

Replica trick and string winding

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We apply the replica trick to compute the entropy of a cylinder amplitude in string theory. We focus on the contribution from nonperturbative winding modes and impose tadpole cancellation to understand the correct prescription for integrating over moduli. Choosing the entangling surface to cut longitudinally over the whole length of the cylinder, we obtain an answer that is interpreted as the entropy of a density matrix. We recast this result in target space language, in both the open and closed string picture.

DOI: [10.1103/PhysRevD.96.026009](https://doi.org/10.1103/PhysRevD.96.026009)**I. INTRODUCTION**

The replica trick allows us to compute path integrals associated to density matrices. It is employed in the most diverse fields, from statistical physics and machine learning to conformal field theory, especially in relation with the study of entanglement; see, e.g., [1–5]. In general, given a region A at some fixed time in a manifold \mathcal{C} , the replica trick allows us to compute the q th power of the reduced density matrix ρ_A . This is the partial trace $\rho_A = \text{tr}_B \rho$ of the full quantum state ρ defined at a fixed time in \mathcal{C} over the degrees of freedom contained in the complement of A , called B here. Explicitly, one has [1,2]

$$\text{tr}(\rho_A^q) = \frac{Z_q(A)}{(Z_1(A))^q}, \quad (1)$$

where $Z_q(A)$ is the path integral over a q -sheeted surface obtained by gluing together q replicas of the original manifold \mathcal{C} after having performed a cut along A , which is called an entangling surface. Equipped with this trace, the associated von Neumann entropy can be computed by analytically continuing q , differentiating with respect to it, and finally setting $q = 1$.

The goal of this paper is to apply the replica trick to a basic string theory object: the cylinder amplitude. We want to compute (1) for the case in which \mathcal{C} is the cylindrical world sheet formed by a propagating string and A is taken to cut longitudinally over the whole length of the cylinder, from one boundary to another. The resulting surface is depicted in Fig. 1. The world-sheet time in the figure parametrizes the compact dimension. It is obvious that, in this case, $Z_1(A)$ is the usual cylinder amplitude, which shall be called Z in what follows. Note that, as the entangling surface at fixed world-sheet time covers the full world sheet, we are not tracing over any external degrees of freedom. In other words, B is empty in this setup. The output of the computation is then the usual von Neumann entropy—rather than some entanglement entropy—of whatever quantum state, either pure or mixed,

has been produced by the path integral at that time. The result for the entropy is nonvanishing if and only if that state is a density matrix.

The above statement, relating the usual thermal entropy with the entanglement entropy in the limit of a vanishing complement of the entangling region, comes from a holographic intuition [6]. For a system of size L , one has

$$\lim_{\epsilon \rightarrow 0} (S_{\text{ent}}(L - \epsilon) - S_{\text{ent}}(\epsilon)) = S_{\text{therm}}. \quad (2)$$

This can be interpreted as the difference in length between a geodesic revolving around the dual black hole and a vanishing geodesic, so that the entropy is produced by the first term alone. From the replica trick point of view, this translates into the physical requirement $\lim_{\epsilon \rightarrow 0} Z_q(\epsilon) = (Z_1(\epsilon))^q$ (for an appropriate cutoff choice).

Strings have both localized, oscillatory modes and extended, topologically inequivalent configurations characterized by winding numbers. Here we focus on the nonperturbative contribution from the latter modes, for the case in which the target space is either a two-dimensional torus or a circle [7]. This is interesting for a variety of reasons. First of all, it is one of the first attempts [8] to use the replica trick on the string world sheet, which is tricky because of the obvious topological nature of the theory. Here this construction is well defined as the cut stretches from one boundary of the world sheet to the other, so it is itself ingrained in the topology. Second, we will

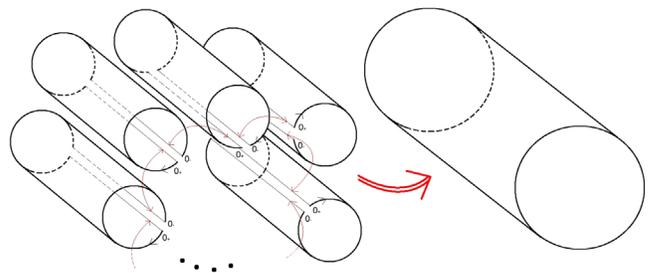


FIG. 1. q cylinders are cut longitudinally and then glued together, resulting in a large cylinder.

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consider nonperturbative contributions that, to the best of our knowledge, have only been studied before, for entanglement entropy, in [17]. Third, we will deal with some subtleties in the definition of moduli on the world-sheet replicas by requiring tadpole cancellation. Finally, the result will be expressed in terms of target space quantities, like the complex structure and the open string moduli, establishing a precise map between the entropy of the cylinder and the geometrical properties of the embedding space. This map will be further checked by deriving the world-sheet results from a target space perspective, both in the open and closed string channels.

II. THE REPLICA TRICK ON THE WORLD SHEET

We start with the world-sheet computation of the entropy. To this scope, we consider open strings stretching between parallel D-branes.

The target space we consider is a two-dimensional torus times noncompact spectating dimensions. The torus has two cycles parametrized by R_1 and $R_2 e^{i\alpha}$, with $R_1, R_2, \alpha \in \mathbb{R}$, and complex structure $\sigma = \frac{R_2}{R_1} e^{i\alpha}$. In terms of target space coordinates $x^a = (x^1, x^2)$, related to the standard complex coordinates (z, \bar{z}) by $z = R_1 x^1 + R_2 e^{i\alpha} x^2$, the Wilson line for the Chan-Paton factor on the i th brane of the stack is $A_i = \theta_i dx^1 - \phi_i dx^2$.

There are a number of equivalent ways to do the computation, related by T-duality. We shall consider D0-branes at positions determined by the Wilson lines. These are obtained by T-dualizing twice the space-filling D2-branes with A_i , as follows. In terms of the coordinates x^a , the metric is found to be

$$g_{11} = R_1^2, \quad g_{12} = R_1 R_2 \cos \alpha, \quad g_{22} = R_2^2, \quad (3)$$

with the complex structure σ and the torus area $t_2 = R_1 R_2 \sin \alpha$ being given by $\sigma = \frac{g_{12}}{g_{11}} + i \frac{\sqrt{\det(g)}}{g_{11}}$ and $t_2 = \sqrt{\det(g)}$. The introduction of a B-field $b = b_{12} dx^1 \wedge dx^2$, with $b_{12} = R_1 R_2 \cos \alpha$, complexifies the area: $t = t_1 + it_2 = b_{12} + i \sqrt{\det(g)}$. A sequence of two T-dualities along x^1 and x^2 maps (3) to the dual metric

$$\tilde{g}_{11} = \frac{1}{R_1^2}, \quad \tilde{g}_{12} = \tilde{b}_{12} = -\frac{\cos \alpha}{R_1 R_2}, \quad \tilde{g}_{22} = \frac{1}{R_2^2}. \quad (4)$$

The D2-brane with Wilson line A_i is mapped onto a D0-brane at position θ_i along x^1 and $-\phi_i$ along x^2 . An open string with Chan-Paton factors i, j at the end points extends between two D0-branes localized at $(\theta_i, -\phi_i)$ and $(\theta_j, -\phi_j)$, respectively.

The cylinder world sheet \mathcal{C} can be parametrized by $\ell_1 \in [0, 1]$, stretching between the two D0-branes, and $\ell_2 \in [0, 2\pi s)$, which is attached on the D0-branes at

$\ell_1 = 0, 1$. There are two independent winding numbers along ℓ_1 : the string can in fact wrap m times along R_1 and n times along $R_2 e^{i\alpha}$. Due to the cyclic identification of the sheets along the entangling surface, one can only have one pair of winding numbers (n, m) common to all sheets. The classical maps X^a from the world sheet (ℓ_1, ℓ_2) to the target space are

$$X^1 = \ell_1(m + \theta_{ij}), \quad X^2 = \ell_1(n - \phi_{ij}), \quad (5)$$

with $\theta_{ij} \equiv \theta_i - \theta_j$ and $\phi_{ij} \equiv \phi_i - \phi_j$. The corresponding path integral weight $e^{-S_{\text{cl}}}$ can be obtained by evaluating

$$S_{\text{cl}} = \int d^2 \ell \partial_{\ell_1} X^a \partial_{\ell_1} X^b \tilde{g}_{ab} = \sin^2 \alpha \frac{2\pi s}{t_2 \sigma_2} |n + \sigma m - u_{ij}|^2, \quad (6)$$

with $u_{ij} \equiv \phi_{ij} + \sigma \theta_{ij}$.

Computing the winding number contribution to the amplitude Z for \mathcal{C} is simple. The prescription is to integrate (6) over s and to sum over the winding numbers:

$$Z = \sum_{m,n} \int_0^\infty \frac{ds}{4s} \exp \left(-\sin^2 \alpha \frac{2\pi s}{t_2 \sigma_2} |n + \sigma m - u_{ij}|^2 \right). \quad (7)$$

Evaluating the integral and the sums is a well-known procedure; see, e.g., [18]. A Poisson resummation in n and m needs to be performed. Discarding the $n = m = 0$ term, one can sum the series and obtain

$$Z = -\frac{1}{2} \log |e^{\pi i \theta_{ij}^2 \sigma} \theta_1(u_{ij} | \sigma) \eta(\sigma)^{-1}|, \quad (8)$$

in terms of the modular functions θ_1 and η [19].

A couple of comments are now in order. First, the mechanism of tadpole cancellation plays an important role here. Given an appropriate choice of gauge group, it cancels the divergent $n = m = 0$ terms among the cylinder, the Möbius strip and the Klein bottle, leaving a finite series for nonvanishing Wilson lines [18]. Second, the $n = m = 0$ modes would be the only ones depending on the prefactors in the action (6), but this dependence disappears from (8).

The amplitude $Z_q(A)$ comes from considering q cylinders, cutting them longitudinally and gluing the cuts together, thus obtaining a big cylinder out of q rectangles, as shown in Fig. 1. To compute this, one needs to understand the correct prescription for integrating over moduli. At first sight, there appear to be two possibilities, which depend on the order in which the operations of cutting/gluing cylinders and integrating/modding out the world-sheet metric by the $\text{diff} \times \text{Weyl}$ symmetries are carried out.

Suppose one first integrates on each copy of the cylinder and then cuts and glues them together. The first operation yields q integrals over moduli from $\text{diff} \times \text{Weyl}$ -inequivalent metrics. When cutting and gluing the

cylinders, the fields on adjacent sheets should be identified across the cuts, so that the moduli also get identified, leaving a single surviving modulus. Analogously, the measure factors of $1/4s$ appearing in each integral from translation invariance along the cylinder are reduced to a single one, as translation along a sheet moves the cut and leads to the same translation along all other cuts. In (6), we now have a factor of qs , from the sum of q actions with the same modulus and winding numbers. This first integration prescription leads to

$$Z_q(A) = \sum_{m,n} \int_0^\infty \frac{ds}{4s} \exp\left(-\sin^2 \alpha \frac{2\pi qs}{t_2 \sigma_2} |n + \sigma m - u_{ij}|^2\right). \quad (9)$$

The other possible choice is to first cut and glue the q cylinders and then to integrate over $\text{diff} \times \text{Weyl}$ -inequivalent metrics. In this case, one first constructs the big cylinder and then deals with the path integral over the world-sheet metric, leading to a single modulus measuring the entire circumference of the cylinder. This second option gives just the usual cylinder (7). We note at this point that the factor of q that appears in the exponent in (9) can be simply reabsorbed by a rescaling of the prefactor. As seen above, the final result will be independent of this prefactor after tadpole cancellation has been applied. In summary, the two procedures are equivalent [20].

The Renyi entropy (1) is easily obtained [23]:

$$\text{tr}(\rho^q) = \left(-\frac{1}{2} \log |e^{\pi i \theta_{ij}^2 \sigma} \theta_1(u_{ij}|\sigma) \eta(\sigma)^{-1}|\right)^{1-q}. \quad (10)$$

The entropy of ρ is given by differentiating this expression with respect to q and setting in the end $q = 1$, which results in

$$S^{\text{torus}} = \log \left(-\frac{1}{2} \log |e^{\pi i \theta_{ij}^2 \sigma} \theta_1(u_{ij}|\sigma) \eta(\sigma)^{-1}|\right). \quad (11)$$

This quantity S^{torus} measures the nonperturbative contribution to the entropy of the quantum state ρ . This result would be trivially zero if ρ were pure. We can therefore conclude that this state is a density matrix. We shall discuss the nature of this density matrix below, when we recast it in a target space language in terms of open and closed string channels.

Let us now specialize to the case when the target space is not a torus, but a circle of radius R . We can obtain the cylinder amplitude Z from (7) by considering a purely imaginary complex structure $\alpha = \pi/2$, a Wilson line $-\phi_i$ along the x^2 direction, and by removing the winding number m . After an obvious renaming of R_2 , the amplitude reads

$$Z = \sum_n \int_0^\infty \frac{ds}{4s} \exp\left(-\frac{2\pi s}{R^2} (n - \phi_{ij})^2\right), \quad (12)$$

whose evaluation is straightforward. One first Poisson resums and then does the integral, with the $n = 0$ term removed as before, obtaining

$$Z = \sum_{n \neq 0} \frac{1}{4|n|} e^{2\pi i n \phi_{ij}} = -\frac{1}{2} \log |1 - e^{2\pi i \phi_{ij}}|. \quad (13)$$

The entropy of ρ on the target space circle becomes

$$S^{\text{circle}} = \log \left(-\frac{1}{2} \log |1 - e^{2\pi i \phi_{ij}}|\right). \quad (14)$$

Note that the result is not defined for values of ϕ_{ij} such that (13) is negative, namely, $1/6 < \phi_{ij} < 5/6$. To avoid this, we introduce by hand a cutoff by inserting a μ (with $1/2 \geq \mu \geq 0$) in the argument of the second log in (14). The appearance of such constants is common in path integral computations of replicated surfaces coming from singularities at the branch points (in our case at the boundary of the cylinder) [2], although the usual understanding as a UV cutoff is unclear here.

III. TARGET SPACE DERIVATION OF THE DENSITY MATRIX

We want to interpret these results from a target space point of view, using a construction for open and closed strings summarized in Fig. 2. The main idea is to select an appropriate target space density matrix which reproduces the world-sheet expressions.

A. Open string picture

We initially look at open strings, corresponding to space-filling D-branes with Wilson lines. Limiting our attention to the center of mass of the strings and suppressing their oscillations, as done above in (5), we can think of them as particles charged under the Wilson lines. To further simplify things, we consider a circle instead of a torus target space, thus aiming to reproduce (14). The density matrix we consider is a sum over projectors, each corresponding to a particle of momentum k , charged under the i ,

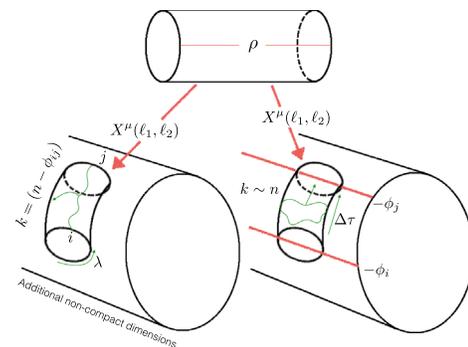


FIG. 2. Target space interpretation of the world-sheet density matrix ρ , in the open (left). The open string stretches between two D1-branes and carries momentum $k = n - \phi_{ij}$, whereas the closed string stretches between two D0-branes and carries momentum $k \sim n$.

j Wilson lines, and that has propagated along a path of length $2\pi\lambda$; see Fig. 2 (left). The path integral weight for such a particle is given by $\frac{1}{4\lambda} \exp(-2\pi\lambda k^2)$ (see, e.g., [24]) and we get

$$\rho = \frac{1}{\text{tr}\rho_u} \rho_u, \quad \rho_u = \sum_n \int_0^\infty \frac{d\lambda}{4\lambda} e^{-2\pi\lambda k^2} |n, \lambda\rangle \langle n, \lambda|. \quad (15)$$

We change variables to $\gamma(\lambda) = \frac{1}{4} \log \lambda$. We emphasize that we are integrating over the particle world-line length, rather than the modulus of the cylinder world sheet. The states $|n, \gamma\rangle$ are taken to be orthonormal: $\langle n, \gamma | n', \gamma' \rangle = \delta_{nn'} \delta(\gamma - \gamma')$. The momentum k is quantized on the circle and shifted by the Wilson line $k^2 = (n - \phi_{ij})^2 / R^2$. To compute the entropy we first construct ρ^q , whose trace is

$$\text{tr}(\rho^q) = \frac{1}{(\text{tr}\rho_u)^q} \sum_n \int_{-\infty}^\infty d\gamma e^{-\frac{2\pi q \lambda(\gamma)(n - \phi_{ij})^2}{R^2}}. \quad (16)$$

The normalization $(\text{tr}\rho_u)^q$ is similarly evaluated. Remembering that the q in the exponential of (16) can be rescaled away thanks to tadpole cancellation, one finds that in terms of the original λ variable

$$\text{tr}(\rho^q) = \left(\sum_n \int_0^\infty \frac{d\lambda}{4\lambda} e^{-\frac{2\pi\lambda}{R^2}(n - \phi_{ij})^2} \right)^{1-q}, \quad (17)$$

which leads precisely to the world-sheet expression (14) for S^{circle} .

B. Closed string picture

A complementary approach to recover S^{circle} from the target space is to consider closed strings; see Fig. 2 (right). The target space circle will be in the T-dual picture with respect to the previous section, where branes are positioned on their Wilson line values $-\phi_i$ and $-\phi_j$. The density matrix ρ will be a linear combination of projectors corresponding to all possible propagations of a closed string between the i th and j th brane, with all possible momenta (depending on an integer n) and winding numbers m around the circle

$$\rho_u = \frac{\pi R^2}{2} \sum_{n,m} \int_{\substack{X(\tau_i) = \phi_i \\ X(\tau_f) = \phi_j + m}} dxd(\Delta\tau) |n, m; x, \Delta\tau\rangle \langle n, m; x, \Delta\tau| M. \quad (18)$$

Here $M = |\langle B, \tau_f | B, \tau_i \rangle|^2$, while x and $\Delta\tau \equiv \tau_f - \tau_i$ represent, respectively, the position at $\tau = 0$ and the time of propagation for the classical part of the Euclidean string map: $X(\tau) = x - i\tau k$, in units of α' . The quantity $\langle B, \tau_f | B, \tau_i \rangle$ is the scalar product between the two boundary states for the closed string cylinder, representing the two D-branes at path times τ_i and τ_f (for the closed string

world sheet, this is the longitudinal coordinate). We also have that $|B, \tau_i\rangle = e^{(\tau_i - \tau_f)(L_0 + \tilde{L}_0)} |B, \tau_f\rangle$ and $\langle B, \tau | B, \tau \rangle = 1$. Orthonormality in all indices is assumed. We can then repeat the procedure of the previous section. The only difference is that the weights in (16) will be replaced by the ones in (18).

The first step in the evaluation of this amplitude is to enforce the conditions on the embedding $X(\tau)$ to start and finish on the proper brane, after having wound up around the circle m times. These conditions are delta functions that will fix the initial position of the map x and $\Delta\tau$ (with the origin for τ irrelevant by translational symmetry). The condition at τ_i translates into a delta $\delta(X(\tau_i) - \phi_i)$. The condition at τ_f becomes instead $\delta(X(\tau_f) - \phi_j - m)$, which can be transformed into

$$\delta(X(\tau_f) - \phi_j - m) = \delta\left(\Delta\tau + i \frac{\phi_{ij} - m}{k}\right) \frac{1}{|k|}. \quad (19)$$

We can now use that $L_0 = \tilde{L}_0 = k^2 / 4R^2$ and the momentum k is quantized around the circle, $k = 2\pi R^2 n$ for some integer n . Putting everything together, we obtain $\langle B, \tau_f | B, \tau_i \rangle = e^{i\pi n(\phi_{ij} - m)}$ and $|B, \tau_f | B, \tau_i \rangle^\dagger = e^{i\pi n(\phi_{ij} + m)}$. Combining this with (19), one obtains $e^{2\pi i n \phi_{ij}} / 2\pi R^2 |n|$. After removing the $n = 0$ term, the final result for the Renyi entropy reads

$$\text{tr}(\rho_A^q) = \left(\sum_{n \neq 0} \frac{1}{4|n|} e^{2\pi i n \phi_{ij}} \right)^{1-q}, \quad (20)$$

whose corresponding entropy agrees with (14).

IV. DISCUSSION

We have computed the entropy associated to the winding modes of a string wrapping a torus, finding a nonvanishing result that depends only on the complex structure of the torus and on the position of the branes on which the string end points are attached. It is worth emphasizing that this entropy does not originate from localized degrees of freedom, but from extended, topologically inequivalent configurations. This is the first time, to the best of our knowledge, that the replica trick has been applied to nonperturbative states and for a string world sheet. We have chosen a prototypical problem to clarify important issues. The first one is about the correct moduli prescription on the q -sheeted cylinder, which has been solved by applying the tadpole cancellation mechanism. It would be interesting to understand if this idea can be generalized to more complicated Riemann surface topologies. Moreover, although the result we have obtained is nonsingular, a cutoff has to be introduced in order to guarantee the reality and positivity of the entropy. Exploring the origin and justification for the cutoff is certainly something important to look at with more care. Finally, we have reproduced the result from a density matrix constructed using target space quantities, both in the open and closed string channels, making transparent the origin of the entropy we have found.

It would also be interesting to obtain the full result for the entropy, including perturbative contributions, as the natural expectation of the entropy being the one associated with a canonical ensemble, with inverse temperature given by the cylinder radius, is spoiled by the necessity to integrate over this radius, i.e., the modulus of the cylinder. However, to do this computation, some nontrivial issues with the replica trick on the cylinder need to be sorted out first. These include possible boundary contributions [25]; the fact that string theory is conformal only locally, so that the usual CFT computations should be adapted somehow; the topological structure of the world sheet that may create problems in defining nontopological entangling regions; and the still unresolved issue on how to sum over spin structures [15,22]. Finally, a last challenge would be the target space interpretation of this full result.

On a more speculative level, it would be worthwhile to explore the implications of our computation to the understanding of the density of string states and the transitions related to winding modes. Perhaps our entropy could be used as an order parameter for the Hagedorn behavior [26]. We hope to come back to these explorations soon.

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