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DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS.
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A note on the smoothing problem in Chow's theorem

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Abstract
This paper concerns a solution of the smoothing problem in Chow-Rashevskii's connectivity theorem proposed in [1].

1 Introduction and objectives

Let M be a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle $\mathscr{D}$ of TM. The well-known Chow-Rashevskii's connectivity theorem (see [2] and generalizations by P. Stefan in [5, 6] and by H. Sussmann in [7]) asserts that, if $\mathscr{D}$ is bracket-generating, any two points in the same connected component of M may be connected by a sectionally smooth path tangent to $\mathscr{D}$. The question of whether or not any two points in M may be connected by a smooth horizontal immersion was posed by R. Bryant and L. Hsu in [1] and affirmatively answered by M. Gromov in [3], who named the problem as "the smoothing problem in Chow's theorem".

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The purpose of this note is to present an alternative approach to Gromov's solution by means of a method that, to our taste, seems to be more geometrically intuitive. Besides, it conveys some additional information on the connectivity problem: we prove in Theorem 2 and its Corollary 3 that, if the distribution $\mathscr{D}$ is bracket-generating, any two points in a connected open set $\mathcal{U} \subset \mathsf{M}$ may be connected on $\mathcal{U}$ by a smooth horizontal 1-immersion with arbitrary given initial and final velocities in $\mathscr{D}$. Our method is quite simple: given $p, q \in \mathcal{U}$, $v_p \in \mathscr{D}_p \setminus \{0\}$ and $v_q \in \mathscr{D}_q \setminus \{0\}$, we apply the orbit theorem to show that v_p and v_q may be connected on $(\mathscr{D}|_{\mathcal{U}})^*$ (i.e. $\mathscr{D}|_{\mathcal{U}}$ with the zero section removed) by means of a sectionally smooth curve whose smooth arcs are integral curves of second order vector fields on $\mathscr{D}$, i.e. local smooth sections of $\tau_{\mathscr{D}} : \mathsf{T}\mathscr{D} \to \mathscr{D}$ whose integral curves are lifts of smooth curves on M. It then follows that the projection on M of this sectionally smooth curve is a horizontal 1 immersed curve connecting p and q on $\mathcal{U}$, whose initial and final velocities coincide with v_p and v_q, respectively. This method may also be applied in case the linear subbundle $\mathscr{D}$ is not bracket-generating: we prove in Theorem 3 that, if $\mathscr{D}$ satisfies Sussmann's necessary and sufficient condition for reachability given in theorem 7.1 of [7], then any two points in the same connected component of M may be connected by a smooth horizontal 1-immersion with arbitrary given initial and final velocities in $\mathscr{D}$.

2 Preliminaries and notation

2.1 Smooth distributions

We denote the tangent bundle of a finite dimensional paracompact smooth[1] manifold M by $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$. Following the notation and definitions in [7], a *distribution* $\mathscr{D}$ on M is a family $\{\mathscr{D}_x\}_{x\in\mathsf{M}}$ of linear subspaces of each fiber of the tangent bundle $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$. The distribution $\mathscr{D}$ is called *smooth* if $\mathscr{D}_x$ varies smoothly with $x \in \mathsf{M}$, in the sense that there exists a set $\mathcal{D}$ of locally defined smooth vector fields on M such that, for each $x \in \mathsf{M}$, $\mathscr{D}_x = \operatorname{span}\{V(x) \mid V \in \mathcal{D}, x \in \operatorname{dom} V\}$ (with the convention that the linear span of the empty set is $\{0\}$). If that is the case, we say that the smooth distribution $\mathscr{D}$ is generated by $\mathcal{D}$. Equivalently, and perhaps more naturally, the distribution $\mathscr{D}$ is smooth if there exists a subsheaf $\mathcal{D}$ of the sheaf $\mathcal{C}^{\infty}_{\mathsf{TM}}$ of germs of smooth sections of TM (considered as a sheaf of $\infty(\mathsf{M})$-modules) such that, for each $x \in \mathsf{M}$, $\mathscr{D}_x = \{V(x) \mid V \in \mathcal{D}_x\}$ (where $\mathcal{D}_x$ denotes the stalk of $\mathcal{D}$ over x). We avoid, however, the use of sheaves, in order to keep the notation and formalism compatible with that of [7] and [5, 6]. Note that the rank of $\mathscr{D}_x$ depends on x, i.e. $\mathscr{D}$ need not be a linear subbundle of TM (but we do assume that as a hypothesis for our main results). If $\mathcal{D}$ is a set of locally defined smooth vector fields on M, we denote by $[\mathcal{D}]$ the smooth distribution generated by $\mathcal{D}$.

We say that V is a (local) smooth section of a smooth distribution $\mathscr{D}$ if it is a smooth (local) section of $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$ defined on an open set $\mathcal{U} \subset \mathsf{M}$ such that, for all $x \in \mathcal{U}$, $V(x) \in \mathscr{D}_x$. We denote the set of such local smooth sections by $\Gamma^{\infty}_{\mathrm{loc}}(\mathscr{D})$; it is clear that the smooth distribution $\mathscr{D}$ is generated by $\Gamma^{\infty}_{\mathrm{loc}}(\mathscr{D})$.

[1] *smooth* in this paper means "C^{∞}"

Given two locally defined smooth vector fields on M, their Lie bracket is a well-defined smooth vector field on the intersection of their domains. We say that a set of locally defined smooth vector fields $\mathcal{D}$ on M is *involutive* if it is closed by the operation of taking Lie brackets. Any set $\mathcal{D}$ of locally defined smooth vector fields $\mathcal{D}$ on M is contained in a smallest involutive set of locally defined smooth vector fields on M, which we denote by $\mathcal{D}_*$. Indeed, the family $\mathcal{F}$ of all involutive sets of locally defined smooth vector fields containing $\mathcal{D}$ is nonempty (since $\Gamma^\infty_{\mathsf{loc}}(\mathsf{TM})$ is such a set) and $\cap\mathcal{F}$ does the work. We say that a smooth distribution $\mathscr{D}$ on M is *involutive* if so is $\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$.

We say that a smooth distribution $\mathscr{D}$ on M is *bracket-generating* if the smooth distribution generated by $\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})_*$ coincides with TM.

2.2 Orbits of local groups of diffeomorphisms and distributions

A *local group of diffeomorphisms* G on M is a set of smooth diffeomorphisms defined on open subsets of M which is closed under compositions and under taking inverses, i.e. if $\phi : \mathcal{U} \to \mathcal{V}$ and $\psi : \mathcal{U}' \to \mathcal{V}'$ belong to G, then both $\phi^{-1} : \mathcal{V} \to \mathcal{U}$ and $\psi\circ\phi : \phi^{-1}(\mathcal{U}' \cap \mathcal{V}) \to \psi(\mathcal{U}' \cap \mathcal{V})$ belong to G (note that the diffeomorphism with empty domain, that is, the empty set, is allowed). If G is a set of locally defined smooth diffeomorphisms on M, there exists a smallest local group of diffeomorphisms G_* which contains G: we take the intersection $\cap\mathcal{F}$ of the family $\mathcal{F}$ of all local groups of diffeomorphisms which contain G (note that $\mathcal{F}$ is nonempty, since the set of all locally defined diffeomorphisms on M is such a local group). We call G_* the *local group of diffeomorphisms generated by* G.

Let G be a local group of diffeomorphisms on M. We define an equivalence relation on M by $x \sim y$ if $x = y$ or if there exists $\phi \in \mathsf{G}$ such that $x \in \operatorname{dom} \phi$ and $\phi(x) = y$. The equivalence classes of this relation are called *orbits of* G. Note that, if $x \in \mathsf{M}$ and there is no $\phi \in \mathsf{G}$ such that $x \in \operatorname{dom} \phi$, the orbit of x is the singleton of x. If G is a set of locally defined smooth diffeomorphisms on M, we define the *orbits of* G as the orbits of G_*.

Given a locally defined smooth vector field X on M, we denote by $(X_t)_{t\in\mathbb{R}}$ the local one-parameter group of diffeomorphisms associated with X. If $\mathcal{D}$ is a set of locally defined smooth vector fields on M, we denote by $\Theta\mathcal{D}$ the set of locally defined smooth diffeomorphisms on M given by

$$\Theta\mathcal{D} = \cup_{X\in\mathcal{D},t\in\mathbb{R}} X_t,$$

and by $\Psi\mathcal{D}$ the local group of diffeomorphisms on M generated by $\Theta\mathcal{D}$, i.e. the set of all finite compositions of local diffeomorphisms in $\Theta\mathcal{D}$ (we are borrowing here the notation from [5, 6]). We define the *orbits of* $\mathcal{D}$ as the orbits of $\Theta\mathcal{D}$. If $\mathscr{D}$ is a smooth distribution on M, we define the *orbits of* $\mathscr{D}$ as the orbits of $\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$.

We say that a smooth distribution $\mathscr{D}$ on M is *invariant* by a local group of diffeomorphisms G on M if, for each $x \in \mathsf{M}$, each $v \in \mathscr{D}_x$ and each $\phi \in \mathsf{G}$ such that $x \in \operatorname{dom} \phi$, we have $\phi_* v \in \mathscr{D}_{\phi(x)}$, where ϕ_* denotes the tangent map of ϕ. We say that a smooth distribution $\mathscr{D}$ on M is *invariant* by a set G of locally defined smooth diffeomorphisms on M if it is invariant by G_*. We say that $\mathscr{D}$ is *invariant* by a set $\mathcal{D}$ of locally defined smooth vector fields on M if $\mathscr{D}$ is invariant by $\Psi\mathcal{D}$.

Given $\mathscr{D}$ and $\mathscr{D}'$ distributions on M, we say that $\mathscr{D} \subset \mathscr{D}'$ if, for all $x \in \mathsf{M}$, $\mathscr{D}_x \subset \mathscr{D}'_x$.

Given a smooth distribution $\mathscr{D}$ on M and a local group of diffeomorphisms G on M, there exists a smallest smooth distribution $\mathscr{D}^{\mathsf{G}}$ on M which contains $\mathscr{D}$ and is invariant by G: if $\mathscr{D}$ is generated by the set of locally defined smooth vector fields $\mathcal{D}$, $\mathscr{D}^{\mathsf{G}}$ is the distribution generated by the set of locally defined smooth vector fields $\{\phi_* X \mid \phi \in \mathsf{G}, X \in \mathcal{D}\}$, where $\phi_* X$ denotes the pushforward of X by ϕ (which is a locally defined smooth vector field on M). Consequently, if $\mathcal{D}$ is a set of locally defined smooth vector fields on M, there exists a smallest smooth distribution $\mathsf{P}_{\mathcal{D}}$ (this time we are borrowing the notation from [7]) on M which contains the distribution $[\mathcal{D}]$ generated by $\mathcal{D}$ and which is invariant by $\mathcal{D}$, i.e. it is invariant by $\Psi\mathcal{D}$. The smooth distribution $\mathsf{P}_{\mathcal{D}}$ is generated by $\{\phi_* X \mid \phi \in \Psi\mathcal{D}, X \in \mathcal{D}\}$.

We can finally enunciate a version of the so-called *orbit theorem*. The following statement is a subset of the the more general statements contained in [7] (Theorem 4.1) and [5] (Theorems 1 and 5).

Theorem 1 (orbit theorem) *Let* M *be a finite dimensional paracompact smooth manifold and* $\mathcal{D}$ *a set of locally defined smooth vector fields on* M. *Then each orbit* S *of* $\mathcal{D}$ *is an immersed smooth submanifold of* M *such that, for each* $x \in S$, *the tangent space of* S *at* x *coincides with* $\mathsf{P}_{\mathcal{D}}(x)$.

It was actually proved in [5] that each orbit S of $\mathcal{D}$ admits a unique smooth manifold structure which turns it into a *leaf* of M, i.e. a smooth immersed submanifold with the property that, for each locally connected topological space N and each continuous map $f : N \to \mathsf{M}$ with image contained in S, the induced map $f : N \to S$ is continuous. Besides, the partition of M determined by the orbits of S is a *foliation with singularities* (cf. definition on page 700 of [5]). In particular, $\mathsf{P}_{\mathcal{D}}$ is an involutive distribution (that was also proved in [7]). It then follows that (recall that $\mathcal{D}_*$ denotes the smallest involutive subset of locally defined smooth vector fields on M containing $\mathcal{D}$) we have inclusions

$$[\mathcal{D}] \subset [\mathcal{D}_*] \subset \mathsf{P}_{\mathcal{D}}.$$

Indeed, the first inclusion is clear, and the second inclusion follows from the inclusion $\mathcal{D}_* \subset \Gamma^\infty_{\mathrm{loc}}(\mathsf{P}_{\mathcal{D}})$ (since, by the involutiveness of the distribution $\mathsf{P}_{\mathcal{D}}$, $\Gamma^\infty_{\mathrm{loc}}(\mathsf{P}_{\mathcal{D}})$ is an involutive set of locally defined smooth vector fields containing $\mathcal{D}$, hence it must contain $\mathcal{D}_*$) and from the fact that $\mathsf{P}_{\mathcal{D}}$ is generated by $\Gamma^\infty_{\mathrm{loc}}(\mathsf{P}_{\mathcal{D}})$. We therefore conclude that, if $\mathscr{D}$ is a smooth bracket-generating distribution on M and $\mathcal{D} = \Gamma^\infty_{\mathrm{loc}}(\mathscr{D})$, then

$$[\mathcal{D}_*] = \mathsf{P}_{\mathcal{D}} = \mathsf{TM}.$$

In particular, if M is connected, $\mathscr{D}$ admits a unique orbit which coincides with M. We have thus proved the following version of Chow-Rashevskii's connectivity theorem. We say that a sectionally smooth curve on M is *horizontal* with respect to $\mathscr{D}$ if all of its tangent vectors belong to $\mathscr{D}$.

Corollary 1 (Chow-Rashevskii) *Let* M *be a finite dimensional paracompact connected smooth manifold and* $\mathscr{D}$ *a smooth bracket-generating distribution on* M. *Then* M *is* $\mathscr{D}$*-connected, i.e. any two points in* M *may be connected by a sectionally smooth curve on* M *horizontal with respect to* $\mathscr{D}$.

The converse to Chow-Rashevskii's theorem fails, i.e. the bracket-generating condition is not necessary for $\mathscr{D}$-connectivity (see [4], page 24).

A necessary and sufficient condition for $\mathscr{D}$-connectivity may be obtained as a direct consequence of the following corollary of theorem 1 (cf. theorem 7.1 in [7]).

Corollary 2 (Sussmann's condition for $\mathcal{D}$-connectivity) *Let* M *be a finite dimensional paracompact connected smooth manifold and* $\mathcal{D}$ *a set of locally defined smooth vector fields on* M. *Then* M *is* $\mathcal{D}$*-connected (i.e.* M *is an orbit of* $\mathcal{D}$*) if, and only if,*

$$\mathsf{P}_{\mathcal{D}} = \mathsf{TM}.$$

2.3 Fiber and parallel derivatives

Our last ingredient is a computational tool. Given a smooth linear subbundle $\mathscr{D}$ of TM, we shall need to compute Lie brackets of vector fields in $\mathfrak{X}(\mathscr{D})$. That could be accomplished by means of local charts on M and local trivializations of the vector bundle $\pi_{\mathscr{D}} : \mathscr{D} \to \mathsf{M}$, but in that case the computations we need to perform become rapidly messy. Instead, we compute by means of a method introduced in [8] and summarized below.

Let $\pi_E : E \to \mathsf{M}$ be a smooth vector bundle over M endowed with a connection $\nabla^E : \mathfrak{X}(\mathsf{M}) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$. The connection ∇^E defines a horizontal subbundle $\mathrm{Hor}(E)$ of $\mathsf{T}E$, where $(\forall v_q \in E)\mathrm{Hor}_{v_q}(E)$ is the image of the *horizontal lift at* v_q, $\mathrm{H}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_{v_q}E$, defined by $w_q \mapsto \mathsf{T}V \cdot w_q$, where T denotes the tangent map and V is any smooth local section of $\pi_E : E \to \mathsf{M}$ defined on an open neighborhood of q such that $V(q) = v_q$ and $\nabla^E_{w_q} V = 0$. The horizontal lift $\mathrm{H}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathsf{T}_{v_q}E$ is therefore a linear isomorphism onto $\mathrm{Hor}_{v_q}(E)$ whose inverse is the restriction of the tangent map $\mathsf{T}\pi_E$ to $\mathrm{Hor}_{v_q}(E)$. Denoting by $\mathrm{Ver}(E) := \ker \mathsf{T}\pi_E$ the *vertical subbundle* of the tangent bundle $\mathsf{T}E$, we thus have a Whitney sum decomposition

$$\mathsf{T}E = \mathrm{Hor}(E) \bigoplus_E \mathrm{Ver}(E).$$

The *connector* $\kappa_E : \mathsf{T}E \to E$ associated to the connection is given by $X_{v_q} \in \mathsf{T}_{v_q}E \mapsto P_V(X_{v_q}) \in \mathrm{Ver}_{v_q}(E)$ (where P_V is the projection on the vertical subbundle induced by the Whitney sum decomposition above) followed by the inverse of the *vertical lift* $\lambda_{v_q} : E_q \to \mathrm{Ver}_{v_q}(E)$ at v_q (which is the canonical linear isomorphism $E_q \equiv \mathsf{T}_{v_q}(E_q) = \mathrm{Ver}_{v_q}(E)$). Note that, with these definitions:

1) for all $X_{v_q} \in \mathsf{T}E$, $X_{v_q} = \mathrm{H}_{v_q}(\mathsf{T}\pi_E \cdot X_{v_q}) + \lambda_{v_q}(\kappa_E \cdot X_{v_q})$;

2) for all $w_q \in \mathsf{TM}$, for all V smooth local section of $\pi_E : E \to \mathsf{M}$ defined on an open neighborhood of q, we have $\nabla^E_{w_q} V = \kappa_E \cdot \mathsf{T}X \cdot w_q \in E_q$.

Next, we consider two smooth vector bundles $\pi_E : E \to \mathsf{M}$ and $\pi_F : F \to \mathsf{N}$ over paracompact smooth manifolds M and N, respectively, and $b : E \to F$ be a morphism of smooth fiber bundles (i.e. it preserves fibers and is smooth) over $\tilde{b} : \mathsf{M} \to \mathsf{N}$. We denote by $\mathbb{F}b : E \to \mathsf{L}(E, \tilde{b}^* F)$ the *fiber derivative* of b, i.e. the morphism of smooth fiber bundles defined by, for all $v_q, w_q \in E_q$, $\mathbb{F}b(v_q) \cdot w_q \doteq \kappa_F^V \cdot \mathsf{T}b \cdot \lambda_{v_q}(w_q) \in F_{\tilde{b}(q)}$, where κ_F^V denotes the restriction of the connector κ_F to the vertical subbundle (that is, κ_F^V is the inverse of the vertical lift). We don't need the connections to define the fiber derivative; what we need them for is to define the *parallel derivative* $\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$. That is a smooth fiber bundle morphism defined by, for all $v_q \in E$ and all $z_q \in \mathsf{T}_q\mathsf{M}$,

$$\mathbb{P}b(v_q) \cdot z_q \doteq \kappa_F \cdot \mathsf{T}b \cdot \mathrm{H}_{v_q}(z_q) \in F_{\tilde{b}(q)}.$$

The idea in considering these fiber and parallel derivatives is to provide a coordinate-free technique to compute the tangent map of b, allowing its computation at a given element of $\mathsf{T}E$ in terms of its vertical and horizontal components, so that they play a role of "partial derivatives". That is to say, for all $X_{v_q} \in \mathsf{T}E$, the following formulae hold:

$$\begin{aligned} \mathsf{T}\pi_F \cdot \mathsf{T}b \cdot X_{v_q} &= \mathsf{T}\tilde{b} \cdot \mathsf{T}\pi_E \cdot X_{v_q} \\ \kappa_F \cdot \mathsf{T}b \cdot X_{v_q} &= \mathbb{F}b(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}b(v_q) \cdot \mathsf{T}\pi_E \cdot X_{v_q}. \end{aligned}$$

We finally come back to our initial setting, i.e. take M a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle $\mathscr{D}$ of TM. We fix an auxiliary Riemannian metric tensor g on M, which induces a Whitney sum decomposition $\mathsf{TM} = \mathscr{D} \underset{\mathsf{M}}{\oplus} \mathscr{D}^\perp$. We denote by $P : \mathsf{TM} \to \mathscr{D}$ the projection on the first factor determined by this Whitney sum, and by $\nabla^{\mathscr{D}} : \mathfrak{X}(\mathsf{M}) \times \Gamma^\infty(\mathscr{D}) \to \Gamma^\infty(\mathscr{D})$ the connection on the vector bundle $\pi_{\mathscr{D}} : \mathscr{D} \to \mathsf{M}$ given by

$$\nabla^{\mathscr{D}}_X Y := P\nabla_X Y,$$

where ∇ is the Levi-Civita connection of (M, g). Thus, both vector bundles $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$ and $\pi_{\mathscr{D}} : \mathscr{D} \to \mathsf{M}$ are endowed with connections ∇ (Levi-Civita) and $\nabla^{\mathscr{D}}$, with respective connectors and horizontal lifts denoted by κ, H_{v_q} and $\kappa_{\mathscr{D}}, \mathrm{H}^{\mathscr{D}}_{v_q}$. With respect to these connections, the Lie bracket $[X, Y]$ of (possibly locally defined) smooth vector fields, $X, Y \in \mathfrak{X}(\mathscr{D})$ was computed in proposition 1 of [8] by means of the following formulae, given $v_q \in$ dom $X \cap$ dom Y:

$$\begin{aligned}
\kappa_{\mathscr{D}} \cdot [X, Y](v_q) = {} & \mathbb{F}(\kappa_{\mathscr{D}} \circ Y)(v_q) \cdot \kappa_{\mathscr{D}} \cdot X(v_q) + \mathbb{P}(\kappa_{\mathscr{D}} \circ Y)(v_q) \cdot \mathsf{T}\pi_{\mathscr{D}} \cdot X(v_q) - \\
& - \mathbb{F}(\kappa_{\mathscr{D}} \circ X)(v_q) \cdot \kappa_{\mathscr{D}} \cdot Y(v_q) - \mathbb{P}(\kappa_{\mathscr{D}} \circ X)(v_q) \cdot \mathsf{T}\pi_{\mathscr{D}} \cdot Y(v_q) + \\
& + \mathsf{R}^{\mathscr{D}}\big(\mathsf{T}\pi_{\mathscr{D}} \cdot Y(v_q), \mathsf{T}\pi_{\mathscr{D}} \cdot X(v_q)\big) \cdot v_q, \\
\mathsf{T}\pi_{\mathscr{D}} \cdot [X, Y](v_q) = {} & \mathbb{F}(\mathsf{T}\pi_{\mathscr{D}} \circ Y)(v_q) \cdot \kappa_{\mathscr{D}} \cdot X(v_q) + \mathbb{P}(\mathsf{T}\pi_{\mathscr{D}} \circ Y)(v_q) \cdot \mathsf{T}\pi_{\mathscr{D}} \cdot X(v_q) - \\
& - \mathbb{F}(\mathsf{T}\pi_{\mathscr{D}} \circ X)(v_q) \cdot \kappa_{\mathscr{D}} \cdot Y(v_q) - \mathbb{P}(\mathsf{T}\pi_{\mathscr{D}} \circ X)(v_q) \cdot \mathsf{T}\pi_{\mathscr{D}} \cdot Y(v_q),
\end{aligned}$$

where $\mathsf{R}^{\mathscr{D}}$ is the curvature tensor of $\nabla^{\mathscr{D}}$.

We shall need the formulae above in the particular case in which: 1) X is the nonholonomic vector field $\mathrm{X}_{\mathscr{D}}$ of $(\mathsf{M}, \mathsf{g}, \mathscr{D})$, i.e. the vector field given by

$$\mathrm{X}_{\mathscr{D}}(v_q) = \mathrm{H}^{\mathscr{D}}_{v_q}(v_q) = \mathsf{T}P \cdot \mathsf{S}(v_q),$$

where S is the geodesic spray of (M, g); 2) Y is an arbitrary (locally defined) smooth vertical vector field. In this case, the above formulae simplify to, for all $v_q \in \operatorname{dom} Y$:

$$\begin{aligned}
\kappa_{\mathscr{D}} \cdot [\mathrm{X}_{\mathscr{D}}, Y](v_q) &= \mathbb{P}(\kappa_{\mathscr{D}} \circ Y)(v_q) \cdot v_q \\
\mathsf{T}\pi_{\mathscr{D}} \cdot [\mathrm{X}_{\mathscr{D}}, Y](v_q) &= -\kappa_{\mathscr{D}} \cdot Y(v_q).
\end{aligned} \tag{1}$$

3 Statement and proof of the main results

Theorem 2 (Smoothing in Chow's Theorem) *Let* M *be a finite dimensional paracompact connected smooth manifold endowed with a smooth linear subbundle* $\mathscr{D}$ *of* TM. *If* $\mathscr{D}$ *is bracket-generating, then any two points in* M *may be connected by a horizontal curve which is both a* 1 *immersion and sectionally smooth, with arbitrary given initial and final velocities in* $\mathscr{D}$.

Proof It suffices to consider the case $\dim \mathsf{M} \geq 2$, otherwise the thesis is trivial. Then, since $\mathscr{D}$ is bracket-generating, we must have $\operatorname{rk} \mathscr{D} \geq 2$; it then follows that the slit bundle $\mathscr{D}^*$ (i.e. $\mathscr{D}$ with the zero section removed) is a connected open submanifold of $\mathscr{D}$ (the fact that it is connected is a consequence of being the total space of a fiber bundle with fibers and base connected). We may apply the orbit theorem 1 to the paracompact connected smooth manifold $\mathscr{D}^*$ endowed with the set $\mathcal{D}$ of locally defined smooth second order vector fields on $\mathscr{D}^*$, i.e. (noting that $\mathsf{T}(\mathscr{D}^*) = \mathsf{T}\mathscr{D}|_{\mathscr{D}^*}$)

$$\mathcal{D} = \{X \in \Gamma^{\infty}_{\mathsf{loc}}(\mathsf{T}\mathscr{D}|_{\mathscr{D}^*}) \mid \forall v_q \in \operatorname{dom} X, \mathsf{T}\pi_{\mathscr{D}} \cdot X(v_q) = v_q\}.$$

We contend that $\mathsf{P}_{\mathcal{D}} = \mathsf{T}\mathscr{D}|_{\mathscr{D}^*}$. Once we prove this contention, we conclude that each orbit of $\mathcal{D}$ is a connected open submanifold of $\mathscr{D}^*$, which implies, due to the connectedness of $\mathscr{D}^*$, that $\mathscr{D}^*$ is the only orbit of $\mathcal{D}$. That is to say, given $p, q \in \mathsf{M}$ and $v_p \in \mathscr{D}_p \setminus \{0\}$, $v_q \in \mathscr{D}_q \setminus \{0\}$, there exists a sectionally smooth curve in $\mathscr{D}^*$ connecting v_p to v_q, whose smooth arcs are integral curves of vector fields in $\mathcal{D}$, i.e. of second order vector fields. The projection on M of this sectionally smooth curve connects p to q, with initial velocity v_p and final velocity v_q, and it is both a sectionally smooth and a 1-immersed horizontal curve on M. By the arbitrariness of p, q taken in M and of the initial and final velocities in $\mathscr{D}^*$, we have thus reached the thesis.

It remains, therefore, to prove our contention, i.e. that $\mathsf{P}_{\mathcal{D}} = \mathsf{T}\mathscr{D}|_{\mathscr{D}^*}$. Given $v_q \in \mathscr{D}^*$, we must prove that $\mathsf{P}_{\mathcal{D}}(v_q) = \mathsf{T}_{v_q}\mathscr{D}$, which will be done along the steps below. We fix an auxiliary Riemannian metric tensor g on M and use the notation from subsection 2.3 of the preliminaries.

1) Since any local smooth vertical vector field in $\mathfrak{X}(\mathscr{D}^*)$ may be written as a difference of two smooth second order vector fields, i.e. of two vector fields in $\mathcal{D} \subset \Gamma^\infty_{\mathsf{loc}}(\mathsf{P}_{\mathcal{D}})$, and since $\mathsf{P}_{\mathcal{D}}$ is a smooth distribution, we conclude that any local smooth vertical vector field in $\mathfrak{X}(\mathscr{D}^*)$ is a smooth local section of $\mathsf{P}_{\mathcal{D}}$, which implies that the vertical space $\mathrm{Ver}_{v_q}(\mathscr{D})$ is contained in $\mathsf{P}_{\mathcal{D}}(v_q)$.
2) Let $\mathrm{X}_{\mathscr{D}}$ be the nonholonomic vector field of $(\mathsf{M}, \mathsf{g}, \mathscr{D})$ (which is a second order vector field in $\mathfrak{X}(\mathscr{D})$, so that its restriction to the open submanifold $\mathscr{D}^*$ belongs to $\mathcal{D}$) and Y an arbitrary vertical smooth vector field in $\mathfrak{X}(\mathscr{D}^*)$ defined on an open neighborhood of v_q. Then both $\mathrm{X}_{\mathscr{D}}|_{\mathscr{D}^*}$ and Y are sections of $\mathsf{P}_{\mathcal{D}}$; since the latter smooth distribution is involutive, we conclude that the Lie bracket $[\mathrm{X}_{\mathscr{D}}, Y]$ is a section of $\mathsf{P}_{\mathcal{D}}$. But, as we have computed in (1), $\mathsf{T}\pi_{\mathscr{D}} \cdot [\mathrm{X}_{\mathscr{D}}, Y](v_q) = -\kappa_{\mathscr{D}} \cdot Y(v_q)$. It then follows that the vector

$$\mathrm{H}^{\mathscr{D}}_{v_q}\big(-\kappa_{\mathscr{D}} \cdot Y(v_q)\big) = [\mathrm{X}_{\mathscr{D}}, Y](v_q) - \lambda_{v_q}(\kappa_{\mathscr{D}} \cdot [\mathrm{X}_{\mathscr{D}}, Y](v_q))$$

belongs to $\mathsf{P}_{\mathcal{D}}(v_q)$, as both vectors in the second member of the previous equality belong to that space. Since the restriction of $\kappa_{\mathscr{D}}$ to $\mathrm{Ver}_{v_q}(\mathscr{D})$ is a linear isomorphism onto $\mathscr{D}_q$ (it is the inverse of the vertical lift $\lambda_{v_q} : \mathscr{D}_q \to \mathrm{Ver}_{v_q}(\mathscr{D})$), and since the smooth vertical vector field Y in $\mathfrak{X}(\mathscr{D}^*)$ on a neighborhood of v_q was arbitrarily taken, we conclude that

$$\mathrm{H}^{\mathscr{D}}_{v_q}(\mathscr{D}_q) \subset \mathsf{P}_{\mathcal{D}}(v_q).$$

3) It follows from the previous step and from the arbitrariness of the fixed $v_q \in \mathscr{D}^*$ that, for any smooth locally defined vector field $X \in \Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$, the horizontal lift $X^{\mathrm{Hor}} \in \Gamma^\infty_{\mathsf{loc}}(\mathsf{T}\mathscr{D}|_{\mathscr{D}^*})$ defined by

$$w_q \in \mathscr{D}^* \cap \pi_{\mathscr{D}}^{-1}(\operatorname{dom} X) \mapsto \mathrm{H}^{\mathscr{D}}_{w_q}\big(X(q)\big)$$

is a smooth local section of $\mathsf{P}_{\mathcal{D}}$. Moreover, for all $w_q \in \mathscr{D}^* \cap \pi_{\mathscr{D}}^{-1}(\operatorname{dom} X)$, we have $\mathsf{T}\pi_{\mathscr{D}} \cdot X^{\mathrm{Hor}}(w_q) = X(q) = X \circ \pi_{\mathscr{D}}(w_q)$, i.e. the vector fields X^{Hor} and X are $\pi_{\mathscr{D}}$-related. Then so are the Lie brackets of vector fields of this form, i.e. if Y is another smooth locally defined vector field in $\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$, the locally defined vector fields $[X^{\mathrm{Hor}}, Y^{\mathrm{Hor}}]$ and $[X, Y]$ are $\pi_{\mathscr{D}}$-related. As $\mathsf{P}_{\mathcal{D}}$ is involutive, we conclude by induction on k that, for an arbitrary k-tuple $X_1, \ldots, X_k$ in $\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$ defined on an open neighborhood of q, $[\cdots[[X_1^{\mathrm{Hor}}, X_2^{\mathrm{Hor}}], \cdots]X_{k-1}^{\mathrm{Hor}}], X_k^{\mathrm{Hor}}]$ is a smooth local section of $\mathsf{P}_{\mathcal{D}}$ defined on a neighborhood of v_q and the locally defined vector fields

$$[\cdots[[X_1^{\mathrm{Hor}}, X_2^{\mathrm{Hor}}], \cdots]X_{k-1}^{\mathrm{Hor}}], X_k^{\mathrm{Hor}}] \text{ and } [\cdots[[X_1, X_2], \cdots]X_{k-1}], X_k]$$

are $\pi_{\mathscr{D}}$-related. It then follows that the vector

$$\begin{aligned} &\mathrm{H}_{v_q}^{\mathscr{D}}\big([\cdots[[X_1, X_2], \cdots]X_{k-1}], X_k](q)\big) = \\ &\quad = [\cdots[[X_1^{\mathrm{Hor}}, X_2^{\mathrm{Hor}}], \cdots]X_{k-1}^{\mathrm{Hor}}], X_k^{\mathrm{Hor}}](v_q) - \\ &\qquad - \lambda_{v_q}\big(\kappa_{\mathscr{D}} \cdot [\cdots[[X_1^{\mathrm{Hor}}, X_2^{\mathrm{Hor}}], \cdots]X_{k-1}^{\mathrm{Hor}}], X_k^{\mathrm{Hor}}](v_q)\big) \end{aligned}$$

belongs to $\mathsf{P}_{\mathcal{D}}(v_q)$, since both vectors on the second member of the previous equality belong to that space. But, since $\mathscr{D}$ is a bracket-generating distribution, we have

$$\mathsf{T}_q\mathsf{M} = \operatorname{span}\{[\cdots[[X_1, X_2], \cdots]X_{k-1}], X_k](q) \mid k \in \mathbb{N}, X_1, \ldots, X_k \in \Gamma_{\mathrm{loc}}^{\infty}(\mathscr{D})\}.$$

We finally conclude that $\mathrm{Hor}_{v_q}(\mathscr{D}) = \mathrm{H}_{v_q}^{\mathscr{D}}(\mathsf{T}_q\mathsf{M}) \subset \mathsf{P}_{\mathcal{D}}(v_q)$. Thus, in view of step 1, we have

$$\mathsf{T}_{v_q}\mathscr{D} = \mathrm{Hor}_{v_q}(\mathscr{D}) \oplus \mathrm{Ver}_{v_q}(\mathscr{D}) \subset \mathsf{P}_{\mathcal{D}}(v_q),$$

hence the equality holds in the above inclusion and our contention is proved. □

Corollary 3 *Let* M *be a finite dimensional paracompact smooth manifold endowed with a smooth linear subbundle* $\mathscr{D}$ *of* TM. *If* $\mathscr{D}$ *is bracket-generating, then any two points belonging to a connected open subset* $\mathcal{U} \subset \mathsf{M}$ *may be connected by a horizontal curve in* $\mathcal{U}$ *which is both a* 1 *immersion and sectionally smooth, with arbitrary given initial and final velocities in* $\mathscr{D}$.

Proof Apply the previous theorem with $\mathcal{U}$ in place of M and $\mathscr{D}|_{\mathcal{U}}$ in place of $\mathscr{D}$. □

We finally prove that the same smoothness property holds under Sussmann's condition for $\mathscr{D}$-connectivity (corollary 2).

Theorem 3 (smoothness in Sussmann's condition for $\mathscr{D}$-connectivity) *Let* M *be a finite dimensional paracompact connected smooth manifold endowed with a smooth linear subbundle* $\mathscr{D}$ *of* TM *such that* $\mathsf{P}_{\Gamma_{\mathrm{loc}}^{\infty}(\mathscr{D})} = \mathsf{TM}$. *Then any two points in* M *may be connected by a horizontal curve which is both a* 1 *immersion and sectionally smooth, with arbitrary given initial and final velocities in* $\mathscr{D}$.

Proof As in the proof of Theorem 2, it suffices to consider the case $\dim \mathsf{M} \geq 2$, otherwise the thesis is trivial. Then, since $\mathsf{P}_{\Gamma_{\mathrm{loc}}^{\infty}(\mathscr{D})} = \mathsf{TM}$, we must have $\operatorname{rk} \mathscr{D} \geq 2$, so that the slit bundle $\mathscr{D}^*$ is a connected open submanifold of $\mathscr{D}$. Once more we consider the paracompact connected smooth manifold $\mathscr{D}^*$ endowed with the set $\mathcal{D}$ of locally defined smooth second order vector fields on $\mathscr{D}^*$, i.e.

$$\mathcal{D} = \{X \in \Gamma^\infty_{\mathsf{loc}}(\mathsf{T}\mathscr{D}|_{\mathscr{D}^*}) \mid \forall v_q \in \text{ dom } X, \mathsf{T}\pi_{\mathscr{D}} \cdot X(v_q) = v_q\}.$$

We contend that $\mathsf{P}_{\mathcal{D}} = \mathsf{T}\mathscr{D}|_{\mathscr{D}^*}$. Once we prove this contention, the thesis follows from Sussmann's condition 2.

Given $v_q \in \mathscr{D}^*$, we must prove that $\mathsf{P}_{\mathcal{D}}(v_q) = \mathsf{T}_{v_q}\mathscr{D}$, which will be done along the steps below.

1) We fix an auxiliary Riemannian metric tensor g on M. Steps 1) and 2) in the proof of Theorem 2 apply *ipsis litteris*, so that both the vertical subspace $\mathrm{Ver}_{v_q}(\mathscr{D})$ and the horizontal lift $\mathrm{H}^{\mathscr{D}}_{v_q}(\mathscr{D}_q)$ are linear subspaces of $\mathsf{P}_{\mathcal{D}}(v_q)$. Hence, for any smooth locally defined vector field $X \in \Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$, the horizontal lift $X^{\mathrm{Hor}} \in \Gamma^\infty_{\mathsf{loc}}(\mathsf{T}\mathscr{D}|_{\mathscr{D}^*})$ is a smooth local section of $\mathsf{P}_{\mathcal{D}}$.
2) Since $\mathsf{P}_{\mathcal{D}}$ is generated by $\Gamma^\infty_{\mathsf{loc}}(\mathsf{P}_{\mathcal{D}})$, it follows from Theorems 4.1 and 4.2 in [7] that $\mathsf{P}_{\mathcal{D}}$ is $\Gamma^\infty_{\mathsf{loc}}(\mathsf{P}_{\mathcal{D}})$-invariant. Hence, for each $X \in \Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$, we conclude from the previous step that $(X^{\mathrm{Hor}}_t)_{t\in\mathbb{R}}$ preserves $\mathsf{P}_{\mathcal{D}}$.
3) Let $w_q \in \mathsf{T}_q\mathsf{M}$. Since $\mathsf{T}_q\mathsf{M} = \mathsf{P}_{\Gamma^\infty_{\mathsf{loc}}(\mathscr{D})}(q)$, we may take $z_p \in \mathscr{D}$ and finite families $(X_i)_{1\le i\le k}$ of smooth local sections of $\mathscr{D}$ and $(t_i)_{1\le i\le k}$ of real numbers such that $(X_{k,t_k} \circ \cdots \circ X_{1,t_1})_* z_p = w_q$. But, for any for any smooth locally defined vector field $X \in \Gamma^\infty_{\mathsf{loc}}(\mathscr{D})$, the horizontal lift $X^{\mathrm{Hor}} \in \Gamma^\infty_{\mathsf{loc}}(\mathsf{T}\mathscr{D}|_{\mathscr{D}^*})$ is $\pi_{\mathscr{D}}$-related to X; it then follows, recalling that $\mathrm{X}_{\mathscr{D}}$ denotes the nonholonomic vector field of $(\mathsf{M}, \mathsf{g}, \mathscr{D})$, that

$$\begin{aligned}&\mathsf{T}\pi_{\mathscr{D}} \circ (X^{\mathrm{Hor}}_{k,t_k} \circ \cdots \circ X^{\mathrm{Hor}}_{1,t_1})_* \mathrm{X}_{\mathscr{D}}(z_p) = \\ &\quad = (X_{k,t_k} \circ \cdots \circ X_{1,t_1})_* \circ \mathsf{T}\pi_{\mathscr{D}} \cdot \mathrm{X}_{\mathscr{D}}(z_p) = w_q.\end{aligned}$$

We therefore conclude that

$$\begin{aligned}\mathrm{H}^{\mathscr{D}}_{v_q}(w_q) = {} & (X^{\mathrm{Hor}}_{k,t_k} \circ \cdots \circ X^{\mathrm{Hor}}_{1,t_1})_* \mathrm{X}_{\mathscr{D}}(z_p) - \\ & - \lambda_{v_q}\left(\kappa_{\mathscr{D}} \cdot (X^{\mathrm{Hor}}_{k,t_k} \circ \cdots \circ X^{\mathrm{Hor}}_{1,t_1})_* \mathrm{X}_{\mathscr{D}}(z_p)\right).\end{aligned}$$

Hence, $\mathrm{H}^{\mathscr{D}}_{v_q}(w_q)$ belongs to $\mathsf{P}_{\mathcal{D}}(v_q)$, since both vectors on the second member of the previous equality belong to that space, in view of steps 1 and 2. Since $w_q \in \mathsf{T}_q\mathsf{M}$ was arbitrarily taken, we conclude that $\mathrm{Hor}_{v_q}(\mathscr{D}) = \mathrm{H}^{\mathscr{D}}_{v_q}(\mathsf{T}_q\mathsf{M}) \subset \mathsf{P}_{\mathcal{D}}(v_q)$. Thus, $\mathsf{T}_{v_q}\mathscr{D} = \mathrm{Hor}_{v_q}(\mathscr{D}) \oplus \mathrm{Ver}_{v_q}(\mathscr{D}) \subset \mathsf{P}_{\mathcal{D}}(v_q)$, hence the equality holds in the above inclusion and our contention is proved.

□

Delcarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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