World Scientific
www.worldscientific.com

Semisimplicity and separability for pseudocompact algebras

Kostiantyn Iusenko ⓘ*,‡ and John William MacQuarrie ⓘ†,§

Instituto de Matemática e Estatística
Universidade de São Paulo, São Paulo, SP, Brazil

†*Universidade Federal de Minas Gerais, Belo Horizonte*
MG, Brazil
‡*iusenko@ime.usp.br*
§*john@mat.ufmg.br*

We give a self-contained introduction to the wonderfully well-behaved class of pseudocompact algebras, focusing on the foundational classes of semisimple and separable algebras. We give characterizations of such algebras analogous to those for finite dimensional algebras. We give a self-contained proof of the Wedderburn–Malcev Theorem for pseudocompact algebras.

Keywords: Pseudocompact algebra; inverse limit; semisimple algebra; separable algebra.

Mathematics Subject Classification 2020: 16W80, 16D60, 18A30

1. Introduction

Let k be a field. A pseudocompact k-algebra is the inverse limit of an inverse system of finite dimensional associative unital k-algebras, taken in the category of topological k-algebras. Pseudocompact algebras thus have the same role in the world of associative algebras as profinite groups do in the world of groups. They appear naturally in many contexts: for instance, if one wishes to study the representation theory of the profinite group G, one must study the modules for the completed group algebra $k[[G]]$, defined to be the inverse limit of the finite dimensional group algebras $k[G/N]$ where N runs through the open normal subgroups of G. As another example, the category of pseudocompact algebras and continuous algebra homomorphisms is precisely dual to the category of coalgebras and coalgebra homomorphisms — a category central to the study of Hopf algebras.

§Corresponding author.

K. Iusenko & J. W. MacQuarrie

Pseudocompact algebras are incredibly well-behaved, with a theory that in many ways resembles that of finite dimensional algebras. This said, pseudocompact algebras have an unfair reputation for being difficult and technical: a slanderous claim we intend to set right! Part of the problem seems to stem from the fact that the literature on pseudocompact algebras is widely scattered: you might only find the simple result you want about pseudocompact algebras as a special case of a complicated result stated for some wider and more difficult class of algebras, for example.

Here we intend to give a clear and uncomplicated survey of some foundational results in pseudocompact algebras, assuming little more than a working knowledge of basic results of finite dimensional associative algebras, and some undergraduate topology. We focus on the fundamental class of semisimple pseudocompact algebras, and the even better behaved subclass of separable pseudocompact algebras. We will note that, exactly as with finite dimensional algebras, semisimple algebras are a powerful starting place from which to study arbitrary pseudocompact algebras.

The text is organized as follows. In Sec. 2 we collect the definitions and basic properties we will need to study pseudocompact algebras. In Sec. 3 we first discuss the topological Jacobson radical of a pseudocompact algebra, giving it a robust characterization in Proposition 3.2. We then define the notion of semisimplicity for pseudocompact algebras and characterize these algebras in Proposition 3.7. In Sec. 4 we consider separable pseudocompact algebras, characterizing them in Theorem 4.3, before finishing with a self-contained proof of the Wedderburn–Malcev theorem for pseudocompact algebras (Theorems 4.6 and 4.7).

Full disclosure regarding the phrase "self-contained": while the main thread and principal results of the discussion are for the most part honest-to-goodness self-contained, we allow ourselves a little more freedom to refer to external literature in two circumstances. Firstly in examples: frequently examples can be illuminating even without a rigorous understanding of the justifications, and for this reason we include them without going into details, citing external results for proofs and further details. Secondly, our focus is on semisimple and separable algebras, rather than a survey of pseudocompact algebras in general. There are concepts (basic properties of the inverse limit functor, existence of a tensor product, etc.) whose proofs do not add to the discussion, and in these cases we again allow ourselves to cite the literature.

2. Preliminaries

Throughout, let k be a field, treated as a discrete topological ring.

Definition 2.1. A *pseudocompact k-algebra* is a topological k-algebra having a topological basis of 0 consisting of ideals of finite codimension whose intersection is 0, and complete with respect to this topology. Equivalently, a pseudocompact k-algebra is an inverse limit of discrete finite dimensional k-algebras and algebra homomorphisms.

If A is a pseudocompact algebra, say that an A-module U is *discrete* if its topology is discrete and the multiplication $A \times U \to U$ is continuous (where $A \times U$ is given the product topology). A *pseudocompact A-module* is an inverse limit of discrete finite dimensional A-modules. The category of pseudocompact A-modules is an abelian category with exact inverse limits, see for instance [14, Chap. IV, Theorem 3].

We provide several examples.

Example 2.2. Let A be the algebra of formal power series $k[[x]]$ in the variable x. Every ideal of $k[[x]]$ apart from 0 is of the form (x^n) for some $n \in \mathbb{N}$, which has codimension n in A. For each n, the natural map from $k[[x]]/(x^{n+1}) = k[x]/(x^{n+1})$ to $k[[x]]/(x^n)$ sending x to x is a surjective algebra homomorphism. We obtain by composing these maps an inverse system of finite dimensional algebras as follows:

$$\cdots \to k[[x]]/(x^3) \to k[[x]]/(x^2) \to k[[x]]/(x).$$

The inverse limit of this inverse system is $k[[x]]$, and so $k[[x]]$ is pseudocompact. More generally, the ring of power series in any number of variables (finite or infinite) is a pseudocompact algebra.

Example 2.3. A *profinite group* is a compact, Hausdorff, totally disconnected topological group. In other words, a profinite group is an inverse limit in the category of topological groups of an inverse system of discrete finite groups. Profinite groups are most frequently encountered in nature as Galois groups of Galois field extensions: If L is a Galois extension of a field K then L is the union (= direct limit) of the finite intermediate Galois extensions of L/K. If F is such an extension, then $\text{Gal}(F : K)$ is a finite group. Furthermore, an inclusion $F \to F'$ induces a surjective group homomorphism $\text{Gal}(F' : K) \to \text{Gal}(F : K)$ by restricting the domain and codomain of each $\rho : F' \to F'$ to F. Thus, applying $\text{Gal}(- : K)$, the direct system of field extensions becomes an inverse system of finite groups and we define the Galois group $\text{Gal}(L : K)$ of L/K to be the inverse limit of this inverse system. The topology of a profinite group (and by extension, of the algebra we will shortly define) is justified by the generalization of the fundamental theorem of Galois theory to this context: there is a perfect order reversing correspondence between the intermediate extensions of L/K and the *closed* subgroups of the profinite group $\text{Gal}(L : K)$ (see for instance [27, Theorem 2.11.3]). If G is a profinite group and k is a field, then for each continuous finite quotient G/N we may consider the finite dimensional group algebra $k[G/N]$. The inverse system of finite groups induces an inverse system of finite dimensional algebras in the obvious way, and we define the *completed group algebra* $k[[G]]$ to be the inverse limit of this inverse system. Thus $k[[G]]$ is a pseudocompact algebra by definition and it is the natural place to study the representation theory of the profinite group G [4, 20–22, 24, 30, 31].

Example 2.4. A *quiver* Q is a directed graph, with multiple arrows and cycles permitted. From Q we can construct a pseudocompact algebra $k[[Q]]$, the *completed*

path algebra of Q, in analogy with the well-known construction in finite dimensional algebras. The details are not difficult but require some further definitions that would lead us away from our main interest: for the construction in the finite case see [3, Chap. III.1] and in the pseudocompact case see [28, Sec. 8]. As a simple example, the completed path algebra $k[[Q]]$ of the quiver having one vertex and one loop x, is the algebra of formal power series $k[[x]]$ discussed above. As in the finite dimensional case, completed path algebras are fundamental examples for pseudocompact algebras: if k is algebraically closed and A is a pseudocompact k-algebra, then the category of pseudocompact A-modules is equivalent to the category of pseudocompact modules for the algebra $k[[Q]]/I$, where Q is a quiver and I is a suitable closed ideal of $k[[Q]]$ (this follows by duality from [5, Theorem 4.3]).

Example 2.5. For us, an important class of examples of infinite dimensional pseudocompact algebras and modules comes by taking direct products of finite dimensional objects of the same type. Namely, if $\{X_\lambda, \lambda \in \Lambda\}$ is a collection of finite dimensional topological k-vector spaces indexed by a set Λ (each possibly with the structure of an algebra or a module), the corresponding direct product is the topological k-vector space $X := \prod_{\lambda \in \Lambda} X_\lambda$, endowed with the product topology. Additional structure is applied coordinate-wise, making the direct product into a pseudocompact object of the same type. One may check that X can be expressed as the inverse limit of an inverse system of finite direct products as follows. Given any subset F of Λ, denote by X_F the direct product $\prod_{\lambda \in F} X_\lambda$ (so that in particular $X = X_\Lambda$). Let $\mathcal{F}$ be the set of all finite subsets F of Λ, and consider the inverse system $\{X_F, \varphi_{FG}, \mathcal{F}\}$ where $\varphi_{FG} : X_G \to X_F$, defined whenever $F \subseteq G$, is the obvious projection. Then

$$X = \varprojlim_{F \in \mathcal{F}} X_F.$$

While not every pseudocompact object is a product of finite dimensional objects, the product notation can still be utilized for a general pseudocompact object, for the following reason: if X is the inverse limit of the inverse system $\varprojlim_{\lambda \in \Lambda} \{X_\lambda, \varphi_{\lambda\lambda'}\}$ of finite dimensional objects, then

$$X \cong \left\{ (x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_\lambda \mid \varphi_{\lambda\lambda'}(x_{\lambda'}) = x_\lambda, \ \lambda \leq \lambda' \right\},$$

a closed subobject of X_Λ. We do not use this (standard) fact explicitly in this discussion, so we skip the details.

Example 2.6. A *simple* pseudocompact module for the pseudocompact algebra A is a non-zero pseudocompact module not having any closed submodules other than 0 and itself. Any infinite dimensional pseudocompact A-module has a lot of submodules by definition, so that in particular simple pseudocompact A-modules are finite dimensional. Finite dimensional pseudocompact A-modules are necessarily discrete, and hence simple pseudocompact A-modules are simple as abstract A-modules.

It is *not* the case that an arbitrary finite dimensional A-module, given the discrete topology, is necessarily pseudocompact! This is because not every left ideal of finite codimension is open in A. To exhibit an explicit example is tricky, but one may work indirectly as follows: Let $\mathbb{F}_p$ be the field of p elements, and consider $A = \prod_{n \in \mathbb{N}} \mathbb{F}_p$, a countable product of copies of the 1-dimensional algebra $\mathbb{F}_p$. The ideal $I = \bigoplus_{n \in \mathbb{N}} \mathbb{F}_p$ is proper in A, so define M to be a maximal ideal of A containing I. Note that I is dense in the topology of A, and hence so is M. In particular M, being proper and dense, cannot be closed in A. To see that M has codimension 1, consider an element $a = (a_n)_{n \in \mathbb{N}}$ of A. Then $a^p = (a_n{}^p)_{n \in \mathbb{N}} = (a_n)_{n \in \mathbb{N}} = a$. Thus A/M is a field in which every element is a root of $x^p - x$ and hence $A/M \cong \mathbb{F}_p$. We mention in passing that replacing $\mathbb{F}_p$ with an arbitrary field k, the structure of A/M can be much more complicated!

Note that pseudocompact algebras and modules are in particular pseudocompact k-vector spaces. Pseudocompact objects need not of course be compact, since even a finite dimensional k-vector space is not compact when k is infinite. But a very useful weaker version of compactness holds. Recall that an *affine subspace* of a vector space U is a coset of a subspace of U.

Lemma 2.7. *Let U be a pseudocompact vector space. Then U is* linearly compact, *meaning that if $\mathcal{X}$ is a collection of closed affine subspaces of U and if $\bigcap_{X \in \mathcal{X}} X = \varnothing$, then there is a finite subcollection $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots \cap X_n = \varnothing$.*

Proof. It is more-or-less obvious that a finite dimensional k-vector space is linearly compact, so this result is [34, Proposition 4]. $\qquad\square$

Linearly compact algebras and modules are well-studied and there is a huge literature on the subject, see for instance [19, 32, 34] for a general introduction. Certain results we will discuss for pseudocompact objects hold for this larger and more difficult class of objects, but our focus is clarity of exposition rather than generality, and so we will not attempt to present results in the maximum possible generality.

We give some simple examples of useful properties of pseudocompact algebras and modules that follow essentially from Lemma 2.7. Firstly, a pseudocompact vector space V has the discrete topology if, and only if, it has finite dimension: if V is finite dimensional then there is an open subspace W of smallest possible dimension, and this W must contain every open subspace of V. Now since V is Hausdorff and the open subspaces of V form a basis of neighbourhoods of 0, it follows that W must be $\{0\}$ and hence V is discrete. If V is discrete and has infinite dimension, then choose a linearly independent set $\{v_1, v_2, v_3, \ldots\}$ of vectors in V. For each $n \in \mathbb{N}$, define the affine subspace

$$W_n = (v_1 + \cdots + v_n) + \langle v_{n+1}, v_{n+2}, \ldots \rangle.$$

The intersection of a finite number of W_n is non-empty, but the intersection of all the W_n is empty, showing that V is not linearly compact and hence not pseudocompact.

It follows from this observation that a closed subspace W of a pseudocompact vector space V is open if, and only if, it has cofinite dimension: one simply looks at the inverse image of 0 under the continuous projection $V \to V/W$.

Lemma 2.8 (Adaptation of [33, Proposition 0.3.3]). *Let A be a pseudocompact vector space and let $\mathcal{I}$ be a collection of open subspaces of A such that whenever $X, Y \in \mathcal{I}$, there is $Z \in \mathcal{I}$ contained in $X \cap Y$, and such that $\bigcap_{X \in \mathcal{I}} X = 0$. If C is a closed subspace of A, then*

$$C = \bigcap_{X \in \mathcal{I}} (C + X).$$

Proof. If $a \notin C$ then

$$(a + C) \cap \bigcap_{X \in \mathcal{I}} X = \varnothing.$$

By Lemma 2.7 there are $X_1, \ldots, X_n \in \mathcal{I}$ so that

$$(a + C) \cap (X_1 \cap \cdots \cap X_n) = \varnothing.$$

There is $Y \in \mathcal{I}$ such that $Y \subseteq X_1 \cap \cdots \cap X_n$. So $(a + C) \cap Y = \varnothing$, hence $a \notin C + Y$. We thus obtain the inclusion

$$\bigcap_{X \in \mathcal{I}} (C + X) \subseteq C + \bigcap_{X \in \mathcal{I}} X = C,$$

the other being trivial. $\qquad\square$

Corollary 2.9. *Every proper closed (left) ideal of A is contained in a maximal closed (left) ideal of A.*

Proof. Let $\mathcal{I}$ denote the set of open ideals of A. If C is a proper closed (left) ideal of A then, since $C = \bigcap_{X \in \mathcal{I}} (C + X)$ by Lemma 2.8, some $C + X$ is proper in A. Hence C is contained in a proper open left ideal $C + X$ (which is an ideal if C is) and the result follows because $A/(C + X)$ is finite dimensional. $\qquad\square$

We give a (probably well-known) technical lemma. Recall that two proper left ideals of a ring A are said to be coprime when their sum is A.

Lemma 2.10. *Let X, Y, Z be open left ideals of A with Z maximal. If neither A/X nor A/Y have composition factors isomorphic to A/Z, then $X \cap Y$ and Z are coprime.*

Proof. As Z is maximal, we need only check that $X \cap Y \nsubseteq Z$. If it were, then we would have a projection $A/(X \cap Y) \to A/Z$ and hence A/Z would appear as a composition factor of $A/(X \cap Y)$. But the composition factors of $A/(X \cap Y)$ are the factors of A/X and the factors of $X/(X \cap Y) \cong (X + Y)/Y$, so A/Z does not appear. $\qquad\square$

Lemma 2.11. *Let A be a pseudocompact algebra and $\mathcal{X}$ the set containing one representative of each isomorphism class of simple pseudocompact left A-module and no other elements. The module $M = \prod_{X \in \mathcal{X}} X$ is cyclic.*

Proof. Take as our representative of each isomorphism class the module A/I with I some maximal closed left ideal of A. We claim there is a surjective module map $A \to M$. As $\varprojlim$ is exact, it is sufficient to check that the natural map

$$A \to \prod_{i=1}^{n} A/I_i$$

is surjective for each finite set of $A/I_1, \ldots, A/I_n \in \mathcal{X}$, and for this we follow the proof of the Chinese remainder theorem. If $n = 1$, surjectivity is immediate, so suppose the result holds given fewer than n left ideals. The left ideals $I_1 \cap \cdots \cap I_{n-1}$ and I_n are coprime by the previous lemma and so we can write $1 = x + y$ with $x \in I_1 \cap \cdots \cap I_{n-1}$ and $y \in I_n$. Given $(b + (I_1 \cap \cdots \cap I_{n-1}), c + I_n)$, the element $a = by + cx$ maps onto it, so the map $A \to A/(I_1 \cap \cdots \cap I_{n-1}) \oplus A/I_n$ is onto. But by induction, the map

$$A/(I_1 \cap \cdots \cap I_{n-1}) \to A/I_1 \oplus \cdots \oplus A/I_{n-1}$$

is onto, and hence the map $A \to A/I_1 \oplus \cdots \oplus A/I_n$ is onto, as required. $\qquad\square$

Lemma 2.12. *Let A be a pseudocompact algebra, Ω an indexing set and $V = \prod_{\omega \in \Omega} S_\omega$ a product of simple pseudocompact A-modules. If U is a closed submodule of V then U is a continuous direct summand of V.*

Proof. This proof is dual to the abstract case (see for instance [29, Proposition 1.7.1]). Denote by $\mathcal{X}$ the set of subsets Λ of Ω such that the composition

$$U \xrightarrow{\iota} V \xrightarrow{\pi_\Lambda} \prod_{\lambda \in \Lambda} S_i$$

is surjective, ordered by inclusion. Then $\mathcal{X} \neq \varnothing$ because $\varnothing \in \mathcal{X}$. Let $\mathcal{C}$ be a chain in $\mathcal{X}$. Using the notation from Example 2.5, we have natural projection maps $\varphi_\Lambda = \varphi_{\Lambda\Omega}$ for each $\Lambda \in \mathcal{C}$ and these maps induce a so-called map of inverse systems (cf. [27, §1.1]) $\{\varphi_\Lambda \iota : U \to \prod_{\lambda \in \Lambda} S_\lambda\}$. Each of these maps is onto and hence the limit map

$$\varprojlim \pi_\Lambda \iota : U \to \varprojlim{}_{\Lambda \in \mathcal{C}} \prod_{\lambda \in \Lambda} S_\lambda = \prod_{\lambda \in \cup\Lambda} S_\lambda$$

is onto by the exactness of $\varprojlim$. It follows that $\cup\Lambda$ is an upper bound for $\mathcal{C}$ in $\mathcal{X}$. So by Zorn's lemma, $\mathcal{X}$ contains a maximal element Λ_0.

We claim that the composition $\pi_{\Lambda_0}\iota$ is injective (and hence an isomorphism). If not, then $U \cap \prod_{\lambda \in \Omega \setminus \Lambda_0} S_\lambda \neq 0$ and hence there is an element $u = (u_\lambda)$ of U such that $u_\lambda = 0$ for every $\lambda \in \Lambda_0$ but $u_\omega \neq 0$ for some $\omega \notin \Lambda_0$. Abusively denoting by S_ω and $\prod_{\lambda \in \Lambda_0} S_\lambda$ the images of S_ω and $\prod_{\lambda \in \Lambda_0} S_\lambda$ under $\pi_{\Lambda_0 \cup \{\omega\}}$, we

note that $S_\omega \cap \pi_{\Lambda_0 \cup \{\omega\}}(U) \neq 0$ and hence, S_ω being simple, $S_\omega \subseteq \pi_{\Lambda_0 \cup \{\omega\}}(U)$. Given $z \in \prod_{\lambda \in \Lambda_0} S_\lambda$, we know since $\pi_{\Lambda_0} \iota$ is onto that there is an element $a \in U$ such that $\pi_{\Lambda_0 \cup \{\omega\}}(a) = z + z'$ with $z' \in S_\omega$. But $z' = \pi_{\Lambda_0 \cup \{\omega\}}(b)$ for some b and hence $z = \pi_{\Lambda_0 \cup \{\omega\}}(a - b)$. Thus S_ω and $\prod_{\lambda \in \Lambda_0} S_\lambda$ are contained in $\mathrm{Im}(\pi_{\Lambda_0 \cup \{\omega\}} \iota)$, showing that $\pi_{\Lambda_0 \cup \{\omega\}} \iota$ is surjective and contradicting the maximality of Λ_0.

It follows that $\pi_{\Lambda_0} \iota$ is an isomorphism, so that ι splits via $(\pi_{\Lambda_0} \iota)^{-1} \pi_{\Lambda_0}$. $\qquad\square$

The Krull–Schmidt theorem [18, Corollary 19.22] tells us that a finitely generated module for a finite dimensional algebra can be decomposed as a direct sum of finitely many indecomposable modules in an essentially unique way. We have just seen in Lemma 2.11 that a finitely generated module for a pseudocompact algebra may well be an infinite product of indecomposable modules. The following very useful analogue of the Krull–Schmidt theorem tells us that this is as bad as it can get, and that decompositions of a finitely generated module as a product of indecomposables are essentially unique, in a precise way.

Proposition 2.13 ([23, Proposition 3.5 and before]). *Let A be a pseudocompact algebra and M a finitely generated pseudocompact A-module.*

(1) *M is isomorphic to a direct product of indecomposable A-modules.*
(2) *(the exchange property) If Λ is a set and if X is a continuous direct summand of the module $Y = \prod_{\lambda \in \Lambda} Y_\lambda$, where each Y_λ is an indecomposable direct summand of M, then there exists a subset Ω of Λ with the property that*

$$Y = X \oplus \prod_{\lambda \in \Omega} Y_\lambda.$$

Given closed ideals I, I' of the pseudocompact algebra A, denote by $I \cdot I'$ the topological closure of the abstract product of I and I'. Given a closed ideal I and $n \geq 2$, define recursively the closed ideal I^n to be $I^{n-1} \cdot I$.

Example 2.14. It is intuitively clear that the abstract product of two ideals need not be closed, but we did not encounter explicit examples in the literature so we present one here. Let $A = k[[x_1, x_2, \ldots]]$ be the pseudocompact algebra of power series in infinitely many variables x_i. Let I be the closed ideal of A generated by the x_i. The element $x = \sum_{i \in \mathbb{N}} x_i{}^2$ is in I^2, but it cannot be written as a finite sum of monomials, and hence is not an element of the abstract product of I with itself.

Note that $I = J(A)$ (cf. §3.1), so this is an example of a pseudocompact algebra A for which $J^2(A)$ is not the abstract product of $J(A)$ with itself.

3. The Jacobson Radical and Semisimple Pseudocompact Algebras

The main result of this section is a long and familiar (to those who work with finite dimensional algebras) characterization of semisimple pseudocompact algebras. We extend the fundamental Artin–Wedderburn theorem, showing that a semisimple

pseudocompact algebra is a product of matrix algebras over finite dimensional division algebras. As with finite dimensional algebras, the Jacobson radical is a useful tool to measure the failure of semisimplicity of a pseudocompact algebra. So we start by defining a topological version of the Jacobson radical and compare the definition with the classical one. For finite dimensional algebras there are a huge number of equivalent descriptions of the Jacobson radical, which may not be equivalent for an arbitrary algebra. We observe that (following a common pattern) the situation for pseudocompact algebras is as equally well behaved as the situation for finite dimensional algebras.

3.1. *The Jacobson radical*

Let A be a pseudocompact algebra. The *topological Jacobson radical* $J(A)$ of A is the intersection of the maximal closed left ideals of A (which by Lemma 2.8 is the intersection of the maximal open left ideals of A).

Example 3.1. (1) Suppose that $A = k[[x]]$. Then A has a unique maximal closed left ideal, generated by x, hence $J(A) = (x)$ has codimension 1 in A. Compare this with the polynomial algebra $k[x]$, whose Jacobson radical is 0.

(2) Consider the pseudocompact algebra

$$A = \prod_{\mathbb{N}} k = k_0 \times k_1 \times \cdots,$$

the product of a countable number of copies of the algebra k. For each $n \in \mathbb{N}$, the closed ideal

$$I_n = k_0 \times \cdots \times k_{n-1} \times 0 \times k_{n+1} \times \cdots$$

has codimension 1 and so is maximal. As $\bigcap_{n \in \mathbb{N}} I_n = 0$, it follows that $J(A) = 0$.

(3) If G is a profinite group, and char k does not divide the order of any continuous finite quotient group of G, then $J(k[[G]]) = 0$ — this follows using Lemma 3.3 below and Maschke's theorem for finite groups (see for instance [8, Theorem 15.6]).

(4) Generalizing the first example, let Q be any quiver. Then $J(k[[Q]])$ is the closed ideal generated by the arrows of Q. This fact follows by [17, Lemma 2.12], whose proof, given the results of the next section, does not require the corresponding semisimple algebra to be finite dimensional. Such a clean description of the radical is in stark contrast with the case of the abstract path algebra kQ, even if Q is finite with loops or cycles. The radical of the abstract path algebra kQ is awkward to describe: it is the ideal with k-basis the so-called "regular paths" in Q (we could not find a reference for precisely this fact but it follows from the more general [6, Proposition 1.3]). We suggest that this difference provides strong evidence that the completed path algebra is a better tool than the abstract path algebra to study finite dimensional algebras as quotients of path algebras.

The following proposition gives a characterization of $J(A)$. We say that a closed ideal I of A is *pronilpotent* if $\bigcap_{n \in \mathbb{N}} I^n = 0$. An element a of a (topological) algebra A is a (*topological*) *non-generator* if whenever X is a subset of A and the (closed) left ideal generated by $X \cup \{a\}$ is A, then the (closed) left ideal generated by X is already A. In the following proof and elsewhere, the notation $I \lhd_O A$ indicates that I is an open ideal of A.

Proposition 3.2. *Let A be a pseudocompact algebra. The following subsets of A are equal*:

(1) *The intersection of the maximal closed left ideals of A*;
(2) *The intersection of the maximal closed right ideals of A*;
(3) *The intersection of the maximal closed two sided ideals of A*;
(4) *The intersection of the (not necessarily closed) maximal left ideals of A*;
(5) $\{x \in A \,|\, 1 - yx$ *is left invertible for any* $y \in A\}$;
(6) $\{x \in A \,|\, 1 - yxz$ *is invertible for any* $y, z \in A\}$;
(7) *The set of all non-generators of A*;
(8) *The set of all topological non-generators of A*;
(9) *The intersection of the annihilators of the simple pseudocompact left A-modules*;
(10) *The smallest closed submodule V of A such that A/V is a product of simple A-modules*;
(11) *The maximal closed pronilpotent ideal.*

Proof. [(4) = (5) = (6) = (7)] is classical (see for instance [18, Lemmas 4.1 and 4.3] for [(4) = (5) = (6)] and [11, p. 65] for [(4) = (7)]).

[(1) = (9)] is also completely standard, given that A/I is a simple pseudocompact A-module whenever I is a maximal closed ideal of A and every pseudocompact simple module is isomorphic to one of this form.

The proof that [(1) = (8)] is just like the proof that [(4) = (7)]: if a is a topological non-generator and M is a maximal closed left ideal, then $a \in M$ because otherwise $\{a\} \cup M$ generates A while M does not. For the reverse inclusion, if the closed left ideal generated by the set X is not A then it is contained in a maximal closed left ideal M by Corollary 2.9. If a is in (1) then $a \in M$ and hence $\{a\} \cup X$ also does not generate A.

[(1) = (3)] follows by the equality of these sets for finite dimensional algebras. Denote by $J(B)$ the intersection of the maximal closed left ideals of B and by $r(B)$ the intersection of the maximal closed two sided ideals of B. Given an open two sided ideal of A, $J(A/I) = r(A/I)$ by the result for finite dimensional algebras [9, Corollary 3.1.8]. By the correspondence theorem, it follows that the intersection of the maximal closed left ideals containing I is equal to the intersection of the maximal closed two sided ideals containing I. Since $\bigcap_{I \lhd_O A} I = 0$, a standard compactness argument shows that every maximal open left ideal L contains an open two sided ideal: We have L^C is closed and $L^C \cap \bigcap_{I \lhd_O A} I = \varnothing$. Hence by

Lemma 2.7 there is a finite set of open ideals $I_1, \ldots, I_n$ such that

$$L^C \cap I_1 \cap \cdots \cap I_n = \varnothing$$

and the open ideal $I_1 \cap \cdots \cap I_n$ is contained in L. Thus

$$J(A) = \bigcap_{I \triangleleft_O A} \bigcap_{\max \text{ left } M \supseteq I} M = \bigcap_{I \triangleleft_O A} \bigcap_{\max \text{ two-sided } M \supseteq I} M = r(A).$$

$[(1) \subseteq (5)]$ Let $x \in (1)$ and $y \in A$ be such that $1 - yx$ is not left invertible. Thus $A(1 - yx) \subsetneq A$. But $A(1 - yx)$ is closed, being finitely generated as a left A-module [4, Lemma 2.1] and so it is contained in some maximal closed left ideal M of A by Corollary 2.9. Hence $1 \in yx + M$, but $yx + M = M$, a contradiction.

The inclusion $[(2) \subseteq (5)]$ is similar because (6) is left–right symmetric, so we can swap between left and right as we choose.

The inclusion $[(4) \subseteq (1)]$ is immediate: a maximal closed left ideal is maximal as an abstract left ideal because an ideal containing an open ideal is open. By left–right symmetry, we obtain $[(4) \subseteq (2)]$ in the same way.

$[(9) \subseteq (10)]$ If $x \in (9)$ and V is a closed submodule with A/V a product of simple modules, then x is in the kernel of the natural projection $A \to A/V$, so $x \in V$.

$[(10) \subseteq (1)]$ Let J denote the intersection of the maximal closed left ideals of A and denote by $\mathcal{M}$ the set of all maximal closed left ideals. The natural projections $A/J \to A/M$ ($M \in \mathcal{M}$) yield a continuous map

$$\gamma : A/J \to \prod_{M \in \mathcal{M}} A/M,$$

whose kernel is 0. By Lemma 2.12, this inclusion splits. The product X of one copy of each isomorphism class of simple A-module is finitely generated by Lemma 2.11, so by Lemma 2.13,

$$\prod_{M \in \mathcal{M}} A/M = \gamma(A/J) \oplus \prod_{M \in \mathcal{N}} A/M$$

for some subset $\mathcal{N}$ of $\mathcal{M}$. Thus A/J is isomorphic to a product of simple modules, and hence $V \subseteq J$.

$[(1) \subseteq (11)]$ Let $J(A)$ denote the intersection of the maximal closed left ideals of A and let I be an open left ideal. Then $J(A) + I \subseteq J(A/I)$ and hence $J(A)^n \subseteq I$ for some n by the nilpotence of the radical for finite dimensional algebras (see for instance [9, Proposition 3.1.9]). Thus $\bigcap_{n \in \mathbb{N}} J(A)^n \subseteq \bigcap_{I \triangleleft_O A} I = 0$.

$[(11) \subseteq (1)]$ Let M be a closed maximal left ideal and P a pronilpotent ideal in A. The module A/M is simple, hence $P(A/M)$ equals 0 because P is pronilpotent. Therefore $P = PA \subseteq M$, so $P \subseteq (1)$. $\qquad \square$

Note that the equality of (1) and (4) in the above proposition says that the topological Jacobson radical of the pseudocompact algebra is equal to the Jacobson radical of A treated as an abstract algebra. As there is no ambiguity, we will from

now on refer to $J(A)$ simply as the Jacobson radical of A. Item (3) is the definition of the Jacobson radical given in [4, p. 444].

The Jacobson radical $J(A)$ is well-behaved with respect to radicals of finite dimensional quotients of A, in the sense that $J(A)$ is the inverse limit of the $J(A/I)$ as I runs through the open ideals of A. The latter implies that surjective homomorphisms between pseudocompact algebras map radicals onto radicals. These two properties were proved, for instance, in [17]. For the sake of completeness we state them here with complete proofs.

Lemma 3.3. *Write $A = \varprojlim_I \{A/I, \alpha_{II'}\}$ as an inverse limit of finite dimensional quotient algebras A/I. Then*

$$J(A) = \varprojlim_I J(A/I).$$

Proof. Since the quotients $\alpha_{II'} : A/I' \to A/I$ are surjective, it is immediate that $\alpha_{II'}(J(A/I')) \subseteq J(A/I)$. Thus, the restriction of the inverse system of A/I to their Jacobson radicals indeed yields an inverse system with inverse limit $\varprojlim J(A/I) \subseteq A$. Since an element $x \in J(A)$ maps into each $J(A/I)$, it follows that $J(A) \subseteq \varprojlim J(A/I)$. On the other hand, given $x \notin J(A)$, there is some open maximal left ideal M not containing x. Working within the cofinal subsystem of A/I with $I \subseteq M$, we see that $x + I \notin J(A/I)$, and hence $x \notin \varprojlim J(A/I)$, so that $\varprojlim J(A/I) \subseteq J(A)$. $\square$

Corollary 3.4. *Let A, B be pseudocompact algebras and let $\alpha : A \to B$ be a continuous surjective algebra homomorphism. Then $\alpha(J(A)) = J(B)$.*

Proof. That $\alpha(J(A)) \subseteq J(B)$ is immediate. To prove the other inclusion, we begin by supposing that A, B are finite dimensional. Recall [3, Proposition 3.5] that the radical $\mathrm{Rad}_B(U)$ of a finitely generated B-module U is given by $J(B)U$ (and similarly for A). We can treat B either as a B-module, or as an A-module via α, and since α is surjective it follows that $\mathrm{Rad}_B B = \mathrm{Rad}_A B$. Thus

$$J(B) = J(B)B = \mathrm{Rad}_B B = \mathrm{Rad}_A B = J(A) \cdot B = \alpha(J(A))B \subseteq \alpha(J(A)).$$

Now let A be general and B finite dimensional. Since B is discrete and α is continuous, we can consider a cofinal subset of open ideals of A contained in the kernel of α. By factorizing α through these quotients we obtain a map of inverse systems $\{\alpha_I : A/I \to B \mid I \lhd_O A, I \leq \mathrm{Ker}(\alpha)\}$. Restricting and corestricting this inverse system to the Jacobson radicals, we obtain a surjective map of inverse systems $\alpha_I : J(A/I) \twoheadrightarrow J(B)$, whose inverse limit is $\alpha : J(A) \to J(B)$ by Lemma 3.3. It is onto by the exactness of $\varprojlim$.

Finally, allowing both A and $B = \varprojlim \{B/K, \beta_{KK'}\}$ to be general, the obvious composition $\beta_K \alpha : A \to B \to B/K$ is a surjective map onto the finite dimensional algebra B/K and hence it restricts to a surjection $J(A) \twoheadrightarrow J(B/K)$ for each K. We obtain in this way a surjective map of inverse systems and the result follows from the exactness of $\varprojlim$. $\square$

3.2. *Topologically semisimple algebras*

Recall that a (classically) semisimple algebra is an associative algebra for which every left ideal has a complement. By the Wedderburn–Artin theorem [18, Theorem 3.5], a semisimple k-algebra is a direct product of a finite number of matrix algebras over k-division algebras.

Definition 3.5. Let A be a topological algebra. We say that A is ($left$) *topologically semisimple* if for any closed left ideal I in A there exists a closed left ideal L such that $I \oplus L = A$.

Example 3.6. Let Λ be an infinite set and consider the algebra $A = \prod_{\lambda \in \Lambda} k$. Then A is not (classically) semisimple: the (non-closed) ideal $I = \bigoplus_{\lambda \in \Lambda} k$ has no complement, because every non-zero ideal of A intersects I. But it follows from Proposition 3.7 that A *is* topologically semisimple.

The following is a Wedderburn–Artin theorem for pseudocompact algebras. Several of the equivalences here are presented for a wider class of algebras in [16, Theorem 3.10].

Proposition 3.7. *Let A be a pseudocompact algebra. The following conditions are equivalent*:

(1) *A is ($left$) topologically semisimple*;
(2) *$J(A) = 0$*;
(3) *The free left A-module A is isomorphic to a product of simple modules*;
(4) *Every left pseudocompact A-module is projective*;
(5) *Every left pseudocompact A-module is injective*;
(6) *A is an inverse limit of finite-dimensional semisimple algebras*.
(7) *A is isomorphic to a product of full matrix algebras $\prod_{\lambda \in \Lambda} M_{n_\lambda}(D_\lambda)$, where Λ is a set, the n_λ are positive integers and the D_λ are finite dimensional k-division algebras. This product is unique up to permutation of the terms and isomorphism of the division algebras D_λ.*

Proof. $[(1) \Rightarrow (2)]$ As A is topologically semisimple and $J(A)$ is closed, $A = J(A) \oplus I$, for some closed left ideal I. If $J(A) \neq 0$ then $I \neq A$ so I is contained in a closed maximal left ideal M. But M does not contain $J(A)$, since otherwise $A = J(A) + I \subseteq M$, a contradiction.

$[(2) \Rightarrow (3)]$ This follows by Proposition 3.2 because $A = A/J(A)$ is a product of simple A-modules.

$[(3) \Rightarrow (4)]$ Any left pseudocompact A-module U is an inverse limit of discrete finite dimensional A modules. An inverse limit of projective A-modules is a projective A-module by [4, Corollary 3.3], so it is enough to show that any discrete finite dimensional A-module is projective. Given such a U, let $\gamma : A^n \twoheadrightarrow U$ be a continuous surjection. The kernel of γ is open in $A^n = \prod_{\lambda \in \Lambda} S_\lambda$ (S_λ simple), and so γ factors through some product T of finitely many S_λ. The map $A^n \to T$ splits

and the map $T \to U$ splits by the finite dimensional version of this result. Hence γ splits, showing that U is projective.

[(4) $\Rightarrow$ (5)] Given a pseudocompact module U and an injective map of modules $\gamma : U \to V$, the short exact sequence $U \to V \to V/\gamma(U)$ splits since the third module is projective. So U is injective.

[(5) $\Rightarrow$ (1)] Suppose that I is a closed left ideal in A. The short exact sequence $I \to A \to A/I$ splits because I is injective. Hence $A \cong I \oplus A/I$ and A is (left) topologically semisimple.

[(2) $\Rightarrow$ (6)] Suppose that $J(A) = 0$. Write $A = \varprojlim \{A_\lambda, \varphi_{\lambda\lambda'}\}$ with each A_λ a finite-dimensional algebra and each $\varphi_{\lambda\lambda'}$ a surjective algebra map. By Corollary 3.4 we have $J(A_\lambda) = \varphi_\lambda(J(A)) = 0$, hence each A_λ is semisimple.

[(6) $\Rightarrow$ (7)] Denote by C a complete set of centrally primitive central idempotents of A, which exists by applying [14, IV. §3, Corollaries 1 and 2] to the center of A. By the same results we have $A = \prod_{e \in C} Ae$ as an algebra. We must check that each Ae is a matrix algebra. Write $A = \varprojlim \{A_\lambda, \varphi_{\lambda\lambda'}\}$ as an inverse limit of finite dimensional semisimple quotient algebras of A. Denoting by C_λ the corresponding complete set for A_λ, we have that the C_λ form a direct system via the maps $\gamma_{\lambda\lambda'} : C_\lambda \to C'_\lambda$ sending $f \in C_\lambda$ to the unique element c of C'_λ such that $\varphi_{\lambda\lambda'}(c)f \neq 0$. Furthermore, $C = \varinjlim \{C_\lambda, \gamma_{\lambda\lambda'}\}$ by [22, Proposition 6.4], whose proof goes through for arbitrary fields. The maps $\gamma_{\lambda\lambda'}$ are injective because, as the A_λ are semisimple, the image of an irreducible factor of A'_λ under $\varphi_{\lambda\lambda'}$ is either irreducible or 0. Hence the maps $\gamma_\lambda : C_\lambda \to C$ are injective. It follows that $A_\lambda \varphi_\lambda(e)$ is 0 or a matrix algebra. Consider the cofinal inverse system of A_λ such that $A_\lambda \varphi_\lambda(e) \neq 0$. Within this cofinal inverse system the maps $\varphi_{\lambda\lambda'}|_{A'_\lambda \varphi_{\lambda'}(e)}$ are injective, because the kernel of such a map is a proper ideal of a matrix algebra, so must be 0. It follows that $\varphi_\lambda|_{Ae}$ is injective, so that Ae is isomorphic to a unital subalgebra of $M_n(D)$, where D is a finite k-division algebra. Hence Ae has the required form. Unicity of the decomposition follows from unicity in the finite dimensional case, given that the set C defined at the start of this paragraph is unique: if C' is another complete set of centrally primitive central idempotents, then the images of C and C' in any finite dimensional quotient algebra of A are equal, and hence $C = C'$. The implication (6) $\Rightarrow$ (7) also follows from the more general [32, Theorem 29.7].

[(7) $\Rightarrow$ (2)] For any $\lambda' \in \Lambda$ the closed ideal $M_{\lambda'} = \prod_{\lambda \in \Lambda, \lambda \neq \lambda'} M_{n_\lambda}(D_\lambda)$ is maximal in the algebra $A = \prod_{\lambda \in \Lambda} M_{n_\lambda}(D_\lambda)$, so that $J(A) \subseteq \bigcap_{\lambda' \in \Lambda} M'_\lambda = 0$. $\qquad\square$

Remark 3.8. By [14, IV. §3, Corollary 3], idempotents can be lifted modulo $J(A)$ in a pseudocompact algebra A: that is, for any idempotent f of $A/J(A)$, there is an idempotent $e \in A$ such that $e + J(A) = f$. Since $A/J(A)$ is a product of matrix algebras by the above characterization, every non-zero ideal of $A/J(A)$ contains a non-zero idempotent, and hence by [25, Proposition 1.4], every ideal of A not contained in $J(A)$ contains a non-zero idempotent. It now follows from [25, Proposition 2.1] that when e is a primitive idempotent of a pseudocompact algebra, the algebra eAe is local (see [13, Lemma 4.2] for a different proof of this fact).

Example 3.9. A commutative finite dimensional algebra is a direct product of local rings (e.g. [2, Theorem 8.7]), and the same is true for pseudocompact algebras. Indeed, let A be a commutative pseudocompact algebra. By [14, IV. §3, Corollaries 1 and 2], there is a set of primitive idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ in A such that

$$A = \prod_{\lambda \in \Lambda} Ae_\lambda.$$

By Remark 3.8, each $Ae_\lambda = e_\lambda Ae_\lambda$ is local.

Example 3.10. From Part (2) of Proposition 3.7 and Item (3) of Example 3.1, one obtains a presumably well-known version of Maschke's theorem for profinite groups: if G is a profinite group and k is a field whose characteristic does not divide the order of any continuous finite quotient group of G, then $k[[G]]$ is topologically semisimple.

4. Separable Pseudocompact Algebras and the Wedderburn–Malcev Theorem

Separable algebras are a particularly well-behaved class of semisimple algebras whose definition becomes important when the field k is not algebraically closed, and which have a literature to themselves (see for example [26, Chap. 10] or [12]). Having the notion of semisimple pseudocompact algebra, we are in a strong position to define and characterize separable pseudocompact algebras in a precise way.

Having done so, we will prove a Wedderburn–Malcev theorem for pseudocompact algebras: if A is a pseudocompact algebra for which $A/J(A)$ is separable, then A contains a closed subalgebra S such that $A = S \oplus J(A)$ (the "Wedderburn part"), and any two such subalgebras are conjugate (the "Malcev part").

4.1. *Separable pseudocompact algebras*

In what follows, by $\widehat{\otimes}_k$ we denote the *completed tensor product* over k, which is defined by a universal property analogous to that of the abstract tensor product in the category of pseudocompact modules (see [4, Sec. 2] for the details). Completed tensor products have similar properties to abstract tensor products, but if V, W are pseudocompact k-vector spaces, then $V \otimes_k W = V \widehat{\otimes}_k W$ if, and only if, at least one of V, W is finite dimensional [23, Proposition 2.2].

Definition 4.1. A pseudocompact algebra A (over a field k) is called *separable* if $A \widehat{\otimes}_k E$ is a semisimple algebra over E for every finite extension field E of k.

Remark 4.2. A finite dimensional algebra A is usually said to be separable if $A \otimes_k E$ is semisimple for every extension field E of k. A finite dimensional algebra that is separable in this sense is clearly separable in our sense, and the converse is also true. Indeed, it is enough to check this for finite dimensional simple algebras. By the usual characterization of separable algebras (see for example [12, Theorem

4.5.7]), if $M_n(D)$ is not separable then D is a division algebra whose centre Z is a non-separable finite field extension of k (whose characteristic must therefore be a prime number p). We will check that $Z \otimes_k M_n(D)$ is not semisimple. Let $S : k$ be the separable closure of the extension $Z : k$. Hence $Z : S$ is a (non-trivial) purely inseparable extension, so for any $\alpha \in Z \backslash S$ there is $n \in \mathbb{N}$ such that $\alpha^{p^n} \in S$ [15, Theorem V.6.4]. One checks that $x = \alpha \otimes 1 - 1 \otimes \alpha$ is nilpotent ($x^{p^n} = 0$) in $Z \otimes_S Z$. This implies (by the Wedderburn–Artin theorem) that $J(Z \otimes_S Z) \neq 0$. But now the surjection $Z \otimes_k Z \twoheadrightarrow Z \otimes_S Z$ implies by Corollary 3.4 that $J(Z \otimes_k Z) \neq 0$. An element $0 \neq y \in J(Z \otimes_k Z)$ has torsion, and hence the element $y \otimes 1 \in (Z \otimes_k Z) \otimes_Z D \cong Z \otimes_k D$ has torsion, so that $J(Z \otimes_k D) \neq 0$. Finally

$$J(Z \otimes_k M_n(D)) = J(M_n(Z \otimes_k D)) = M_n(J(Z \otimes_k D)) \neq 0.$$

We generalize some further concepts from finite dimensional to pseudocompact algebras. A *pseudocompact derivation* of the pseudocompact algebra A is a pseudocompact A-bimodule T together with a continuous linear map $d : A \to T$ having the property that

$$d(ab) = ad(b) + d(a)b.$$

An *inner pseudocompact derivation of A* is a pseudocompact derivation of the form $a \mapsto ua - au$, for some fixed $u \in T$ (such a map is easily checked to be a derivation). Consider $A \widehat{\otimes}_k A$ as an A-bimodule as follows:

$$a \cdot (b \widehat{\otimes} c) := ab \widehat{\otimes} c, \quad (b \widehat{\otimes} c) \cdot a := b \widehat{\otimes} ca.$$

Observe that $A \widehat{\otimes}_k A$ is the free A-bimodule of rank 1 (freely generated by $1 \widehat{\otimes} 1$). This done, we can treat the multiplication of A as a continuous A-bimodule homomorphism $m : A \widehat{\otimes}_k A \to A$ given on pure tensors as $b \widehat{\otimes} c \mapsto bc$. Finally, a *separability idempotent for A* (if it exists) is an element $p \in A \widehat{\otimes}_k A$ with the following properties:

$$m(p) = 1, \quad a \cdot p = p \cdot a \quad \forall a \in A.$$

Theorem 4.3. *The following statements concerning a pseudocompact algebra A over a field k are equivalent*:

(1) *A is a separable algebra*;
(2) *A is isomorphic to a product of full matrix algebras $\prod_{\lambda \in \Lambda} M_{n_\lambda}(D_\lambda)$ with each D_λ a finite k-division algebra whose center is a separable field extension of k*;
(3) *A is an inverse limit of separable finite-dimensional algebras*;
(4) *A is projective as an A-bimodule*;
(5) *The multiplication map $m : A \widehat{\otimes}_k A \to A$ splits as a homomorphism of A-bimodules*;
(6) *A has a separability idempotent p*;
(7) *Every pseudocompact derivation $d : A \to T$ is inner*;
(8) *The universal derivation $f : A \to \mathrm{Ker}(m)$ defined by $a \mapsto 1 \otimes a - a \otimes 1$ is inner.*

Proof. [(1) $\Rightarrow$ (2)] If A is separable then in particular it is semisimple, so by Proposition 3.7 a product of matrix algebras $A = \prod M_{n_\lambda}(D_\lambda)$, where each D_λ is a finite dimensional k-division algebra. If some D_λ did not have the form given in (2) then by [26, Proposition 10.7], $M_{n_\lambda}(D_\lambda)$, hence also A, would not be separable.

[(2) $\Rightarrow$ (3)] $\prod_\Lambda M_{n_\lambda}(D_\lambda) \cong \varprojlim_{\Omega \subseteq \Lambda, |\Omega| < \infty} \prod_\Omega M_{n_\lambda}(D_\lambda)$ and each of these finite products is separable, by [26, Proposition 10.7].

[(3) $\Rightarrow$ (1)] Write $A = \varprojlim A_\lambda$ with each A_λ finite dimensional and separable. For a finite extension field E of k we have

$$A \widehat{\otimes}_k E = \varprojlim (A_\lambda \widehat{\otimes}_k E)$$

by [4, Sec. 2]. Each $A_\lambda \widehat{\otimes}_k E$ is semisimple so $A \widehat{\otimes}_k E$ is semisimple by Proposition 3.7. Thus A is separable.

[(3) $\Rightarrow$ (4)] Write $A = \varprojlim A_\lambda$ with each A_λ finite dimensional and separable, with surjective maps. By the finite dimensional version of this result (see for instance [26, Sec. 10.2]) each A_λ is a projective A_λ-bimodule. Recall that an A-bimodule is the same thing as a left $A \widehat{\otimes}_k A^{\mathrm{op}}$-module — the equivalence may be checked as with abstract (bi)modules. We have $A \widehat{\otimes}_k A^{\mathrm{op}} = \varprojlim A_\lambda \widehat{\otimes}_k A_\lambda^{\mathrm{op}}$ and the maps remain surjective, hence A is a projective $A \widehat{\otimes}_k A^{\mathrm{op}}$-module by [4, Corollary 3.3]

[(4) $\Rightarrow$ (5)] This is immediate: if A is projective as a bimodule then any continuous bimodule homomorphism onto A splits, and in particular m splits.

[(5) $\Rightarrow$ (6)] Let $\gamma : A \to A \widehat{\otimes}_k A$ split m and write $p = \gamma(1)$. Then

$$m(p) = m\gamma(1) = 1,$$
$$a \cdot p = a\gamma(1) = \gamma(a \cdot 1) = \gamma(1 \cdot a) = \gamma(1)a = p \cdot a$$

since γ is a bimodule hom.

[(6) $\Rightarrow$ (3)] This is essentially immediate. Given an open ideal I denote by $m_I : A/I \otimes_k A/I \to A/I$ the multiplication map, and let p be a separability idempotent of A. The commutativity of the square

$$
\begin{array}{ccc}
A \widehat{\otimes}_k A & \xrightarrow{\ m\ } & A \\
{\scriptstyle \pi \widehat{\otimes} \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
A/I \otimes_k A/I & \xrightarrow[\ m_I\ ]{} & A/I
\end{array}
$$

shows that the element $p_I = (\pi \widehat{\otimes} \pi)(p)$ is a separability idempotent of A/I:

$$m_I(p_I) = m_I(\pi \widehat{\otimes} \pi)(p) = \pi m(p) = \pi(1) = 1,$$
$$(m_I \widehat{\otimes} \mathrm{id})((a + I) \widehat{\otimes} p_I) = (m_I \widehat{\otimes} \mathrm{id})(\pi \widehat{\otimes} \pi \widehat{\otimes} \pi)(a \widehat{\otimes} p)$$
$$= (\pi \widehat{\otimes} \pi)(m \widehat{\otimes} \mathrm{id})(a \widehat{\otimes} p)$$
$$= (\pi \widehat{\otimes} \pi)(\mathrm{id} \widehat{\otimes} m)(p \widehat{\otimes} a)$$
$$= (\mathrm{id} \widehat{\otimes} m_I)(p_I \widehat{\otimes} (a + I)).$$

[(6) $\Rightarrow$ (7)] By definition, a pseudocompact A-bimodule T comes equipped with actions

$$\nu : A\widehat{\otimes}_k T \to T, \quad \mu : T\widehat{\otimes}_k A \to T,$$

which in compact notation are just $\nu(a\widehat{\otimes}t) = at$ and $\mu(t\widehat{\otimes}a) = ta$. In this language, d being a derivation translates into

$$dm = \nu(\mathrm{id}\widehat{\otimes}d) + \mu(d\widehat{\otimes}\mathrm{id}). \qquad (*)$$

It is easy to confirm (by checking on pure tensors) that the following diagram commutes:

$$
\begin{array}{ccccc}
A\widehat{\otimes}_k A\widehat{\otimes}_k A & \xrightarrow{m\widehat{\otimes}\mathrm{id}} & A\widehat{\otimes}_k A & \xrightarrow{\mathrm{id}\widehat{\otimes}d} & A\widehat{\otimes}_k T \\
\Big\downarrow{\scriptstyle \mathrm{id}\widehat{\otimes}\mathrm{id}\widehat{\otimes}d} & & & & \Big\downarrow{\scriptstyle \nu} \\
A\widehat{\otimes}_k A\widehat{\otimes}_k T & \xrightarrow{\mathrm{id}\widehat{\otimes}\nu} & A\widehat{\otimes}_k T & \xrightarrow{\nu} & T
\end{array}
$$

Given $a \in A$, applying the diagram to the element $a\widehat{\otimes}p$ we obtain the equation

$$a\nu(\mathrm{id}\widehat{\otimes}d)(p) = \nu(\mathrm{id}\widehat{\otimes}d)(ap). \qquad (\dagger)$$

Define the element $u = \mu(d\widehat{\otimes}\mathrm{id})(p) \in T$. For any $a \in A$ we have

$$
\begin{aligned}
ua - au &= \mu(d\widehat{\otimes}\mathrm{id})(p)a - a\mu(d\widehat{\otimes}\mathrm{id})(p) \\
&= \mu(d\widehat{\otimes}\mathrm{id})(ap) - adm(p) + a\nu(\mathrm{id}\widehat{\otimes}d)(p) & (pa = ap \text{ and } (*)) \\
&= \mu(d\widehat{\otimes}\mathrm{id})(ap) + a\nu(\mathrm{id}\widehat{\otimes}d)(p) & (dm(p) = d(1) = 0) \\
&= \mu(d\widehat{\otimes}\mathrm{id})(ap) + \nu(\mathrm{id}\widehat{\otimes}d)(ap) & ((\dagger)) \\
&= (\mu(d\widehat{\otimes}\mathrm{id}) + \nu(\mathrm{id}\widehat{\otimes}d))(ap) \\
&= dm(ap) & ((*) \text{ again}) \\
&= d(am(p)) \\
&= d(a)
\end{aligned}
$$

and hence d is inner.

[(7) $\Rightarrow$ (8)] The universal derivation is pseudocompact, so this is immediate.

[(8) $\Rightarrow$ (6)] This is identical to the finite version. The universal derivation is inner, so write $f(a) = ua - au$ for some $u \in \mathrm{Ker}(m)$ and define $p = 1\widehat{\otimes}1 - u$. Then p is a separability idempotent:

$$m(p) = 1 - m(u) = 1.$$

$$a \cdot p - p \cdot a = a\widehat{\otimes}1 - au - 1\widehat{\otimes}a + ua = f(a) - f(a) = 0 \quad \forall\, a \in A. \qquad \square$$

Corollary 4.4. *Let A be a separable pseudocompact algebra and I a closed ideal of A. Then A/I is separable.*

Proof. By Theorem 4.3, A is a direct product of algebras $\prod_{\lambda \in \Lambda} M_{n_\lambda}(\Delta_\lambda)$, where Δ_λ is a k-division algebra whose center is a separable extension of k. Treating the ideal I as a left A-module, it has a unique decomposition of the form $I = \prod_{\lambda \in \Lambda} 1_{M_{n_\lambda}(\Delta_\lambda)} \cdot I = \prod_{\lambda \in \Lambda} (M_{n_\lambda}(\Delta_\lambda) \cap I)$ — this can be checked directly, or it follows from [13, Proposition 4.3]. But $M_{n_\lambda}(\Delta_\lambda) \cap I$ is an ideal of $M_{n_\lambda}(\Delta_\lambda)$ and is hence either $M_{n_\lambda}(\Delta_\lambda)$ or 0. It follows that $I = \prod_{\lambda \in \Omega} M_{n_\lambda}(\Delta_\lambda)$, for some subset Ω of Λ. The algebra A/I is thus isomorphic to $\prod_{\lambda \in \Omega \setminus \Lambda} M_{n_\lambda}(\Delta_\lambda)$, which is separable by Theorem 4.3. □

4.2. *The Wedderburn splitting theorem*

If A is a pseudocompact algebra, then by Proposition 3.7, $A/J(A)$ is a topologically semisimple pseudocompact algebra. Even for finite dimensional algebras, the canonical projection $A \to A/J(A)$ need not split as an algebra homomorphism. But when $A/J(A)$ is *separable* in the sense of Definition 4.1, it always does — this is a version for pseudocompact algebras of the famous Wedderburn splitting theorem. To our knowledge, a direct proof of this result does not exist in the literature. The existence of the splitting when $A/J^2(A)$ is finite dimensional is a result of Curtis [7, Theorem 1]. The result in full generality has been proved for coalgebras by Abe [1, Theorem 2.3.11], and so follows for pseudocompact algebras by duality. We present a complete proof of this important result for pseudocompact algebras (Theorem 4.6). Our proof is essentially dual to Abe's, but we think it is worth presenting for several reasons: the proof becomes accessible to the reader not familiar with the language of coalgebras; Abe's proof itself employs duality, and dualizing twice seems rather unnatural; finally, because Abe passes rather quickly over a subtle point that we feel benefits from more attention (namely, the dual of Lemma 4.5, which is treated as self-evident).

Lemma 4.5. *Let A be a pseudocompact algebra, let I be a closed ideal of A, and suppose that the canonical projection from A to $A/J(A) = A/J$ has a splitting $s : A/J \to A$ as an algebra homomorphism. Then $s(I + J) \subseteq I$.*

Proof. First note that it sufficient to check this for I open, because if the claim holds for open ideals and I is closed, then

$$s(I + J) = s\left(\bigcap_{L \triangleleft_O A} (I + L) + J \right) \subseteq \bigcap_{L \triangleleft_O A} s((I + L) + J) \subseteq \bigcap_{L \triangleleft_O A} I + L = I.$$
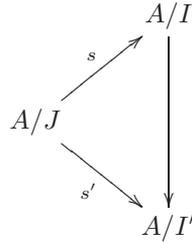
Suppose from now on that I is open. Then for some n, $J^n(A) \subseteq I$. The closed subspace $(I + J)/J$ is an ideal of A/J and hence, by Proposition 3.7, is a direct product of matrix algebras $M_n(\Delta)$. Since s is an algebra homomorphism, it is sufficient to show that s sends the identity of each such matrix algebra inside I. So fix such a factor $M_n(\Delta)$ and $x \in I$ such that $x + J = 1_{M_n(\Delta)}$. By the definition of s, $s(x + J) = x + j$, for some $j \in J$. Using that $x + J$ is idempotent, we have

$$s(x + J) = s((x + J)^n) = s(x + J)^n = (x + j)^n.$$

Expanding $(x + j)^n$, every term is an element of I since I is an ideal, except the final term j^n, which is an element of $J^n \subseteq I$. So $s(x + J) \in I$ as required. $\qquad\square$

Theorem 4.6. *Let A be a pseudocompact algebra such that A/J is separable. The canonical projection $A \to A/J$ splits continuously as an algebra homomorphism.*

Proof. We consider the set $\mathcal{F}$ of pairs (I, s), where I is a closed ideal of A contained in J and $s : A/J \to A/I$ is a splitting of the natural projection $A/I \to A/J$. Then $\mathcal{F}$ is non-empty, since it contains (J, id). Order $\mathcal{F}$ by declaring $(I, s) \leq (I', s')$ whenever $I \subseteq I'$ and the diagram

$$
\begin{array}{ccc}
 & & A/I \\
 & \nearrow{\scriptstyle s} & \big\downarrow \\
A/J & & \\
 & \searrow{\scriptstyle s'} & \\
 & & A/I'
\end{array}
$$

commutes, where the vertical map is the canonical projection. A chain $\mathcal{C}$ in $\mathcal{F}$ is by definition a totally ordered chain of closed ideals I together with a map of inverse systems

$$\{s : A/J \to A/I \,|\, (I, s) \in \mathcal{C}\},$$

which yields a unique map $\varprojlim s : A/J \to A/\bigcap_{\mathcal{C}} I$. Then $(\bigcap_{\mathcal{C}} I, \varprojlim s)$ is a lower bound for $\mathcal{C}$ and so by Zorn's lemma, $\mathcal{F}$ contains a minimal element (I_0, s_0). Our task is to prove that $I_0 = 0$. Suppose it were not. Then there is an open ideal I of A such that $I \cap I_0$ is properly contained in I_0. We will find a splitting $s' : A/J \to A/(I \cap I_0)$ such that $(I \cap I_0, s') < (I_0, s_0)$, contradicting the minimality of (I_0, s_0) and completing the proof.
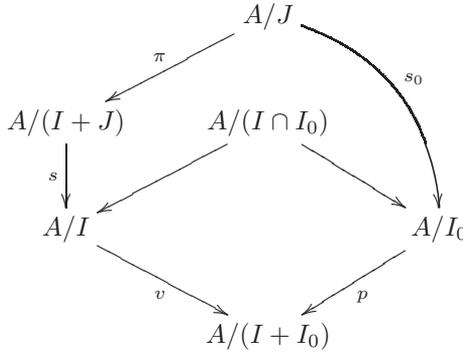
We construct two splittings of the canonical projection map $q : A/(I + I_0) \to A/(I + J)$.

- First let s be an algebra splitting of $A/I \to A/(I+J) = (A/I)/J(A/I)$. This splitting exists by the Wedderburn splitting theorem for finite dimensional algebras [8, Theorem 72.19], which applies since $A/(I + J)$ is separable by Corollary 4.4. Composing s with the canonical projection $v : A/I \to A/(I + I_0)$ we obtain a splitting vs of q.
- Second, since A/J is semisimple, by Part (7) of Proposition 3.7 the kernel $(I + J)/J$ of the canonical projection $\pi : A/J \to A/(I + J)$ has a unique complement S in A/J. Denote by $\iota : A/(I+J) \to A/J$ the (non-unital) splitting of π, so that $S = \mathrm{Im}(\iota)$. If $p : A/I_0 \to A/(I + I_0)$ is the canonical projection, then the map $ps_0\iota$ is another splitting of q.

The Malcev uniqueness theorem for finite dimensional algebras implies that there is an element $a \in J(A/(I + I_0))$ such that

$$(1 + a)vs(1 + a)^{-1} = ps_0\iota.$$

Let a' be an element of $J(A/I)$ that projects onto a and replace s with $(1+a')s(1+a')^{-1}$. We may thus assume from now on that $vs = ps_0\iota$. Put these maps together in the following diagram:

$$
\begin{array}{ccccc}
 & & A/J & & \\
 & \swarrow{\scriptstyle \pi} & & \searrow{\scriptstyle s_0} & \\
A/(I+J) & & A/(I \cap I_0) & & \\
\downarrow{\scriptstyle s} & & & & \\
A/I & & & & A/I_0 \\
 & \searrow{\scriptstyle v} & & \swarrow{\scriptstyle p} & \\
 & & A/(I + I_0) & &
\end{array}
$$

The lower square is a pullback diagram, and so if the outer shape commutes, then there is a unique map $s' : A/J \to A/(I \cap I_0)$ making the diagram commute, and this s' will be the splitting we require. In order to check that the outer shape commutes, write $A/J = \operatorname{Im}(\iota) \times (I + J)/J$. Given $x = \iota(y)$ in the first factor, we have

$$vs\pi(x) = vs\pi\iota(y) = ps_0\iota\pi\iota(y) = ps_0\iota(y) = ps_0(x).$$

An element x of the second factor is sent to 0 by π, so we need to check that $ps_0(I + J) \subseteq I + I_0$. But $s_0(I + J) \subseteq I + I_0$ by Lemma 4.5, so we are done. □

4.3. *The Malcev uniqueness theorem*

Now that we have the "Wedderburn part" of the fundamental Wedderburn–Malcev theorem, we turn to the "Malcev part", which says that the splitting of Theorem 4.6 is unique up to conjugation by a unit of the form $1 + w$, where w is an element of the radical of A. The Malcev uniqueness theorem for pseudocompact algebras was proved by Eckstein [10, Theorem 17]. We provide a simple proof that closely mimics the argument for finite dimensional algebras (cf. for instance [8, Theorem 72.19]).

Theorem 4.7 (Malcev uniqueness theorem). *Suppose that the algebra A/J is separable, and let S_1, S_2 be two closed subalgebras of A such that $S_1 \oplus J(A) = A = S_2 \oplus J(A)$. There is $w \in J$ such that*

$$S_1 = (1 - w)S_2(1 - w)^{-1}.$$

K. Iusenko & J. W. MacQuarrie

Proof. Let $s_1, s_2 : A/J \to A$ be splittings of the canonical projection $A \to A/J$ with images S_1, S_2, respectively. We define an A/J-bimodule structure on J by

$$x \cdot j := s_1(x)j, \quad j \cdot x := js_2(x) \quad (x \in A/J, j \in J).$$

The continuous function $d : A/J \to J$ given by $d(x) = s_1(x) - s_2(x)$ is well-defined, because

$$\pi d(x) = \pi s_1(x) - \pi s_2(x) = x - x = 0,$$

so that $d(x) \in J$. Furthermore, d is a derivation:

$$d(xy) = s_1(xy) - s_1(x)s_2(y) + s_1(x)s_2(y) - s_2(xy)$$
$$= s_1(x)(s_1(y) - s_2(y)) + (s_1(x) - s_2(x))s_2(y)$$
$$= x \cdot d(y) + d(x) \cdot y.$$

By Theorem 4.3, there is $w \in J$ such that

$$d(x) = s_1(x) - s_2(x) = x \cdot w - w \cdot x = s_1(x)w - ws_2(x)$$

for all $x \in A/J$, so that

$$s_1(x)(1 - w) = (1 - w)s_2(x).$$

The element $1 - w$ is invertible by Proposition 3.2 and so

$$\mathrm{Im}(s_1) = (1 - w)\mathrm{Im}(s_2)(1 - w)^{-1}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

ORCID

Kostiantyn Iusenko https://orcid.org/0000-0003-1360-9985
John William MacQuarrie https://orcid.org/0000-0002-6503-4602

References

[1] E. Abe, *Hopf Algebras*, Cambridge Tracts in Mathematics, Vol. 74 (Cambridge University Press, Cambridge, 1980) [translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka].

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley Publishing Co., Reading, MA, 1969).

[3] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, Vol. 36 (Cambridge University Press, Cambridge, 1997) [corrected reprint of the 1995 original].

[4] A. Brumer, Pseudocompact algebras, profinite groups and class formations, *J. Algebra* **4** (1966) 442–470.

[5] W. Chin and S. Montgomery, Basic coalgebras, in *Modular Interfaces* (*Riverside, CA*, 1995), AMS/IP Studies in Advanced Mathematics, Vol. 4 (American Mathematical Society, Providence, RI, 1997), pp. 41–47.

[6] F. U. Coelho and S.-X. Liu, Generalized path algebras, in *Interactions Between Ring Theory and Representations of Algebras* (*Murcia*), Lecture Notes in Pure and Applied Mathematics, Vol. 210 (Dekker, New York, 2000), pp. 53–66.

[7] C. W. Curtis, The structure of non-semisimple algebras, *Duke Math. J.* **21** (1954) 79–85.

[8] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Pure and Applied Mathematics, Vol. XI (Interscience Publishers, New York, 1962).

[9] Y. A. Drozd and V. V. Kirichenko, *Finite-Dimensional Algebras* (Springer-Verlag, Berlin, 1994) [translated from the 1980 Russian original and with an appendix by Vlastimil Dlab].

[10] F. Eckstein, The Malcev theorem, *J. Algebra* **12** (1969) 372–385.

[11] B. Farb and R. Keith Dennis, *Noncommutative Algebra*, Graduate Texts in Mathematics, Vol. 144 (Springer-Verlag, New York, 1993).

[12] T. J. Ford, *Separable Algebras*, Graduate Studies in Mathematics, Vol. 183 (American Mathematical Society, Providence, RI, 2017).

[13] R. J. Franquiz Flores and J. W. MacQuarrie, Block theory and Brauer's first main theorem for profinite groups, *Adv. Math.* **397** (2022) 108121.

[14] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. Fr.* **90** (1962) 323–448.

[15] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics, Vol. 73 (Springer-Verlag, New York, 1980) [reprint of the 1974 original].

[16] M. C. Iovanov, Z. Mesyan and M. L. Reyes, Infinite-dimensional diagonalization and semisimplicity, *Isr. J. Math.* **215**(2) (2016) 801–855.

[17] K. Iusenko and J. W. MacQuarrie, The path algebra as a left adjoint functor, *Algebras Representation Theory* **23**(1) (2020) 33–52.

[18] T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, Vol. 131 (Springer-Verlag, New York, 1991).

[19] S. Lefschetz, *Algebraic Topology*, American Mathematical Society Colloquium Publications, Vol. 27 (American Mathematical Society, New York, 1942).

[20] J. W. MacQuarrie, Green correspondence for virtually pro-p groups, *J. Algebra* **323**(8) (2010) 2203–2208.

[21] J. W. MacQuarrie, Modular representations of profinite groups, *J. Pure Appl. Algebra* **215**(5) (2011) 753–763.

[22] J. W. MacQuarrie and P. Symonds, Brauer theory for profinite groups, *J. Algebra* **398** (2014) 496–508.

[23] J. W. MacQuarrie, P. Symonds and P. A. Zalesskii, Infinitely generated pseudocompact modules for finite groups and Weiss' theorem, *Adv. Math.* **361** (2020) 106925.

[24] O. V. Mel'nikov, Subgroups and the homology of free products of profinite groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**(1) (1989) 97–120.

[25] W. K. Nicholson, *I*-rings. *Trans. Am. Math. Soc.* **207** (1975) 361–373.

[26] R. S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics, Vol. 88 (Springer-Verlag, New York, 1982) [Studies in the History of Modern Science, Vol. 9].

[27] L. Ribes and P. Zalesskii, *Profinite Groups*, 2nd edn., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A Series of Modern Surveys in Mathematics, Vol. 40 [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics, Vol. 40] (Springer-Verlag, Berlin, 2010).

[28] D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, *Colloq. Math.* **90**(1) (2001) 101–150.

[29] B. Stenström, *Rings of Quotients*: *An Introduction to Methods of Ring Theory*, Die Grundlehren der Mathematischen Wissenschaften, Vol. 217 (Springer-Verlag, New York, 1975).

[30] P. Symonds, Permutation complexes for profinite groups, *Comment. Math. Helv.* **82**(1) (2007) 1–37.

[31] P. Symonds, Double coset formulas for profinite groups, *Commun. Algebra* **36**(3) (2008) 1059–1066.

[32] S. Warner, *Topological Rings*, North-Holland Mathematics Studies, Vol. 178 (North-Holland Publishing Co., Amsterdam, 1993).

[33] J. S. Wilson, *Profinite Groups*, London Mathematical Society Monographs, New Series, Vol. 19 (The Clarendon Press, Oxford University Press, New York, 1998).

[34] D. Zelinsky, Linearly compact modules and rings, *Am. J. Math.* **75** (1953) 79–90.