) and series systems, where $T_1, \ldots, T_n$ are the common component lifetimes. Clearly, S is less than T, that is, $S \le T$, and so Eq. (3) becomes*

$$\varepsilon(S,T) = E[|T-S|]$$

as a distance between T and the series system S.

Also, using the same arguments as above, we have

$$\varepsilon(T,S) = E\left[\int_0^S A_s\,ds\right] = E\left[1_{\{T\le S\}}(S-T)\right] = E[1_{\{T\le S\}}|S-T|]. \tag{4}$$

EXAMPLE 2.2: *Now, let T and $S = \max\{T_1, \ldots, T_n\}$ be lifetimes of coherent (mixed) and parallel systems, where $T_1, \ldots, T_n$ are the common component lifetimes. Evidently, S is greater than T, that is, $S \ge T$, and so Eq. (4) becomes*

$$\varepsilon(S,T) = E[|T-S|]$$

as a distance between T and the parallel system S.

Eqs. (3) and (4) enable us to define a symmetric generalization of Kumar and Taneja's inaccuracy measure.

DEFINITION 2.3: *If S and T are continuous positive random variables defined in a complete probability space $(\Omega, \Im, P)$, the CRI measure is*

$$\begin{aligned} CRI_{S,T} = CRI_{T,S} &= \varepsilon(S,T) + \varepsilon(T,S) \\ &= E\left[\int_0^T B_s\, ds\right] + E\left[\int_0^S A_s\, ds\right] \\ &= E[|T-S|]. \end{aligned} \quad \textbf{(5)}$$

The definition $CRI_{S,T} = \varepsilon(S,T) + \varepsilon(T,S)$ in terms of compensator processes is suitable to work for stochastic dependence; for example, suppose we observe a coherent system lifetime in a complete information level, that is, only through their dependent component lifetimes.

$CRI_{S,T}$ can then be seen as a dispersion measure when using a lifetime S asserted by the experimenter information of the true lifetime T. Provided that we identify random variables that are equal almost everywhere, $CRI_{S,T}$ is a metric in the L^1 space of random variables, and as a metric, it possesses several useful properties.

EXAMPLE 2.4 (Using empirical distribution to approximate $\mathrm{CRI}_{S,T}$): *When the experimenter's information, S, of the true lifetime T, is one selected from a set of possible system lifetimes, $S_1, S_2, \ldots, S_n, \ldots$, which are independent and identically distributed as S, with lifetime G, we can then use the random lifetime defined by*

$$Y_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{S_i \le t\}},$$

which is an unbiased and consistent nonparametric estimator of the distribution function $G(t)$, that is,

$$E[Y_n(t)] = \frac{1}{n}\sum_{i=1}^{n} E[1_{\{S_i \le t\}}] = G(t)$$

and

$$Var(Y_n(t)) = E[(Y_n(t) - G(t))^2] = \frac{1}{n^2}\, Var\left(\sum_{i=1}^{n} 1_{\{S_i \le t\}}\right) = \frac{G(t)(1-G(t))}{n},$$

which goes to 0 as n goes to infinity. Hence, $Y_n(t)$ converges in quadratic mean to S and, consequently, converges in probability to S which does imply convergence in distribution to S, as $n \to \infty$.

As $f(Y_n) = |Y_n| \le 1$ is a bounded and continuous function and $Y_n(t)$ converges in distribution to S, we have

$$E[|T - Y_n|] \to E[|T-S|].$$

As in Kumar and Taneja [21], we can extend $\mathrm{CRI}_{S,T}$ to time varying forms corresponding to residual lifetimes in the set $\{t < S \wedge T\}$. In this regard, we observe that

$$E\left[\int_t^T B_x\,dx\right] = E\left[\int_t^T\left(\int_0^x dB_y\right)dx\right]$$
$$= E\left[\int_0^t\left(\int_t^T dx\right)dB_y\right] + E\left[\int_t^T\left(\int_y^T dx\right)dB_y\right]$$
$$= E\left[(T-t)\int_0^t d1_{\{S\le y\}}\right] + E\left[\int_t^T (T-y)\,d1_{\{S\le y\}}\right]$$
$$= E[(T-t)1_{\{S\le t\}}] + E[(T-S)1_{\{t<S\le T\}}] = E[|T-S|1_{\{t<S\le T\}}]. \quad \textbf{(6)}$$

We also have

$$E\left[\int_t^S A_x\,dx\right] = E[|S-T|1_{\{t<T\le S\}}]. \quad \textbf{(7)}$$

These yield the following definition.

Definition 2.5: *If S and T are continuous positive random variables in a complete probability space $(\Omega, \Im, P)$, the dynamic CRI measure at time t, in the set $\{t < S\wedge T\}$, is*

$$DCRI^t_{S,T} = DCRI^t_{T,S} = E[1_{\{t<S\wedge T\}}|T-S|]. \quad \textbf{(8)}$$

2.2. Characterization Problem

If T and S are two absolutely continuous lifetimes such that A_t is the $\Im_t$-compensator of $1_{\{T\le t\}}$ and $B_t = \alpha A_t$, $0<\alpha\le 1$, is the $\Im_t$- compensator of $1_{\{S\le t\}}$, we then say that T and S satisfy the proportional risk hazard process.

Theorem 2.6: *If T and S satisfy the proportional risk hazard process, then the dynamic cumulative residual inaccuracy measure $DCRI^t_{S,T} < \infty$ uniquely determines the distribution function of T.*

Proof: Let T_1 and S_1 be two absolutely continuous lifetimes such that A^1_t is the $\Im_t$-compensator of $1_{\{T_1\le t\}}$ and $B^1_t = \alpha^1 A^1_t$, $0<\alpha^1\le 1$, is the $\Im_t$- compensator of $1_{\{S_1\le t\}}$. Further, let T_2 and S_2 be two absolutely continuous lifetimes such that A^2_t is the $\Im_t$-compensator of $1_{\{T_2\le t\}}$ and $B^2_t = \alpha^2 A^2_t$, $0<\alpha^2\le 1$, is the $\Im_t$-compensator of $1_{\{S_2\le t\}}$. Then, using Eq. (8) in the set $\{t<S_1\wedge T_1\}\cap\{t<S_2\wedge T_2\}$, we have

$$\mathrm{DCRI}^t_{S_1,T_1} = \mathrm{DCRI}^t_{S_2,T_2} \leftrightarrow E\left[\int_t^{S_1} A^1_t\,dt + \int_t^{T_1}\alpha^1 A^1_t\,dt\right]$$
$$= E\left[\int_t^{S_2} A^2_t\,dt + \int_t^{T_2}\alpha^2 A^2_t\,dt\right].$$

However, for $i=1,2$, we have

$$E\left[\alpha^i\int_t^{T_i} A^i_s\,ds\right] = \alpha^i E\left[\int_0^t (T_i-t)\,dA^i_s\right] + \alpha^i E\left[\int_t^{T_i}(T_i-s)\,dA^i_s\right] = 0.$$

Without loss of generality, using the Optimal Sampling Theorem, in the set $\{t < S = S_1 \bigwedge S_2\}$, we have

$$\begin{aligned}
&\mathrm{DCRI}^S_{S_1,T_1} = \mathrm{DCRI}^S_{S_2,T_2}\\
&\quad \leftrightarrow E\left[\int_0^S A^1_t\,dt\right] = E\left[\int_0^S A^2_t\,dt\right]\\
&\quad \leftrightarrow E\left[\int_0^\infty 1_{\{t<S\}} A^1_t 1_{\{A^1_t > A^2_t\}}\,dt\right] + E\left[\int_0^\infty 1_{\{t<S\}} A^1_t 1_{\{A^1_t \le A^2_t\}}\,dt\right]\\
&\quad = E\left[\int_0^\infty 1_{\{t<S\}} A^2_t 1_{\{A^1_t > A^2_t\}}\,dt\right] + E\left[\int_0^\infty 1_{\{t<S\}} A^2_t 1_{\{A^1_t \le A^2_t\}}\,dt\right]\\
&\quad \leftrightarrow E\left[\int_0^\infty 1_{\{t<S\}} (A^1_t - A^2_t) 1_{\{A^1_t > A^2_t\}}\,dt\right] = E\left[\int_0^\infty 1_{\{t<S\}} (A^2_t - A^1_t) 1_{\{A^1_t \le A^2_t\}}\,dt\right]\\
&\quad \leftrightarrow \int_0^\infty E[1_{\{t<S\}} |A^1_t - A^2_t| 1_{\{A^1_t > A^2_t\}}]\,dt = \int_0^\infty E[1_{\{t<S\}} |A^1_t - A^2_t| 1_{\{A^1_t \le A^2_t\}}]\,dt\\
&\quad \leftrightarrow \int_0^\infty E[1_{\{t<S\}} |A^1_t - A^2_t| (1_{\{A^1_t > A^2_t\}} - 1_{\{A^1_t \le A^2_t\}})]\,dt = 0. \qquad (9)
\end{aligned}$$

In Eq. (9), $\{A^1_t > A^2_t\} \cap \{A^1_t \le A^2_t\} = \emptyset$, the integrand is positive, and we have $A^1_t = A^2_t$. Then, $P(T_1 \le t) = E[A^1_t] = E[A^2_t] = P(T_2 \le t)$, and thus $\mathrm{DCRI}_{S,T}$ uniquely determines the distribution function of T. ■

2.3. Calculating CRI Measure

To calculate the CRI measure, we can write

$$\mathrm{CRI}_{S,T} = E[|T - S|] = E[S \vee T] - E[S \wedge T],$$

where $S \vee T = \max\{S, T\}$ and $S \wedge T = \min\{S, T\}$.

The lifetime distribution of the series system $S \wedge T$ is completely characterized by the $\Im_t$-compensator $C_t = A_t + B_t$, and so

$$\begin{aligned}
E[S \wedge T] &= \int_0^\infty P(S \wedge T > t)\,dt = \int_0^\infty E[P(S \wedge T > t|\Im_t)]\,dt\\
&= E\left[\int_0^\infty \exp\{-(A_t + B_t)\}\,dt\right].
\end{aligned}$$

Also, the reliability function of a parallel system $S \vee T$ is given by

$$P(S \vee T > t|\Im_t) = e^{-A_t} + e^{-B_t} - e^{-(A_t+B_t)}.$$

To deal with stochastic dependence, the martingale approach is quite convenient. Given a simple $\Im_t$-submartingale point process, there exists an unique $\Im_t$-compensator process, expressed in terms of conditional probabilities given the available information, that facilitates the analytical work under stochastic dependence.

THEOREM 2.7: *Let S and T be totally inaccessible $\Im_t$-stopping times representing two lifetimes, with compensator processes $(B_t)_{t\ge 0}$ and $(A_t)_{t\ge 0}$, respectively. Then, the CRI measure is*

$$CRI_{S,T} = E\left[\int_0^\infty (e^{-A_t} + e^{-B_t} - 2e^{-(A_t+B_t)})\,dt\right]. \qquad (10)$$

Proof: We have

$$\begin{aligned}\mathrm{CRI}_{S,T} &= E[S \vee T] - E[S \wedge T] \\ &= \int_0^\infty [P(S \vee T > t) - P(S \wedge T > t)]\, dt \\ &= E\left[\int_0^\infty \{(e^{A_t} + e^{B_t} - 1)e^{-(A_t+B_t)} - e^{-(A_t+B_t)}\}\, dt\right] \\ &= E\left[\int_0^\infty \{e^{-A_t} + e^{-B_t} - 2e^{-(A_t+B_t)}\}\, dt\right].\end{aligned}$$

■

Example 2.8: *Let the lifetime T of a coherent system have Weibull distributions with shape parameter $\beta = 2$ and scale parameter θ_1. Further, let the lifetime S of a coherent system, independently of T, be asserted by the experimenter to follow a Weibull distribution with $\beta = 2$ and θ_2. Then, T and S have deterministic compensators $A_t = (t \wedge T/\theta_1)^2$ and $B_t = (t \wedge S/\theta_2)^2$. In the set $\{t < S \wedge T\}$, we then find*

$$\begin{aligned}&E\left[\int_0^\infty \{e^{-A_t} + e^{-B_t} - 2e^{-(A_t+B_t)}\}\, dt\right] \\ &= \int_0^\infty e^{-(t/\theta_1)^2}\, dt + \int_0^\infty e^{-(t/\theta_2)^2}\, dt - 2\int_0^\infty e^{-(t/\theta_1)^2} e^{-(t/\theta_2)^2}\, dt \\ &= \frac{\sqrt{\pi}\theta_1}{2} + \frac{\sqrt{\pi}\theta_2}{2} - 2\frac{\sqrt{\pi}}{2}\sqrt{\frac{\theta_1^2\theta_2^2}{\theta_1^2 + \theta_2^2}}.\end{aligned}$$

In particular, for $\theta_1 = 1$ and $\theta_2 = 0.5$, we thus have $E[|T - S|] = 0.53$.

3. SIGNATURE POINT PROCESS

Let T be the lifetime of a mixed system consisting of n independent and identically distributed components with lifetimes, $T_1, \ldots, T_n$, having a continuous distribution F. Then, a well-known mixture representation of the system reliability at time t is [31,32]

$$P(T > t) = \sum_{k=1}^{n} P(T = T_{(k)})P(T_{(k)} > t), \tag{11}$$

where $T_{(1)}, \ldots, T_{(n)}$ are the order statistics of the component lifetimes. System signatures have many extensions and applications in engineering reliability; see Samaniego [31].

For coherent systems with i.i.d. components, preservation results in their signatures in three different senses stochastic ordering, hazard rating ordering and likelihood ratio ordering have been established by Kochar *et al.* [19] and Navarro *et al.* [23].

In our context, for dealing with lifetimes of two coherent systems, S and T, with n stochastically dependent common components, the martingale approach will be convenient and then the results in Section 2 can be extended in the complete information level. For this purpose, we consider the signature point process introduced by Bueno [10].

In the general setup, at the complete information level, we assume to observe the vector $(T_1, \ldots, T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j$, with i, j being in $C = \{1, \ldots, n\}$, the index set of components. Note that the lifetimes can be dependent, but simultaneous failures are assumed to be not possible.

The evolution of components over time defines a point process given through the failure times. We denote by $T_{(1)} < T_{(2)} < \cdots < T_{(n)}$ the ordered lifetimes of $T_1, T_2, \ldots, T_n$, as they appear over time and, as a convention, we set $T_{(n+1)} = T_{(n+2)} = \cdots = \infty$. Then, the sequence $(T_n)_{n\geq 1}$ defines a point process; see Bremaud [8].

The mathematical description of the observations, namely the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, 1 \leq i \leq n, 0 < s < t\}$$

satisfies the Dellacherie conditions of right continuity and completeness.

Intuitively, at each time t, the observer knows if the event $\{T_{(i)} \leq t\}$ has either occurred or not, and if it had occurred, then the observer knows exactly the $T_{(i)}$ value. We assume that $T_1, \ldots, T_n$ are totally inaccessible $\Im_t$-stopping times.

Under the complete information level, the representation of the point process $P(T \leq t \,|\, \Im_t)$, as information flows continuously over time, is as given in the following result.

THEOREM 3.1: *Let $T_1, \ldots, T_n$ be totally inaccessible $\Im_t$-stopping times representing the lifetimes of components of a coherent system with lifetime T. Then,*

$$P(T \leq t \,|\, \Im_t) = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}}. \tag{12}$$

PROOF: From the total probability law, we have

$$\begin{aligned} P(T \leq t|\Im_t) &= \sum_{k=1}^{n} P(\{T \leq t\} \cap \{T = T_{(k)}|\Im_t) \\ &= \sum_{k=1}^{n} E[1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}}|\Im_t]. \end{aligned}$$

As T and $T_{(k)}$ are $\Im_t$-stopping times and due to the fact that the event $\{T = T_{(k)}\} \in \Im_{T_{(k)}}$ (see Dellacherie [11]), where

$$\Im_{T_{(k)}} = \{A \in \Im_\infty : A \cap \{T_{(k)} \leq t\} \in \Im_t, \forall t \geq 0\},$$

$\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\}$ is $\Im_t$-measurable. Hence,

$$P(T \leq t|\Im_t) = \sum_{k=1}^{n} E[1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}}|\Im_t] = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}}.$$

The above decomposition allows us to define the point signature process upon observing the lifetimes of components. ■

DEFINITION 3.2: *The vector*

$$(1_{\{T=T_{(k)}\}}, 1 \leq k \leq n)$$

is defined as the point signature process of the system lifetime T, at the component level.

Remark 3.3: *As* $P(T_i = T_j) = 0$, *the collection* $\{\{T = T_{(i)}\}, 1 \le i \le n\}$ *defines a partition of* Ω *and* $\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} = 1$. *We, therefore, have*

$$P(T > t \,|\, \Im_t) = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}>t\}}. \tag{13}$$

Remark 3.4: *Using Eq. (13), we can calculate the system reliability as*

$$E[P(T > t|\Im_t)] = E\left[\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}>t\}}\right] = \sum_{k=1}^{n} P(\{T = T_{(k)}\} \cap \{T_{(k)} > t\}).$$

If the component lifetimes are totally inaccessible, independent and identically distributed, we will then have

$$P(T > t) = \sum_{k=1}^{n} P(T = T_{(k)}) P(T_{(k)} > t),$$

recovering the classical result of Samaniego [30].

4. CRI MEASURE FOR COHERENT SYSTEMS

4.1. Definition of Measure

The signature concept is an important tool for studying coherent systems with independent and identically distributed continuous lifetimes. The extended concept of point process signature allows us to incorporate component's stochastic dependence. Furthermore, as $P(T_{(k)} > t|\Im_{T_{(k)}}) = e^{-A_t((k))}$, it allows us to work with compensator processes. In this regards, we have the following Theorem:

Theorem 4.1: *Let* $T_1, \ldots, T_n$ *be totally inaccessible* $\Im_t$*-stopping times representing the lifetimes of common components of two coherent systems with lifetimes* T *and* S. *Then, the cumulative residual accuracy measure of* S *and* T, *at the component level, that is, observing all* T_i, $1 \le i \le n$, *is*

$$\begin{aligned} CRI_{S,T} = \int_0^\infty \Bigg\{ & \sum_{k=1}^{n} P(T_{(k)} > t|T = T_{(k)}) P(T = T_{(k)}) \\ & + \sum_{j=1}^{n} P(T_{(j)} > t|S = T_{(j)}) P(S = T_{(j)}) \\ & - 2\sum_{i=1}^{n} P(T_{(i)} > t|S \wedge T = T_{(i)}) P(S \wedge T = T_{(i)}) \Bigg\}. \end{aligned} \tag{14}$$

PROOF: As in Section 2.2, we can write

$$\mathrm{CRI}_{T,S} = \int_0^\infty E[P(S \vee T > t|\Im_t) - P(S \wedge T > t|\Im_t)]\,dt$$
$$= \int_0^\infty E[P(T > t|\Im_t) + P(T > t|\Im_t) - 2P(S \wedge T > t|\Im_t)]\,dt.$$

Using the point process signature representation in Eq. (13), we then have

$$\mathrm{CRI}_{T,S} = \int_0^\infty E\left[\sum_{k=1}^n 1_{\{T=T_{(k)}\}}1_{\{T_{(k)}>t\}} + \sum_{j=1}^n 1_{\{S=T_{(j)}\}}1_{\{T_{(j)}>t\}} - 2\sum_{i=1}^n 1_{\{S\wedge T=T_{(k)}\}}1_{\{T_{(k)}>t\}}\right] dt,$$

proving Eq. (14). ■

REMARK 4.2: *A nonhomogeneous Poisson process (NHPP) is generated by record values of random variables independent and identically distributed. Let $T_1, \ldots, T_n$ be the lifetimes of components of a coherent system, which are subject to failures according to a well-known counting process $(N_t)_{t\geq 0}$ with deterministic compensator $\Lambda(t)$, called NHPP. The lifetime of the k-th occurrence of the NHPP, S_k, has its survival function as*

$$\overline{G}_k(t) = P(S_k > t) = P(N(t) < k) = \sum_{j=0}^{k-1} \frac{(\Lambda(t))^j}{j!}\, e^{-\Lambda(t)}, \quad k = 0, 1, \ldots,$$

where $\Lambda(t) = E[N_t] = -\ln \overline{G}(t)$ and $\overline{G}(t)$ is the reliability function of the first occurrence time.

The reliability function $\overline{G}_k(t)$ is also the reliability function of upper record values in a sequence of independent nonnegative random variables $T_1, T_2, \ldots,$ generated from $G(t)$, the distribution function of the time to first failure; see Arnold *et al.* [2,3]. In other words, a NHPP is essentially a record-counting process subject to its mean value function $\Lambda(t)$ being continuous and tending to ∞ as $t \to \infty$. Therefore, the sequence of occurrence times can be considered as record values of a sequence of independent and identically distributed random variables each having distribution function G, in which case the events $\{T = T_{(k)}\}$, $1 \leq k \leq n$, and the inter-arrival times are independent; see Randles and Wolfe [27].

EXAMPLE 4.3: *As in Example 2.8, let $T_1, \ldots, T_n$ be the lifetimes of components of a coherent system, with system lifetime T, being subject to failures according to a Weibull process with parameters β and θ_1. Let the lifetime S asserted by the experimenter follow a Weibull process with parameters β and θ_2. Then, $S \wedge T$ follows a Weibull process with parameters β and $\theta_1^\beta\theta_2^\beta/(\theta_1^\beta + \theta_2^\beta)$. In practice, we consider the ordered lifetimes $T_{(1)}, \ldots, T_{(n)}$ with a conditional reliability function given by*

$$\overline{F_i}(t_i|t_1, \ldots, t_{i-1}) = \exp\left[-\left(\frac{t_i}{\theta}\right)^\beta + \left(\frac{t_{i-1}}{\theta}\right)^\beta\right]$$

for $0 \leq t_{i-1} < t_i$, where t_i are the ordered observations. Upon considering $T_1, \ldots, T_n$ as record values of independent and identically distributed random variables, the events

$\{T = T_{(i)}\}$ and $\{T_{(i)} > t\}$ are independent, and so $P(T_{(i)} > t | T = T_{(i)}) = P(T_{(i)} > t)$. Then, by using Theorem 4.1, we have

$$
\begin{aligned}
CRI_{S,T} = \int_0^\infty & \left[\sum_{k=1}^n s_k^T \exp\left\{ -\left(\frac{t_k}{\theta_1}\right)^\beta + \left(\frac{t_{k-1}}{\theta_1}\right)^\beta \right\} \right. \\
& + \sum_{j=1}^n s_j^S \exp\left\{ -\left(\frac{t_j}{\theta_2}\right)^\beta + \left(\frac{t_{j-1}}{\theta_2}\right)^\beta \right\} \\
& \left. - 2\sum_{i=1}^n s_i^{S\wedge T} \exp\left\{ -\left(\frac{t_i}{\theta_1^\beta\theta_2^\beta/(\theta_1^\beta+\theta_2^\beta)}\right)^\beta + \left(\frac{t_{i-1}}{\theta_1^\beta\theta_2^\beta/(\theta_1^\beta+\theta_2^\beta)}\right)^\beta \right\} \right] dt,
\end{aligned}
$$

where s_k^T, s_j^S, and $s_i^{S\wedge T}$, $1 \le i, j, k \le n$, are the components of the systems signature vectors s^T, s^S, and $s^{S\wedge T}$ of the lifetimes T, S, and $S \wedge T$, respectively.

4.2. Comparisons of Systems

Toomaj *et al.* [35] mention that physical system's structures can restrict the use of stochastic ordering for pairwise comparisons. It is well-known that the reliability function of any arbitrary coherent system is in between that of a series system and of a parallel system almost everywhere, that is, $T_{(1)} \le^{\text{st}} T \le^{\text{st}} T_{(n)}$, for any system lifetime T. Hence, instead of just pairwise comparisons of systems, Toomaj *et al.* [35] suggested to look for a system with distribution closer to the distribution of a parallel system and far from the distribution of a series system.

Considering the CRI measure as a metric, we have the following:

$$
\begin{aligned}
\text{CRI}_{T_{(n)},T_{(1)}} &\ge \text{CRI}_{T,T_{(1)}}; \\
\text{CRI}_{T_{(n)},T_{(1)}} &\ge \text{CRI}_{T_{(n)},T}; \\
|\text{CRI}_{T,T_{(1)}} - \text{CRI}_{T,T_{(n)}}| &\le \text{CRI}_{T,T_{(1)}} + \text{CRI}_{T,T_{(n)}} = E[T - T_{(1)}] + E[T_{(n)} - T] \\
&= E[|T_{(n)} - T_{(1)}| = \text{CRI}_{T_{(n)},T_{(1)}}.
\end{aligned}
$$

So, following up on the idea of Toomaj *et al.* [35], to verify if the system distribution is closer to a parallel system distribution and produce great reliability, we propose the following symmetric distance measure for a coherent system lifetime T and the lifetime of a parallel (series) system:

$$
\text{DS}(T) = \frac{\text{CRI}_{T,T_{(1)}} - \text{CRI}_{T_{(n)},T}}{\text{CRI}_{T_{(n)},T_{(1)}}}. \tag{15}
$$

The idea is to verify if the system distribution is closer to a parallel system distribution to produce great reliability. We readily have

$$
|\text{DS}(T)| \le 1.
$$

We can note that the system is parallel if, and only if, $\text{DS}(T) = 1$, while the system is series if, and only if, $\text{DS}(T) = -1$. Thus, we can say that if $\text{DS}(T)$ is close to 1, the distribution of T is closer to the distribution of a parallel system, but if $\text{DS}(T)$ is close to -1, the distribution of T is closer instead to the distribution of a series system.

Furthermore, we can define an ordering between lifetimes, T_1 and T_2, of two coherent systems. We say T_2 is more preferable than T_1 in DS, denoted by $T_1 \leq_{\mathrm{DS}} T_2$, if $\mathrm{DS}(T_1) \leq \mathrm{DS}(T_2)$.

If $T_1 \leq_{\mathrm{DS}} T_2$, we can say that the distribution of T_2 is closer to a parallel system distribution than the distribution of T_1, and therefore, T_2 is more reliable than T_1.

As an application, we consider the notions of stochastic (st), hazard rate (hr), and likelihood ratio (lr) orderings; see Shaked and Shanthikumar [33] for details.

THEOREM 4.4: *Let T_1 and T_2 be the lifetimes of two coherent systems with n independent and identically distributed (or exchangeable) common components, and with signatures $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, respectively. If any one of $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$, $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$, and $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$ holds, then $T_1 \leq_{DS} T_2$.*

PROOF: It is clear that Eq. (15) can be written as

$$\mathrm{DS}(T) = \frac{2E[T] - E[T_{(n)}] - E[T_{(1)}]}{E[T_{(n)}] - E[T_{(1)}]}.$$

Because the systems have common components, it is sufficient to prove that $E[T_1] \leq E[T_2]$. As $\mathbf{s}_1 \leq_{\mathrm{st}} \mathbf{s}_2$, $T_1 \leq_{\mathrm{st}} T_2$ by Theorem 2.1 in Navarro *et al.* [23], implying the result partially. It is well-known that $\mathbf{s}_1 \leq_{\mathrm{lr}} \mathbf{s}_2$ implies $\mathbf{s}_1 \leq_{\mathrm{hr}} \mathbf{s}_2$, which in turn implies $\mathbf{s}_1 \leq_{\mathrm{st}} \mathbf{s}_2$. Hence, the theorem proved. ■

EXAMPLE 4.5: *Let T_1 and T_2 be the lifetimes of two coherent systems with three independent and identically distributed (or exchangeable) common components, with signatures $\boldsymbol{s}_1 = (0.5, 0.2, 0, 3)$ and $\boldsymbol{s}_2 = (0.4, 0.2, 0.4)$, respectively. In this case, it can be shown that $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$, $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$, and $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$, and so by Theorem 4.4, we can conclude that $T_1 \leq_{DS} T_2$.*

5. JOINT SIGNATURE POINT PROCESS AND ASYMPTOTIC RELIABILITY

5.1. Joint Signature Point Process

For two coherent systems with shared components, Navarro *et al.* [24] defined and studied a new measure, called the joint signature, under the i.i.d. assumption on component lifetimes, and they also investigated some ordering properties. Zarezadeh *et al.* [37] provided further discussions on joint signatures of several coherent systems with some shared components. Here, we extend the signature point process in Section 3 to the joint signature point process to consider stochastic dependence and analyze cumulative residual stochastic measure for coherent systems under the double stochastic Poisson process.

In a general setup, we consider lifetimes of coherent systems, T and S, with corresponding component lifetimes $T_1, \ldots, T_n$ and $S_1, \ldots, S_m$, which are finite positive random variables defined in a complete probability space $(\Omega, \Im, P)$. We again assume that $P(T_i = T_j) = 0, P(S_k = S_l) = 0$, for all $i \neq j$, $k \neq l$, and also that relations between random variables and measurable sets, respectively, always hold with probability 1, meaning that the term P-a.s. is suppressed. Thus, the lifetimes of components can be dependent, but simultaneous failure is assumed to be not possible.

The mathematical description of our observations, that is, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, 1_{\{S_{(j)}>s\}}, 1 \leq i, j \leq n, 0 < s < t\}$$

satisfies the Dellacherie conditions of right continuity and completeness. Intuitively, this means at each time t, the observer knows if the events $\{T_{(i)} \leq t\}(\{S_{(j)} \leq t\})$ have occurred or not and if they had occurred, then the values of $T_{(i)}(S_{(j)})$ will be known exactly.

We now assume that $T_1, \ldots, T_n,\ S_1, \ldots, S_m$ are totally inaccessible $\Im_t$-stopping times. The evolution of components over time define a point process given through the failure times: we denote by $T_{(1)} < \cdots < T_{(n)}$ $(S_{(1)} < \cdots < S_{(m)})$ the ordered lifetimes $T_1, \ldots, T_n$ $(S_1, \ldots, S_m)$ as they appear over time. Also, as a convention, we define $T_{(n+1)} = T_{(n+2)} = \cdots = S_{(m+1)} = S_{(m+2)} = \cdots = \infty$, indicating that the sequences $(T_{(n)})_{n\geq 1}$ and $(S_{(m)})_{m\geq 1}$ define non-explosive point processes; see Bremaud [8].

Under the above conditions, it is well-known that a coherent system fails exactly at the failure of one of its components. This results in the following theorem.

Theorem 5.1: *Let $T_1, \ldots, T_n,\ S_1, \ldots, S_m$ be totally inaccessible $\Im_t$-stopping times representing the components lifetimes of two coherent systems with lifetimes T and S, respectively. Then,*

$$P(T \leq t, S \leq s|\Im_{t\vee s}) = \sum_{i=1}^{n}\sum_{j=1}^{m} 1_{\{T=T_{(i)},S=S_{(j)}\}} 1_{\{T_{(i)}\leq t, S_{(j)}\leq s\}}. \quad \textbf{(16)}$$

Proof: Because $\{T = T_{(i)}, S = S_{(j)}\}$ defines a partition of Ω, it follows from the total probability law that

$$\begin{aligned} P(T \leq t, S \leq s|\Im_{t\vee s}) &= \sum_{i=1}^{n}\sum_{j=1}^{m} P(T \leq t, S \leq s, T = T_{(i)}, S = S_{(j)}|\Im_{t\vee s}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} E[1_{\{T_{(i)}\leq t, S_{(j)}\leq s\}} 1_{\{T=T_{(i)},S=S_{(j)}\}}|\Im_{t\vee s}]. \end{aligned}$$

However, we have (see Dellacherie [11])

$$\{T = T_{(i)}\} \in \Im_{T_{(i)}} = \{A \in \Im_\infty : \{A \cap \{T_{(i)} \leq t\}\} \in \Im_t, \forall t > 0\},$$
$$\{S = S_{(j)}\} \in \Im_{S_{(j)}} = \{A \in \Im_\infty : \{A \cap \{S_{(j)} \leq s\}\} \in \Im_s, \forall s > 0\},$$

and so

$$\{T = T_{(i)}\} \cap \{T_{(i)} \leq t\} \in \Im_t \subseteq \Im_{t\vee s}$$

and

$$\{S = S_{(j)}\} \cap \{S_{(j)} \leq s\} \in \Im_s \subseteq \Im_{t\vee s}.$$

Hence,

$$\{T = T_{(i)}\} \cap \{T_{(i)} \leq t\} \cap \{S = S_{(j)}\} \cap \{S_{(j)} \leq s\}$$

is an $\Im_{t\vee s}$-measurable set implying Eq. (16):

$$P(T \leq t, S \leq s|\Im_{t\vee s}) = \sum_{i=1}^{n}\sum_{j=1}^{m} 1_{\{T_{(i)}\leq t, S_{(j)}\leq s\}} 1_{\{T=T_{(i)},S_{(j)}=S\}}.$$

The above decomposition allows us to define the joint signature point process as follows. ■

DEFINITION 5.2: *The vector* $(1_{\{T=T_{(i)},S=S_{(j)}\}}, 1 \le i \le n, 1 \le j \le m)$ *is defined as the joint signature point process of the bivariate lifetime* (T, S).

REMARK 5.3: *Because* $P(T_i = T_j) = 0$, $P(S_k = S_l) = 0$ *for all* $i \ne j$, *the collection* $\{\{T = T_{(i)}\}, 1 \le i \le n\}$ *is a partition of* Ω *and* $\sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} = 1$. *Also, the collection* $\{\{S = S_{(j)}\}, 1 \le j \le m\}$ *is a partition of* Ω *with* $\sum_{j=1}^{m} 1_{\{S=S_{(j)}\}} = 1$. *Therefore,*

$$\sum_{j=1}^{m} 1_{\{T=T_{(i)},S=S_{(j)}\}} = \sum_{j=1}^{m} 1_{\{T=T_{(i)}\}} 1_{\{S=S_{(j)}\}} = 1_{\{T=T_{(i)}\}}.$$

Then, the vector $(1_{\{T=T_{(i)}\}}, 1 \le i \le n)$ *is defined as the marginal signature point process of the coherent system with lifetime* T. *We also have Eq. (12):*

$$P(T \le t|\Im_t) = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} 1_{\{T_{(i)} \le t\}}.$$

The joint conditional reliability function of (S, T), *defined as* $P(T > t, S > s|\Im_{t \vee s})$, *can be given as follows.*

COROLLARY 5.4: *Let* $T_1, \ldots, T_n$, $S_1, \ldots, S_m$ *be totally inaccessible* $\Im_t$*-stopping times representing the lifetimes of components of two coherent systems with lifetimes* T *and* S, *respectively. Then,*

$$P(T > t, S > s|\Im_{t \vee s}) = \sum_{i=1}^{m} \sum_{j=1}^{n} 1_{\{T=T_{(i)},S=S_{(j)}\}} 1_{\{T_{(i)} > t, S_{(j)} > s\}}. \tag{17}$$

PROOF: One can easily obtain this result using Theorem 5.1 and Remark 5.3, observing the equality

$$1\{T > t, S > s\} = 1 - 1\{T \le t\} - 1\{S \le s\} + 1\{T \le t, S \le s\}.$$

■

REMARK 5.5: *Using Eq. (17), we can calculate the "joint reliability of systems" as*

$$\begin{aligned} P(T > t, S > s) &= E[P(T > t, S > s|\Im_{t \vee s})] \\ &= E\left[\sum_{i=1}^{n} \sum_{j=1}^{m} 1_{\{T=T_{(i)},S=S_{(j)}\}} 1_{\{T_{(i)} > t, S_{(j)} > s\}}\right] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} P(\{T = T_{(i)}, S = S_{(j)}\} \cap \{T_{(i)} > t, S_{(j)} > s\}). \end{aligned}$$

As in Randles and Wolfe [27], if the components $T_1, \ldots, T_n$, $S_1, \ldots, S_m$ *are all independent and identically distributed with a continuous distribution* F, *the events* $\{T = T_{(i)}, S = S_{(j)}\}$ *and* $\{T_{(i)} \le t, S_{(j)} \le s\}$ *are independent, and we then have in this case*

$$P(T > t, S > s) = \sum_{k=1}^{n} \sum_{i=1}^{m} P(T = T_{(i)}, S = S_{(j)}) P(T_{(i)} > t, S_{(j)} > s).$$

Remark 5.6: *Navarro et al. [25] considered coherent systems with common shared independent and identically distributed component lifetimes* $T_1, \ldots, T_n$ *with a continuous distribution function* F*, and introduced a "bivariate signature matrix." For the sake of clarity, we can think of two systems with* n_1 *and* n_2 *components and having* $n_{1,2}$ *components in common, so that there are* $n = n_1 + n_2 - n_{1,2}$ *components in total. If* $n_{1,2} = 0$*, the lifetimes of the two systems become independent and the joint distribution of their lifetimes is then simply the product of their marginal distributions. Navarro et al. [25] also considered the case in which these systems are based just on some of these component lifetimes and not on all of them; so, these systems may share all, some, or none of these components. They then considered the random vector* $\boldsymbol{I} = (I_1, I_2)$ *with*

$$\boldsymbol{I} = (i, j) \quad \textit{whenever } T = T_{(i)} \text{ and } S = T_{(j)}$$

with joint probability mass function of $\boldsymbol{I}$*, denoted by* $p_{i,j} = P(\boldsymbol{I} = (i, j))$*, with*

$$p_{i,j} = \frac{|A_{i,j}|}{n!},$$

where $|A_{i,j}|$ *is the size of the set*

$$A_{i,j} = \{\sigma \in \wp_n : T = T_{(i)} \textit{ and } S = T_{(j)} \textit{ whenever } T_{\sigma(1)} < \cdots < T_{\sigma(n)}\}$$

and $\wp_n$ *is the set of permutations of the set* $\{1, \ldots, n\}$*. The matrix* $\boldsymbol{P} = (p_{i,j})$ *has been termed the bivariate signature matrix (BSM) associated with* (S, T) *by Navarro et al. [25]. Also,* $s_j^S = \sum_{i=1}^n p_{i,j}$ *defines the univariate signature (marginal) of the coherent system corresponding to lifetime* S *and* $s_i^T = \sum_{j=1}^n p_{i,j}$ *similarly defines the univariate signature (marginal) of the coherent system signature corresponding to lifetime* T*.*

If we consider the component lifetimes in $\{T_1, \ldots, T_n\}$*, we can write*

$$G(t, s) = P(T \le t, S \le s) = \sum_{i=1}^n \sum_{j=1}^n p_{i,j} F_{i,j}(t, s),$$

where $p_{i,j} = P(T = T_{(i)} S = T_{(j)})$*,* $F_{i,j}(t, s) = P(T_{(i)} \le t, T_{(j)} \le s)$*, and* $G(t, s)$ *is the joint distribution function of the system lifetimes. It should be noted that* G *can have a singular part in the set* $\{T = S\}$*, in which case we have*

$$F_{i,i} = P(T_{(i)} \le t, T_{(i)} \le s) = F_{(i)}(t \wedge s),$$

and we can then continue to use the above decomposition.

5.2. Compensator Process and Asymptotic Reliability

The point process $N_t((i)) = 1_{\{T_{(i)} \le t\}}$ is an $\Im_t$-submartingale, that is, $T_{(i)}$ is $\Im_t$-measurable and $E[N_t((i)) | \Im_s] \ge N_s((i))$ for all $0 \le s \le t$. Then, from Doob–Meyer decomposition, there exists an unique $\Im_t$-predictable process, denoted by $(A_t((i)))_{t \ge 0}$, called the $\Im_t$-compensator of $N_t((i))$, with $A_0((i)) = 0$ and such that $M_t((i)) = N_t((i)) - A_t((i))$ is a zero mean uniformly integrable $\Im_t$-martingale. We assume that T_i, $1 \le i \le n$ are totally inaccessible $\Im_t$-stopping times and, under this assumption, $A_t((i))$ is continuous. As $N_t((i))$ can only count in the time interval $(T_{(i-1)}, T_{(i)}]$, the corresponding compensator differential $dA_t((i))$ must vanish outside this interval.

The $\Im_t$-compensator of $P(T \le t | \Im_t)$, where T is the system lifetime, is as given in the following theorem.

THEOREM 5.7: *Let $T_1, \ldots, T_n$ be totally inaccessible $\Im_t$-stopping times representing the lifetimes of components of a coherent system with lifetime T. Then, the $\Im_t$-submartingale $P(T \le t|\Im_s)$, $0 \le s \le t$, has $\Im_t$-compensator*

$$\sum_{i=1}^{n} \int_0^t 1_{\{T=T_{(i)}\}}\, dA_s((i)). \tag{18}$$

PROOF: Let us consider the deterministic process

$$1_{\{T=T_{(i)}\}}(w,s) = 1_{\{T=T_{(i)}\}}(w).$$

It is left continuous and, therefore, $\Im_t$-predictable, implying that (see [8])

$$\int_0^t 1_{\{T=T_{(i)}\}}(w)\, dM_s((i))$$

is an $\Im_t$-martingale.

Because a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{i=1}^{n} \int_0^t 1_{\{T=T_{(i)}\}}\, dM_s((i)) = \sum_{i=1}^{n} \int_0^t 1_{\{T=T_{(i)}\}}\, d1_{\{T_{(i)} \le s\}} - \sum_{i=1}^{n} \int_0^t 1_{\{T=T_{(i)}\}}\, dA_s((i))$$

is an $\Im_t$-martingale. As the compensator is unique, the proof gets completed. ■

We consider the definition of a $\Im_t$-double stochastic Poisson process.

DEFINITION 5.8: *A point process N_t, adapted to a history $(\Im_t)_{t\ge 0}$, is said to be a $\Im_t$-double stochastic Poisson process, directed by A_t if, for all $t \ge s \ge 0$ and all $u \in \Re$,*

$$E\{\exp[iu(N_t - N_s)|\Im_s\} = \exp[(e^{iu} - 1)(A_t - A_s)],$$

where $(A_t)_{t\ge 0}$ is a finite, nonnegative $\Im_0$-measurable process. Also, the above expression yields, for all $t \ge s \ge 0$ and all $k \ge 0$,

$$P(N_t - N_s = k|\Im_s) = \exp[-(A_t - A_s)]\frac{[A_t - A_s]^k}{k!},$$

and N_t is said to be a $\Im_t$-double stochastic Poisson process or a $\Im_t$-conditional Poisson process.

If we have $A_t = \int_0^t \lambda_s\, ds$, where λ_t is a nonnegative $\Im_0$-measurable process with $\int_0^t \lambda_s\, ds < \infty$, then λ_t is called the intensity process.

If $A_t = A$, where A is some nonnegative $\Im_0$-measurable random variable, then N_t is called a homogeneous double stochastic Poisson process.

If $A_t = A(t)$, where $A(t)$ is a deterministic function of time, then $N_t = N(t)$ is called a nonhomogeneous Poisson process.

We now apply Brown's Theorem (see Ref. [9]) in the signature point process representation of a coherent system.

THEOREM 5.9 (Brown [9: *)] Let $(\Im_t^n)_{n\ge 1}$ be a sequence of history defined on a common probability space $(\Omega, \Im, P)$, $(N_t^n)_{n\ge 1}$ be a sequence of a simple point processes $\Im_t^n$-adapted, for each n, and $(A_t^n)_{n\ge 1}$ be the sequence of $\Im_t^n$-compensators of $(N_t^n)_{n\ge 1}$. Let $(A_t)_{t\ge 0}$ be a* *cumulative process defined on* $(\Omega, \Im, P)$, *with continuous trajectories and such that for each* $t > 0$,

(i) A_t *is* $\Im_0^n$*-measurable, for every* $n = 1, 2, \ldots$;
(ii) $A_t^n \to A_t$, *in probability, when* $n \to \infty$.

Then, N_t^n *converges weakly to a double stochastic Poisson process directed by* A_t.

Now, we apply Theorem 5.9 to calculate the asymptotic reliability of a coherent system.

Corollary 5.10: *Let* $T_1, \ldots, T_n$ *be totally inaccessible* $\Im_t$*-stopping times representing lifetimes of components of a coherent system with lifetime* T. *Consider a component level filtration given by*

$$\Im_t^n = \sigma\{1_{\{T_{(i)}>s\}},\ 1 \le i \le n,\ 0 < s < t\},$$

and the point process

$$N_t^n = P(T \le t | \Im_t^n) = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} 1_{\{T_{(i)} \le t\}}$$

with $\Im_t^n$*-compensator*

$$A_t^n = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} A_t((i)).$$

If, for all $t \ge 0$, $A_t^n \to A_t$, *in probability when* $n \to \infty$, *where* A_t *has a continuous sample path and is* $\Im_0^n$*-measurable, then* N_t^n *converges weakly to a double stochastic Poisson process directed by* A_t.

Proof: As we have, for $k \le n$, $\{T_{(i)} = T\} \in \Im^n_{T_{(i)}}, \{T_{(i)} = T\} \cap \{T_{(i)} \le t\} \in \Im_t^n, \forall t \ge 0$, N_t^n is $\Im_t^n$-adapted and the proof follows readily from Brown's Theorem. We denote this limit by

$$A_t^T = \sum_{i=1}^{\infty} 1_{\{T=T_{(i)}\}} A_t((i)).$$

$(A_t^T)_{t\ge 0}$ is the $\Im_t$-compensator of $(N_t^T)_{t\ge 0}$, where

$$N_t^T = \lim_{n\to\infty} N_t^n = \sum_{i=1}^{\infty} 1_{\{T=T_{(i)}\}} 1_{\{T_{(i)} \le t\}},$$

also denoted by $(T_n)_{n\ge 1}$. ■

Remark 5.11: *To give a meaning and interpretation for system signature of "infinite order," we mention the following result of Navarro et al. [23]: Given an arbitrary coherent system with lifetime* T *and distribution function* $F_T(t)$, *with* n *i.i.d. components, there exists, for any integer* $m > n$, *an equivalent (equal in law) coherent system with* m *i.i.d. components with the same distribution function* $F_T(t)$. *Formally, Navarro et al. [23] established the following result.*

THEOREM 5.12 (Navarro *et al.* [23]): *Let* $\boldsymbol{s} = (s_1, \ldots, s_k)$ *be the signature of an arbitrary coherent system of order* k*. Then, for any integer* $n > k$*, the system with signature* $\boldsymbol{s}$ *is equivalent to the* n*-component system with signature* $\boldsymbol{s}^* = (s_1^*, \ldots, s_n^*)$ *given by*

$$\boldsymbol{s}^* = \sum_{i=1}^{k} s_i \sum_{j=i}^{n+i-k} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} \boldsymbol{s_{j:n}},$$

where $\boldsymbol{s_{j:n}} = (0, \ldots, 0, 1, 0, \ldots, 0)$ *is the signature vector of a* j*-out-of-*n*:F system.*

It is important to note that

$$\sum_{j=i}^{n+i-k} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} = 1$$

and

$$\lim_{n\to\infty} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} = 0.$$

EXAMPLE 5.13: *Let* T_1, T_2, T_3*, and* T_4 *be independent and identically distributed component lifetimes with distribution function* F*. Let* S *and* T *be the lifetimes of the following coherent systems with a single-shared component:* $S = \wedge\{T_1 \vee T_2, T_1 \vee T_2, T_2 \vee T_3\}$ *and* $T = T_3 \wedge T_4$*, where we use the following notation* $T_i \wedge T_j = \min\{T_i, T_j\}$ *and* $T_i \vee T_j = \max\{T_i, T_j\}$*. Navarro et al. [25] have then calculated the probability distribution of the random pair* (I_1, I_2) *by using the definition in Remark 5.6, as the matrix* $\boldsymbol{P}$ *given by*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a two-component system, the system signature corresponding to lifetime T *is* $(1, 0)$*. As a three-component system, its signature is* $(\frac{2}{3}, \frac{1}{3}, 0)$*, and as a four-component system its signature, determined from* $\boldsymbol{P}$*, is* $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0)$*.*

Similarly, as a three-component system, the system signature corresponding to lifetime S *is* $(0, 1, 0)$*. As a four-component system, its signature, determined from* $\boldsymbol{P}$*, is* $(0, \frac{1}{2}, \frac{1}{2}, 0)$*.*

EXAMPLE 5.14: *To illustrate the asymptotic procedure, we consider the well-known Cesaro Summability Condition:*

(I) If $0 < p(n,k) < 1$*, for all* n *and* $1 \le k \le n$*, then the sum* $\sum_{k=1}^{n} p(n,k)$ *and the product* $\Pi_{k=1}^{n}(1 - p(n,k)$*, either converges or diverges;*

(II) If $0 < p(n,k) < 1$*, for all* n*,* $1 \le k \le n$ *and*

$$\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{n} p(n,k) = \lambda t,$$

for fixed t *and some* $\lambda > 0$*, as* $n \to \infty$*.*

Then,

$$\prod_{k=1}^{\infty}(1-p(n,k)) = \exp(-\lambda t),$$

for fixed t and some $\lambda > 0$.

We now assume that the components of coherent systems are subject to failure according to an NHPP. This property characterizes a minimal repair process which means that, at each failure in the set $\{T = T_{(i)}\}$ the system is repaired and it continues to work with the same failure rate as it had immediately before failure. In this cases, we characterize the coherent system through its compensator process

$$A_t^n = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} A_t((i)).$$

It is well-known that (see Ref. [1]), in the absolutely continuous case in $\{t < T_{(i)}\}$, $A_t((i)) = -\ln(1 - F_{(i)}(t|\Im_{t-}))$. So, $A_t((i)) = \sum_{j=1}^{\infty}(1/j)P(T_{(i)} \le t|\Im_{t-})^j$ and

$$A_t^n = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} \sum_{j=1}^{\infty} \frac{1}{j} P(T_{(i)} \le t|\Im_{t-})^j.$$

We can thus state conditions under which we apply the Cesaro Summability Condition to coherent system, for example:

If $F_{(k)}(t|\Im_{t-})$ are absolutely continuous and

$$A_t^n = \sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} \sum_{j=1}^{\infty} \frac{1}{j} P(T_{(i)} \le t|\Im_{t-})^j \to \lambda t,$$

for fixed t and some $\lambda > 0$, as $n \to \infty$, then the coherent system converges to a Poisson process.

Under the above conditions and the second part of Cesaro Summability Condition, we can conclude that the asymptotic reliability is equal to

$$\prod_{i=1}^{\infty} P(T_{(i)} > t|\Im_{t-}) = \exp(-\lambda t).$$

We consider a coherent system of identically distributed components where, for fixed t and some λ, the failure probability of an ordered component depends on the size n of the system and its position, i, and tends to zero at the rate $\lambda t/n^{1/i}$. It then follows that

$$P(T_{(i)} \le t|\Im_{t-}) = \left[\frac{\lambda t}{n^{1/i}} + o\left(\frac{1}{n^{1/i}}\right)\right]^i.$$

Therefore,

$$-\sum_{i=1}^{n} 1_{\{T=T_{(i)}\}} \sum_{j=1}^{\infty} \frac{1}{j} \left\{\left[\frac{\lambda t}{n^{1/i}} + o\left(\frac{1}{n^{1/i}}\right)\right]^i\right\}^n \to (\lambda t)^k,$$

as $n \to \infty$. Then, the coherent system converges to a Weibull process. This, in fact, provides a motivation to asymptotically extend the inaccuracy measure for a double stochastic Poisson process.

6. CRI MEASURE

6.1. CRI Measure for Coherent Systems under the Double Stochastic Poisson Process

We now define CRI measure between the coherent system modeled by double stochastic Poisson processes as follows.

DEFINITION 6.1: *Let $(N_t^T)_{t\geq 0}$, defined by the sequence $(T_n)_{n\geq 1}$, be a double stochastic Poisson process modeling a coherent system with lifetime T and $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$, and similarly $(N_t^S)_{t\geq 0}$, defined by the sequence $(S_n)_{n\geq 1}$, be a double stochastic Poisson process modeling a coherent system with lifetime S and $\Im_t$-compensator processes $(B_t^S)_{t\geq 0}$. Then, the CRI measure between N_t^T and N_t^S, as in* (6), *is*

$$CRI_{S,T} = E\left[\int_0^T B_s^S\, ds\right] + E\left[\int_0^S A_s^T\, ds\right].$$

THEOREM 6.2: *Let $T_1, \ldots, T_n, \ldots,\ S_1, \ldots, S_m, \ldots$ be totally inaccessible $\Im_t$-stopping times representing the lifetimes of components of two coherent systems with lifetimes T and S, respectively. Then, the CRI measure of N_t^T and N_t^S at the component level, that is, observing $T_{(i)}(i\geq 1)$ and $S_{(i)}(i\geq 1)$, is*

$$CRI_{S,T} = E\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}|S_{(k)} - T_{(j)}|\right]. \tag{19}$$

PROOF: We have

$$\mathrm{CRI}_{S,T} = E\left[\int_0^T B_s^S\, ds\right] + E\left[\int_0^S A_s^T\, ds\right].$$

Now, using the results of Section 5, we have

$$E\left[\int_0^T B_s^S\, ds\right] = E\left[\int_0^T \left(\int_0^s dB_t^S\right) ds\right] = E\left[\int_0^T \left(\int_t^T ds\right) dB_t^S\right] = E\left[\int_0^T (T-t)\, dB_t^S\right].$$

As $dB_t^S = \sum_{i=1}^{\infty} 1_{\{S=S_{(i)}\}}\, dB_t((i))$, we have

$$\begin{aligned} E\left[\int_0^T B_s^S\, ds\right] &= E\left[\int_0^T (T-t)\sum_{i=1}^{\infty} 1_{\{S=S_{(i)}\}}\, dB_t((i))\right] \\ &= E\left[\sum_{i=1}^{\infty}\int_0^T (T-t) 1_{\{S=S_{(i)}\}} d1_{\{S_{(i)}\leq t\}}\right] \\ &= E\left[\sum_{i=1}^{\infty} 1_{\{S=S_{(i)}\}}(T - S_{(i)}) 1_{\{S_{(i)}\leq T\}}\right]. \end{aligned} \tag{20}$$

Using the same argument, we also obtain

$$E\left[\int_0^S A_s^T\, ds\right] = E\left[\sum_{j=1}^{\infty} 1_{\{T=T_{(j)}\}}(S - T_{(j)}) 1_{\{T_{(j)}\leq S\}}\right]. \tag{21}$$

Hence, by adding Eqs. (20) and (21), we obtain

$$\begin{aligned}\mathrm{CRI}_{S,T} &= E\left[\sum_{i=1}^{\infty} 1_{\{S=S_{(i)}\}}(T - S_{(i)})1_{\{S_{(i)}\le T\}} + \sum_{i=1}^{\infty} 1_{\{T=T_{(i)}\}}(S - T_{(i)})1_{\{T_{(i)}\le S\}}\right.\\ &\quad \left. + \sum_{i=1}^{\infty} 1_{\{T=T_{(i)}\}}\left(\sum_{j=1}^{\infty} 1_{\{S=S_{(j)}\}}\right)(S - T_{(i)})1_{\{T_{(i)}\le S\}}\right]\\ &= E\left[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}|(T_{(j)} - S_{(i)}|1_{\{S_{(i)}\le T_{(j)}\}}\right.\\ &\quad \left. + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{T=T_{(i)},S=S_{(j)}\}}|S_{(j)} - T_{(i)}|1_{\{T_{(i)}\le S_{(j)}\}}\right]\\ &= E\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}|S_{(i)} - T_{(j)}|\right].\end{aligned}$$

■

Remark 6.3: *The interpretation of the CRI measure between the double stochastic Poisson process is retained. We note that (19) can be written as*

$$\begin{aligned}CRI_{S,T} &= E\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}|S_{(i)} - T_{(j)}|\right]\\ &= E\left[|S-T|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}\right]\\ &= E|S-T|.\end{aligned}$$

Thus, $CRI_{T,S}$ can be seen as a dispersion measure when using a coherent system lifetime S asserted by the experimenter's information of the true coherent system lifetime T.

Corollary 6.4: *Let $(N_t^T)_{t\ge 0}$, defined by the sequence $(T_n)_{n\ge 1}$, be a deterministic NHPP modeling a coherent system with lifetime T subject to minimal repairs, and similarly let $(N_t^S)_{t\ge 0}$, defined by the sequence $(S_n)_{n\ge 1}$, be a deterministic NHPP modeling a coherent system with lifetime S subject to minimal repairs, independently of T. Then, the CRI measure of N_t^T and N_t^S at the component level is given by*

$$\begin{aligned}CRI_{S,T} &= \sum_{i=1}^{\infty}\int_0^{\infty} s_i^S P(S_{(i)} > t)\,dt + \sum_{j=1}^{\infty}\int_0^{\infty} s_j^T P(T_{(j)} > t)\,dt\\ &\quad - 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\infty} p_{i,j} P(S_{(i)} > t, T_{(j)} > t)\,dt. \qquad \textbf{(22)}\end{aligned}$$

PROOF: As in Eq. (19), we have

$$\begin{aligned}
\mathrm{CRI}_{S,T} &= E\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(i)},T=T_{(j)}\}}|S_{(i)}-T_{(j)}|\right] \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E\{E[1_{\{S=S_{(i)},T=T_{(j)}\}}|S_{(i)}-T_{(j)}||S=S_{(i)},T=T_{(j)}]\} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E\{1_{\{S=S_{(i)},T=T_{(j)}\}}E[|S_{(i)}-T_{(j)}||S=S_{(i)},T=T_{(j)}]\} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(S=S_{(i)},T=T_{(j)})E[|S_{(i)}-T_{(j)}|] \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}E[(S_{(i)}\vee T_{(j)})-(S_{(i)}\wedge T_{(j)})] \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\int_0^{\infty}[P((S_{(i)}\vee T_{(j)})>t)-P((S_{(i)}\wedge T_{(j)})>t)]\,dt \\
&= \sum_{i=1}^{\infty}\int_0^{\infty} s_i^S P(S_{(i)}>t)\,dt+\sum_{j=1}^{\infty}\int_0^{\infty} s_j^T P(T_{(j)}>t)\,dt \\
&\quad -2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\infty} p_{i,j}P(S_{(i)}>t,T_{(j)}>t)\,dt.
\end{aligned}$$

■

EXAMPLE 6.5: *As in Example 4.3, let $T_1,\ldots,T_n,\ldots$ be the lifetimes of components of a coherent system with lifetime T that are subject to failure according to a Weibull process with parameters β and θ_1. Similarly, let $S_1,\ldots,S_m,\ldots$ be the lifetimes of components of a coherent system with lifetime S, asserted by the experimenter, that are subject to failure according to a Weibull process with parameters β and θ_2. Then, $S\wedge T$ follows a Weibull process with parameters β and $\theta_1^{\beta}\theta_2^{\beta}/(\theta_1^{\beta}+\theta_2^{\beta})$. In practice, we consider the ordered lifetimes $T_1,\ldots,T_n,\ldots$ with a conditional reliability function*

$$\overline{F_i}(t_i|t_1,\ldots,t_{i-1})=\exp\left[-\left(\frac{t_i}{\theta}\right)^{\beta}+\left(\frac{t_{i-1}}{\theta}\right)^{\beta}\right]$$

for $0\le t_{i-1}<t_i$, where t_i are the ordered observations. Now, considering $T_1,\ldots,T_n,\ldots$, $S_1,\ldots,S_m,\ldots$ as record values of independent and identically distributed random variables, we can apply Corollary 6.4 to obtain

$$CRI_{S,T}=\int_0^{\infty}\left\{\sum_{i=1}^{\infty} s_i^T\exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta}+\left(\frac{t_{i-1}}{\theta_1}\right)^{\beta}\right]\right.$$

$$+\sum_{j=1}^{\infty} s_j^S \exp\left[-\left(\frac{t}{\theta_2}\right)^{\beta} + \left(\frac{s_{j-1}}{\theta_2}\right)^{\beta}\right]$$

$$-2\sum_{i=1}^{\infty} s_i^{S\wedge T} \exp\left[-\left(\frac{t}{\theta_1^{\beta}\theta_2^{\beta}/(\theta_1^{\beta}+\theta_2^{\beta})}\right)^{\beta} + \left(\frac{t_{i-1}}{\theta_1^{\beta}\theta_2^{\beta}/(\theta_1^{\beta}+\theta_2^{\beta})}\right)^{\beta}\right]\Bigg\} dt,$$

where s_k^T,s_j^S, and $s_i^{S\wedge T}$ are the components of vectors s^T, s^S, and $s^{S\wedge T}$ of the ("infinite order") coherent system signatures with lifetimes T, S, and $S \wedge T$, respectively.

6.2. Dynamic CRI Measure for Coherent Systems under the Double Stochastic Poisson Process

It is of interest to extend the concept to a time varying form corresponding to a residual process after a fixed time t. In this regard, based on Definition 2.5, we can provide the following definition.

DEFINITION 6.6: *Let $(N_t^T)_{t\geq 0}$, defined by the sequence $(T_n)_{n\geq 1}$, be a double stochastic Poisson process with $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$, and similarly let $(N_t^S)_{t\geq 0}$, defined by the sequence $(S_n)_{n\geq 1}$, be a double stochastic Poisson process with $\Im_t$-compensator processes $(B_t^S)_{t\geq 0}$. Then, the dynamic CRI measure between N_t^T and N_t^S is given by*

$$DCRI_{S,T}^t = E\left[\int_t^T B_s^S\, ds\right] + E\left[\int_t^S A_s^T\, ds\right]$$

$$= E\left[\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_i,T=T_n\}} |T_n - S_i| 1_{\{t<T_n\wedge S_i\}}\right]. \quad \textbf{(23)}$$

The above definition is consistent in the way that it can be proved using the same arguments as in the proof of Theorem 6.2 upon replacing the integral domain from $(0,\infty)$ to (t,∞).

REMARK 6.7: *Let $(N_t^T)_{t\geq 0}$ and $(N_t^S)_{t\geq 0}$ be double stochastic Poisson processes with $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$ and $(B_t^S)_{t\geq 0}$, respectively. We say that N_t^T and N_t^S satisfy the proportional risk hazard process if $B_t^S = \alpha A_t^T$, $\forall t \geq 0$ for some α, $0 < \alpha < 1$.*

THEOREM 6.8 (Characterization Problem): *If N_t^T and N_t^S satisfy the proportional risk hazard process, then the dynamic CRI measure $DCRI_{S,T}^t < \infty$ uniquely determines the double stochastic Poisson process.*

PROOF: The proof follows along the lines similar to those of Theorem 2.6 and is therefore not presented here for the sake of brevity. ■

7. CONCLUDING REMARKS

In this work, we have introduced a symmetric CRI measure between two lifetimes as a metric. Next, we have used a point process martingale approach to introduce the signature point process. We have then extended these results to coherent systems and have also defined a measure to compare the relative performance of different coherent systems. In the second part of the paper, we have defined a joint signature point process for two coherent systems which will be useful in calculating the residual cumulative inaccuracy measure between two coherent systems. We have also provided an asymptotic reasoning for extending the introduced inaccuracy measure to a double stochastic Poisson process. We have then defined the cumulative inaccuracy measure for coherent systems under a double stochastic Poisson process and paid special attention to a nonhomogeneous Poisson process for modeling a minimal repair coherent system. We have presented some examples to illustrate the calculation of the inaccuracy measure between minimal repair point processes.

It will be of great interest to generalize this work in two directions: first, to deal with the case of several coherent systems simultaneously sharing some components as discussed in the work of Zarezadeh *et al.* [37], and secondly, to handle the notion of ordered system signatures as discussed in the works of Balakrishnan and Volterman [6] (for binary state systems) and He *et al.* [15] (for multi-state systems). Work is currently under progress on these problems and we hope to report the findings in a future paper.

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