Hyperbolic disordered ensembles of random matrices

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Using the recently introduced simple procedure of dividing Gaussian matrices by a positive random variable, a family of random matrices is generated characterized by a behavior ruled by the generalized hyperbolic distribution. The spectral density evolves from the semicircle law to a Gaussian-like behavior while concomitantly, the local fluctuations show a transition from the Wigner-Dyson to the Poisson statistics. Long range statistics such as number variance exhibit large fluctuations typical of nonergodic ensembles.

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I. INTRODUCTION

In a recent paper [1], the concept of disorder has been introduced in random matrix theory (RMT) to denote ensembles of random matrices generated by superimposing an external source of randomness to the fluctuations of the Gaussian ensembles. The method generates matrices which preserve unitary invariance, although at the price of introducing correlations among matrix elements. Their statistical properties are amenable to analytical derivation. Families of ensembles recently proposed in the literature fit into this category, for instance, Refs. [2,3] in the Wigner Gaussian case and Refs. [4–6] in the case of Wishart matrices. Disordered ensembles can be considered as an instance of the so-called superstatistics [7], aside from the fact that their density distributions are normalized differently, as explained below.

As a consequence of having operating an extra source of randomness, the ergodicity property of the standard ensembles, characterized by the equivalence between spectral averages taken along the spectrum and averages extracted from the spectra of a set of matrices, is broken. This creates an ambiguity in measuring its collective spectral fluctuations, although for those which concern individual eigenvalues such as extreme value statistics, it remains well defined. This kind of statistics has been the object of many studies [8] in the last two decades. In this direction, it was shown how the behavior of the largest eigenvalue described by the Tracy-Widom distribution [8] is affected by disorder [5,9].

Despite the great success of the RMT Gaussian ensemble (see [10] for recent reviews), it has the features of an abstract mathematical model such as, for example, an eigenvalue density defined in a compact support. Parallel to its development, there was a search for models whose randomness would be closer to physical situations. This is the case of the two-body random ensemble (TBRE) proposed in the context of the shell model of nuclear physics [11] and of the more general k-body embedded Gaussian ensembles (EGE) proposed to many-body systems [12]. In them, the Hamiltonian model is made randomic by taking the strength of the residual interaction among particles from a Gaussian distribution. The main features of these ensembles are correlations among matrix elements, Gaussian eigenvalue density, and nonergodicity [13]. Nevertheless, under restricted conditions that take into account the nonergodicity ambiguity, their spectral fluctuations are supposed to follow the same fluctuations pattern of the Gaussian matrices, namely, the Wigner-Dyson statistics [14]. Disordered ensembles may provide a simple way to understand features of these physically motivated ensembles.

Families of disordered ensembles have, so far, been considered mostly in association with an external randomness governed by the gamma distribution [2,3]. This leads to ensembles in which statistics are dominated by power-law behavior that implies strong nonergodicity. This has also been considered the case of the inverse gamma distribution [5] which contains singularities. Our purpose here is to study an ensemble whose auxiliary random variable is taken from a generalized inverse Gaussian (GIG), which acting on the Gaussian matrices generates disordered matrices distributed according to a generalized hyperbolic (GH) distribution. This kind of distribution was introduced in 1977 by Barndorff-Nielsen [15] and since then has found many applications, especially in finance [16].

II. DISORDERED ENSEMBLES

We start by reviewing general aspects of the formalism of the disordered ensemble of Ref. [1]. The idea is to construct matrices by the relation

$$H(\xi) = \frac{H_V}{\sqrt{\xi/\bar{\xi}}}, \tag{1}$$

where H_V is a random matrix of dimension N with joint density probability distribution

$$P_V(H_V) = \frac{1}{Z_f} \exp\left[-\frac{\beta}{2} \mathrm{tr}\, V(H_V)\right], \tag{2}$$

and ξ is a positive random variable with a normalized density probability distribution $w(\xi)$ with average $\bar{\xi}$ and variance σ_ξ^2. In Eq. (2), $f = N + \beta N(N-1)/2$ is the number of independent matrix elements, β is the Dyson index $\beta = 1,2,4$, and $V(x)$ is a confining potential which makes normalization

$$Z_f = \int dH \exp\left[-\frac{\beta}{2} \mathrm{tr}\, V(H)\right] \tag{3}$$

finite with respect to the measure $dH = \prod_{i=1}^N dH_{ii} \prod_{j>i} \prod_{k=1}^{\beta} \sqrt{2} dH_{ij}^k$. Rewritten as $H_V = H(\xi)\sqrt{\xi/\bar{\xi}}$, Eq. (1) means that the joint distribution of the matrix elements of the disordered ensemble is obtained from Eq. (2) by averaging over the ξ variable, namely, as

$$P(H) = \int d\xi\, w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{f/2} \frac{1}{Z_f} \exp\left[-\frac{\beta}{2} \mathrm{tr}\, V\left(\sqrt{\frac{\xi}{\bar{\xi}}} H\right)\right]. \tag{4}$$

Equation (4) translates the relation (1) among the random quantities into a relation among their density distribution functions. Despite the resemblance of Eq. (4) with distributions constructed in the context of the superstatistics formalism [7], we note that here the two distributions are integrated normalized, while in the superstatistics case the normalization is performed after the integration.

While Eq. (4) puts in evidence the correlations among matrix elements, the relation (1), which is expressed in terms of the random quantities themselves, gives a clearer picture of the model and at the same time, provides an efficient algorithm to generate the disordered matrices. We also note that in [1], an alternative procedure has been described in which matrices of the ensemble are generated taking into account correlations among their elements (see the next section).

Turning now to eigenvalues and eigenvectors, we observe that by diagonalizing the matrices in Eq. (1), the relation

$$D(\xi) = \frac{D_V}{\sqrt{\xi/\bar{\xi}}} \tag{5}$$

is obtained, where $D(\xi)$ and D_V are diagonal matrices whose elements are the respective eigenvalues of $H(\xi)$ and H_V. Rewriting Eq. (5) as $D_V = D(\xi)\sqrt{\xi/\bar{\xi}}$, as done with Eq. (1), the relation

$$P(E_1,\ldots,E_N) = \int d\xi\, w(\xi) \left(\frac{\xi}{\bar{\xi}}\right)^{N/2} P_V(x_1,x_2,\ldots,x_N) \tag{6}$$

is derived, where $x_i = \sqrt{\xi/\bar{\xi}}\, E_i$ and

$$P_V(x_1,\ldots,x_N) = K_N^{-1} \exp\left[-\frac{\beta}{2}\sum_{k=1}^{N} V(x_k)\right] J(x_1,x_2,\ldots,x_N). \tag{7}$$

In Eq. (7), K_N is a normalization constant and $J(x_1,x_2,\ldots,x_N)$ is the eigenvalue part of the Jacobian of the transformation from matrix elements to eigenvalues and eigenvectors. From Eq. (6), statistical measures of the generalized family can be calculated by weighting, with the $w(\xi)$ distribution, the corresponding measures of the original ensemble with the eigenvalues multiplied by the factor $\xi/\bar{\xi}$.

Considering particular choices of the distribution $w(\xi)$, the most used so far, both in the Gaussian and the Wishart cases, was the gamma distribution

$$w(\xi) = \exp(-\xi)\xi^{\bar{\xi}-1}/\Gamma(\bar{\xi}), \tag{8}$$

with variance $\sigma_\xi = \sqrt{\bar{\xi}}$. Considering the case of the Wigner ensemble in which $V(H) = H^2$, when Eq. (8) is substituted into Eq. (4), the integrals are readily performed to give the joint distribution

$$P(H;\bar{\xi}) = \frac{1}{Z_f}\frac{\Gamma(\bar{\xi}+f/2)}{\Gamma(\bar{\xi})}\left(1+\frac{\beta}{2\bar{\xi}}\mathrm{tr}\, V(H)\right)^{-\bar{\xi}-f/2}. \tag{9}$$

To illustrate the power-law behavior induced by the above gamma distribution, take out of the f-independent elements, the density distribution of a given one denoted by h (see the next section for the definition of the h variable), after integrating over the $f-1$ others, we obtain

$$p(h;\bar{\xi}) = \left(\frac{\beta}{2\pi\bar{\xi}}\right)^{1/2}\frac{\Gamma(\bar{\xi}+1/2)}{\Gamma(\bar{\xi})}\left(1+\frac{\beta}{2\bar{\xi}}h^2\right)^{-\bar{\xi}-1/2}. \tag{10}$$

Since for large $|h|$, $p_\beta(h;\bar{\xi}) \sim 1/|h|^{2\bar{\xi}+1}$, the distribution (10) does not have moments of order superior to $2\bar{\xi}$. This fact makes the value $\bar{\xi} = 1$ critical since below it Eq. (10) does not have a second moment [2]. We note that, apart from the lack of independence, the marginal distribution of the matrix elements has the same kind of distribution, namely, one with an asymptotic power-law behavior, as does that of the ensemble of Lévy matrices [19].

Another distribution considered in [5] was the inverse distribution obtained by making $\xi = 1/\nu$, which leads to

$$w(\nu) = \frac{\bar{\nu}}{\Gamma(1/\bar{\nu})}\exp\left(-\frac{1}{\nu}\right)\nu^{-(1/\nu)-2}. \tag{11}$$

Again, the integral in Eq. (4) can be performed to give

$$P(H;\bar{\nu}) = \frac{2}{Z_f\Gamma(1/\bar{\nu})}\left(\sqrt{\frac{\bar{\nu}}{2}\beta\,\mathrm{tr}\, V(H)}\right)^{1+(1/\bar{\nu})-f/2} \times K_{f/2-1/\bar{\nu}-1}\left(\sqrt{\frac{\bar{\nu}}{2}\beta\,\mathrm{tr}\, V(H)}\right), \tag{12}$$

which shows, as a drawback, the presence of a singularity at the origin.

Our present purpose is to study the model generated by taking the disorder variable out of the generalized inverse Gaussian

$$w(a,b,\lambda;\xi) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})}\xi^{\lambda-1}\exp\left[-\frac{1}{2}\left(\frac{a}{\xi}+b\xi\right)\right], \tag{13}$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind. This is a three parameter probability distribution where a and b are positive and λ is real. It contains the above distributions (8) and (11) as special cases. In fact in the limit $a \to 0$, apart from scaling, the gamma distribution [Eq. (8)] is recovered, while in the limit $b \to 0$ the inverse distribution is obtained. The GIG first moment is

$$\bar{\xi} = \sqrt{\frac{a}{b}}\frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \tag{14}$$

and its variance is

$$\sigma_\xi^2 = \left(\frac{b}{a}\right)\left[\frac{K_{\lambda+2}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} - \frac{K_{\lambda+1}^2(\sqrt{ab})}{K_\lambda^2(\sqrt{ab})}\right]. \tag{15}$$

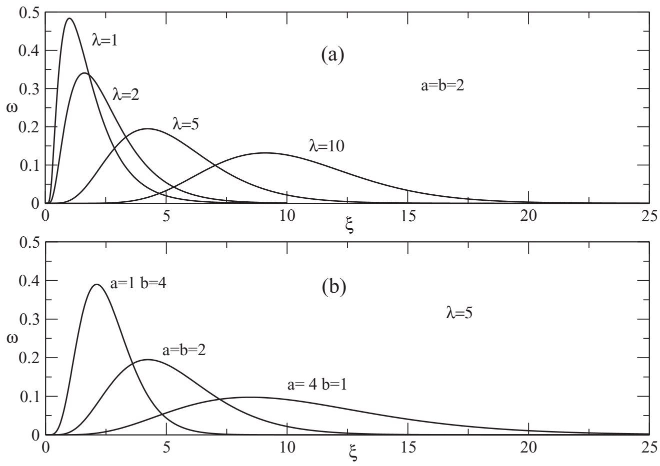

FIG. 1. The GIG distribution, Eq. (13), is plotted for the indicated values of the parameters: In (a) the parameters a and b are kept constant while in (b) $\lambda = 5$ and the ratio a/b is varied keeping $ab = 4$.

By taking the ratio

$$\frac{\bar{\xi}}{\sigma_\xi} = \frac{b}{a}\sqrt{\frac{K_{\lambda+2}(\sqrt{ab})K_\lambda(\sqrt{ab})}{K^2_{\lambda+1}(\sqrt{ab})} - 1}, \tag{16}$$

we have a better parameter to understand the effect of the disorder generated by the GIG distribution. First we remark that by changing the sign of λ in Eq. (16), the symmetry property $K_{-\nu}(x) = K_\nu(x)$ of the Bessel functions makes the above ratio symmetric with respect to the value $\lambda = -1$. Secondly, for large orders, the Bessel functions can be approximated by the dominant term of their uniform approximation [17]

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}}\frac{\exp(-\mu\nu)}{(1+z^2)^{1/4}}, \tag{17}$$

where

$$\mu = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}. \tag{18}$$

Using this, the Bessel functions in Eq. (16) can be approximated for large λ as

$$K_\lambda(\sqrt{ab}) \sim \sqrt{\frac{\pi}{2\lambda}}\left(\frac{\lambda}{\sqrt{ab}}\right)^\lambda \exp(-\lambda), \tag{19}$$

where, in the presence of one, the square of the term $\sqrt{ab}/\lambda$ has been neglected. As a consequence, with $a = b$, the ratio (16) becomes practically constant when λ increases, as indeed can be seen in Fig. 1. On the other hand, by fixing λ and the product ab, the GIG also becomes more localized as the ratio b/a decreases (see Fig. 1). From this analysis, we expect larger disorder for small values of the parameter λ or large values of the ratio b/a, with the value $\lambda = -1$ playing a special role in this generalized family.

III. HYPERBOLIC DISORDERED GAUSSIAN ENSEMBLE

The Wigner ensemble is obtained by making $V(H) = H^2$ with H a Hermitian matrix. The number of independent matrix elements is of course $f = N + \beta N(N-1)/2$ and it is convenient to consider the set of f-independent elements denoted by h_n with $n = 1,2,3,\ldots,f$. They are defined such that first N ones are equal to the diagonal elements; that is, $h_i = H_{ii}$. The remaining ones, the off-diagonal elements, are scaled as $h_n = \sqrt{2}H^k_{ij}$, with $n = N+1, N+2,\ldots,f$. In terms of these f-independent variables, the joint distribution [Eq. (4)], together with Eq. (13), becomes

$$P_f(h_1,\ldots,h_f) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})}\int_0^\infty d\xi\, \xi^{\lambda-1}\left(\frac{\beta\xi}{2\pi\bar{\xi}}\right)^{f/2} \times \exp\left[-\frac{1}{2}\left(\frac{a}{\xi} + b\xi\right) - \sum_{n=1}^{f}\frac{\beta\xi h_n^2}{2\bar{\xi}}\right]. \tag{20}$$

Each h_n variable is treated equally in Eq. (20) and that is the reason to introduce this set of variables in which the inconvenient factor of $\sqrt{2}$ multiplying the off-diagonal ones has been absorbed in the new variable. The integration over the extra variable ξ makes the distribution in Eq. (20) a correlated distribution of the set of variables, but the above distribution has the important property that removing by integration any one variable, the distribution of the remnant others has the same form, namely, we have

$$P_{f-1}(h_1,\ldots,h_{f-1}) = \int dh_f\, P_f(h_1,\ldots,h_f). \tag{21}$$

This property is called scale-invariant occupancy of phase space [18]. Of course, it is a manifestation of the uncorrelation among the matrix elements of the original ensemble. In this way, one can expect this property to hold in general for the disordered ensembles. On the other hand, it is not expected to hold for generalizations of distributions constructed using the strict superstatistics recipe.

As occurred with the gamma distribution and its inverse, after substituting Eq. (13) in Eq. (4), the integrals are performed and we find the joint distribution of matrix elements and the distribution of a generic matrix element h to be given, respectively, by the generalized hyperbolic distributions

$$P(a,b,\lambda;H) = \left(\frac{\sqrt{a/b}}{2\pi\bar{\xi}}\right)^{f/2}\frac{1}{\left(1+\frac{1}{b\bar{\xi}}\sum_{i=1}^f h_i^2\right)^{\lambda_f/2}} \times \frac{K_{\lambda_f}\left[\sqrt{ab\left(1+\frac{1}{b\bar{\xi}}\sum_{i=1}^f h_i^2\right)}\right]}{K_\lambda(\sqrt{ab})} \tag{22}$$

and

$$p(a,b,\lambda;h) = \frac{(a/b)^{1/4}}{\sqrt{2\pi\bar{\xi}}}\frac{1}{\left(1+\frac{h^2}{b\bar{\xi}}\right)^{\lambda_1/2}}\frac{K_{\lambda_1}\left[\sqrt{ab\left(1+\frac{h^2}{b\bar{\xi}}\right)}\right]}{K_\lambda(\sqrt{ab})}, \tag{23}$$

where with $1 \leqslant k \leqslant f$, $\lambda_k = \lambda + k/2$. The scale-invariant property has been used to write Eq. (23). Comparing these GH equations with the correspondent equations for the gamma disorder, we see that the GH ones contain in themselves the gamma ones but multiplied by the Bessel function that decays exponentially for large values of $|h|$. This leads to a more regular behavior as becomes clear from the expression

$$\overline{h^n} = \frac{(2\bar{\xi}b/a)^{n/2}}{K_\lambda}\frac{\Gamma[(n+1)/2]}{\sqrt{\pi}}K_{\lambda-n/2}(\sqrt{ab}) \quad (24)$$

for the moment of arbitrary nth order of Eq. (23). From the symmetry property of the Bessel function, i.e., $K_{-\nu}(x) = K_\nu(x)$, we conclude that Eq. (23) has all moments, resulting in milder fluctuations.

Consider the identity

$$P_f(h_1,...,h_f) = \frac{P_f(h_1,...,h_f)}{P_f(h_1,...,h_{f-1})}\cdots\frac{P_f(h_1,...,h_k)}{P_f(h_1,...,h_{k-1})} \cdots\frac{P_2(h_1,h_2)}{P_1(h_1)}P_1(h_1) \quad (25)$$

in which by definition each ratio gives the conditional probability $p(h_k|h_1,...,h_{k-1})$ of the last element h_k in the numerator argument once the preceding $k-1$ ones have been determined. Using the property of scale invariance, this probability can be written in terms of the Bessel functions as

$$p(h_k|h_1,h_2,...,h_{k-1}) = \frac{(a/b)^{1/4}}{\sqrt{2\pi\bar{\xi}}}\frac{\left(1+\frac{1}{b\bar{\xi}}\sum_1^{k-1}h_i^2\right)^{\lambda_{k-1}/2}}{\left(1+\frac{1}{b\bar{\xi}}\sum_1^{k}h_i^2\right)^{\lambda_k/2}} \times\frac{K_{\lambda_k}\left[\sqrt{ab\left(1+\frac{1}{b\bar{\xi}}\sum_1^k h_i^2\right)}\right]}{K_{\lambda_k}\left[\sqrt{ab\left(1+\frac{1}{b\bar{\xi}}\sum_1^{k-1} h_i^2\right)}\right]}. \quad (26)$$

A more compact expression is obtained doing the following:

$$1+\frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2+\frac{h_k^2}{b\bar{\xi}} = \left(1+\frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2\right) \times\left[1+\frac{h_k^2}{b\bar{\xi}\left(1+\frac{1}{b\bar{\xi}}\sum_{i=1}^{k-1}h_i^2\right)}\right] = \frac{b_{k-1}}{b}\left(1+\frac{h_k^2}{b_{k-1}\bar{\xi}}\right), \quad (27)$$

where

$$b_k = b\left(1+\frac{1}{b\bar{\xi}}\sum_{i=1}^{k}h_i^2\right), \quad (28)$$

with $b_0 = b$. With these definitions the probability distribution of one element h_k, once $k-1$ other ones have already been sorted, is

$$p(h_k|h_1,h_2,...,h_{k-1}) = \frac{(a/b_{k-1})^{1/4}}{\sqrt{2\pi\bar{\xi}}}\frac{1}{\left(1+\frac{h_k^2}{b_{k-1}\bar{\xi}}\right)^{\lambda_k/2}} \times\frac{K_{\lambda_k}\left[\sqrt{ab\left(1+\frac{h_k^2}{b_{k-1}\bar{\xi}}\right)}\right]}{K_{\lambda_{k-1}}(\sqrt{ab_{k-1}})}. \quad (29)$$

With k running from 2 to f, these equations form a set of univariate distributions which can sequentially be used to generate a matrix of the ensemble in which correlations among elements are taken into account.

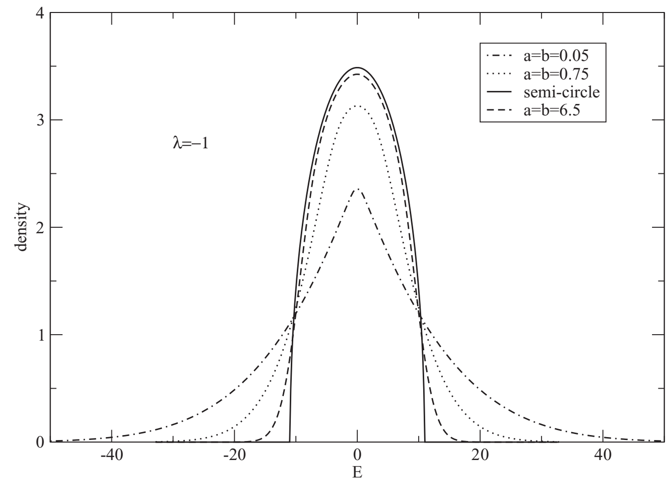

FIG. 2. The transition of the density from the semicircle to Gaussian-like for the indicated values of the parameters.

Integrating Eq. (6) over all eigenvalues but one and multiplying by N, the eigenvalue density is expressed in terms of Wigner's semicircle law density $\rho_G(E) = \sqrt{2N-E^2}/\pi$ of the Gaussian ensemble [14] as

$$\rho(E) = \frac{1}{\pi}\int_0^{\xi_{\max}} d\xi\, w(\xi)\left(\frac{\xi}{\bar{\xi}}\right)^{1/2}\sqrt{2N-\frac{\xi}{\bar{\xi}}E^2}, \quad (30)$$

where $\xi_{\max} = 2N\bar{\xi}/E^2$. From Eq. (30), we find that at the origin

$$\rho(0) = \frac{\sqrt{2N}}{\pi}\frac{\overline{\sqrt{\xi}}}{\sqrt{\bar{\xi}}} = \rho_G(0)\frac{\overline{\sqrt{\xi}}}{\sqrt{\bar{\xi}}}, \quad (31)$$

and since the ratio between the two averages is less than one, the disordered density is, at the origin, smaller than the semicircle value. When $|E| \to \infty$, $\xi_{\max} \to 0$, and therefore the interval of integration in Eq. (30) collapses and a crude approximation to the integral is

$$\rho(E) \sim \exp\left(-\frac{aE^2}{2N\bar{\xi}}\right)E^{-2\lambda-3}, \quad (32)$$

which shows that the density has Gaussian tails. In Fig. 2, it is shown that with the parameter λ fixed at the value -1, the density undergoes a transition from the semicircle to a Gaussian-like shape, which becomes more and more deformed as $a = b$ decreases.

By integrating the density [Eq. (30)] from the origin to a value E, we obtain the function

$$N(E) = \frac{N}{2}\left(1-\int_0^{\xi_{\max}} d\xi\, w(\xi)\left[1-\frac{2}{N}N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}E\right)\right]\right), \quad (33)$$

where

$$N_G(E) = \frac{N}{2}\left(\arcsin\frac{E}{\sqrt{2N}}+\frac{E}{\sqrt{2N}}\sqrt{1-\frac{E^2}{2N}}\right). \quad (34)$$

The functions $N(E)$ and $N_G(E)$ count the average number of eigenvalues in the interval $(0,E)$ for the the disordered and the Gaussian ensembles, respectively.

To measure spectral fluctuations, the dependence on the density is removed by the transformation $x = N(E)$ which maps the actual spectrum into a new one with unit density. Starting with short range correlations, we define the spacing function that gives the probability $E(0,s)$ that the interval $(-\frac{s}{2},\frac{s}{2})$ is empty. To calculate this function we integrate Eq. (6) over all eigenvalues outside the interval $(-\frac{\theta}{2},\frac{\theta}{2})$ to obtain with $s = 2N(\frac{\theta}{2})$,

$$E(0,s) = \int_0^\infty d\xi\, w(\xi) E_G\left[0, 2N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)\right], \tag{35}$$

where for the Gaussian spacing we use the Wigner surmise for the Gaussian Orthogonal Ensemble (GOE) case ($\beta = 1$)

$$E_G(0,s) = \operatorname{erfc}\left(\sqrt{\frac{\pi}{4}}s\right). \tag{36}$$

From this function the nearest-neighbor distribution (NND) is derived as $p(s) = \frac{d^2E(0,s)}{ds^2}$, which gives

$$p(s) = \begin{cases} -\frac{\rho'(\theta/2)E'(0,s)}{2\rho^3(\theta/2)} + \frac{1}{2\rho^2(\theta/2)}\int_0^{\xi_m} d\xi\, w(\xi)\left(\frac{\xi}{\bar{\xi}}\right)\rho_G'\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)E_G'\left[0,2N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)\right] \\ +\frac{1}{\rho^2(\theta/2)}\int_0^{\xi_m} d\xi\, w(\xi)\left(\frac{\xi}{\bar{\xi}}\right)\rho_G^2\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)p_G\left[0,2N_G\left(\sqrt{\frac{\xi}{\bar{\xi}}}\frac{\theta}{2}\right)\right]. \end{cases} \tag{37}$$

The above equations are exact and the spacings presented below in Fig. 3 were calculated with them. To understand these results we derive approximated spacings considering that for large N, the two counting functions $N(E)$ and $N_G(E)$ can be replaced at the center of the spectrum by $N(E) = \rho(0)E$ and $N_G(E) = \rho_G(0)E$. With these approximations, Eq. (35) becomes

$$E(0,s) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})}\int_0^\infty d\xi\, \exp\left[-\frac{1}{2}\left(\frac{a}{\xi} + b\xi\right)\right]\xi^{\lambda-1} \times \operatorname{erfc}\left[\frac{\sqrt{\xi\pi}\,\rho_W(0)}{2\sqrt{\bar{\xi}}\,\rho(0)}s\right] \tag{38}$$

and its derivative $F(0,s) = -dE(0,s)/ds$,

$$F(0,s) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})}\frac{\rho_W(0)}{\rho(0)\sqrt{\bar{\xi}}}\int_0^\infty d\xi \times \exp\left\{-\frac{1}{2}\left[\frac{a}{\xi} + \left(b + \frac{\pi\rho_W^2(0)s^2}{2\bar{\xi}\rho^2(0)}\right)\xi\right]\right\}\xi^{\lambda-(1/2)}. \tag{39}$$

We note that using Eq. (31), it can be verified that the condition $F(0,0) = 1$ is satisfied. We can go further by expressing Eq. (39) in terms of Bessel functions as

$$F(0,s) = \frac{(b/a)^{\lambda/2}}{K_\lambda(\sqrt{ab})}\frac{\rho_W(0)}{\rho(0)\sqrt{\bar{\xi}}}\frac{K_{\lambda_1}[\sqrt{ab\alpha(s)}]}{[\alpha(s)]^{\lambda_1/2}}, \tag{40}$$

where

$$\alpha(s) = 1 + \frac{\pi\rho^2(0)}{2b\rho_G^2(0)\bar{\xi}}s^2. \tag{41}$$

Finally, in the same approximation the NND has the expression

$$p(s) = \frac{K_\lambda^2(\sqrt{ab})}{K_{\lambda_1}^3(\sqrt{ab})}\frac{K_{\lambda_3}[\sqrt{ab\alpha(s)}]}{[\alpha(s)]^{\lambda_3/2}}\frac{\pi}{2}s. \tag{42}$$

Replacing now the Bessel function by its asymptotic behavior, assuming that as a function of s its argument is large, we find the NND decays as

$$p(s) \sim \exp\left(-\sqrt{ab\left[1 + \frac{\pi\rho^2(0)}{2b\rho_G^2(0)\bar{\xi}}s^2\right]}\right). \tag{43}$$

This decaying can present two limiting situations: First, if a and b increase, the second term inside the square root becomes smaller than one, in such a way that a Gaussian decay is obtained by expanding the square root. In the second situation, b decreases and makes the second term much greater than one such that it becomes the dominant term leading to an exponential decay.

Therefore, as a function of its parameters, this model constitutes a family which locally describes intermediate cases between the Wigner-Dyson and the Poisson statistics. This is illustrated in Figs. 3 and 4 where the cumulative NND, $F(s) = 1 - F(0,s)$, and NND are plotted by the indicated values of the parameters. We see that as the density goes from the semicircle to the Gaussian-like shape, concomitantly the spacing moves

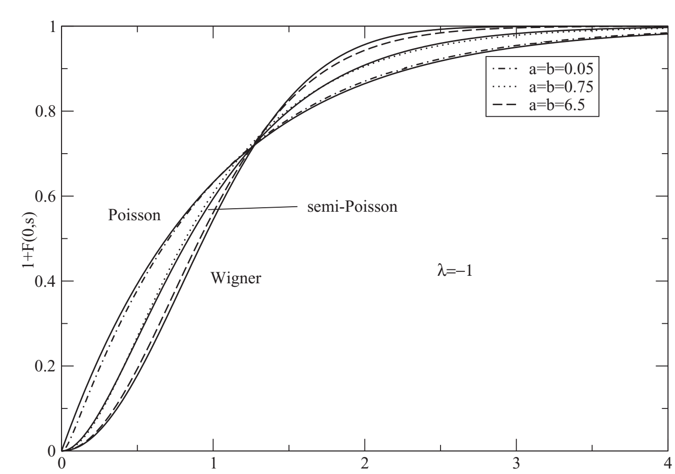

FIG. 3. The transition of the spacing $F(s)$ from Wigner-Dyson to Poisson statistics is shown for the same values of the parameters in Fig. 2.

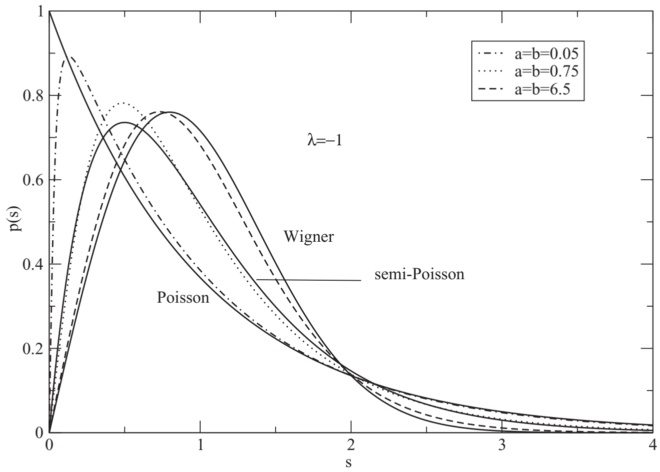

FIG. 4. The nearest-neighbor distribution (NND) calculated with the approximated equation (42) for the same values of the parameters in Figs. 2 and 3.

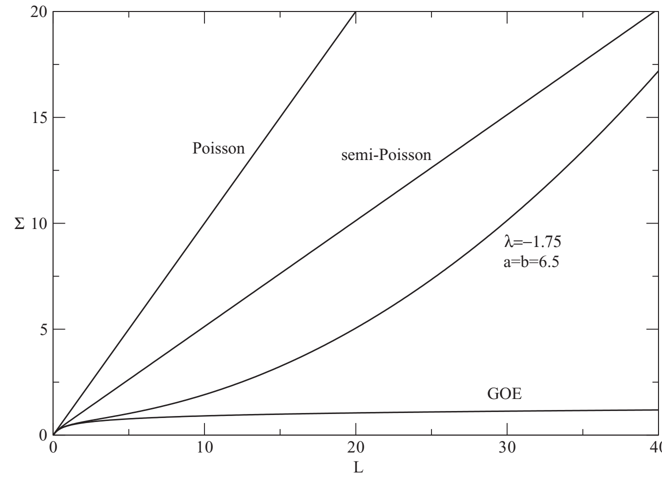

FIG. 5. The number variance with small level of disorder is shown and compared with the GOE, the Poisson, and the semi-Poisson predictions.

from the Wigner surmise to the Poisson distribution $1 - \exp(-s)$. Further, at an intermediate point of the transition, which coincides with the density approaching a Gaussian shape, the cumulative NND approaches the so-called semi-Poisson distribution given by $1-(1+2s)\exp(-2s)$ [20].

A statistics that measures spectral long range correlations and also estimates nonergodicity [1,21], is the variance $\Sigma^2(L)$ of the number of eigenvalues in the interval $[-\theta/2,\theta/2]$, with $L = 2N(\theta/2)$. In [1] it is shown that this variance is given by

$$\Sigma^2(L) = \int d\xi\, w(\xi) \left\{ \Sigma_G^2 \left[\left(2N_G \sqrt{\frac{\xi}{\bar{\xi}}} \frac{\theta}{2} \right) \right] - 2N_G \left(\sqrt{\frac{\xi}{\bar{\xi}}} \frac{\theta}{2} \right) + 4N_G^2 \left(\sqrt{\frac{\xi}{\bar{\xi}}} \frac{\theta}{2} \right) \right\} + L - L^2, \quad (44)$$

where Σ_G^2 is Gaussian number variance. In the limit of large spectra, we again use the linear approximations of the two counting functions to obtain

$$\Sigma^2(L) = \Sigma_G^2 \left[\frac{\rho_G(0)}{\rho(0)} L \right] + \left[\frac{\rho_G(0)}{\rho(0)} - 1 \right] L^2, \quad (45)$$

where using the fact that $\Sigma_G^2(x)$ is a logarithmic smooth function of its argument, the integral was asymptotically performed. Equation (45) shows that the effect of disorder in the number variance statistics is twofold: (i) it enhances it by rescaling the argument of the Gaussian expression and (ii) it introduces a super-Poissonian quadratic term that affects large intervals. These effects are illustrated in Fig. 5, in which Eq. (45) is calculated with parameter values which give a density and a spacing distribution closest to the Wigner-Dyson ones in Figs. 2 and 3.

To study the behavior of the largest eigenvalues in the limit of large matrix size N, one introduces the scaled variable

$$s(E) = \sqrt{2} N^{2/3} \left[\frac{E}{\sqrt{2N}} - 1 \right] \quad (46)$$

in terms of which the probability $E_{G,\beta}(0,s)$ that the infinite interval (s,∞) is empty is known for the three symmetry classes [8]. For the unitary class, $\beta = 2$,

$$E_{G,2}(s) = \exp\left[-\int_s^\infty (x-s) q^2(x) dx \right], \quad (47)$$

where $q(s)$ satisfies the Painlevé II equation

$$q'' = sq + 2q^3 \quad (48)$$

with boundary condition

$$q(s) \sim \mathrm{Ai}(s) \qquad \text{when } s \to \infty, \quad (49)$$

where $\mathrm{Ai}(s)$ is the Airy function. For the orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) classes the probabilities are given, respectively, by

$$[E_{G,1}(s)]^2 = E_2(0,s) \exp[\mu(s)] \quad (50)$$

and

$$[E_{G,4}(s)]^2 = E_2(0,s) \cosh^2 \frac{\mu(s)}{2}, \quad (51)$$

where

$$\mu(s) = \int_s^\infty q(x) dx. \quad (52)$$

The above equations give a complete description of the fluctuations of the eigenvalues at the edge of the spectra of the Gaussian ensembles. When perturbed, it has been shown [9] that for the three symmetry classes the above probabilities modify to

$$E_\beta(\lambda_{\max} < t) = \int d\xi\, w(\xi) E_{G,\beta}[S(\xi,t)], \quad (53)$$

with the argument of $S(\xi,t)$ obtained by plugging in the above ξ variance, namely,

$$S(\xi,t) = \sqrt{2} N^{2/3} \left[\frac{t}{\sqrt{2N}} \sqrt{\frac{\xi}{\bar{\xi}}} - 1 \right]. \quad (54)$$

Equations (53) and (54) give a complete analytical description of the behavior of the largest eigenvalue once the function $w(\xi)$ is chosen. In [9], asymptotic results, when the limit $N \to \infty$ is taken, have been derived without specifying $w(\xi)$. To do this, it was assumed that the localization of $w(\xi)$, given by the ratio $\sigma_w/\bar{\xi}$, was dependent on the matrix size N. Considering the distribution $w(\xi)$ independent of N keeping the ratio $\frac{t}{\sqrt{2N}}$ fixed when the matrix size N increases, for the three invariant ensembles, the function $E_{G,\beta}$ becomes a step function centered at $\xi = 2N\bar{\xi}/t^2$. Therefore, in this regime, the probability distribution for the largest eigenvalue converges to

$$E_\beta\,(\lambda_{\max} < t) = \int_{2N\bar{\xi}/t^2}^{\infty} d\xi\; w\,(\xi)\,, \tag{55}$$

with density

$$\frac{dE_\beta\,(t)}{dt} = \frac{4N\bar{\xi}\,w(2N\bar{\xi}/t^2)}{t^3}, \tag{56}$$

which shows a GIG distribution for the square of the largest eigenvalue.

IV. HYPERBOLIC DISORDERED WISHART MATRICES

Taking now $V = X^\dagger X$ with X being a rectangular matrix X of size $(M \times N)$, the Wishart ensemble is defined by the joint density distribution

$$P_W(X) = \left(\frac{\beta}{2\pi}\right)^{f/2} \exp\left(-\frac{\beta}{2}\mathrm{tr}(X^\dagger X)\right), \tag{57}$$

where $f = \beta M N$. From Eq. (57), the elements of X are Gaussian distributed. The eigenvalues of the random matrix $V = X^\dagger X$ have joint density distribution

$$P_W(x_1,x_2,\ldots,x_N) = K_N \exp\left(-\frac{\beta}{2}\sum_{k=1}^{N} x_k\right)\prod_{i=1}^{N} \times x_i^{(\beta/2)(1+M-N)-1}\prod_{j>i} \mid x_j - x_i \mid^\beta\,. \tag{58}$$

This ensemble was introduced by the statistician Wishart [22] and plays an important role in statistical analysis [23] where $V = X^\dagger X$ is a covariant matrix. More recently, it appeared associated in the chiral random ensemble [24] when the square of its eigenvalue is taken.

It can be shown that, for large matrices, the density of these eigenvalues approaches the Marchenko-Pastur density [25]

$$\rho_{\mathrm{MP}}(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}, \tag{59}$$

in which, with $c = \sqrt{\frac{M}{N}}$, $x_\pm = N(c \pm 1)^2$. The counting function obtained integrating this density is

$$N_W(x) = \frac{1}{4\pi}\left[-4\sqrt{x_- x_+}\,\arctan\sqrt{\frac{x_+(x - x_-)}{x_-(x_+ - x)}} + (x_+ + x_-)\arccos\left(\frac{x_+ + x_- - 2x}{x_+ - x_-}\right) + 2\sqrt{(x_+ - x)(x - x_-)}\right]. \tag{60}$$

As in the Wigner case, disorder is introduced in the ensemble by defining new matrices through the relation

$$X(\xi) = \frac{X_W}{\sqrt{\xi/\bar{\xi}}}, \tag{61}$$

which replaced in Eq. (57) leads to an ensemble with joint density distribution of matrix elements

$$P(X) = \int d\xi\; w(\xi)\left(\frac{\beta\xi}{2\pi\bar{\xi}}\right)^{f/2} \exp\left(-\frac{\beta\xi}{2\bar{\xi}}\mathrm{tr}(X^\dagger X)\right). \tag{62}$$

After substituting Eq. (13) in Eq. (62), as occurred in the Wigner case, integrals can be performed and we get

$$P(a,b,\lambda;H) = \left(\frac{\sqrt{a/b}}{2\pi\bar{\xi}}\right)^{f/2}\frac{1}{\left(1+\frac{\mathrm{tr}(X^\dagger X)}{b\bar{\xi}}\right)^{\lambda_f/2}} \times \frac{K_{\lambda_f}\left[\sqrt{ab\left(1+\frac{\mathrm{tr}(X^\dagger X)}{b\bar{\xi}}\right)}\right]}{K_\lambda(\sqrt{ab})}. \tag{63}$$

The joint distribution of eigenvalues is given by Eq. (6) with P_V replaced by P_W. Eigenvalue measures are therefore obtained by averaging those of the Wishart matrices and, for instance, the eigenvalue density is given by

$$\rho(x) = \frac{1}{2\pi x}\int_{\xi_-}^{\xi_+} d\xi\; w(\xi)\left(\frac{\xi}{\bar{\xi}}\right)^{1/2} \times \sqrt{\left(x_+ - x\sqrt{\frac{\xi}{\bar{\xi}}}\right)\left(x\sqrt{\frac{\xi}{\bar{\xi}}} - x_-\right)}, \tag{64}$$

where

$$\xi_\pm = \bar{\xi}\left(\frac{x_\pm}{x}\right)^2. \tag{65}$$

Of course, the above density extends beyond the Marchenko-Pastur limits $x_\pm$ as can be seen in Fig. 6. By taking the limit $x \to \infty$, the integration interval collapses and a crude approximation to the integral gives the exponential decay

$$\rho(x) \sim \exp\left[-\frac{ax^2}{2\bar{\xi}(x_+^2 + x_-^2)}\right]\Big/ x. \tag{66}$$

For the counting function, after integrating the density we find

$$N(x) = \int_{\xi_-}^{\xi_+} d\xi\; w(\xi) N_W\left(\sqrt{\frac{\xi}{\bar{\xi}}}x\right) + N\left[1 - \int_0^{\xi_+} d\xi\; w(\xi)\right]. \tag{67}$$

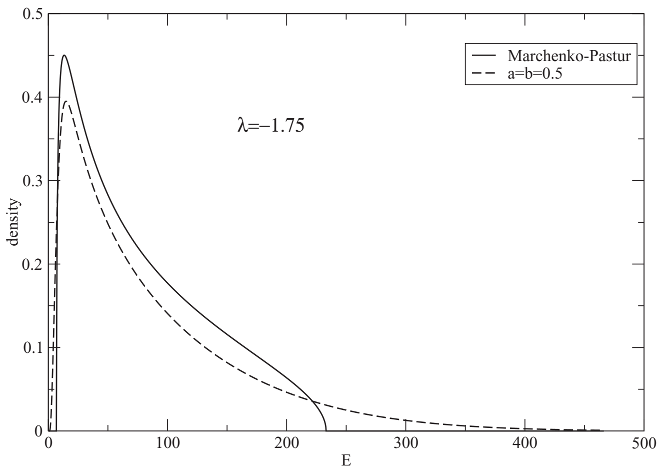

FIG. 6. The density of the disordered Wishart matrices of size $M = 80$ and $N = 40$ is compared with the Marchenko-Pastur density for the indicated values of the parameters.

The Wishart ensemble belongs to a class of random ensembles whose spectral fluctuations are derived from a general formalism based on orthogonal polynomials (the Laguerre ones) and shares the same universal statistical properties [26]. This universality in the case of the Wigner-Dyson statistics is manifested after mapping the eigenvalues into variables with density one. We consider two spacing functions which give the probability of an interval to be empty: one at the bulk and the other at the inferior extreme of the spectrum.

At the bulk, we take the interval $(\bar{x} - \theta/2, \bar{x} + \theta/2)$ where $\bar{x} = M$ is the average position of the Marchenko-Pastur density. The spacing or gap function after introducing disorder is related to the unperturbed one by

$$E(0,s) = \int_0^\infty d\xi\, w(\xi) \left(E_G \left\{ 0, N_W \left[\sqrt{\frac{\xi}{\bar{\xi}}} \left(M + \frac{\theta}{2} \right) \right] \right\} - E_G \left\{ 0, N_W \left[\sqrt{\frac{\xi}{\bar{\xi}}} \left(M - \frac{\theta}{2} \right) \right] \right\} \right), \qquad (68)$$

where the spacing s is calculated with Eq. (67) as

$$s = N\left(x = M + \frac{\theta}{2}\right) - N\left(x = M - \frac{\theta}{2}\right). \qquad (69)$$

Comparing Eqs. (68) and (35), we conclude that these spacings have the same behavior.

To study the behavior of the smallest eigenvalues in the limit of large matrix size N, one introduces the scaled variable $s = x/4N$ in terms of which the probability $E_{W,\beta}(0,s)$ that the interval $(0,s)$ is empty has been derived in Refs. [27,28] for $\beta = 2$ and [29] for $\beta = 1$. When perturbed, the above probabilities modify to

$$E(0,s) = \frac{(b/a)^{\lambda/2}}{2K_\lambda(\sqrt{ab})} \int_0^\infty d\xi\, \xi^{\lambda-1} \exp\left[-\frac{1}{2}\left(\frac{a}{\xi} + b\xi\right)\right] \times E_{W,\beta}\left(s\sqrt{\frac{\xi}{\bar{\xi}}}\right), \qquad (70)$$

which shows how the smallest eigenvalue distribution is deformed in the presence of a hyperbolic disorder. This probability becomes simple when $M = N$ with the Marchenko-Pastur density diverging at the origin. In this case the probability $E_{W,\beta}(0,s)$ assumes a simple exponential form, particularly for $\beta = 1$.

V. CONCLUSION

We have investigated the effect of superimposing an extra source of randomness governed by a generalized inverse Gaussian to the Gaussian fluctuations of the Wigner and the Wishart ensembles. The result is an ensemble of random matrices ruled by the generalized hyperbolic distribution, which contains as particular cases, disordered ensembles previously studied. Like spacing distribution, the spectral density and short range statistics show a transition from Wigner-Dyson to Poisson statistics, approaching universal critical statistics, namely, semi-Poisson statistics, at an intermediate point of the transition. However, differently from the short range statistics, long range statistics of the hyperbolic ensemble show a super-Poissonian behavior with large fluctuations. The combination of semi-Poisson at short range and super-Poisson at long range have been observed in the nonergodic embedded Gaussian models of many-body systems [13]. Therefore the present hyperbolic model is more suited to give a simple way to model features associated with the nonergodicity of physically motivated ensembles.

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