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by

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# ASYMPTOTIC EFFICIENCY OF THE LIKELIHOOD RATIO CONDITIONAL TEST FOR MULTINOMIAL DISTRIBUTIONS

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SUMMARY

Asymptotic Efficiency of the Likelihood Ratio
Conditional Test for Multinomial Distributions

This paper studies the non-local asymptotic optimality of the likelihood ratio statistic, for suitable composite hypotheses, when the test is performed conditionally given a sufficient statistic. Asymptotic optimality is discussed in the framework of Bahadur efficiency. The results concern multinomial models with appropriate composite hypotheses. The conditional test, using the likelihood ratio statistic $-(1/n)\log \lambda_n$, is shown to possess the same asymptotic optimality as does the unconditional likelihood ratio test. Sufficient conditions for this equivalence are presented and shown to hold for a variety of multinomial examples having importance in Statistics and Genetics.

## 1. Introduction

Bahadur considers a measure of asymptotic efficiency of a test statistic $T_n$, the rate at which the level attained by $T_n$ tends to zero under a fixed alternative. Another equivalent way considers the rate at which the significance level $\alpha_n$ of the test tends to zero when we fix the power at fixed alternative, as a measure of asymptotic efficiency of the test (Bahadur, 1960a, 1960b, 1967, 1971).

It is shown in Bahadur (1965, 1967, 1971), under certain conditions that, for testing $\theta \varepsilon \Theta_0$ against $\theta \varepsilon \Theta - \Theta_0$ in the independent identically distributed (i.i.d.) case the level attained by any test sequence cannot tend to zero at a rate faster than $(\rho(\theta))^n$ when $\theta$ obtains, where $\Theta$ is the parameter space, $\Theta_0$ is a subset of $\Theta$, and $\rho$ is a parametric function defined in terms of Kullback-Leibler (1968) information number. It is also proved that the level attained by the likelihood ratio statistics do tend to zero at the optimal exponential rate.

Still the i.i.d. case, Bahadur and Bickel (1970) have shown that $(\rho(\theta))^n$ remains the optimal exponential rate even when $L_n$ is the conditional level attained by $T_n$, given that a conditioning statistics $U_n$ has been observed.

Bahadur and Raghavachari (1970) proved the statement above without the assumption of independent identically distributed observations and they have found some sufficient conditions for a conditional testing sequence to be optimal in the sense mentioned above. But, there is not equivalent statement with respect to the likelihood ratio statistics when conditional levels are used.

In Sections 2 and 3 of this paper, we prove a theorem that gives a sufficient condition under which the Likelihood Ratio Conditional (LRC) test for testing $H: \underline{p} \varepsilon \Omega_0$ against $K: \underline{p} \varepsilon \Omega - \Omega_0$, based on a multinomial observation $\underline{Z}^{(n)} = (n_1/n, n_2/n, \ldots n_k/n)$ with $\sum_{i=1}^{k} n_i = n$, is asymptotically optimal in the sense that the conditional significance level tends to zero at the optimal exponential rate when we fix the power bounded away from zero and one at a fixed alternative where $\Omega = \{\underline{p} = (p_1, \ldots, p_k)\ p_i \geq 0,\ \sum_{i=1}^{k} p_i = 1\}$ and $\Omega_0 = \{\underline{p} \varepsilon \Omega\ p_i > 0;\ p_i = g_i(\underline{v});\ i=1,2,\ldots,k;\ \underline{v} \subset R^m,\ m < k-1\}$. The functions $g_i$ are such that there exists the maximum likelihood estimate of $\underline{v}$ for each sample size. In Section 4 we verify this sufficient condition for some genetic models.

## 2. The Likelihood Ratio Conditional Test for Multinomial Distributions

The notation is the following

$X = \{(1,0,\ldots,0);\ (0,1,0,\ldots,0); \ldots\ldots ; (0,0,\ldots,0,1)\} \subset R^k$ is the sample space for one multinomial observation. $\mathcal{B}$ is the $\sigma$-field of Boral sets of $X$.

$\Omega = \{\underline{p} = (p_1, p_2, \ldots, p_k)\ |\ p_i \geq 0;\ \sum_{i=1}^{k} p_i = 1\}$ is the parameter space, $\Omega_0$ is a subset of $\Omega$. For each $\underline{p} \varepsilon \Omega$, $P(\cdot|\underline{p})$ is a probability measure on $\mathcal{B}$, absolutely continuous with respect to counting measure $\lambda$ on $X$. That is

$$dP(\underline{x}_1|\underline{p}) = f(\underline{x}_1|\underline{p})\, d\lambda$$

where $\underline{x}_1 = (x_{11}, x_{21}, \ldots, x_{k1}) \varepsilon X$, and

$$f(\underline{x}_1|\underline{p}) = \prod_{i=1}^{k} p_i^{x_{i1}} \quad \text{for } \underline{p} \varepsilon \Omega .$$

By convention, $p_i^{x_{i1}} = 1$ if $p_i = x_{i1} = 0$. For a fixed $\underline{p} \varepsilon \Omega$ and any $\underline{p}_o \varepsilon \Omega_o$, consider the functions $K(\underline{p}, \underline{p}_o)$ and $J(\underline{p})$ given by

$$(2.1) \qquad K(\underline{p}, \underline{p}_o) = E_{\underline{p}}[\log(f(\underline{x}_1|\underline{p})/f(\underline{x}_1|\underline{p}_o))] = I(\underline{p}, \underline{p}_o)$$

$$(2.2) \qquad J(\underline{p}) = \inf\{K(\underline{p}, \underline{p}_o) \mid \underline{p}_o \varepsilon \Omega_o\} = I(\underline{p}, \Omega_o)$$

where for any two points $\underline{a}$ and $\underline{b}$ in $\Omega$,

$$(2.3) \qquad I(\underline{a}, \underline{b}) = \sum_{i=1}^{k} a_i \log(a_i/b_i)$$

By convention $a_i \log(a_i/b_i) = 0$ if $a_{\underline{i}} = 0$. We note that $I(\underline{a}, \underline{b}) > 0$ unless $\underline{a} = \underline{b}$, and $I(\underline{a}, \underline{b}) < \infty$ unless $b_i = 0$ and $a_i > 0$ for some $1 \leq i \leq k$. Let $s = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \ldots)$ be an infinite sequence of independent observations on $\underline{x} \varepsilon X$. Then $(S, A) = (X^{(\infty)}, \mathcal{B}^{(\infty)})$ is the sample space of $s$. For each positive integer $n$ let $\underline{Z}^{(n)} = (Z_1^{(n)}, Z_2^{(n)}, \ldots, Z_k^{(n)})$ be a measurable transformation from $(S, A)$ into $(\Omega^{(n)}, \mathcal{B}_n)$ where

$$\Omega^{(n)} = \{\underline{z}^{(n)} = (z_1^{(n)}, \ldots, z_k^{(n)}) \mid z_i^{(n)} = n_i/n,\ n_i = \sum_{j=1}^{n} x_{ij},\ i = 1, \ldots, k,\ \sum_{i=1}^{k} n_i = n\}$$

and $\mathcal{B}_n$ is the $\sigma$-field of Borel sets of $\Omega^{(n)}$. Note that $\Omega^{(n)}$ is in one-to-one correspondence with the set of lattice points $\underline{z}^{(n)}$ of $\Omega$. For mathematical convenience we consider $\Omega$ as our sample space where we extend from $\Omega^{(n)}$ by attributing probability zero to all points of $\Omega$ that are not in $\Omega^{(n)}$. For each $\underline{p} \varepsilon \Omega$, $P_n(.|\underline{p})$ denotes a probability measure on ${}_n$ absolutely continuous with respect to counting measure $\mu$ on $\Omega^{(n)}$. That is, $dP_n(\underline{z}^{(n)}|\underline{p}) = f_n(\underline{z}^{(n)}|\underline{p})d\mu$, where

$$f_n(\underline{z}^{(n)}|\underline{p}) = (n!) \prod_{i=1}^{k} (p_i^{n_i}/n_i!) \quad \text{for} \quad \underline{p} \in \Omega$$

we can write

$$(2.4) \qquad f_n(\underline{z}^{(n)}|\underline{p}) = f_n(\underline{z}^{(n)}\,|\,\underline{x}^{(n)})\exp\{-nI(\underline{z}^{(n)}, \underline{p})\}$$

where $I(\underline{z}^{(n)}, \underline{p})$ was defined in (2.3).

Based on an observation $\underline{z}^{(n)}$ consider the problem of testing $H$: $\underline{p}\in\Omega_o$ against $K$: $\underline{p}\in\Omega-\Omega_o$, where $\Omega_o=\{\underline{p}\in\Omega|p_i>0;\ p_i=g_i(\underline{v});\ i=1,2,\ldots,k;\ \underline{v}\subset R^m;\ m<k-1\}$, and $g_i(\underline{v})$ are such that there exists the maximum likelihood estimate $\hat{\underline{v}}$ of $\underline{v}$ based on an observation $\underline{z}^{(n)}$ of the random vector $\underline{Z}^{(n)}$. Let $\hat{T}_n(\underline{z}^{(n)})$ be the test statistic defined by

$$(2.5) \qquad \hat{T}_n(\underline{z}^{(n)}) = -(1/n)\log \lambda_n(\underline{z}^{(n)}) = I(\underline{z}^{(n)}, \Omega_o),$$

where

$$\lambda_n(\underline{z}^{(n)}) = \{\sup\{f_n(\underline{z}^{(n)}|\underline{p})\,|\,\underline{p}\in\Omega_o\}\}\Big/\{\sup\{f_n(\underline{z}^{(n)}\,|\underline{p})\,|\,\underline{p}\in\Omega\}\}$$
$$= \exp\{-nI(\underline{z}^{(n)}, \Omega_o)\}$$

$$(2.6) \qquad I(\underline{z}^{(n)}, \Omega_o) = \inf\{I(\underline{z}^{(n)}, p)\,|\,\underline{p} \in \Omega_o\}.$$

Let $\underline{U}^{(n)}$ be a minimal sufficient statistic for $\underline{v}$ and $V^{(n)}$ the set of all possible values of $\underline{U}^{(n)}$. For each $\underline{u}^{(n)}\varepsilon V^{(n)}$ let $\Omega(\underline{u}^{(n)})=\{\underline{a}\varepsilon\Omega|\underline{u}^{(n)}(\underline{a})=\underline{u}^{(n)}\}$ be the elements of a partition of $\Omega$ induced by $\underline{U}^{(n)}$.

Remark. To facilitate writing, in the remaining part of this paper, we will use $\underline{Z}$ for $\underline{Z}^{(n)}$, $\underline{z}$ for $\underline{z}^{(n)}$, $\underline{u}$ for $\underline{u}^{(n)}$ and $\underline{U}$ for $\underline{U}^{(n)}$.

For each $\underline{p}\varepsilon\Omega$ the distribution of $\underline{U}$ and the conditional distribution of $\underline{Z}$ given $\underline{u}$ are respectively

$$h_n(\underline{u}|\underline{p}) = \sum_{\underline{z}\varepsilon\Omega(\underline{u})} f_n(\underline{z}|\underline{p}) \qquad \underline{u} \varepsilon V^{(n)}.$$

$$\overset{\underline{Z}|u}{f_n}(\underline{z}|\underline{p}) = \begin{cases} f_n(\underline{z}|\underline{p})/h_n(\underline{U}|\underline{p}) & \text{if } h_n(\underline{u}|\underline{p}) \neq 0,\ \underline{z}\ \varepsilon\ \Omega(\underline{u}) \\ 0 & \text{if } h_n(\underline{u}|\underline{p}) = 0 \end{cases}$$

when $\underline{p}\ \varepsilon\ \Omega_0$ the conditional distribution above is free of parameters. Then we abbreviate $\overset{\underline{Z}|u}{f_n}(\underline{z}|\underline{p})$ to $\overset{\underline{Z}|u}{f_n}(\underline{z})$ for $\underline{p}\varepsilon\Omega$.

Definition. For testing $H$: $\underline{p}\varepsilon\Omega_o$ against $K$: $\underline{p}\varepsilon\Omega-\Omega_o$ based on an observation $\underline{z}$, the LRC test $\{\hat{T}_n(\underline{z})\ \underline{u}\}$ given that we have observed a value $\underline{u}$ of the random vector $\underline{U}$, has critical function given by

$$\phi(\underline{z}|\underline{u}) = \begin{cases} 1 & \text{when} \quad I(\underline{z}, \Omega_o) > C_n(\underline{u}) \\ \gamma_n(\underline{u}) & \text{when} \quad I(\underline{z}, \Omega_o) = C_n(\underline{u}) \\ 0 & \text{when} \quad I(\underline{z}, \Omega_o) < C_n(\underline{u}) \end{cases}$$

where the functions $C_n$ and $\gamma_n$ are determined by

$$(2.7) \qquad \hat{\alpha}_n = \sum_{I(\underline{z},\Omega_o)>C_n(\underline{u})} \overset{\underline{Z}|u}{f_n}(\underline{z}) + \gamma_n(\underline{u}) \sum_{I(\underline{z},\Omega_o)=C_n(\underline{u})} \overset{\underline{Z}|u}{f_n}(\underline{z})$$

for all $\underline{u}$.

The test has Neyman structure with respect to $\underline{u}$.

Let $\hat{\beta}_n$ be the probability, under $\underline{p}\varepsilon\Omega-\Omega_0$, of not rejecting H without randomization; that is,

$$\hat{\beta}_n = \sum_{z\varepsilon W_n} f_n(z|p) = P_n(W_n|\underline{p})$$

where

$$(2.8) \qquad W_n = \bigcup_{u\varepsilon V^{(n)}} W_n(\underline{u}) \quad \text{and} \quad W_n(\underline{u}) = \{\underline{z}\varepsilon\Omega(u)|I(\underline{z},\Omega_0)<C_n(\underline{u})\}$$

We assume that for any $0<\beta<1$ and $p\varepsilon\Omega-\Omega_0$, there exist functions $C_n(u)$ and $\gamma_n(\underline{u})$, such that $\hat{\beta}_n \to \beta$ as $n \to \infty$, and $\hat{\alpha}_n$ is constant for all $\underline{u}$.

Under this assumption, for any test statistic $\{T_n|\underline{U}\}$ having the structure described above with $I(\underline{z}, \Omega_0)$ replaced by $T_n(\underline{z})$ we can conclude from Bahadur and Bickel (1966) and Bahadur and Raghavachari (1971) that

$$(2.9) \qquad \liminf_{n\to\infty} (1/n) \log \alpha_n > -J(\underline{p}) = -I(\underline{p}, \Omega_0)$$

where $\alpha_n$ is the conditional significance level of the test sequence $\{T_n|\underline{u}\}$.

In the next section it will be proved that under certain conditions the (LRC) test $\{T_n|\underline{u}\}$ is asymptotically optimal for testing $H_0$: $\underline{p}\varepsilon\Omega_0$ against K: $\underline{p}\varepsilon\Omega-\Omega_0$, that is

$$\lim (1/n) \log \hat{\alpha}_n = -J(p),$$

where $\hat{\alpha}_n$ is given by (2.7).

## 3. Asymptotic Properties of the LRC Test

The proofs of statements that will be made in this section are based on some properties of the function $I(a,b)$ and an important theorem on multinomial distribution proved by Hoeffding (1965) that by conveniency we have restated in Appendix 1 of this paper.

Theorem. Let $\Omega_0$ be as defined in Section 2. Let $\{\hat{T}_n|\underline{u}\}$ be the LRC test for testing $H: p\varepsilon\Omega_0$ against $K: \underline{p}\varepsilon\Omega-\Omega_0$ satisfying the assumption that for any fixed alternative $\underline{p}$ and any $0<\beta<1$, there exist functions $C_n(\underline{u})$ and $\gamma_n(\underline{u})$ such that $\hat{\beta}_n\to\beta$ as $n\to\infty$ and $\hat{\alpha}_n$ is constant for all $\underline{u}$. Suppose also that the conditional null distribution satisfies the following condition.

For each $\underline{u}$ and any subset $A_n$ of $\Omega(\underline{u})=\{\underline{a}\varepsilon\Omega\ U(\underline{a})=\underline{u}\}$

$$(3.1)\quad P_n^{\underline{Z}|\underline{u}}(A_n) = \sum_{Z\varepsilon A_n} f^{\underline{Z}|\underline{u}}(\underline{z}) = \mathrm{ex}\{-n\ I(A_n^{(n)}|,\Omega_0) + O(\log n)\}$$

where $A_n^{(n)}$ is the set of lattice points $\underline{z}$ contained in $A_n$ and for any subsets $A$ and $B$ of $\Omega$ $I(A,B)=\inf\{I(\underline{a},\underline{b})|\underline{a}\varepsilon A \text{ and } \underline{b}\varepsilon B\}$.

Then $\{\hat{T}_n|u\}$ is an asymptotically optimal test, that is

$$(3.2)\qquad \lim_{n\to\infty} (1/n)\log\hat{\alpha}_n = -J(\underline{p})$$

under the alternative $p$.

Remark. For each particular form of the functions $g_i(\underline{u})$ we can conclude if the LRC test is asymptotically optimal by verifying condition (3.1). In Section 4 we prove that this condition is satisfied by several important multinomial models.

The proof of the theorem is based on the following Lemmas.

Lemma 3.1. For each $n$ condition (3.1) implies that

(3.3) $$\hat{\alpha}_n \leq \exp\{-nC_n(\underline{u}) + O(\log n)\} \quad \text{for all } \underline{u}.$$

Proof.

Let $W_n^*(\underline{u}) = \{z\varepsilon\Omega(\underline{u}) \mid I(\underline{z},\Omega_o) \geq C_n(\underline{u})\}$ and $W_n^{*(n)}(\underline{u})$ the set lattice points of $W_n^*(u)$. Hence

(3.4) $$I(W_n^{*(n)}, \Omega_o) \geq I(W_n^*, \Omega) \geq C_n(\underline{u}).$$

By (2.7) we have

$$\hat{\alpha}_n \leq \sum_{I(Z,\Omega_o)\geq C_n(\underline{u})} \bar{f}_n^{Z|u}(\underline{z}) = P_n^{Z|u}(\bar{W}_n^*(\underline{u}))$$

Therefore (3.1), (3.4) implies (3.3).

Lemma 3.2. Let $W_n$ be given as in (2.8) and let all assumptions of the theorem be satisfied. Then for each fixed alternative $p\varepsilon\Omega-\Omega_o$, the condition $\hat{\beta}_n \to \beta$ as $n\to\infty$, implies that there is at least one point $\underline{y}_n$ such that

(3.5) $$\underline{y}_n \varepsilon \bar{W}_n \quad \text{and} \quad \underline{y}_n \to \underline{p} \quad \text{as} \quad n\to\infty.$$

If $\underline{p}\not\varepsilon\bar{W}_n$ for all large values of $n$ any $\underline{y}_n$ satisfying (3.5) is in the boundary of $W_n$.

Remark. For any subset $A$ of $\Omega$, $A'=\Omega-A$ is the complement of $A$. $\bar{A}$ denotes the closure of $A$. The boundary of $A$ is $\bar{A}\cap\bar{A}'$.

Proof.

We recall that $\hat{\beta}_n = P_n(W_n/\underline{p})$ where $W_n = \bigcup_{u\varepsilon V(n)} W_n(\underline{u})$ and $W_n(\underline{u})=\{z\varepsilon\Omega(\underline{u}) \mid I(\underline{z},\Omega_o)<C_n(\underline{u})\}$. Using the result on large deviation probabilities in multinomial distributions given in Appendix 1, we have

$$\hat{\beta}_n = \exp\{-nI(W_n^{(n)},\underline{p}) + O(\log n)\} .$$

where $W_n^{(n)}$ is the set of the lattice points $\underline{z}$ of $W_n$. Then for any $0<\beta<1$ and a fixed alternative $\underline{p}\varepsilon\Omega-\Omega_o$, the condition $\hat{\beta}_n \to \beta$ implies that $I(W_n^{(n)},\underline{p})\to 0$ as $n\to\infty$. Since $0\leq I(W_n,\underline{p})\leq I(W_n^{(n)},\underline{p})$, we have

(3.6) $$I(W_n,\underline{p}) \to 0 \quad \text{as} \quad n \to \infty.$$

For each $n$, $W_n$ is a subset of $\Omega$, then by $P_5$ and $P_6$ of Appendix 1, there is at least one point $\underline{y}_n$ such that

(3.7) $$\underline{y}_n \; \varepsilon \; \bar{W}_n \quad \text{and} \quad I(\underline{y}_n,\underline{p}) = I(W_n,\underline{p})$$

Hence for any $\underline{y}_n$ satisfying (3.7) $I(y_n,p)\to 0$ as $n\to\infty$ and using P.1 and P.2 of Appendix 1, we have $\underline{y}_n\to p$ as $n\to\infty$, for otherwise there is a contradiction.

Lemma 3.3. Let all assumptions of the theorem be satisfied. Then for each $y_n$ satisfying (3.5), there is $\underline{u}_o$ such that

(3.8) $$\underline{y}_n \; \varepsilon \; \bar{W}_n(\underline{u}_o) \quad \text{and} \quad \liminf_{n\to\infty} C_n(\underline{u}_o) \geq J(\underline{p}) = I(\underline{p},\Omega_o)$$

where $\underline{p}$ is a fixed alternative.

Proof

Since $W_n$ is the union of $W_n(\underline{u})$ for all $\underline{u}$, then for each $\underline{y}_n\varepsilon\bar{W}_n$ there is a $\underline{u}_o$ such that $\underline{y}_n\varepsilon\bar{W}_n(\underline{u}_o)$ and hence $I(y_n\Omega_o)\leq C_n(\underline{u}_o)$. By P.8 of Appendix 1, $I(\cdot,\Omega_o)$ is continuous, then

$$\liminf C_n(\underline{u}) \geq I(\underline{p},\Omega_o) = J(\underline{p})$$

Proof of Theorem.

By lemmas 3.2 and 3.3 we have that for a fixed alternative $\underline{p}\varepsilon\Omega-\Omega_o$ there is at least one $\underline{u}_o$ such that $\liminf_{n\to\infty}(C_n(\underline{u}_0)\geq J(\underline{p})$.
By lemma 3.1 we have that for each $n$ $\hat{\alpha}_n\leq\exp\{-nC_n(\underline{u}_o)+O(\log n)\}$.
Hence

$$\limsup_{n\to\infty}(1/n)\log\hat{\alpha}_n \leq -J(p) ,$$

By applying (2.9) to $\hat{\alpha}_n$ we have

$$-J(\underline{p}) \leq \liminf_{n\to\infty}(1/n)\log\hat{\alpha}_n \leq \limsup_{n\to\infty}(1/n)\log\hat{\alpha}_n \leq -J(\underline{p})$$

Therefore the theorem is proven.

## 4. Probabilities of Large Deviations for Conditional Distributions

In this section we verify condition (3.1) for several multinomial models arising from practical situations. We then conclude that for those models the LRC test defined in Section 2, is asymptotically optimal. Here we use the term 'model' to denote any particular choice of the subset $\Omega_o$ and the statistic $\underline{U}$.

Model I

Suppose that the individuals in a population can be described as belonging to one of $r$ different categories, $B_1,B_2,\ldots,B_r$ with respect to some attribute $B$ and to one of $s=m(m+1)/2$ categories $A_1A_1,A_1A_2,\ldots A_1A_m$, $A_2A_2,A_2A_3,\ldots,A_2A_m,\ldots,A_mA_m$ generated by a diploid genetic system with $m$ different genes in one locus. The probability distribution of observed frequencies $n_{ijk}(1\leq i\leq r,\ 1\leq j\leq k\leq m)$ in a sample of size $n$ from this population is assumed to follow a multinomial distribution with parameter space $\Omega=\{\underline{p}|p_{ijk}>0,\ \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=j}^{m} p_{ijk}=1\}$

We consider the problem of testing simultaneously whether the attribute $B$ and genotypes are independent and if the distribution of genotypes follows the Hardy-Weinberg equilibrium law.

Thus we can formally write $H: \underline{p}\varepsilon\Omega_o \subset \Omega$ where

$$\Omega_o = \{\underline{p}\varepsilon\Omega / p_{ijk} = \gamma_i \rho_j \rho_k (2-\delta_{jk}),\ 1 \le i \le r,\ 1 \le j \le k \le m\} .$$

Here $\delta_{jk}$ is the Kronecker delta (it is equal to 1 if j=k and 0 otherwise), $\gamma_i$ is the probability associated with category $B_i$, and $\rho_\ell$ the probability associated with gene $A_\ell$ $(\sum_{\ell=1}^{m} \rho_\ell = 1)$.

Under the null hypothesis H the frequency distribution is given by

$$f_n(\underline{z}/\underline{p}) = \frac{(n!)\ 2^{(\sum_{i=1}^{r} \sum_{j=1}^{m} \sum_{k=j}^{m} n_{ijk})}}{\prod_{i=1}^{r} \prod_{j=1}^{m} \prod_{k=j}^{m} n_{ijk}!} \left(\prod_{i=1}^{r} \gamma_i^{n_{i..}}\right)\left(\prod_{\ell=1}^{m} \rho_\ell^{n_\ell}\right)$$

where

$$z_{ijk} = n_{ijk}/n,\ n_{i...} = \sum_{j=1}^{m} \sum_{k=j}^{m} n_{ijk},\quad p\ \varepsilon\ \Omega_o$$

and

$$n_\ell = \sum_{j=1}^{\ell} n_{.j\ell} + \sum_{j=\ell}^{m} n_{.\ell j}\ (n_{.jk} = \sum_{i=1}^{r} n_{ijk}) .$$

Under H the sufficient statistic for the parameters are $(n_{1..}, n_{2..}, \ldots, n_{r..})$ and $(n_1, n_2, \ldots, n_m)$ with independent multinomial distribution $(n, (\gamma_1, \ldots, \gamma_r))$ and $(n, (\rho_1, \ldots, \rho_m))$ respectively. The null conditional distribution is given by

$$(4.1)\quad f_n^{\underline{Z}|\underline{u}}(\underline{z}) = \left[\prod_{i=1}^{m+r} (nu_i)!\; 2^{n\sum_{i=1}^{r}\sum_{j=1}^{m}\sum_{k=j}^{m} z_{ijk}}\right] \Big/ 2(n)! \prod_{i=1}^{r}\prod_{j=1}^{m}\prod_{k=j}^{m} (nz_{ijk})!$$

where $u_i = n_{i..}/n$, $i=1,2,\ldots,r$; $u_{r+\ell} = n_\ell/n$, $\ell=1,2,\ldots,m$, and $\underline{z}\varepsilon\Omega(\underline{u}) = \{\underline{a}\varepsilon\Omega\ U^{(n)}(\underline{a}) = \underline{u}\}$. Substituting in Stirling's formula $n! = (2\Pi n)^{1/2} n^n \exp\{-n+\theta/12n\}$ $0<\theta<1$, which is valid for $n\geq 1$, we obtain

$$(4.2)\quad f_n^{\underline{Z}|\underline{u}}(\underline{z}) = G_n(\underline{z}|\underline{u}) \exp\{-nI(\underline{z}, \hat{p}(\underline{u})) + K_n(\underline{z}|\underline{u})\}$$

where

$$G_n(\underline{z}|\underline{u}) = \Big[\prod_{\substack{i=1\\ nu_i>1}}^{m+r} (2\Pi nu_i)^{1/2}\Big] \Big/ \Big[(4\Pi n)^{1/2} \prod_{\substack{i=1\\ nz_{ijk}>1}}^{r} \prod_{j=1}^{m} \prod_{k=j}^{m} (2\Pi nz_{ijk})^{1/2}\Big],$$

$$K_n(\underline{z}|\underline{u}) = (1/12) \sum_{\substack{i=1\\ nu_i>1}}^{m+r} (\theta_i/nu_i) - (\theta/2n) - \sum_{\substack{i=1\\ nz_{ijk}>1}} \sum_{j=1} \sum_{k=j} (\theta_{ijk}/nz_{ijk})$$

$(x^x = 1$ if $x=0)$

and the elements of $\hat{p}(\underline{u})$ are the maximum likelihood estimates of $p_{ijk}$ under the null hypotheses, that is

$$\hat{p}_{ijk} = u_i(2-\delta_{jk})(u_{r+j}\, u_{r+k})/u$$

The following theorem is concerned with probabilities of large deviations for the conditional distribution (4.1).

Theorem. Let $\underline{u}$ be given and for any subset $A_n$ of $\Omega(u)$ let $P_n^{\underline{Z}|\underline{u}}(A_n) = {}_{z\varepsilon A_n} f_n^{\underline{Z}|\underline{u}}(z)$, where $f_n^{\underline{Z}|\underline{u}}(z)$ is given by (4.1). Then

$$(4.3) \qquad C_0 \cdot n^{-C_1} < P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) . \exp\{nI(A_n^{(n)}, \hat{\underline{p}}(\underline{u})\} < \binom{n+rs-1}{rs-1} . D_0 n^{-D_1}$$

where $C_0, C_1, D_0$ and $D_1$ are positive constants not depending on $n$. Hence

$$(4.4) \qquad P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) = \exp\{-nI(A_n^{(n)}, \hat{\underline{p}}(\underline{u})) + O(\log n)\}$$

uniformly for $A_n \subset \Omega(\underline{u})$, and $\underline{u}$.

The proof of this theorem is based on the following Lemma, which is proved in Appendix 2.

Lemma 4.1. For any $\underline{u}$ and $\underline{z}$ in $\Omega(\underline{u})$ we have

$$(4.5) \qquad C_0 \cdot n^{-C_1} < G_n(\underline{z}|\underline{u}) . \exp\{K_n(\underline{z}|\underline{u})\} < D_0 \cdot n^{-D_1}$$

where $C_0$, $C_1$, $D_0$ and $D_1$ are positive constants not depending on $n$ and $G_n(\underline{z}|\underline{u})$, $K_n(\underline{z}|\underline{u})$ were defined before.

Proof of the Theorem

If $A_n$ is empty then $f_n^{\underline{Z}|\underline{u}}(\bar{A}_n)=0$ and (4.2.6) is true since $I(A_n^{(n)}, \hat{\underline{p}}(\underline{u})) = +\infty$ if $A_n$ is empty.

Assume that $A_n$ is not empty. For each $\underline{z}$ in $A_n$ the expression (4.2) implies that

$$f_n^{\underline{Z}|\underline{u}}(\underline{z}) < G_n(\underline{z}|\underline{u}) . \exp\{-nI(A_n^{(n)}, \hat{\underline{p}}(\underline{u})) + K_n(\underline{z}|\underline{u})\}$$

where

$$I(A_n^{(n)}, \hat{\underline{p}}(\underline{u})) = \inf\{I(\underline{z}, \underline{p}(\underline{u})) \mid \underline{z} \in A_n^{(n)}\}$$

Then by lemma 4.2.1 we have

$$f_n^{\underline{Z}|\underline{u}}(\underline{z}) \le D_0 . n^{-D_1} . \exp\{-nI(A_n^{(n)}, \underline{\hat{p}}(\underline{u}))\}$$

The number of points $\underline{z}$ in $\Omega$ is the binomial coefficient $\binom{n+rs-1}{rs-1}$. Hence

$$P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) = \sum_{\underline{z}\varepsilon A_n} f_n^{\underline{Z}|\underline{u}}(\underline{z}) \le \binom{n+rs-1}{rs-1} . D_0 . n^{-D_1} . \exp\{-nI(A_n^{(n)}, \underline{\hat{p}}(\underline{u}))\}$$

and this proves the second inequality of (4.3). On the other hand, lemma (4.1) implies that

$$P_n^{\underline{Z}|\underline{u}}(\underline{z}) \ge C_o . n^{-C_1} . \exp\{-nI(\underline{z}, \underline{\hat{p}}(\underline{u}))\} .$$

Then

$$P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) \ge C_0 . n^{-C_1} \sum_{\underline{z}\varepsilon A_n} \exp\{-nI(\underline{z}, \underline{\hat{p}}(\underline{u}))\}$$

and therefore

$$P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) \ge C_0 . n^{-C_1} . \exp\{-nI(A_n^{(n)}, \underline{\hat{p}}(\underline{u}))\}$$

Thus we have completed the proof of (4.3). Clearly (4.3) implies (4.4). Hence for any subset $A_n$ of $\Omega(\underline{u})$ we have

$$P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) - \exp\{-nI(A_n^{(n)}, \underline{\hat{p}}(\underline{u})) + 0\ (\log n)\}$$

where for any $\underline{z}\varepsilon\Omega(\underline{u})$, $\underline{\hat{p}}(\underline{u})$ is the maximum likelihood estimate of $\underline{p}\varepsilon\Omega_o$.

Therefore

$$I(\underline{z}, \hat{\underline{p}}(\underline{u})) = I(\underline{z}, \Omega_o) \quad \text{for any} \quad \underline{z}\varepsilon\Omega(\underline{u}).$$

Then for any subset $A_n$ of $\Omega(\underline{u})$ we can write

$$P_n^{\underline{Z}|\underline{u}}(A_n) = \exp\{-nI A_n^{(n)}, \Omega_o) + 0 (\log n)\}$$

and the condition (3.1) is satisfied. We conclude then that the (LRC) test is asmyptotically optimal for this model.

Using the same technique as in Model.I, the (LRC) test is optimal for the following multinomial models.

Model II

Suppose we classify families with $m$ children according to the number of male children and according to $r$ categories $B_1,\dots,B_r$ of an attribute $B$. The probability distribution of observed frequencies in a sample of size $n$ from this population is assumed to follow the multinomial distribution.

We consider the problem of testing the hypothesis that the sex distribution is binomial and the same for each category $B_i$. We can formally write $H: \underline{p}\varepsilon\Omega_0\subset\Omega$ where

$$\Omega_o = \{\underline{p}\varepsilon\Omega | p_{ij} = p_{i.} \binom{m}{j} p^j (1-p)^{(m-j)}, \quad i=1,\dots,r, \ j=0,1,\dots,m\}$$

Here $0 < p < 1$ is the probability of male children in that population, and $p_{i.}$ is the probability associated with category $B_i$.

Under $H$ the frequency distribution is given by

$$f_n(\underline{z}|\underline{p}) = \frac{(n!) \prod_{j=0}^{m} \binom{m}{j}^{n_{.j}}}{\prod_{i=1}^{r} \prod_{j=0}^{m} n_{ij}!} \left(\prod_{i=1}^{r} p_i^{n_{i.}}\right) p^{n_1} (1-p)^{n_2}$$

where

$$n_{i.} = \sum_{j=0}^{m} n_{ij} \qquad n_{.j} = \sum_{i=1}^{r} n_{ij} \qquad \underline{p}\varepsilon\Omega_o$$

$$n_1 = \sum_{j=0}^{m} jn_{.j} \qquad n_2 = \sum_{j=0}^{m} (m-j)n_{.j}\ .$$

Then $(n_{1.}, n_{2.}, \ldots, n_{r.})$ and $(n_1, n_2)$, are, under H, sufficient statistic for the parameters, and they are independent with respective distributions:

$(n_{1.}, \ldots, n_{r.})$: Multinomial $(n, (p_{1.}, \ldots, p_{r.}))$

$(n_1, n_2)$: Binomial $(mn, p)$

The null conditional distribution is given by

$$f_n^{\underline{Z}|\underline{u}}(\underline{z}) = \frac{\prod_{i=1}^{r+2} (nu_i)! \prod_{j=0}^{m} \binom{m}{j}^{nz_{.j}}}{(mn)! \prod_{i=1}^{r} \prod_{j=0}^{m} (nz_{ij})!}$$

where $u_i = n_{i.}/n$, $i=1,2,\ldots,r$, $u_{r+k} = n_k$, $k=1.2.$

As in Model I, it can be proved that for any subset $A_n$ of $\Omega(\underline{u})$ we have

$$P_n^{\underline{Z}|\underline{u}}(A_n) = \exp\{-nI(A_n^{(n)}, \Omega_o) + 0\ (\log n)\}$$

where $I(A_n^{(n)}, \Omega_o) = I(A_n^{(n)}, \hat{\underline{p}}(\underline{u}))$ and the elements of $\hat{\underline{p}}(\underline{u})$ are given by

$$\hat{p}_{ij}(\underline{u}) = u_i \binom{m}{j} (u_{r+1}/m)^j (u_{r+2}/m)^{m-j}$$

Therefore the (LRC) test is asymptotically optimal for this model.

Model III

Suppose that individuals of a population can be described as having one of the two phenotypes, normal (AA or Aa) and affected (aa) of a genetic system with two genes A and a in one locus where the gene A is dominant over a. Then the expected frequencies under the Hardy-Weinberg equilibrium are $q^2$ for affected and $1-q^2$ for normal, where $0 < q < 1$ is the probability associated with gene a.

Suppose we have a sample of $n$ couples from this population and we are interested in testing the hypothesis that the probability distribution of phenotypes in the two sexes are independent and they are in Heardy-Weinberg equilibrium with the same probability $q$ associated with gene a. The sample data can be represented in the following 2x2 contingency table.

| | Normal | Affected | Total |
|---|---|---|---|
| Normal | $n_{11}$ | $n_{12}$ | $n_{1.}$ |
| Affected | $n_{21}$ | $n_{22}$ | $n_{2.}$ |
| Total | $n_{.1}$ | $n_{.2}$ | $n$ |

Under the general model the frequency distribution is assumed to follow a multinomial distribution. The null hypothesis can be expressed by $H: p \varepsilon \Omega_o$ where

$$\Omega_o = \{p \varepsilon \Omega | p_{11} = (1-q^2)^2,\ p_{12} = p_{21} = q^2(1-q^2),\ p_{22} = q^4\}$$

Under the null hypothesis the sufficient statistic for the parameters is $(n_1, n_2)$ where

$$n_1 = 2n_{11}+n_{12}+n_{21} \quad \text{and} \quad n_2 = 2n_{22}+n_{12}+n_{21}$$

The null conditional distribution is given by

$$f_n^{\underline{Z}|\underline{u}}(\underline{z}) = \frac{(n!) \prod_{i=1}^{2} (nu_i)!}{(2n)! \prod_{i=1}^{2} \prod_{j=1}^{2} (nz_{ij})!}$$

where

$$z_{ij} = n_{ij}/n, \quad i = 1,2 \quad j = 1,2 \quad \text{and} \quad u_i = n_i/n \quad i = 1,2$$

Using the same technique as before, it can be proved that for any subset $A_n$ of $\Omega(\underline{u})$ we have

$$P_n^{\underline{Z}|\underline{u}}(\bar{A}_n) = \exp\{-nI(A_n^{(n)}, \Omega_o) + 0 (\log n)\}$$

Therefore the (LRC) test is asymptotically optimal for this model.

Model IV

Consider the problem of testing the hypothesis of independence in a (rxs) contingency table. In the usual notation the null hypothesis is $H: p\varepsilon\Omega_o$ where

$$\Omega_o = \{\underline{p}\varepsilon\Omega \ p_{ij} = (p_{i.})(p_{.j}), \quad i = 1,\ldots,r, \ j = 1,\ldots,s\}$$

Here

$$p_{i.} = \sum_{j=1}^{s} p_{ij} \quad \text{and} \quad p_{.j} = \sum_{i=1}^{r} p_{ij}$$

Under $H$ the marginals $(n_{1.},\dots,n_{r.})$ and $(n_{.1},\dots,n_{.s})$ are sufficient for the parameters and the null conditional distribution is given by

$$f_n^{\underline{Z}|u}(\underline{z}) = [\prod_{i=1}^{r+s} (nu_i)!]/[(n!)\prod_{i=1}^{r}\prod_{j=1}^{s}(nz_{ij})!]$$

where $u_i = n_{i.}/n$, $i = 1,2,\dots,r$ and $u_{r+k} = n_{.k}/n$, $k = 1,2,\dots,s$.

As in preceding sections, for any subset $A_n$ of $\Omega(\underline{u})$ we can write

$$P_n^{\underline{Z}|u}(\bar{A}_n) = \exp\{-nI(A_n^{(n)}, \Omega_o) + O(\log n)\},$$

and therefore the (LRC) test is asymptotically optimal for this model.

## APPENDIX 1

Let $\Omega$ be the simplex defined in Section 2, that is

$$\Omega = \{\underline{p} = (p_1,p_2,\ldots,p_k) | p_i \geq 0; \sum_{i=1}^{k} p_i = 1\}$$

For any subset $A$ of $\Omega$, $A' = \Omega - A$ is the complement of $A$. We denote the closure of $A$ by $\bar{A}$. The boundary of $A$ is $\bar{A} \cap \bar{A}'$. Let $S_o$ and $S(\underline{p})$ (for $\underline{p} \varepsilon \Omega$) be subsets of $\Omega$ defined by

$$S_o = \{\underline{p} \varepsilon \Omega | p_i > 0; i=1,2,\ldots,k\}$$
$$S(\underline{p}) = \{\underline{a} \varepsilon \Omega | a_i = 0 \text{ if } p_i = 0\}$$

Following we give some properties of the function $I(\underline{a}, \underline{b})$ and its infima which are proved by Hoeffding (1965a).

($1^o$) $I(\underline{a}, \underline{p}) = \sum_{i=1}^{k} a_i \log(a_i/p_i)$ ($\underline{a}$ and $\underline{p}$ are points of $\Omega$)

P.1 $0 \leq I(\underline{a}, \underline{p}) \leq \infty$, for any $\underline{a}$ and $\underline{p}$ in $\Omega$.
$I(\underline{a}, \underline{p}) = 0$ if and only if $\underline{a} = \underline{p}$.
$I(\underline{a}, \underline{p}) < \infty$ if and only if $\underline{a}$ is in $S(\underline{p})$.

P.2 For each $\underline{p} \varepsilon S_o$, $I(., \underline{p})$ is continuous and bounded in $S(\underline{p})$. For each $\underline{p} \varepsilon S_o$, $I(., \underline{p})$ is continuous and bounded in $\Omega$.

P.3 For each $\underline{a} \varepsilon \Omega$, $I(\underline{a}, .)$ is continuous in $\Omega$.

P.4 For each $\underline{a} \varepsilon \Omega$, $I(\underline{a}, .)$ and $I(., \underline{p})$ are convex in $\Omega$.

($2^{o}$) $I(A, \underline{p}) = \inf\{I(\underline{a}, \underline{p})\ \underline{a}\varepsilon A\}$ ($A$ is a nonempty subset of $\Omega$)

<u>P.5</u> For each $\underline{p}\varepsilon S_0$ there is at least one point $\underline{a}$ such that

(A1.1) $\underline{a}\ \varepsilon\ \overline{A}$ and $I(\underline{a}, \underline{p}) = I(A, \underline{p})$

If $\underline{p}\varepsilon A$ then $I(A, \underline{p}) = 0$ and (A1.1) is satisfied only for $\underline{a} = \underline{p}$. If $\underline{p}\not\varepsilon\overline{A}$ then $I(A, \underline{p}) > 0$ and any point $\underline{a}$ which satisfies (A1.1) is in the boundary of $A$.

<u>P.6</u> Let $\underline{p}\not\varepsilon S_0$ and suppose that $B = A \cap S(\underline{p})$ is not empty, then $I(B, \underline{p}) = I(A, \underline{p}) < \infty$. In this case the statements <u>P.5</u> are true with $A$ replaced by $B$.

($3^{o}$) $I(\underline{a}, \Omega_0) = \inf\{I(\underline{a}, \underline{p})\ \underline{p}\varepsilon\Omega_0\}$ ($\Omega_0$ is a nonempty subset of $\Omega$)

The maximum likelihood estimate of $\underline{p}$ under $\underline{p}\varepsilon\Omega_0$, based on the observation $\underline{z}^{(n)}$ is a point $\hat{\underline{p}}(\underline{z}^{(n)})$ which maximizes $f_n(\underline{z}^{(n)}\ \underline{p})$ for $\underline{p}\varepsilon\Omega_0$. From (2.4) we can see that $\hat{\underline{p}}(\underline{z}^{(n)})$ minimizes $I(\underline{z}^{(n)}, \underline{p})$ for $\underline{p}\varepsilon\Omega_0$. Then

(A1.2) $$I(\underline{z}^{(n)}, \hat{\underline{p}}(\underline{z}^{(n)})) = I(\underline{z}^{(n)}, \Omega_0).$$

By extension we define $\hat{\underline{p}}(\underline{a})$ for any point $\underline{a}\varepsilon\Omega$ as a point $\hat{\underline{p}}(\underline{a})$ of $\Omega_0$ for which

(A1.3) $$I(\underline{a}, \hat{\underline{p}}(\underline{a})) = I(\underline{a}, \Omega_0)$$

<u>P.7</u> For each $\underline{a}\varepsilon\Omega$ there is at least one point $\hat{\underline{p}}(\underline{a})$ such that

(A1.4) $\hat{\underline{p}}(\underline{a})\ \varepsilon\ \overline{\Omega}_0$ and $I(\underline{a}, \hat{\underline{p}}(\underline{a})) = I(\underline{a}, \Omega_0)$

If $\underline{a}\varepsilon\overline{\Omega}_0$ then $I(\underline{a}, \Omega_0) = 0$, and (A1.4) is satisfied only for $\hat{\underline{p}}(\underline{a}) = \underline{a}$. If $\underline{a}\not\varepsilon\overline{\Omega}_0$ then $I(\underline{a}, \Omega_0) > 0$ and any point satisfying (A1.4) is in the boundary of $\Omega_0$.

<u>P.8</u> If $\Omega_0 C S_0$, then $I(., \Omega_0)$ is continuous

P.9 $I(\cdot, \Omega_o)$ is continuous in $S_o$.

(4°) $I(A, \Omega_o) = \inf\{I(\underline{a}, \underline{p})\ \underline{a}\varepsilon A,\ \underline{p}\varepsilon\Omega_o C S_o\}$

P.10 Assume that $A$ and $\Omega_o$ are not empty subsets of $\Omega$ and such that $I(A, \Omega_o) > 0$. Then there is at least one point $\underline{a}$ such that

(A1.5) $\underline{a} \ \varepsilon \ \overline{A}$ and $I(\underline{a}, \Omega_o) = I(A, \Omega_o)$

and any point $\underline{a}$ satisfying (A1.5) is in the boundary of A.

Now let $A^{(n)}$ be the set of lattice points $\underline{z}^{(n)}$ of $AC\Omega$. Let $P_n(A|P)$ be the multinomial probability defined in Section 2. The following theorem proved by Hoeffding (1965) is concerned with probabilities of large deviations of $\underline{z}^{(n)}$ from its mean $\underline{p}$.

Theorem For any subset $A$ of $\Omega$ and any point $\underline{p}$ in $\Omega$ we have

$$C_o n^{-((k-1)/2)}\exp\{-nI(A^{(n)},\underline{p})\Sigma \le P_n(A|\underline{p}) \le \binom{n+k-1}{k-1}\exp\{-nI(A^{(n)},\underline{p})\}$$

where $C_o$ is a positive constant not depending on $n$. Hence

$$P_n(A\ \underline{p}) = \exp\{-nI(A^{(n)}, \underline{p}) + 0\ (\log n)\} < \exp\{-nI(A, \underline{p}) + 0\ (\log n)\}$$

uniformly for $A$ contained in $\Omega$ and $p\varepsilon\Omega$.

Here we give the proof of Lemma 4.1.

Consider $G_n(z\ u)$ and $k_n(z\ u)$ as defined in Lemma 4.1. Let $NumG_n$ be the numerator and $DenG_n$ the denominator of $G_n(z\ u)$. Then

$$1 < NumG_n < (2\pi 3n^{(m+r)/2}$$

$$1 < DenG_n < (2)^{\frac{1}{2}(2\pi n)(rs+1)/2}$$

Hence

(A2.1) $$(2)^{\frac{1}{2}}.(\pi)^{-(rs+1)/2}.h^{-(rs+1)/2} < G_n(\underline{z}\ \underline{u}) < (6\pi)^{(m+r)/2}.n^{(m+r)/2}$$

uniformly for $\underline{u}$ and $\underline{z}\varepsilon\Omega(\underline{u})$.

On the other hand $K_n(\underline{z}\ \underline{u})$ satisfies the follwing inequality

(A2.2) $$-(1/12)(rs+1) < K_n(\underline{z}\ \underline{u}) < (1/12)(m+r)$$

uniformly for $\underline{u}$ and $\underline{z}\varepsilon\Omega(\underline{u})$.

Using (A2.1) and (A2.2) we obtain (4.1) where

$$C_o = (2)^{\frac{1}{2}}.(2\pi)^{-(rs+1)/2}.\exp\{-(1/12)(rs+1)\}$$

$$C_1 = (rs+1)/s$$

$$D_o = (6\pi)^{(m+r)/2}.\exp\{(1/12)(m+r)\}$$

$$D_1 = (m+r)/2$$

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