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# Attractors for parabolic problems with nonlinear Boundary conditions

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**Abstract**

In this work we prove global existence, boundedness and dissipativeness for systems of weakly coupled reaction-diffusion equations with dispersion and non-linear boundary conditions in a bounded smooth domain $\Omega$.

We work in $X^\alpha \cap L^\infty(\Omega)$ and in $H^1(\Omega)$. In the first case we do not require any growth condition on the non-linear terms and on the second the results are proved under rather general growth assumptions.

This is done under very few hypotheses on the reaction term and on the non-linear flux through the boundary. We allow a source of concentration in the boundary provided that the dissipation of concentrations inside the domain is large enough or vice-versa. The hypotheses on the non-linear reaction and on the non-linear flux of concentration through the boundary are natural and easy to verify in many applications. The tools employed are comparison principles, invariant regions and the La Salle's invariance theory.

# Attractors For Parabolic Problems with Nonlinear Boundary Conditions

by

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## 1 Introduction and Statement of the Results

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Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^N$. In this paper we study the long time behavior of solutions of weakly coupled reaction diffusion systems with dispersion of the form

$$\left.\begin{array}{l} u_t - \mathrm{Div}(a(x)\nabla u) + \sum_{j=1}^N B_j(x)\frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0, \qquad \text{on} \quad \Omega, \\ \dfrac{\partial u}{\partial \vec{n}_a} + g(u) = 0, \qquad \text{on} \quad \Gamma = \partial\Omega. \end{array}\right\} \tag{1.1}$$

where $u = (u_1, \cdots, u_m)^\top$, $m \geq 1$, $a(x) = \mathrm{diag}(a_1(x), \ldots, a_m(x))$, $a_i \in C^1(\overline{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq m$, $\frac{\partial u}{\partial \vec{n}_a} = \langle a\nabla u, \vec{n}\rangle$, $\vec{n}$ is the outward normal to $\Gamma = \partial\Omega$, $B_j = \mathrm{diag}(b_j^1, \ldots, b_j^m)$ is continuous on $\bar{\Omega}$, $1 \leq j \leq N$, $\lambda \in \mathbb{R}$ and $f = (f_1, \cdots, f_m)^\top : \mathbb{R}^m \to \mathbb{R}^m$, $g = (g_1, \cdots, g_m)^\top : \mathbb{R}^m \to \mathbb{R}^m$. Note that the coupling appears both on the reaction term, $f$, and in the boundary conditions through the (nonlinear) radiation term $g$.

The starting point for the study of the long time behavior of the solutions of (1.1) is having a good functional framework in which constructing the solutions. In this direction one needs to have uniqueness and regularity of solutions to define a well-behaved non-linear semigroup in a suitable Banach space. Then, one tries to prove that the dynamical system is dissipative and possesses a global attractor that captures the asymptotic behavior. Therefore, one first needs to choose in which space to work. In making such a decision there are several alternative ways to proceed. A first approach is based on the fact that for the scalar case ($m = 1$) and without dispersion terms ($B_j = 0$) there is a natural energy associated to the solutions, given by

$$V(\phi) = \frac{1}{2}\int_\Omega a|\nabla\phi|^2 + \frac{\lambda}{2}\int_\Omega |\phi|^2 + \int_\Omega F(\phi) + \int_\Gamma G(\phi).$$

(where $F$ and $G$ are primitives of $f$ and $g$ respectively) which is dissipated as time increases. Therefore, one may think of using $L^2(\Omega)$ as an underlying space and the properties of the second order elliptic operator in this space to prove the well posedness of (1.1). Once this is done, the use of the space $H^1(\Omega)$ turns out naturally in view of the energy $V$. The problem then becomes that one must impose some restrictions in the growth of the non-linear terms in order to ensure that the integrals above are finite. This restrictions become more stringent as the dimension $N$ increases. The restrictions can be somewhat relaxed by working in suitable sup-spaces of Sobolev spaces $H^s(\Omega)$, appearing as the fractional power spaces associated to the second order elliptic operator. In some ranges for $N$ and $s$, no growth assumptions are needed on the non-linearities.

Alternatively, one may assume that there are invariant regions, as in [27, 38], and then work with bounded functions $u$, again getting rid of growth assumptions on $f$ and $g$.

Another approach is using the properties of the elliptic operator taking $L^p(\Omega)$ for $p > N$ as underlying space, see [1, 2, 3]. The reason for this is that, in this case, the fractional power spaces become embedded in space of continuous functions and then no growth assumptions are needed in $f$ and $g$ in order to have the non-linear semigroup well defined.

Once this first step is completed satisfactorily the problem becomes proving the dissipativeness of the flow induced by the equations in such a space. One has to obtain estimates on the solution for the norm of the space in which the dynamical system is set, that allows one to prove the existence of attractors, for example by using some of the general results in [16]. At this point it is common the case in which the natural energy $V$ described above, does not give any information about the dissipativeness in the right norm.

We adopt here an intermediate approach. We stick to the $L^2(\Omega)$ setting, which possesses some technical advantages and ease, and then consider either the case in which one works in the "energy space", $H^1(\Omega)$, with the natural growth restrictions on the non-linearities, or work in a different space, which is essentially $H^s(\Omega) \cap C(\overline{\Omega})$, for some $s$, and, since the functions are bounded, again no growth assumptions are needed.

In the first case, we use the natural energy $V$ to prove the existence of the attractor, while in the second we have to use some "contracting regions" to obtain dissipativeness in the proper norm.

In order to have dissipativeness some remarks are needed. For this, we interpret $u$ as a heat distribution in the body $\Omega$. Hence, we assume for the moment that $u \geq 0$. In this situation, note that for ranges in which $f$ is positive we have "absorption" of heat, while when $f$ is negative we have "sources" of heat. The same holds for $g$; when $g$ is positive we have a flow of heat through the boundary of $\Omega$ that extracts heat form the body, while in the opposite case, heat is flowing inside $\Omega$.

When one of the terms is absent, some results are available. First, if $g = 0$, then it is well known that if $f$ has the wrong sign, solutions may develop "explosions" in finite time, [12, 24], while for the good sign dissipativeness follows, [16, 4, 8]. Also, note that solutions of the ode $\dot{u} + \lambda u + f(u) = 0$ are constant in space solutions of (1.1), if $g = 0$, and that solutions of this equation may blow-up in finite time. Similarly, when $f = 0$, if $g$ has the wrong sign, explosions may occur in finite time, [25, 40, 26], while in the opposite case boundedness of solutions follows [40, 41].

When both $f$ and $g$ are non-zero we expect to have some kind of competition between both mechanisms and only if the right sign is dominant we can expect to obtain the desired dissipativeness.

Therefore, in what follows we will impose some growth and/or sign assumptions on the non-linear terms under which we will prove both the well posedness and dissipativeness of the semigroup. We will restrict ourselves to the case $N \leq 3$, but it will be clear form the proofs that everything works the same for any dimensions. Only the growth conditions we impose below are affected by the value of $N$.

To be more precise, we assume the following. Concerning growth conditions, we assume that $f, g : \mathbb{R}^m \to \mathbb{R}^m$ are $C^1$ and $C^2$ functions, respectively, satisfying:

$N = 2$ and for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$|f(u) - f(v)| \leq c_\eta(e^{\eta|u|^2} + e^{\eta|v|^2})|u - v|, \tag{1.2}$$

for every $u, v \in \mathbb{R}^m$, or $N = 3$ and

$$|f(u) - f(v)| \leq L(1 + |u|^2 + |v|^2)|u - v| \tag{1.3}$$

for all $u, v \in \mathbb{R}^m$, and

$N = 2$ and for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$\begin{aligned} |g(u) - g(v)| &\leq c_\eta(e^{\eta|u|^2} + e^{\eta|v|^2})|u - v|, \\ |Dg(u) - Dg(v)| &\leq c_\eta(e^{\eta|u|^2} + e^{\eta|v|^2})|u - v|, \end{aligned} \tag{1.4}$$

for every $u, v \in \mathbb{R}^m$, or $N = 3$ and

$$\begin{aligned} |g(u) - g(v)| &\leq L(1 + |u|^{\rho+1} + |v|^{\rho+1})|u - v|, \\ |Dg(u) - Dg(v)| &\leq L(1 + |u|^{\rho} + |v|^{\rho})|u - v|, \end{aligned} \tag{1.5}$$

for all $u, v \in \mathbb{R}^m$ and some $\rho \leq 1$. Observe that, no growth assumptions are made for $N = 1$. Also, observe that these assumptions are satisfied in the scalar case, $m = 1$, if the following conditions hold

$$\lim_{|s|\to\infty} \frac{|f'(s)|}{e^{\eta|s|^2}} = 0, \forall\, \eta > 0, \text{ if } N = 2, \quad |f'(s)| \leq L(1 + |s|^2), \text{ if } N = 3 \tag{1.6}$$

and

$$\lim_{|s|\to\infty} \frac{|g''(s)|}{e^{\eta|s|^2}} = 0, \forall\, \eta > 0, \text{ if } N = 2, \quad |g''(s)| \leq L(1 + |s|^\rho),\ \rho \leq 1, \text{ if } N = 3 \tag{1.7}$$

Also note that a function satisfying (1.4) or (1.5) also satisfies (1.2) or (1.3), respectively. Hence, the conditions on the boundary nonlinear term are stronger than the conditions on the reaction term.

Concerning sign assumptions, we assume the following. Let $f = (f_1, \cdots, f_m)^\top : \mathbb{R}^m \to \mathbb{R}^m$, $g = (g_1, \cdots, g_m)^\top : \mathbb{R}^m \to \mathbb{R}^m$ be smooth functions satisfying:

There exists $\xi^0 = (\xi_1^0, \cdots, \xi_m^0)$ such that $\xi_i^0 > 0$ and there are constants $c_i^0$ and $d_i^0$ with

$$\frac{\lambda s_i + f_i(s)}{s_i} > c_i^0, \text{ and } \frac{g_i(s)}{s_i} > d_i^0, \text{ for all } |s_i| > \xi_i^0 \tag{1.8}$$

uniformly in $s_j$, $j \neq i$, where $c_i^0$ and $d_i^0$ are such that the first eigenvalue, $\mu_1^0$ of the problem

$$\begin{aligned} &-\mathrm{Div}(a\nabla v) + \textstyle\sum_{j=1}^N B_j(x)\frac{\partial v}{\partial x_j} + c^0 v = \mu v, \quad \text{on } \Omega \\ &\frac{\partial v}{\partial \vec{n}_a} + d^0 v = 0, \quad \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{1.9}$$

for $c^0 = (c_1^0, \ldots, c_m^0)$ and $d^0 = (d_1^0, \ldots, d_m^0)$, is positive.

Observe that either $c_i^0$ or $d_i^0$ can be negative. This allows for one of the non-linear terms to have the wrong sign for large values of its argument. In such a case, the above hypothesis implies that the wrong term is compensated by the other; see Section 6. Also, observe that this hypothesis, which can be seen as a principle of linearized dissipativeness "at infinity", just allows for a sub-linear behavior of the non-linearities in the region of bad sign, excluding then the cases of polynomial non-linearities having bad sign. Finally, note that these assumptions have been previously used in [22], working for $N = 1$ and $m = 1$, where he proves the system is Morse-Smale. See also [30].

Before we proceed any further let us introduce some notation. Let $X = L^2(\Omega, \mathbb{R}^m)$ and $A : D(A) \subset X \to X$ be the operator $A = \mathrm{diag}(A_1, \cdots A_m)$ defined by $D(A) = D(A_1) \times \cdots \times D(A_m)$ with

$$D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial\phi}{\partial \vec{n}_{a_i}} = 0\},$$

$$A_i\phi = -\mathrm{Div}(a_i(x)\nabla\phi) + \textstyle\sum_{j=1}^N b_j^i(x)\frac{\partial\phi}{\partial x_j} + \lambda\phi,$$

where $\dfrac{\partial\phi}{\partial \vec{n}_{a_i}} = a_i\langle\nabla\phi, \vec{n}\rangle$, $i = 1, \ldots, m$.

We can define the fractional powers $A_i^\alpha$ and $A^\alpha$ of $A_i$ and $A$ respectively, see [21], and the fractional power spaces $X_i^\alpha := D(A_i^\alpha)$ and $X^\alpha := D(A^\alpha)$ endowed with the graph norm, $\alpha \in \mathbb{R}$, where $X^\alpha = (X^{-\alpha})'$, if $\alpha < 0$. In this case we can always view $A$ as a sectorial operator with compact resolvent from $X^{\alpha+1}$ into $X^\alpha$ which is positive and self adjoint if $B_j \equiv 0$, for $j = 1, \ldots, N$. In fact, from the diagonal structure of $A$, we have $X^\alpha = X_1^\alpha \times \cdots \times X_m^\alpha$.

These spaces provide a natural framework for solving (1.1), see [21]. It is worth noting at this point that for every $i = 1, \ldots, m$, $X_i^\alpha$ is a closed subspace of the Sobolev space $H^{2\alpha}(\Omega)$, for $\alpha \geq 0$ and in particular $X_i^{1/2} = H^1(\Omega)$, i.e. the energy space.

Let us now review some of the known results using this framework. For the case of no dispersion ($B_j = 0$), linear boundary conditions, $g = 0$, scalar equations ($m = 1$), and $N = 3$, the existence of a global attractor, $\mathcal{A}$, in the energy space $H^1(\Omega)$, for problem (1.1), under conditions (1.3) and the dissipativeness assumption

$$\liminf_{|u|\to\infty} \frac{f(u) + \lambda u}{u} > 0$$

is a well known result, [16]. Also, if $g = 0$, $m = 1$ and $N = 2$, the existence of a global attractor for (1.1) in the energy space $H^1(\Omega)$, under the same dissipativeness assumption, has been established in [7], assuming the non-linearity satisfies

$$|f'(u)| \leq ce^{|u|^\rho}, \text{ for some } \rho < 2$$

Therefore, condition (1.2) slightly relaxes this assumption, for $N = 2$, while (1.8) and (1.9) give a generalized dissipativeness condition.

Working outside the energy space, i.e. in $X^\alpha$ for $\alpha > 1/2$, and for the case $g = 0$, in [15], it is proved the existence of a local attractor for (1.1), which coincides with the embedding of the attractor for $\dot{u} + \lambda u + f(u) = 0$ into the subspace of constant functions of $X^\alpha$, $\alpha > \frac{3}{4}$, under the additional assumption that the diffusion coefficient $a(x)$ is large enough. See also [18, 19, 6] for the case of linear boundary conditions, i.e. $g(u) = c(x)u$, with $c(x) = \mathrm{diag}(c_1(x), \ldots, c_m(x))$.

However, these techniques can only be extended to global attractors if some a priori bound, in $X^\alpha$, on the size of an absorbing set could be obtained and only if the diffusion coefficient is large (see [7, 5]). These a priori bounds are obtained in [8] and [4], always for the case $g = 0$ and $B_j = 0$, also working in $X^\alpha$, for $\alpha > \frac{3}{4}$ and $N = 3$ and assuming $f$ verifies: there exists $\xi^0 = (\xi_1^0, \cdots, \xi_m^0)$ such that $\xi_i^0 > 0$ and

$$\frac{\lambda s_i + f_i(s)}{s_i} > 0, \text{ for all } |s_i| > \xi_i^0$$

In [4] the author uses a technique of "contracting rectangles" to obtain the desired estimates in $X^\alpha$ and the existence of attractors.

Note that in all these articles the authors work in $X^\alpha$ for $1 > \alpha > \frac{3}{4}$ and $N = 3$ and impose no growth conditions in the nonlinearities. The reason for this is the embedding $X^\alpha \subset C(\overline{\Omega}, \mathbb{R}^m)$. However, for $\alpha > \frac{3}{4}$, the space $X^\alpha$ incorporates the boundary condition $\frac{\partial u}{\partial \vec{n}_a} = 0$, [13], and therefore, it cannot be the right space to work in when $g \neq 0$.

However, for $N = 1$ and $3/4 > \alpha > 1/4$ or for $N = 2$ and $3/4 > \alpha > 1/2$, we have $X^\alpha \subset C(\overline{\Omega}, \mathbb{R}^m)$ and $X^\alpha$ does not incorporate any boundary condition. Therefore, we work in this range of $\alpha$, for $N = 1, 2$ and in this case again no growth condition is imposed on $f$ and $g$. On the other hand, as soon as $N \geq 3$, $X^\alpha$ is never included in $C(\overline{\Omega}, \mathbb{R}^m)$, when $\frac{3}{4} > \alpha > \frac{1}{2}$. Therefore, to avoid growth conditions, in the case $N \geq 3$, we will work in $Y = X^\alpha \cap C(\overline{\Omega}, \mathbb{R}^m)$ and prove the well posedness of the problem in this space.

## 2 Notations and background results

Concerning functional spaces, we will use the standard Sobolev spaces $H^1(\Omega)$ and , $W^{1,p}(\Omega)$ and the spaces of traces $H^{1/2}(\Gamma)$ and $W^{1-1/p,p}(\Gamma)$. Also, we will denote by $H^{-s}$ the dual space of $H^s$, either on $\Omega$ or $\Gamma$. Note that this symbol is usually reserved to denote the dual space of $H^s_0$. However, this notation should produce no confusion. The duality pairing between these spaces will be denoted $< \cdot, \cdot >_{-s,s}$. In particular, the scalar product in $L^2$ will be denoted by $< \cdot, \cdot >$. If there is no possible confusion, we will not indicate if the spaces or duality products are referred to functions on $\Omega$ or $\Gamma$. When required, we will write $< \cdot, \cdot >_\Omega$ and $< \cdot, \cdot >_\Gamma$ to differentiate both cases. The symbol $\| \cdot \|$ will always represent the norm in $L^2(\Omega)$.

We will denote by $\gamma$ the trace operator defined on $H^s(\Omega)$, with values in $H^{s-1/2}(\Gamma)$, for $s > 1/2$. Moreover, for a given function $f \in H^s(\Omega)$, we will identify its trace, $\gamma(f) \in H^{s-1/2}(\Gamma)$, with the linear form $\gamma(f) \in H^{-1/2}(\Gamma) \subset H^{-1}(\Omega)$, such that for every $\phi \in H^1(\Omega)$

$$< \gamma(f), \phi >_{-1,1} \stackrel{def}{=} < f, \phi >_\Gamma \stackrel{def}{=} \int_\Gamma \gamma(f)\gamma(\phi)$$

that is, we use the embedding $H^{s-1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma) \subset H^{-1}(\Omega)$. Even more, from the trace theorem, the trace operator $\gamma : W^{1,q}(\Omega) \to W^{1-\frac{1}{q},q}(\Gamma)$ is bounded, [14].

We will also consider the normal derivative operator, relative to the diffusion operator $-Div(a(x)\nabla u)$, defined as follows: if

$$u \in Z \stackrel{def}{=} \{z \in H^1(\Omega), \ -Div(a(x)\nabla z) \in L^2(\Omega)\}$$

then $\frac{\partial u}{\partial \vec{n}_a} \in H^{-1/2}(\Gamma)$ and it is defined as

$$< \frac{\partial u}{\partial \vec{n}_a}, \gamma(v) >_{-1/2,1/2} = -\int_\Omega -Div(a(x)\nabla u)v + \int_\Omega a\nabla u \nabla v = \int_\Omega Div(av\nabla u) \tag{2.1}$$

for every $v \in H^1(\Omega)$.

Under these conditions, and assuming $\lambda > 0$, we introduce the canonical isometric isomorphism between $H^1(\Omega)$ and its dual, $H^{-1}(\Omega)$, such that for every $u, \phi \in H^1(\Omega)$

$$< L_0(u), \phi >_{-1,1} = \int_\Omega a\nabla u \nabla \phi + \lambda \int_\Omega u\phi \tag{2.2}$$

Note that we can then write (2.1) as

$$< \frac{\partial u}{\partial \vec{n}_a}, \gamma(v) >_{-1/2,1/2} = < L_0(u), v >_{-1,1} - < -Div(a(x)\nabla u) + \lambda u, v > \tag{2.3}$$

Also, we consider in $H^1(\Omega)$ the scalar product

$$a(u, v) = \int_\Omega a\nabla u \nabla v + \lambda \int_\Omega uv = < L_0(u), v >_{-1,1} \tag{2.4}$$

which gives a norm equivalent to the usual one.

We define in $L^2(\Omega)$ and $A_0 : D(A_0) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$
\begin{aligned}
D(A_0) &= \{u \in H^2(\Omega) : \frac{\partial u}{\partial \vec{n}_a} = 0, \text{ on } \Gamma\} \\
A_0 u &= -\mathrm{Div}(a\nabla u) + \lambda u, \; u \in D(A).
\end{aligned}
\tag{2.5}
$$

with $\lambda > 0$.

Then, we have the following well known result

**Proposition 2.1** *The operator $A_0$ defined above is positive, selfadjoint and has compact resolvent in $L^2(\Omega)$. In particular, it is a sectorial operator in $L^2(\Omega)$ and its fractional power spaces verify $X^\alpha \subset H^{2\alpha}(\Omega)$, for $\alpha \geq 0$ and in particular*

$$
X^1 = D(A_0), \quad X^{1/2} = H^1(\Omega), \quad X^0 = L^2(\Omega), \quad X^{-1/2} = H^{-1}(\Omega)
$$

*where we have set $H^{-1}(\Omega) \stackrel{def}{=} (H^1(\Omega))'$.*

*The restriction of $L_0$ to $L^2(\Omega)$, i.e. the restriction of $L_0$ to the domain $D = \{u \in H^1(\Omega),$ such that $L_0 u \in L^2(\Omega)\}$, coincides with $A_0$ and $D = D(A_0)$. Moreover, $L_0$ is a sectorial operator in $H^{-1}(\Omega)$ with domain $H^1(\Omega)$.*□

Note that all the above remains true, except for $A_0$ to be positive, if $\lambda \in \mathbb{R}$. Also, when $\lambda > 0$, the norm in $X^{1/2} = H^1(\Omega)$ is given by $\int_\Omega a|\nabla u|^2 + \lambda \int_\Omega |u|^2$, which is precisely the norm in $H^1(\Omega)$ given by the bilinear form (2.4).

By standard perturbation results, [21, Theorem 1.4.8], we have

**Proposition 2.2** *The operator, $A$, in $L^2(\Omega)$ given by $D(A) = D(A_0)$ and*

$$
Au = A_0 u + \sum_{j=1}^{N} b_j(x) \frac{\partial u}{\partial x_j}
$$

*is a sectorial operator with compact resolvent and the same fractional power spaces than those of $A_0$. In particular $A$ can be extended in a unique way to an operator $L$ from $H^1(\Omega)$ to its dual $H^{-1}(\Omega)$, given by the bilinear form*

$$
< L(u), \phi >_{-1,1} = \int_\Omega a\nabla u \nabla \phi + \sum_{j=1}^{N} \int_\Omega b_j(x) \frac{\partial u}{\partial x_j}\phi + \lambda \int_\Omega u\phi
\tag{2.6}
$$

*for every $u, \phi \in H^1(\Omega)$. Moreover, $L$ is a sectorial operator in $H^{-1}(\Omega)$ with domain $H^1(\Omega)$.*□

Let us now define $X = L^2(\Omega, \mathbb{R}^m)$ and $A : D(A) \subset X \to X$ be the operator $A = \mathrm{diag}(A_1, \cdots A_m)$ defined by $D(A) = D(A_1) \times \cdots \times D(A_m)$ with

$$
D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial \vec{n}_{a_i}} = 0\},
$$

$$
A_i \phi = -\mathrm{Div}(a_i(x)\nabla \phi) + \textstyle\sum_{j=1}^{N} b_j^i(x) \frac{\partial \phi}{\partial x_j} + \lambda \phi,
$$

where $\dfrac{\partial \phi}{\partial \vec{n}_{a_i}} = a_i \langle \nabla \phi, \vec{n} \rangle$, $i = 1, \ldots, m$.

Then, $A$ is a sectorial operator in $X$ and from the diagonal structure of $A$, we have $X^\alpha = X_1^\alpha \times \cdots \times X_m^\alpha$. In particular, $-A$ generates an analytic semigroup on $X^\alpha$ such that satisfies the following estimates

$$
\begin{aligned}
\|e^{-At} u_0\|_{X^\alpha} &\leq M e^{-\delta t} \|u_0\|_{X^\alpha}, \quad t \geq 0, \\
\|e^{-At} u_0\|_{X^\alpha} &\leq M e^{-\delta t} t^{-\alpha} \|u_0\|_X, \quad t > 0,
\end{aligned}
\tag{2.7}
$$

for some $\delta \in \mathbb{R}$, $M \geq 1$.

In particular, if $B_j \equiv 0$, $j = 1, \ldots, N$, in order to have $\delta > 0$, $\lambda$ can be taken any positive number. On the other hand, if dispersion is present, $\lambda$ has to be taken large enough for the semigroup to decay exponentially.

From the previous results, we can extend $A$ to the operator $L = \mathrm{diag}(L_1, \cdots, L_m)$, between $X^{1/2} = H^1(\Omega, \mathbb{R}^m)$ and its dual, $X^{-1/2}$, which is also sectorial. The analytic semigroup generated by $-L$ in $X^{-1/2}$, $e^{-Lt}$, is the unique extension of $e^{-At}$ to this space and is such that (2.7) also holds.

As shown in [35, 36], for solving problems with non-homogeneous boundary conditions, it is natural to consider a special class of elements $h \in H^{-1}(\Omega)$ defined as

$$< h, \phi >_{-1,1} = < f, \phi >_\Omega + < g, \gamma(\phi) >_\Gamma$$

for every $\phi \in H^1(\Omega)$, where $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$. So, for short, $h \stackrel{def}{=} f_\Omega + g_\Gamma$.

All along the paper and specially in proofs, we will denote by $c_i$ generic positive constants, whose values are irrelevant for the results.

# 3 Local well posedness

In this section we consider the local well posedness of problems (1.1). To that end, we will make some hypotheses on the non-linearities $f$ and $g$ that will allow us to make use of general abstract results for parabolic evolution equations, described below.

Assume $A$ is a sectorial operator in a Hilbert space $X$. We can define the fractional powers $A^\alpha$ of $A$, and the fractional power spaces $X^\alpha := D(A^\alpha)$, endowed with the graph norm, $\alpha \in \mathbb{R}$, where $X^{-\alpha} = (X^\alpha)'$, for $\alpha > 0$. Even more, $A$ is sectorial in $X^\alpha$ with domain $X^{\alpha+1}$, for any $\alpha$, see [2, 3, 21, 34].

**Theorem 3.1** *With the above notations, assume $h : X^\alpha \to X^\beta$ is locally Lipschitz and bounded on bounded sets, where $0 \le \alpha - \beta < 1$. Then, the abstract parabolic problem*

$$\begin{aligned} &u_t + Au + h(u) = 0 \\ &u(0) = u_0 \in X^\alpha \end{aligned} \tag{3.1}$$

*has a unique locally defined solution, given by the Variation of Constants Formula*

$$u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)}h(u(s))\,ds.$$

*where $e^{-At}$ denotes the analytic semigroup generated by $-A$. Moreover, $u$ verifies,*

$$u \in C([0,T), X^\alpha) \cap C(0,T,X^{\beta+1}), \quad u_t \in C(0,T,X^\gamma)$$

*for every $\gamma < \beta + 1$ and the equation is verified in $X^\beta$. Even more, either the solution is defined for all $t \ge 0$ or it blows up, in $X^\alpha$ norm, in finite time.* □

We will apply this result to the operator $L$ introduced in the previous section and for this, we consider non-linear mappings of the form

$$h(u) := f_\Omega(u) + g_\Gamma(u)$$

where, at least, $f_\Omega : X^\alpha \to L^2(\Omega, \mathbb{R}^m)$ and $g_\Gamma : X^\alpha \to H^{-1/2}(\Gamma, \mathbb{R}^m)$, for some $\alpha \ge 0$. Note that $h$ acts on test functions $\phi \in H^r(\Omega, \mathbb{R}^m)$, for $r > 1/2$, as $< h(u), \phi > = < f_\Omega(u), \phi >_\Omega + < g_\Gamma(u), \gamma(\phi) >_\Gamma$.

Depending on extra regularity properties of $g_\Gamma$, to be made precise below, we will get

$$h : X^\alpha \to X^\beta$$

for suitably chosen $\alpha > 0$ and $\beta \le 0$. But first, we will show some natural a priori requirements on the exponents $\alpha$ and $\beta$. Recall that for the abstract result we need $0 \le \alpha - \beta < 1$.

On the one hand, since we want to give account of non-homogeneous terms on the boundary, i.e. we consider the case $g \ne 0$, that implies $\beta < 0$. Otherwise, we can always take $\beta = 0$. Since, from the results in [35], we are interested in reading the equation in $H^{-1}(\Omega, \mathbb{R}^m)$, then we need $0 > \beta \geq -1/2$. Also, as shown below, for obtaining energy estimates on the solution, we are interested in having enough regularity to have $u, u_t \in H^1(\Omega, \mathbb{R}^m)$, for $t > 0$, and for this one needs, according to the smoothing effect in Theorem 3.1, $\beta + 1 > 1/2$ and then $\beta > -1/2$.

On the other hand, if we want the non-linear term $g(u)$ to depend on the values of $u$ on $\Gamma$, then we must have $\alpha > 1/4$ in order to have the trace of $u$ well defined; otherwise we can take $\alpha \geq 0$. In case we want to have initial data at least in $H^1(\Omega, \mathbb{R}^m)$, we must require $\alpha \geq 1/2$.

Finally, note that, in the case of non-zero terms on the boundary, there is another natural upper bound for $\alpha$ and $\beta$. In fact, $\alpha < 3/4$, since for $\alpha \geq 3/4$ the space $X^\alpha$ incorporates the boundary condition $\frac{\partial u}{\partial \vec{n}_a} = 0$. Also, from the smoothing result in Theorem 3.1, we must also have $\beta + 1 < 3/4$, i.e. $\beta < -1/4$.

Summarizing, if $g \neq 0$, then

$$3/4 > \alpha > 1/4, \quad -1/4 > \beta > -1/2, \quad \text{and} \quad 0 \leq \alpha - \beta < 1$$

while if $g = 0$, we have the standard case $\beta = 0$, $0 \leq \alpha < 1$.

Then we have

**Theorem 3.2** *Assume $f_\Omega$, $g_\Gamma$, $\alpha$ and $\beta$ are as above and*

$$h : X^\alpha \to X^\beta$$

*is locally Lipschitz and bounded on bounded sets. Then, for every $u_0 \in X^\alpha$, there exists a unique, locally defined solution of*

$$u_t + L(u) + h(u) = 0, \quad u(0) = u_0 \tag{3.2}$$

*given by*

$$u(t) = e^{-Lt}u_0 - \int_0^t e^{-L(t-s)} h(u(s))\, ds.$$

*which verifies*

$$u \in C([0, T), X^\alpha) \cap C(0, T, X^{\beta+1}), \quad u_t \in C(0, T, X^\gamma)$$

*for every $\gamma < \beta + 1$ and*

$$\int_\Omega u_t \phi + \int_\Omega a \nabla u \nabla \phi + \sum_{j=1}^N \int_\Omega B_j(x) \frac{\partial u}{\partial x_j} \phi + \lambda \int_\Omega u\phi + \int_\Omega f(u)\phi + < g(u), \gamma(\phi) >_\Gamma = 0 \tag{3.3}$$

*for every $\phi \in H^1(\Omega, \mathbb{R}^m)$. In particular, we have*

$$\begin{aligned} & u_t - Div(a(x)\nabla u) + \sum_{j=1}^N B_j(x) \frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0, \quad \text{on } \Omega \\ & \frac{\partial u}{\partial \vec{n}_a} + g(u) = 0, \quad \text{on } \Gamma \end{aligned} \tag{3.4}$$

*and either the solution is defined for all $t > 0$ or it blows up, in $X^\alpha$ norm, in finite time.*

*In particular, assume*

$$\begin{aligned} & f_\Omega : X^\alpha \to L^2(\Omega, \mathbb{R}^m) \\ & g_\Gamma : X^\alpha \to L^2(\Gamma, \mathbb{R}^m) \text{ or even } H^{-r}(\Gamma, \mathbb{R}^m) \text{ with } 0 \leq r < 1/2 \end{aligned}$$

*are locally Lipschitz non-linear functions, for $\alpha + \frac{r}{2} < 3/4$. Then, there exists $\beta$ such that $-1/4 > \beta > -1/2$ verifying all the above.*

**Proof:** From Theorem 3.1, we have the existence and regularity parts of the statement. Since, $\beta > -1/2$, the equation

$$u_t + L(u) + h(u) = 0$$

holds as an equality in $X^\beta$ and also in $H^{-1}(\Omega, \mathbb{R}^m)$. But from the smoothing effect in Theorem 3.1 and since $\beta + 1 > 1/2$, for $t > 0$, $u(t) \in H^1(\Omega, \mathbb{R}^m)$ and $u_t(t) \in L^2(\Omega, \mathbb{R}^m)$ and then form the previous equation we get (3.3), by taking a test function $\phi \in H^1(\Omega, \mathbb{R}^m)$. Furthermore, we read the equation as a diagonal elliptic system

$$L(u) = -(u_t + f(u))_\Omega - g_\Gamma(u)$$

and, from elliptic regularity theory, we get (3.4).

Note that if $g_\Gamma$ takes values in $L^2(\Gamma, \mathbb{R}^m)$ then $h = f_\Omega + g_\Gamma$ is well defined acting on test functions in $H^s(\Omega, \mathbb{R}^m) \supset X^{s/2}$ for every $s > 1/2$, therefore $h \in X^{-s/2}$ and we can take $-1/2 < \beta = -s/2 < -1/4$. Finally, if $g_\Gamma$ takes values in $H^{-r}(\Gamma, \mathbb{R}^m)$, with $0 < r < 1/2$, then $h$ is well defined acting on test functions in $H^{r+1/2}(\Omega, \mathbb{R}^m) \supset X^{r/2+1/4}$ and therefore $h \in X^{-r/2-1/4}$ and we can take $-1/2 < \beta = -r/2 - 1/4 < -1/4$. In order to apply Theorem 3.1, we need $\alpha - \beta = \alpha + \frac{r}{2} + \frac{1}{4} < 1$, which leads to the condition $\alpha + \frac{r}{2} < 3/4$.□

Observe that if the assumptions of Theorem 3.2 are verified, then this result allows one to define a non-linear semigroup $S(t)$ in $X^\alpha$, such that $S(t)u_0 = u(t)$, the unique solution of (1.1).

The next two subsections are devoted to show that these assumptions are verified for a class of non-linearities in some $X^\alpha$ space.

## 3.1 The case $H^1(\Omega, \mathbb{R}^m)$, $N \leq 3$ and growth conditions

Assume now that $f, g : \mathbb{R}^m \to \mathbb{R}^m$ are $C^1$ and $C^2$ functions, respectively, satisfying (1.2) or (1.3) and (1.4) or (1.5) respectively. We denote by $f_\Omega$ and $g_\Omega$ the composition Nemitsky operators defined by $f$ and $g$, for functions defined on $\Omega$, while we denote by $g_\Gamma$ the Nemitsky operator defined by $g$ for functions defined on $\Gamma$. Observe that if the trace of $u$ and $g_\Omega(u)$ are defined then

$$\gamma(g_\Omega(u)) = g_\Gamma(\gamma(u))$$

We will show below that, under the growth assumptions (1.2)–(1.5), the maps $f_\Omega$ and $g_\Gamma$ are such that $h = f_\Omega + g_\Gamma$, verifies the assumptions of Theorem 3.2 above in $H^1(\Omega, \mathbb{R}^m)$.

For the case $N = 2$, we will make use of the following result due to N. S. Trudinger, [39, 29].

**Lemma 3.1** *There exist two positive constants $\sigma$ and $K$ such that if $\|u\|_{H^1(\Omega)} \leq 1$ then,*

$$\|e^{\sigma|u(\cdot)|^2}\|_{L^2(\Omega)} \leq K. \tag{3.5}$$

*Furthermore, the constant $\sigma$ is bounded above by $2\pi$.*□

With this result we obtain the following.

**Lemma 3.2** *Assume $f$ verifies (1.2), (1.3), then the mapping*

$$f_\Omega : H^1(\Omega, \mathbb{R}^m) \to L^p(\Omega, \mathbb{R}^m)$$

*is Lipschitz continuous and bounded on bounded subsets of $H^1(\Omega, \mathbb{R}^m)$, for $p = \infty$, if $N = 1$, for any $1 \leq p < \infty$, if $N = 2$ and for $p = 2$ if $N = 3$.*

**Proof:** The case $N = 1$ is obvious, since for $\phi, \psi$ in $H^1(\Omega, \mathbb{R}^m)$ such that $\|\phi\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$, we get

$$\|f(\phi) - f(\psi)\|_{L^\infty(\Omega,\mathbb{R}^m)} \leq c_1(r)\|\phi - \psi\|_\infty \leq c_2(r)\|\phi - \psi\|_{H^1(\Omega)}$$

where we used that the derivative of $f$ is bounded on bounded sets and the embedding $H^1(\Omega, \mathbb{R}^m) \subset L^\infty(\Omega, \mathbb{R}^m)$.

We now prove the case $N = 2$. Let $r > 0$ and $\phi, \psi$ be functions in $H^1(\Omega, \mathbb{R}^m)$ such that $\|\phi\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$ and $\|\psi\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$. Choose $\eta > 0$ such that for $p \geq 1$, $\eta < \frac{\sigma}{2pr^2}$, with $\sigma$ as in Lemma 3.1. Then, from (1.2), there exists $c_\eta > 0$ such that $|f(u) - f(v)|^p \leq c_\eta^p (e^{\eta|u|^2} + e^{\eta|v|^2})^p |u - v|^p$, for every $u, v \in \mathbb{R}^m$ and then

$$\|f(\phi) - f(\psi)\|^p_{L^p(\Omega,\mathbb{R}^m)} \leq c_\eta^p \int_\Omega \left[e^{\eta|\phi(x)|^2} + e^{\eta|\psi(x)|^2}\right]^p |\phi(x) - \psi(x)|^p \, dx \leq$$

$$\leq c_\eta^p \left(\int_\Omega \left[e^{\eta|\phi(x)|^2} + e^{\eta|\psi(x)|^2}\right]^{2p} dx\right)^{\frac{1}{2}} \left(\int_\Omega |\phi(x) - \psi(x)|^{2p} \, dx\right)^{\frac{1}{2}} \leq$$

$$\leq c_1 \left(\int_\Omega \left[e^{\eta|\phi(x)|^2} + e^{\eta|\psi(x)|^2}\right]^{2p} dx\right)^{\frac{1}{2}} \|\phi - \psi\|^p_{H^1(\Omega,\mathbb{R}^m)}$$

where we have used the embedding $H^1(\Omega, \mathbb{R}^m) \subset L^{2p}(\Omega, \mathbb{R}^m)$. But, from Lemma 3.1 and the choice of $\eta$, $|\phi|$ and $|\psi|$ verify $\|e^{2p\,\eta|u(\cdot)|^2}\|_{L^2(\Omega)} \leq K$, and this concludes the case $N = 2$, since the integral term above is bounded by a constant depending only on $K$.

The case $N = 3$, is much simpler and well known but the proof is given for completeness. From (1.3), we have $|f(u) - f(v)|^2 \leq L^2(1 + |u|^2 + |v|^2)^2 |u - v|^2$ for all $u, v \in \mathbb{R}^m$ and then if $r > 0$, $\|\phi\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$, $\|\psi\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$, we have

$$\|f(\phi) - f(\psi)\|^2_{L^2(\Omega,\mathbb{R}^m)} \leq L^2 \int_\Omega (1 + |\phi|^2 + |\psi|^2)^2 |\phi - \psi|^2 \leq$$

$$\leq L^2 \left(\int_\Omega (1 + |\phi|^2 + |\psi|^2)^3\right)^{\frac{2}{3}} \left(\int_\Omega |\phi - \psi|^6\right)^{\frac{1}{3}} \leq c_2(r)\|\phi - \psi\|^2_{H^1(\Omega,\mathbb{R}^m)}$$

where we have used Hölder's inequality and the embedding $H^1(\Omega, \mathbb{R}^m) \subset L^6(\Omega, \mathbb{R}^m)$.$\square$

Assume now $g$ verifies (1.4), (1.5). As noted above, then $g$ also verifies (1.2) and (1.3), respectively, and then $g_\Omega$ verifies Lemma 3.2. However, the function $g_\Omega$ has better properties. As observed above, if the trace of $g_\Omega(u)$ is defined, we have $\gamma(g_\Omega(u)) = g_\Gamma(\gamma(u))$. Now, we have

**Lemma 3.3** *If $g$ verifies (1.4), (1.5), the map*

$$g_\Omega : H^1(\Omega, \mathbb{R}^m) \to W^{1,q}(\Omega, \mathbb{R}^m)$$

*is Lipschitz continuous and bounded on bounded subsets of $H^1(\Omega\, \mathbb{R}^m)$ for any $1 \leq q \leq 2$, if $N = 1$, for any $1 \leq q < 2$, if $N = 2$ and for any $1 \leq q \leq 6/(4 + \rho)$, if $N = 3$.*

**Proof:** The case $N = 1$ follows as the case $N = 2$ below, with $q = 2$ and $p = \infty$. If $N = 2$, we first show that $g_\Omega : H^1(\Omega, \mathbb{R}^m) \to W^{1,q}(\Omega, \mathbb{R}^m)$ is a bounded map, for any $1 \leq q < 2$. Since, (1.4) implies that $Dg$ also verifies Lemma 3.2, then if $\|u\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$, we get $\|g(u)\|_{L^q(\Omega,\mathbb{R}^m)} \leq c_1(r, q)$, $\|Dg(u)\|_{L^q(\Omega,\mathbb{R}^{m^2})} \leq c_2(r, q)$ and $\|Dg(u) - Dg(v)\|_{L^q(\Omega,\mathbb{R}^{m^2})} \leq c_3(r, q)\|u - v\|_{H^1(\Omega,\mathbb{R}^m)}$ for any $1 \leq q < \infty$.

Observe that now, if $1 \leq q < 2$ and $p$ is chosen such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$,

$$\|Dg(u)\, \nabla u\|_{L^q(\Omega,\mathbb{R}^m)} \leq \|Dg(u)\|_{L^p(\Omega,\mathbb{R}^{m^2})} \|\nabla u\|_{L^2(\Omega,\mathbb{R}^m)} \leq r c_2(r, p) = c_4(r, q).$$

Therefore, $g_\Omega$ is a bounded map from $H^1(\Omega, \mathbb{R}^m)$ into $W^{1,q}(\Omega, \mathbb{R}^m)$, $1 \leq q < 2$.

On the other hand, if again we choose $p$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$, we have

$$\|Dg(u)\nabla u - Dg(v)\nabla v\|_{L^q(\Omega,\mathbb{R}^m)} \leq \|Dg(u)(\nabla u - \nabla v)\|_{L^q(\Omega,\mathbb{R}^m)} + \|(Dg(u) - Dg(v))\nabla v\|_{L^q(\Omega,\mathbb{R}^m)} \leq$$

$$\leq \|Dg(u)\|_{L^p(\Omega,\mathbb{R}^{m^2})}\|\nabla u - \nabla v\|_{L^2(\Omega,\mathbb{R}^m)} + \|Dg(u) - Dg(v)\|_{L^p(\Omega,\mathbb{R}^{m^2})}\|\nabla v\|_{L^2(\Omega,\mathbb{R}^m)} \leq$$

$$\leq c_2(r, p)\|\nabla u - \nabla v\|_{L^2(\Omega,\mathbb{R}^m)} + r c_3(r, p)\|u - v\|_{H^1(\Omega,\mathbb{R}^m)} \leq c_5(r, p)\|u - v\|_{H^1(\Omega,\mathbb{R}^m)}$$

That concludes the case $N = 2$.

The case $N = 3$ is proved in the following way. From the previous Lemma, $g_\Omega$ is a bounded map from $H^1(\Omega, \mathbb{R}^m)$ into $L^q(\Omega, \mathbb{R}^m)$, for any $1 \leq q \leq 2$. Now, we prove that if $\|u\|_{H^1(\Omega,\mathbb{R}^m)} \leq r$, then $\|Dg(u)\nabla u\|_{L^q(\Omega,\mathbb{R}^m)}$ is bounded by a constant depending only on $r$, for some $q \in (1,2)$. For this, note that

$$\|Dg(u)\nabla u\|^q_{L^q(\Omega,\mathbb{R}^m)} = \int_\Omega |Dg(u)|^q |\nabla u|^q \leq \int_\Omega M|\nabla u|^q + L\left(\int_\Omega (1+|u|^{\rho+1})^{\frac{2q}{2-q}}\right)^{\frac{2-q}{2}} \left(\int_\Omega |\nabla u|^2\right)^{\frac{q}{2}}$$

and since $H^1(\Omega, \mathbb{R}^m) \subset L^6(\Omega, \mathbb{R}^m)$, we have to take $(\rho+1)\frac{2q}{2-q} \leq 6$, i.e. $q \leq \frac{6}{4+\rho}$ for the right hand side to be bounded.

For the lipchitzness, we have from (1.5)

$$\|Dg(u) - Dg(v)\|^{\frac{6}{1+\rho}}_{L^{\frac{6}{1+\rho}}(\Omega,\mathbb{R}^{m^2})} \leq L^{\frac{6}{1+\rho}} \int_\Omega (1+|u|+|v|)^{\rho\frac{6}{1+\rho}} |u-v|^{\frac{6}{1+\rho}} \leq$$

$$\leq L^{\frac{6}{1+\rho}} \left(\int_\Omega (1+|u|+|v|)^6\right)^{\frac{\rho}{1+\rho}} \left(\int_\Omega |u-v|^6\right)^{\frac{1}{1+\rho}} \leq c_6(r)\|u-v\|^{\frac{6}{1+\rho}}_{H^1(\Omega,\mathbb{R}^m)} \tag{3.6}$$

where we have used Hölder's inequality and, for the last inequality, the embedding $H^1(\Omega, \mathbb{R}^m) \subset L^6(\Omega, \mathbb{R}^m)$. With this

$$\|Dg(u)\nabla u - Dg(v)\nabla v\|_{L^{\frac{6}{4+\rho}}(\Omega,\mathbb{R}^m)} \leq \|Dg(u)(\nabla u - \nabla v)\|_{L^{\frac{6}{4+\rho}}(\Omega,\mathbb{R}^m)} + \|(Dg(u)-Dg(v))\nabla v\|_{L^{\frac{6}{4+\rho}}(\Omega,\mathbb{R}^m)} \leq$$

$$\leq \|Dg(u)\|_{L^{\frac{6}{1+\rho}}(\Omega,\mathbb{R}^{m^2})} \|\nabla u - \nabla v\|_{L^2(\Omega,\mathbb{R}^m)} + \|Dg(u)-Dg(v)\|_{L^{\frac{6}{1+\rho}}(\Omega,\mathbb{R}^{m^2})} \|\nabla v\|_{L^2(\Omega,\mathbb{R}^m)} \leq$$

$$\leq c_7(r)\|\nabla u - \nabla v\|_{L^2(\Omega,\mathbb{R}^m)} + c_8(r)\|u-v\|_{H^1(\Omega,\mathbb{R}^m)}$$

where we have used $\frac{1+\rho}{6} + \frac{1}{2} = \frac{4+\rho}{6}$ the last inequality follows from estimate (3.6).□

Then, we have

**Lemma 3.4** *If $g$ verifies (1.4), (1.5), then*
*i) If $N = 1$, $g_\Gamma : H^1(\Omega, \mathbb{R}^m) \to H^{1/2}(\Gamma, \mathbb{R}^m)$ is Lipschitz continuous on bounded sets,*
*ii) If $N = 2$, $g_\Gamma : H^1(\Omega, \mathbb{R}^m) \to H^r(\Gamma, \mathbb{R}^m)$ is Lipschitz continuous on bounded sets for $r < \frac{1}{2}$.*
*iii) If $N = 3$ and $\rho < 1$, $g_\Gamma : H^1(\Omega, \mathbb{R}^m) \to H^{-r}(\Gamma, \mathbb{R}^m)$ is Lipschitz continuous on bounded sets, for $\frac{1}{2} > r > \frac{\rho}{2}$.*

**Proof:** Since, from the lemma above, $g_\Omega : H^1(\Omega, \mathbb{R}^m) \to W^{1,q}(\Omega, \mathbb{R}^m)$ is Lipschitz on bounded sets, for some $q$, and using the trace theorem, it suffices to show that $\gamma : W^{1,q}(\Omega, \mathbb{R}^m) \to W^{1-\frac{1}{q},q}(\Gamma, \mathbb{R}^m) \subset H^{-r}(\Gamma, \mathbb{R}^m)$ is bounded, for some $r$ and such $q$.
i) The proof is obvious, since we can take $q = 2$.
ii) If $N = 2$ then $q \in (1,2)$, and $W^{1-\frac{1}{q},q}(\Gamma, \mathbb{R}^m) \subset H^r(\Gamma, \mathbb{R}^m)$ if $\frac{3}{2} - \frac{2}{q} > r$. Hence taking the largest possible value of $q$, we get $r < \frac{1}{2}$.
iii) If $N = 3$, then $q \leq q_0 = \frac{6}{4+\rho}$ and $W^{1-\frac{1}{q},q}(\Gamma, \mathbb{R}^m) \subset H^{-r}(\Gamma, \mathbb{R}^m)$ if $2 - \frac{3}{q} > -r$. Taking again the largest possible value of $q$, we get $r > \frac{\rho}{2}$ and we can take $r < \frac{1}{2}$ iff $\rho < 1$.□

Therefore, we get

**Theorem 3.3** *Assuming the growth conditions (1.2)-(1.5) and $\rho < 1$ if $N = 3$, (1.1) defines a local semigroup in $H^1(\Omega, \mathbb{R}^m)$.*

**Proof:** According to Lemmas 3.2 and 3.4, $h = f_\Omega + g_\Gamma$ is locally Lipschitz and bounded on bounded sets of $X^{1/2}$ into $H^{-r}(\Gamma, \mathbb{R}^m)$, with $r < 1/2$. Therefore, Theorem 3.2 applies with $\alpha = 1/2$.□

## 3.2 The case $H^s(\Omega, \mathbb{R}^m)$, $s<1$, $N \leq 3$ and non-critical growth conditions

In this section we will give a local existence result for initial values in $H^s(\Omega, \mathbb{R}^m)$, for some $s<1$, $N\leq 3$, assuming the non-linearities grow a little slower than in (1.2)–(1.5). Again no assumptions are made for $N=1$. This local existence result, apart of its intrinsic interest, will be of great help when proving the existence of attractors for (1.1) in Section 4.

More precisely, let $f, g : \mathbb{R}^m \to \mathbb{R}^m$ be $C^1$ and $C^2$ functions, respectively such that (1.8) holds. Assume also that $f$ and $g$ satisfy, instead of (1.2)–(1.5)

$$|f(u)-f(v)| \leq L(1+|u|^\gamma+|v|^\gamma)|u-v| \tag{3.7}$$

for all $u, v \in \mathbb{R}^m$ and $\gamma$ arbitrary, if $N=2$ and $\gamma<2$ if $N=3$, and

$$\begin{aligned} |g(u)-g(v)| &\leq L(1+|u|^{\rho+1}+|v|^{\rho+1})|u-v|, \\ |Dg(u)-Dg(v)| &\leq L(1+|u|^{\rho}+|v|^{\rho})|u-v|, \end{aligned} \tag{3.8}$$

for all $u, v \in \mathbb{R}^m$ and some $\rho<1$, for $N=2,3$.

The following results are refinements of Lemmas 3.2 and 3.3.

**Lemma 3.5** *If $f$ verifies (3.7), the map*

$$f_\Omega : H^s(\Omega, \mathbb{R}^m) \to L^p(\Omega, \mathbb{R}^m)$$

*is Lipschitz continuous on bounded subsets for $p=\infty$ if $N=1$ and $1>s>1/2$, for any $1\leq p<\infty$ if $1>s\geq 1-\frac{2}{p(\gamma+1)}$ and $N=2$ and for any $1\leq p<\frac{6}{\gamma+1}$ if $\gamma<5$, $1>s\geq \frac{3}{2}(1-\frac{2}{p(\gamma+1)})$ and $N=3$.*

**Proof:** The case $N=1$ is as in Lemma 3.2, since $H^s(\Omega,\mathbb{R}^m)\subset L^\infty(\Omega,\mathbb{R}^m)$ for $1>s>1/2$.

Let $R>0$ and $u,v\in H^s(\Omega,\mathbb{R}^m)$, $s<1$, be such that $\|u\|_{H^s(\Omega,\mathbb{R}^m)}\leq R$ and $\|v\|_{H^s(\Omega,\mathbb{R}^m)}\leq R$. Then,

$$\begin{aligned} \|f(u)-f(v)\|^p_{L^p(\Omega,\mathbb{R}^m)} &\leq L\int_\Omega (1+|u(x)|^{p\gamma}+|v(x)|^{p\gamma})\,|u(x)-v(x)|^p dx \\ &\leq c_1(1+\|u\|^{p\gamma}_{L^{p\gamma q}(\Omega,\mathbb{R}^m)}+\|v\|^{p\gamma}_{L^{p\gamma q}(\Omega,\mathbb{R}^m)})\|u-v\|^p_{L^{pq'}(\Omega,\mathbb{R}^m)}, \end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Therefore, if $q$ can be chosen such that $pq'$ and $p\gamma q$ are such that we can apply Sobolev inequalities, we get

$$\|f(u)-f(v)\|^p_{L^p(\Omega,\mathbb{R}^m)} \leq c_2\left(1+\|u\|^{p\gamma}_{H^s(\Omega,\mathbb{R}^m)}+\|v\|^{p\gamma}_{H^s(\Omega,\mathbb{R}^m)}\right)\|u-v\|^p_{H^s(\Omega,\mathbb{R}^m)} \leq c_3(R)\|u-v\|^p_{H^s(\Omega,\mathbb{R}^m)}$$

and the result is proved.

For $N=2$, from Sobolev inclusions, $H^s(\Omega,\mathbb{R}^m)\subset L^p(\Omega,\mathbb{R}^m)$ with continuous inclusion for any $p$ such that $s-1\geq -\frac{2}{p}$, i.e. if $\frac{1}{p}\geq\frac{1-s}{2}$. Therefore, the number $q$ can be chosen as above iff $s\geq 1-\frac{2}{p(1+\gamma)}$ which is compatible with $s<1$ for any $p$ and $\gamma$.

Finally, for $N=3$, from Sobolev inclusions, $H^s(\Omega,\mathbb{R}^m)\subset L^p(\Omega,\mathbb{R}^m)$ with continuous inclusion for any $p$ such that $s-\frac{3}{2}\geq -\frac{3}{p}$, i.e. if $\frac{1}{p}\geq\frac{3-2s}{6}$ and the condition on $q$ reduces to $s\geq\frac{3}{2}-\frac{3}{p(\gamma+1)}$, which is compatible with $s<1$ iff $1\leq p<\frac{6}{\gamma+1}$ and $\gamma<5$.□

**Remark 3.1** *Note that for proof above, it suffices $\gamma<5$ if $N=3$, which is a weaker restriction on $f$ than (3.7). Then, we can take $p=2$ if $1>s\geq\frac{3}{2}\frac{\gamma}{\gamma+1}$ and $\gamma<2$ as in (3.7).*

**Lemma 3.6** *If $g$ verifies (3.8), the map*

$$g_\Omega : H^s(\Omega,\mathbb{R}^m)\to W^{1,q}(\Omega,\mathbb{R}^m)$$

*is Lipschitz continuous and bounded on bounded subsets of $H^s(\Omega\,\mathbb{R}^m)$ for $1\leq q\leq\frac{2}{3-2s}$ if $N=1$ and $1>s>1/2$, for $1\leq q\leq\frac{2}{(1-s)(\rho+2)+1}$ if $N=2$ and $1>s\geq\frac{\rho+1}{\rho+2}$ and for $1\leq q\leq\frac{6}{(3-2s)(\rho+2)+2}$ if $1>s\geq\frac{2+3\rho}{2(\rho+2)}$ and $N=3$.*

**Proof:** Note that from the previous Lemma, $g_\Omega : H^s(\Omega, \mathbf{R}^m) \to L^q(\Omega, \mathbf{R}^m)$ is Lipschitz continuous on bounded subsets for $q = \infty$ if $N = 1$ and $1 > s > 1/2$, for any $1 \le q < \infty$ if $1 > s \ge 1 - \frac{2}{q(\rho+2)}$ and $N = 2$ and for any $1 \le q < \frac{6}{\rho+2}$ if $1 > s \ge \frac{3}{2}(1 - \frac{2}{q(\rho+2)})$ and $N = 3$. Now, if $\|u\|_{H^s(\Omega,\mathbf{R}^m)} \le R$ we have

$$\|Dg(u)\nabla u - Dg(v)\nabla v\|_{L^q(\Omega,\mathbf{R}^m)} \le \|Dg(u)(\nabla u - \nabla v)\|_{L^q(\Omega,\mathbf{R}^m)} + \|(Dg(u) - Dg(v))\nabla v\|_{L^q(\Omega,\mathbf{R}^m)}$$

$$\le c_5 \left( \int_\Omega (1 + |u|^{q(1+\rho)})|\nabla u - \nabla v|^q \right)^{\frac{1}{q}} + c_5 \left( \int_\Omega (1 + |u|^{q\rho} + |v|^{q\rho})|u - v|^q |\nabla v|^q \right)^{\frac{1}{q}}$$

We deal with each of the integrals in the right hand side of the above expression separately. For the first term, we get

$$\int_\Omega (1 + |u|^{q(1+\rho)})|\nabla u - \nabla v|^q \le c_6 \left(1 + \|u\|^{q(1+\rho)}_{L^{q(1+\rho)p'}(\Omega,\mathbf{R}^m)}\right) \|\nabla u - \nabla v\|^q_{L^{pq}(\Omega,\mathbf{R}^m)},$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, while for the second term we apply Hölder's inequality with three terms to get

$$\int_\Omega (1 + |u|^{q\rho} + |v|^{q\rho})|u - v|^q |\nabla v|^q \le c_7 \left(1 + \|u\|^{\rho}_{L^{q\rho t}(\Omega,\mathbf{R}^m)} + \|v\|^{\rho}_{L^{q\rho t}(\Omega,\mathbf{R}^m)}\right)^q \|\nabla v\|^q_{L^{pq}(\Omega,\mathbf{R}^m)} \|u - v\|^q_{L^{qr}(\Omega,\mathbf{R}^m)}$$

with $\frac{1}{p} + \frac{1}{r} + \frac{1}{t} = 1$.

Therefore, if $p$ in the first case and $p, r, t$ in the second case can be chosen such that for the exponents $pq$, $q(1+\rho)p'$ and $q\rho t$, $pq$ and $qr$, respectively, we can use the Sobolev inequalities we get

$$\int_\Omega (1 + |u|^{q(1+\rho)})|\nabla u - \nabla v|^q \le c_8 \left(1 + \|u\|^{q(1+\rho)}_{H^s(\Omega,\mathbf{R}^m)}\right) \|u - v\|^q_{H^s(\Omega,\mathbf{R}^m)},$$

and

$$\int_\Omega (1 + |u|^{q\rho} + |v|^{q\rho})|u - v|^q |\nabla v|^q \le c_9 \left(1 + \|u\|^{\rho}_{H^s(\Omega,\mathbf{R}^m)} + \|v\|^{\rho}_{H^s(\Omega,\mathbf{R}^m)}\right)^q \|v\|^q_{H^s(\Omega,\mathbf{R}^m)} \|u - v\|^q_{H^s(\Omega,\mathbf{R}^m)}$$

and we get the result.

As in Lemma 3.5, $H^s(\Omega, \mathbf{R}^m) \subset L^p(\Omega, \mathbf{R}^m)$, for $p = \infty$ if $1 > s > 1/2$ and $N = 1$, for any $p$ such that $\frac{1}{p} \ge \frac{1-s}{2}$ if $N = 2$ and for any $p$ such that $\frac{1}{p} \ge \frac{3-2s}{6}$ if $N = 3$. Also, $H^s(\Omega, \mathbf{R}^m) \subset W^{1,p}(\Omega, \mathbf{R}^m)$ for any $p$ such that $\frac{1}{p} \ge \frac{3-2s}{2}$ if $N = 1$, for any $p$ such that $\frac{1}{p} \ge \frac{2-s}{2}$ if $N = 2$ and for any $p$ such that $\frac{1}{p} \ge \frac{5-2s}{6}$ if $N = 3$.

For $N = 1$ we can take in the estimates above $p = 1$ and $t = r = \infty$ and $p = 1$ respectively, assumed that $q \le \frac{2}{3-2s}$, which is compatible with $q \ge 1$ since $1 > s > 1/2$.

For $N = 2$ it is easy to check that $p$ and $p, r, t$ can be chosen verifying all the above iff $q \le \frac{2}{(1-s)(\rho+2)+1}$ which is compatible with $q \ge 1$ iff $1 > s \ge 1 - \frac{1}{\rho+2} = \frac{\rho+1}{\rho+2}$.

Analogously, for $N = 3$ $p$ and $p, r, t$ can be chosen verifying all the above iff $q \le \frac{6}{(3-2s)(\rho+2)+2}$ which is compatible with $q \ge 1$ iff $s \ge \frac{2+3\rho}{2(\rho+2)}$ which in turn is compatible with $s < 1$ since $\rho < 2$.□

Then, we have, analogously to Lemma 3.4

**Lemma 3.7** *If $g$ verifies (3.8), then*

$$g_\Gamma : H^s(\Omega, \mathbf{R}^m) \to H^{-r}(\Gamma, \mathbf{R}^m)$$

*is Lipschitz continuous on bounded sets, if*
*i) $N = 1$, $1 > s > 1/2$ and $\frac{1}{2} > r > \frac{1-s}{2}$.*
*ii) $N = 2$, $1 > s > \frac{\rho+1}{\rho+2}$ and $\frac{1}{2} > r > (1-s)(\rho+2) - 1/2$.*
*iii) $N = 3$, $1 > s > \frac{3}{2}\frac{\rho+1}{\rho+2}$ and $\frac{1}{2} > r > \frac{(3-2s)(\rho+2)-2}{2}$.*

**Proof:** Since, from the lemma above, $g_\Omega : H^s(\Omega, \mathbb{R}^m) \to W^{1,q}(\Omega, \mathbb{R}^m)$ is Lipschitz on bounded sets, for some $q$, and using the trace theorem, it suffices to show that $\gamma : W^{1,q}(\Omega, \mathbb{R}^m) \to W^{1-\frac{1}{q},q}(\Gamma, \mathbb{R}^m) \subset H^{-r}(\Gamma, \mathbb{R}^m)$ is bounded, for some $r$ and such $q$. Also, $W^{1-\frac{1}{q},q}(\Gamma, \mathbb{R}^m) \subset H^{-r}(\Gamma, \mathbb{R}^m)$ if $1 - \frac{1}{q} > -r$ and $N = 1$, $\frac{3}{2} - \frac{2}{q} > -r$ and $N = 2$ and if $2 - \frac{3}{q} > -r$ if $N = 3$.
i) According to the previous lemma, if $N = 1$ the ranges for $s$ and $q$ are $1 \leq q \leq q_0 = \frac{2}{3-2s}$ and $1 > s > 1/2$. Therefore, taking the largest value of $q$ in the embedding above, we get $\frac{1-s}{2} < r < 1/2$.
ii) For $N = 2$ the ranges are $1 \leq q \leq q_0 = \frac{2}{(1-s)(\rho+2)+1}$ and $1 > s \geq \frac{\rho+1}{\rho+2}$. Proceeding as before we get $(1-s)(\rho+2) - 1/2 < r$, which is less than $\frac{1}{2}$ iff $1 > s > \frac{\rho+1}{\rho+2}$.
iii) For $N = 3$ the ranges are $1 \leq q \leq q_0 = \frac{6}{(3-2s)(\rho+2)+2}$ and $1 > s \geq \frac{2+3\rho}{2(\rho+2)}$. In this case we get $\frac{(3-2s)(\rho+2)-2}{2} < r$, which is less than $\frac{1}{2}$ iff $1 > s > \frac{3(\rho+1)}{2(\rho+2)}$. □

From all this, we get

**Theorem 3.4** *Assuming the growth conditions (3.7)–(3.8), Theorem 3.2 applies with* $\alpha = s/2$ *provided*
*i)* $N = 1$ *and* $1 > s > \frac{1}{2}$.
*ii)* $N = 2$ *and* $1 > s > max\{\frac{\gamma}{\gamma+1}, \frac{\rho+1}{\rho+2}\}$.
*iii)* $N = 3$ *and* $1 > s > \frac{3}{2} max\{\frac{\gamma}{\gamma+1}, \frac{\rho+1}{\rho+2}\}$.
*Therefore, (1.1) defines a local semigroup in* $H^s(\Omega, \mathbb{R}^m)$ *for* $s$ *in the ranges above.*

**Proof:** Note that the restrictions on $s$ determined by $\gamma$ imply, according to Lemma 3.5 that $f$ maps $H^s(\Omega, \mathbb{R}^m)$ into $L^2(\Omega, \mathbb{R}^m)$.

Since $X^{s/2} \subset H^s(\Omega, \mathbb{R}^m)$, from the results above and the proof of Theorem 3.2, we get that the non-linear term maps $X^{s/2}$ into $X^\beta$ for $\beta = -r/2 - 1/4$. Therefore, to apply the general results in Theorem 3.2, we need $s/2 - \beta < 1$, i.e. $s + r < 3/2$.

From the estimates on $r$ given in Lemma 3.7 this condition can be met if respectively
i) $N = 1$ and any $1 > s > \frac{1}{2}$.
ii) $N = 2$ and $s > \frac{\rho}{\rho+1}$ which holds true, since from Lemma 3.7, $1 > s > \frac{\rho+1}{\rho+2} > \frac{\rho}{\rho+1}$.
iii) $N = 3$ and $s > \frac{3\rho+1}{2(\rho+1)}$ which holds true, since from Lemma 3.7, $1 > s > \frac{3}{2}\frac{\rho+1}{\rho+2} > \frac{3\rho+1}{2(\rho+1)}$. □

## 3.3 The fractional power spaces case and no growth conditions

Note that the basic result Theorem 3.2 imposes some conditions on the non-linearities, namely that $h = f_\Omega + g_\Gamma$, maps $X^\alpha$ into $X^\beta$, and this, in turn, imposes some growth conditions on the mappings $f, g : \mathbb{R}^m \to \mathbb{R}^m$. Therefore, in order to obtain an existence result without any growth condition on the non-linearities we will work in

$$Y = X^\alpha \cap C(\overline{\Omega}, \mathbb{R}^m)$$

for $3/4 > \alpha > 1/4$, endowed with the norm

$$\|u\|_Y = \|u\|_{X^\alpha} + \|u\|_\infty$$

Observe that Theorem 3.2 is, in principle, not directly applicable in this context. However, note that then, if $N = 1$ and $\alpha > 1/4$ or if $N = 2$ and $3/4 > \alpha > 1/2$ then, from Sobolev inclusions, $X^\alpha \subset C(\overline{\Omega}, \mathbb{R}^m)$ and then $Y = X^\alpha$ and Theorem 3.2 can be applied. In fact, we have

**Lemma 3.8** *Assume that* $f : \mathbb{R}^m \to \mathbb{R}^m$ *and* $g : \mathbb{R}^m \to \mathbb{R}^m$ *are* $C^1$ *and* $C^2$ *functions, respectively.*
*If* $N = 1$ *and* $3/4 > \alpha > 1/4$ *or if* $N = 2$ *and* $3/4 > \alpha > 1/2$, *then*

$$f_\Omega : X^\alpha \to C(\overline{\Omega}, \mathbb{R}^m), \quad g_\Omega : X^\alpha \to H^1(\Omega, \mathbb{R}^m), \text{ and } g_\Gamma : X^\alpha \to H^{\frac{1}{2}}(\Gamma, \mathbb{R}^m)$$

*are Lipschitz continuous on bounded sets of* $X^\alpha$.

**Proof:** Let $u, v \in X^\alpha$ such that $\|u\|_{X^\alpha} \leq R$ and $\|v\|_{X^\alpha} \leq R$. Hence, we have

$$\|f(u) - f(v)\|_\infty \leq c_1(R)\|u - v\|_\infty \leq c_2(R)\|u - v\|_{X^\alpha}$$

and the same for $g$. Moreover,

$$\|Dg(u)\nabla u - Dg(v)\nabla v\| \leq c_3(R)\|u - v\|_\infty\|\nabla u\| + c_4(R)\|\nabla u - \nabla v\| \leq c_5(R)\|u - v\|_{X^\alpha}$$

and then $g_\Omega : X^\alpha \to H^1(\Omega, \mathbb{R}^m)$ is Lipschitz on bounded sets of $X^\alpha$. Using the trace operator, we get the result.□

**Remark 3.2** *Note that if $g$ is only of class $C^1$, then the same proof above for $f$, but only with points in the boundary, gives the Lipschitzness of $g_\Gamma : X^\alpha \to L^\infty(\Gamma, \mathbb{R}^m)$ and Theorem 3.2 can still be applied. In this case solutions will be less regular.*

As a consequence of the above and Theorem 3.2, we get

**Theorem 3.5** *With the above assumptions, (1.1) defines a local semigroup in $X^\alpha$, if $N = 1$ and $3/4 > \alpha > 1/4$ or if $N = 2$ and $3/4 > \alpha > 1/2$.*

Now we come back to the cases not covered by the previous result, i.e. $N = 2$ and $1/2 \geq \alpha > 1/4$ or $N \geq 3$ and $3/4 > \alpha 1/4$. Assuming $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ are $C^1$ and $C^2$ functions, respectively, and in view of Theorem 3.2, we will look for solutions of (1.1) which are functions $u \in C([0, T], Y)$, that are solutions of the fixed point problem

$$u(t) = e^{-Lt}u_0 - \int_0^t e^{-L(t-s)}h(u(s))\, ds.$$

with $h = f_\Omega + g_\Gamma$. First note that this equation is equivalent to

$$u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)}f_\Omega(u(s))\, ds - \int_0^t e^{-L(t-s)}g_\Gamma(u(s))\, ds$$

as soon as $u_0, f(u(s)) \in L^2(\Omega, \mathbb{R}^m)$.

To prove the existence result we will make use of the following result that has been proven in [36]

**Proposition 3.1** *Let $X$ be a Banach space and $(A, D(A))$ a sectorial operator in $X$ and consider problem*

$$\begin{cases} u_t + Au = g(t) \\ u(0) = u_0 \end{cases} \qquad (3.9)$$

*with $g \in L^p(0, T, X^\beta)$, $u_0 \in X^\beta$ and $1 \leq p \leq \infty$, $T < \infty$.*

*Then, the solution verifies*

*a) $u \in C(0, T, X^\gamma)$ for all $\gamma < \beta + 1/p'$. Moreover, if $u_0 \in X^\gamma$, then $u \in C([0, T], X^\gamma)$.*

*b) The mapping*

$$X^\gamma \times L^p(0, T, X^\beta) \ni (u_0, g) \mapsto u \in C([0, T], X^\gamma)$$

*is Lipschitz. Moreover, if $Re(\sigma(A)) > \delta > 0$ then that holds also for $T = \infty$. In that case $u \in C_b([0, \infty), X^\gamma)$.*□

With this, we can state

**Theorem 3.6** *With the above assumptions (1.1) defines a local semigroup in $Y$. Moreover, solutions are defined for all $t \geq 0$ or they blow-up, in the $Y$ norm, in finite time.*

**Proof:** Define, for $u_0 \in Y$ and $u \in C([0,T],Y)$

$$\mathcal{F}u(t) = e^{-At}u_0 - \int_0^t e^{-A(t-s)} f_\Omega(u(s))\, ds - \int_0^t e^{-L(t-s)} g_\Gamma(u(s))\, ds$$

First, note that, $f_\Omega(u) \in C([0,T], C(\overline{\Omega}, \mathbb{R}^m))$ and $g_\Gamma(u) \in C([0,T], C(\Gamma, \mathbb{R}^m))$ and then, in particular, $h(u) \in C([0,T], X^\beta)$, for any $\beta \in [-1/2, -1/4)$. Therefore, from Proposition 3.1, we get

$$\mathcal{F}(u) \in C([0,T], X^\alpha) \cap C(0,T,X^\gamma), \text{ for every } \gamma < 3/4.$$

Now, observe that since $u_0 \in X^\alpha \cap C(\overline{\Omega}, \mathbb{R}^m)$, from the fact that $e^{-At}$ defines an analytic semigroup in $X^\alpha$, [21], and in $C(\overline{\Omega}, \mathbb{R}^m)$, [28], we get $e^{-At}u_0 \in C([0,T],Y)$ and it s a classical solution of the linear homogeneous problem (both in the equation and in the boundary conditions).

Now, we define $\mathcal{F}_1 u(t) = -\int_0^t e^{-A(t-s)} f_\Omega(u(s))\, ds$ and $\mathcal{F}_2 u(t) = -\int_0^t e^{-L(t-s)} g_\Gamma(u(s))\, ds$. Then, since $f_\Omega(u) \in C([0,T], C(\overline{\Omega}, \mathbb{R}^m))$, the results in [28], combined with Proposition 3.1, give us $\mathcal{F}_1(u) \in C([0,T], C^{2\sigma}(\overline{\Omega}, \mathbb{R}^m))$ for every $\sigma < 1$.

Finally, from a classical result by Pogorzelsky, [31], see also [1], since $g_\Gamma(u) \in C([0,T], C(\Gamma, \mathbb{R}^m))$, we have $\mathcal{F}_2(u) \in C^{\theta,\theta/2}(\overline{\Omega} \times [0,T], \mathbb{R}^m)$, for every $\theta \in (0,1)$ and moreover

$$\|\mathcal{F}_2(u)\|_{\theta,\theta/2} \le C(\theta) \sup_{t,y} \|g_\Gamma(u)(t,y)\|$$

In particular, $\mathcal{F}_2(u) \in C([0,T], C(\overline{\Omega}, \mathbb{R}^m))$.

Putting this information together we get, in particular, $\mathcal{F}(u) \in C([0,T],Y)$. Therefore, it is legitimate to look for fixed points of $\mathcal{F}$ in $C([0,T],Y)$. For this, we will prove that for a given $u_0 \in Y$, $\mathcal{F}$ is a contraction in a suitable closed subset of $C([0,T],Y)$, for small enough $T$. Once, this is done, classical continuation arguments conclude the proof.

More precisely, for a given $u_0 \in Y$, we denote $V = \{u \in C([0,T],Y),\ \|u(t) - u_0\|_Y \le r,\ \forall\, t \in [0,T]\}$, where $r$ and $T$ are to be chosen below.

First we prove that $\mathcal{F}$ maps $V$ into itself. For this, note that for a given $r$, $T$ can be chosen small enough, such that

$$\|e^{-At}u_0 - u_0\| \le r/3$$

for all $t \in [0,T]$, for the norms of both $X^\alpha$ and $C(\overline{\Omega}, \mathbb{R}^m)$, since $e^{-At}$ is a continuous semigroup in both spaces. Also, from [21], and taking any $\beta \in [-1/2, -1/4)$ such that $0 \le \alpha - \beta < 1$,

$$\| \int_0^t e^{-L(t-s)} h(u(s))\, ds \|_{X^\alpha} \le \left( \int_0^t C(t-s)^{-(\alpha-\beta)}\, ds \right) c_1(r)$$

since $h(u(s))$ takes values in a bounded subset of $X^\beta$. Consequently, the expression above can be made smaller that $r/3$, for all $t \in [0,T]$, if $T$ is small enough.

On the other hand, arguing now in $C(\overline{\Omega}, \mathbb{R}^m)$

$$\|\mathcal{F}_1(u)(t)\|_\infty = \| \int_0^t e^{-A(t-s)} f_\Omega(u(s))\, ds\|_\infty \le \int_0^t \|e^{-A(t-s)}\|_{\infty,\infty} \|f(u(s))\|_\infty\, ds \le t c_2(r)$$

and again, this can be made smaller that $r/3$, for all $t \in [0,T]$, if $T$ is small enough. Finally, since $\mathcal{F}_2(u) \in C^{\theta,\theta/2}(\overline{\Omega} \times [0,T], \mathbb{R}^m)$, for every $\theta \in (0,1)$ and $\mathcal{F}_2(u)(0,x) = 0$, we get, for every $x \in \overline{\Omega}$ and $t \in [0,T]$

$$|\mathcal{F}_2(u)(t,x)| \le t^{\theta/2} \|\mathcal{F}_2(u)\|_{\theta,\theta/2} \le t^{\theta/2} C(\theta) \sup_{t,y} \|g_\Gamma(u)(t,y)\| \le t^{\theta/2} c_3(r,\theta)$$

and again, this can be made smaller that $r/3$, uniformly in $x \in \overline{\Omega}$ and $t \in [0,T]$, if $T$ is small enough. Consequently, $\mathcal{F}(V) \subset V$, if $T$ is small enough.

Now, we prove that $\mathcal{F}$ is a contraction in $V$. For this, let $u, v \in V$. Then,

$$\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_{X^\alpha} \le \int_0^t C(t-s)^{-(\alpha-\beta)} \|h(u(s)) - h(v(s))\|_\beta\, ds \le$$

$$\leq \left(\int_0^t C(t-s)^{-(\alpha-\beta)}\,ds\right) c_4(r) \sup_{[0,T]} \|u(s)-v(s)\|_\infty$$

Therefore, for small enough $T$, we get for all $t \in [0,T]$

$$\|\mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|_{X^\alpha} \leq 1/4 \sup_{[0,T]} \|u(s)-v(s)\|_Y$$

On the other hand, note that $\mathcal{F}(u) - \mathcal{F}(v) = \mathcal{F}_1(u) - \mathcal{F}_1(v) + \mathcal{F}_2(u) - \mathcal{F}_2(v)$ and then

$$\|\mathcal{F}_1(u)(t) - \mathcal{F}_1(v)(t)\|_\infty \leq \int_0^t \|e^{-A(t-s)}\|_{\infty,\infty} \|f(u(s)) - f(v(s))\|_\infty\,ds \leq$$

$$\leq tc_5(r) \sup_{[0,T]} \|u(s)-v(s)\|_\infty \leq 1/8 \sup_{[0,T]} \|u(s)-v(s)\|_Y$$

for small enough $T$. Finally, as before, for small enough $T$

$$\|\mathcal{F}_2(u)(t) - \mathcal{F}_2(v)(t)\|_\infty \leq t^{\theta/2} C(\theta) \sup_{t,y} \|g_\Gamma(u)(t,y) - g_\Gamma(v)(t,y)\| \leq$$

$$\leq t^{\theta/2} c_6(r,\theta) \sup_{[0,T]} \|u(s)-v(s)\|_\infty \leq 1/8 \sup_{[0,T]} \|u(s)-v(s)\|_Y$$

Putting all these together, we get $\|\mathcal{F}(u) - \mathcal{F}(v)\|_V \leq 1/2\|u-v\|_V$ and the result is proven.□

Concerning regularity, we have

**Corollary 3.1** *With the above notations, for every $u_0 \in Y$, the solution verifies*

$$u \in C([0,T),Y) \cap C(0,T,X^\gamma), \;\; u_t \in C(0,T,X^\gamma), \;\; \textit{for any } \gamma < 3/4$$

*and (3.3) and (3.4) hold also true, while the solution exist.*

**Proof:** Taking any $\beta \in [-1/2,-1/4)$ such that $0 \leq \alpha - \beta < 1$, i.e. any $\beta \in (\alpha-1,-1/4)$, as in [21, Lemma 3.3.2], we prove $t \mapsto u(t) \in X^\alpha$ is locally Hölder. With this, we get $t \mapsto h(u(t)) \in X^\beta$ is also locally Hölder, since, for small $h$,

$$|f(u(t+h,x)) - f(u(t,x))| \leq c_1|u(t+h,x) - u(t,x)|, \text{ for all } x \in \Omega,$$

$$|g(u(t+h,x)) - g(u(t,x))| \leq c_2|u(t+h,x) - u(t,x)|, \text{ for all } x \in \Gamma$$

Integrating the first inequality in $\Omega$ we get $\|f(u(t+h)) - f(u(t))\|_{L^2(\Omega,R^m)} \leq c_1\|u(t+h) - u(t)\|_{X^\alpha}$, while integrating the second one in $\Gamma$, we get $\|g(u(t+h)) - g(u(t))\|_{L^2(\Gamma,R^m)} \leq c_2\|u(t+h) - u(t)\|_{L^2(\Gamma,R^m)} \leq c_3\|u(t+h) - u(t)\|_{X^\alpha}$.

Now, [21, Theorem 3.2.2] implies that $u$ is a strong solution in $X^\beta$. But even more, [21, Theorem 3.5.2] implies $u_t \in C(0,T,X^\gamma)$, for any $\gamma < \beta + 1$. Since $\beta$ is arbitrary in $(\alpha-1,-1/4)$, we get the result.□

# 4 Existence of global attractors

In this section we prove that under the dissipativeness assumption (1.8), (1.9), the system (1.1) has a global attractor. Note that from the diagonal structure of (1.9), it reduces to solve $m$ scalar eigenvalue problems. From the results in [32, 33, 23], we have that the first eigenvalue of this problem is always real, where by first we mean that all others have greater real part, see Section 6. Also, the first eigenvalue, $\mu_1^0$, is given by the infimum of the first eigenvalues of each scalar eigenvalue problem.

## 4.1 The fractional power spaces case

In this section we first prove that solutions of (1.1) with initial data in $Y = X^\alpha \cap C(\overline{\Omega}, \mathbb{R}^m)$, are globally defined and orbits of bounded subsets of $Y$, under the semigroup determined by (1.1), $S(t)$, are also bounded in $Y$. Moreover, we will show the semigroup is compact.

Second, we will show that the semigroup is dissipative in $Y$ and for this, we will find "contracting regions", similar to the contracting rectangles considered in [4]. With these and the results in [16], we will obtain the existence of a compact attractor in $Y$.

It is worth noting that the whole idea for proving boundedness and dissipativeness is that estimates on the sup norm are transferred to estimates in the $Y$ norm, as shown below.

We first start with the following important remark.

**Lemma 4.1** *With the assumptions of Section 3 and in particular with the notations of Theorems 3.2, 3.5 and 3.6, let $u(t)$ be a local solution of (1.1) in $Y = X^\alpha \cap C(\overline{\Omega}, \mathbb{R}^m)$ defined in a maximal time interval $[0, t_{max})$.*

*If $\|u(t)\|_{C(\overline{\Omega}, \mathbb{R}^m)} \le c_1$, for all $t \in [0, t_{max})$, for some $c_1 > 0$, then, $t_{max} = +\infty$ and $\|u(t)\|_{X^\alpha} \le c_2$, for all $t \ge 0$ and some $c_2$.*

*Even more, if a bounded set $B \subset Y$ is such that $\{S(t)B, t \ge 0\}$ is bounded in $C(\overline{\Omega}, \mathbb{R}^m)$, then $\{S(t)B, t \ge 0\}$ is bounded in $Y$ and $\{S(t)B, t \ge 1\}$ is compact in $Y$.*

**Proof:** First note that by replacing $A$ and $f$ by $A + aI$ and $f - aI$, we can always assume that (2.7) holds with $\delta > 0$. Then, using the variation of constants formula, we obtain, for $u_0 = u(0)$,

$$\|u(t)\|_{X^\alpha} \le M\|u_0\|_\alpha e^{-\delta t} + M \int_0^t e^{-\delta(t-s)}(t-s)^{-(\alpha-\beta)}\|h(u(s)\|_\beta ds.$$

for some $\beta \le 0$. Therefore, for finite $t \le t_{max}$, we have

$$\|u(t)\|_{X^\alpha} \le M\|u_0\|_\alpha e^{-\delta t} + MK \int_0^t e^{-\delta(t-s)}(t-s)^{-(\alpha-\beta)}ds. \tag{4.1}$$

with $K = \sup_{t\in[0,t_{max})}\{\|h(u(t))\|_\beta\}$, which is finite since $\|u(t)\|_{C(\overline{\Omega}, \mathbb{R}^m)} \le c_1$. Therefore, since the norm in $X^\alpha$, and hence the norm in $Y$, remains bounded in finite time, then the solutions must exist for $t \ge 0$. Moreover, the right hand side of (4.1) remains uniformly bounded in time and the result is proved.

Analogously, if $u_0 \in B$, then the right hand side of (4.1) remains uniformly bounded in time and $u_0 \in B$ and consequently, $\{S(t)B, t \ge 0\}$ is bounded in $Y$. Even more, from the fact that $L$ has compact resolvent, $\{S(t)B, t \ge 0\}$ is bounded in $Y$ and the variation of constant formula, we get as in [16, Theorem 4.2.2], that $\{S(t)B, t \ge 1\}$ is compact in $X^\alpha$. On the other hand, since, with the notations in Theorem 3.6, $S(t) = S_1(t) + S_2(t)$, with

$$S_1(t)u_0 = e^{-At}u_0 - \int_0^t e^{-A(t-s)}f_\Omega(u(s))\,ds, \quad S_2(t)u_0 = -\int_0^t e^{-L(t-s)}g_\Gamma(u(s))\,ds$$

and $A$ has compact resolvent in $C(\overline{\Omega}, \mathbb{R}^m)$, again using [16, Theorem 4.2.2], we get $\{S_1(t)B, t \ge 1\}$ is compact in $C(\overline{\Omega}, \mathbb{R}^m)$. Finally, as in Theorem 3.6 and using [31, 1], we have $S_2(\cdot)u_0 \in C^{\theta,\theta/2}(\overline{\Omega} \times [0,\infty), \mathbb{R}^m)$, for every $\theta \in (0,1)$ and

$$\|S_2(\cdot)u_0\|_{\theta,\theta/2} \le C(\theta, B)$$

and the right hand side is uniform in $t$ and $u_0 \in B$. In particular $\{S_2(t)B, t \ge 1\}$ is in a bounded set of $C^\theta(\overline{\Omega}, \mathbb{R}^m)$ and therefore it is compact in $C(\overline{\Omega}, \mathbb{R}^m)$. Hence, $\{S(t)B, t \ge 1\}$ is compact in $Y$.□

To obtain estimates in $C(\overline{\Omega}, \mathbb{R}^m)$ for the solution of (1.1), in what follows we will use comparison results and the notion of invariant regions. We start by defining sub and super-solutions for (1.1).

**Definition 4.1** *A $C^2$ function $\bar{u} : \Omega \times (0,T) \subset \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^m$ is a super-solution of (1.1), if, for $1 \leq i \leq m$, it satisfies*

$$\begin{aligned}
&(\bar{u}_i)_t - Div(a_i \nabla \bar{u}_i) + \textstyle\sum_{j=1}^N b_j^i(x) \frac{\partial \bar{u}_i}{\partial x_j} + \lambda \bar{u}_i + f_i(u_1, \ldots, u_{i-1}, \bar{u}_i, u_{i+1}, \ldots, u_m) \geq 0, \qquad on \quad \Omega, \\
&\frac{\partial \bar{u}_i}{\partial \vec{n}_a} + g_i(u_1, \ldots, u_{i-1}, \bar{u}_i, u_{i+1}, \ldots, u_m) \geq 0, \qquad on \quad \Gamma \\
&\bar{u}_i(0) \geq (u_0)_i.
\end{aligned} \tag{4.2}$$

*where $u = (u_1, \ldots, u_m)$ is the unique solution of (1.1) with initial value $u_0$. Similarly, we define a sub-solution $\underline{u}$ by replacing the $\geq$ sign by the $\leq$ sign in (4.2).*

We then have

**Lemma 4.2** *If $u$ is the solution of (1.1) with initial value $u_0$ and if $\bar{u}$ and $\underline{u}$ are super and sub-solutions of (1.1) in the sense defined above, then*

$$\underline{u}_i(t,x) \leq u_i(t,x) \leq \bar{u}_i(t,x)$$

*for every $1 \leq i \leq m$, while the solutions exist.*

**Proof:** For each $i = 1, \ldots, m$, let $\hat{f}_i(t,x,v) = \lambda v + f_i(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_m)$, and $\hat{g}_i(t,x,v) = g_i(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_m)$. Therefore, $\bar{u}_i$ and $\underline{u}_i$ are super and sub-solutions of

$$\begin{aligned}
&v_t - \mathrm{Div}(a_i \nabla v) + \textstyle\sum_{j=1}^N b_j^i(x) \frac{\partial v}{\partial x_j} + \hat{f}_i(t,x,v) = 0, \quad \text{on } \Omega, \\
&\frac{\partial v}{\partial \vec{n}_{a_i}} + \hat{g}_i(t,x,v) = 0, \quad \text{on } \Gamma \\
&v(0) = (u_0)_i.
\end{aligned}$$

From the results in [30], we get that there exist a solution of the problem above, such that $\underline{u}_i(t,x) \leq v(t,x) \leq \bar{u}_i(t,x)$. But the unique solution of this problem is $v = u_i$ and the result is proved.□

Assume now $f, g$ satisfy (1.8) and (1.9). Let $\mu_i > 0$ and $\varphi_i$ be, respectively, the first positive eigenvalue and normalized eigenfunction of each component of (1.9) and $m_i = \min_{x \in \bar{\Omega}} \varphi_i(x) > 0$. For each $\theta = (\theta_1, \ldots, \theta_m) \in (\mathbb{R}^+)^m$, define

$$\Sigma_\theta = \left\{ u \in Y : |u_i(x)| \leq \theta_i \varphi_i(x), \text{ for all } x \in \bar{\Omega} \right\}. \tag{4.3}$$

Below we prove that $\Sigma_\theta$ is an invariant attracting region.

**Lemma 4.3** *Let $\xi^0$ be as in (1.8). Then, with the notation above, let $\theta_i m_i \geq \xi_i^0$ and for every $t_0 > 0$, denote $v_i^\pm(t) = \pm e^{-\mu_i(t-t_0)} \theta_i \varphi_i$, for $0 \leq t \leq t_0$, and $v^\pm(t) = (v_1^\pm, \ldots, v_m^\pm)$.*

*Then, for any initial data in $Y$, such that $|(u_0)_i(x)| \leq e^{\mu_i t_0} \theta_i \varphi_i(x)$, for all $x \in \bar{\Omega}$, $v^\pm(t)$, are respectively a sub and super-solutions of (1.1) for $0 \leq t \leq t_0$, and consequently*

$$S(t) \Sigma_{e^{\mu t_0} \theta} \subset \Sigma_{e^{-\mu(t-t_0)} \theta} \to \Sigma_\theta$$

*as $t \to t_0$, where $(e^{\mu t_0} \theta)_i = e^{\mu_i t_0} \theta_i$. In particular, for every $M_i > \theta_i$, define $t_0 = t_0(M) = \sup_i \frac{1}{\mu_i} \log(\frac{M_i}{\theta_i}) > 0$, and $\hat{M}_i = e^{\mu_i t_0} \theta_i \geq M_i$, then for $0 \leq t \leq t_0$,*

$$S(t) \Sigma_M \subset S(t) \Sigma_{\hat{M}} \subset \Sigma_{\hat{M} e^{-\mu(t-t_0)} \theta} \to \Sigma_\theta$$

*as $t \to t_0$. In particular, $\Sigma_\theta$ is a positively invariant region for (1.1).*

**Proof:** First take $v_i^+(t) = e^{-\mu_i(t-t_0)} \theta_i \varphi_i$. Then, for $0 \leq t \leq t_0$, $v_i^+(t,x) \geq e^{-\mu_i(t-t_0)} \theta_i m_i > \xi_i^0$ and consequently, from (1.8)

$$\begin{aligned}
0 &= \frac{\partial v_i^+}{\partial t} - \mathrm{Div}(a_i \nabla v_i^+) + \sum_{j=1}^N b_j^i(x) \frac{\partial v_i^+}{\partial x_j} + c_i^0 v_i^+ \leq \\
&\leq \frac{\partial v_i^+}{\partial t} - \mathrm{Div}(a_i \nabla v_i^+) + \sum_{j=1}^N b_j^i(x) \frac{\partial v_i^+}{\partial x_j} + \lambda v_i^+ + f_i(u_1, \ldots, u_{i-1}, v_i^+, u_{i+1}, \ldots, u_m) \\
0 &= \frac{\partial v_i^+}{\partial \vec{n}_a} + d_i^0 v_i^+ \leq \frac{\partial v_i^+}{\partial \vec{n}_a} + g_i(u_1, \ldots, u_{i-1}, v_i^+, u_{i+1}, \ldots, u_m),
\end{aligned}$$

Thus, $v^+(t)$ is a super-solution. The same for $v^-(t)$ with all the inequalities reversed. The rest is obvious.$\square$

As a consequence, we have

**Corollary 4.1** *All solutions of (1.1) with initial data in $Y$, exist for $t \geq 0$, and the semigroup $S(t)$ is well defined on $Y$ for $t \geq 0$. Moreover, for every bounded set $B \subset Y$, $\{S(t)B, t \geq 0\}$ is bounded in $Y$ and $\{S(t)B, t \geq 1\}$ is compact in $Y$. Even more, if $\Sigma_\theta$ is as in Lemma 4.3, then $\Sigma_\theta$ is an absorbing set for $S(t)$.*

**Proof:** Clearly, for any bounded set $B \subset Y$, there exists $\hat{\theta} \in (\mathbb{R}^+)^m$ such that $B \subset \Sigma_{\hat{\theta}}$ and we can assume that $\hat{\theta}_i m_i \geq \xi_i^0$. Then, Lemma 4.3 implies that $S(t)B \subset S(t)\Sigma_{\hat{\theta}} \to \Sigma_\theta$ as $t \to t_0 = t_0(B)$. Therefore, from Lemma 4.1 we get the result.$\square$

Note that $\Sigma_\theta$ is not a bounded set in $Y$ and therefore dissipativeness does not follow from the corollary. However, we have

**Theorem 4.1** *The problem (1.1) has a global attractor $\mathcal{A}$ in $Y = X^\alpha \cap C(\overline{\Omega})$. Furthermore,*

$$\mathcal{A} \subset \Sigma_\theta \tag{4.4}$$

*for every $\theta$, such that $\theta_i m_i \geq \xi_i^0$, for $i = 1, \ldots, m$.*

**Proof:** Since we already have that orbits of bounded subsets of $Y$ under $S(t)$ are bounded in $Y$ and that $S(t)$ is a compact semigroup, it just remains to prove point dissipativeness to prove the existence of a global attractor $\mathcal{A}$ for $\{S(t);\ t \geq 0\}$ in $Y$, see [16, Theorem 3.4.6]. In fact we show below that $S(t)$ is bounded dissipative.

For this, note that from the Corollary above, for any bounded set $B \subset Y$ there exists $t_0 > 0$ such that $S(t)B \subset \Sigma_\theta$, for all $t \geq t_0$. From the variation of constants formula, where as in Lemma 4.1, we can assume without loss of generality that (2.7) holds with $\delta > 0$, we have for any $u_0 \in B$, and $t \geq t_0$

$$\|S(t)u_0\|_{X^\alpha} \leq M\, e^{-\delta(t-t_0)} L(t_0) + M\, P(\theta)\, |\Omega|^{\frac{1}{2}} \int_{t_0}^{t} e^{-\delta(t-s)} (t-s)^{-(\alpha-\beta)} ds, \tag{4.5}$$

where $\|S(s)u_0\|_Y \leq L(t_0)$, for every $s \geq t_0$ and $u_0 \in B$, and $P(\theta) = \sup_{s \in \Sigma^*} \left\{ |\Omega|^{\frac{1}{2}} |f(s)| + |\Gamma|^{\frac{1}{2}} |g(s)| \right\}$, with $\Sigma^* = \{s \in \mathbb{R}^m, |s_i| \leq \theta M_i\}$, with $M_i = \sup_\Omega |\varphi_i(x)|$. Note that $t_0$ and the right hand side above are independent of $u_0 \in B$, and then letting $t \to \infty$ we obtain

$$\limsup_{t\to\infty} \|S(t)u_0\|_{X^\alpha} \leq M\, P(\theta)\, |\Omega|^{\frac{1}{2}} \int_0^\infty e^{-\delta s} s^{-(\alpha-\beta)} ds, \tag{4.6}$$

and since the right hand side of (4.6) does not depend on $u_0 \in B \subset Y$, $S(t)$ is bounded dissipative in $Y$.

Finally, since $\Sigma_\theta$ is absorbing, we get $\mathcal{A} \subset \Sigma_\theta$.$\square$

With this result, the next one follows from [16] and from the fact that a constant equilibrium $u_0$ for (1.1) may only happen if $\lambda u_0 + f(u_0) = g(u_0) = 0$.

**Corollary 4.2** *The elliptic problem*

$$-Div(a\nabla u) + \sum_{j=1}^N B_j(x) \frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0, \quad on\ \Omega$$
$$\frac{\partial u}{\partial \vec{n}_a} + g(u) = 0, \quad on\ \Gamma,$$

*has at least one solution which is non-trivial whenever $\lambda I + f$ and $g$ have no common zeros.*$\square$

## 4.2 The energy space case

### 4.2.1 Gradient systems in the critical growth case

Now we consider (1.1) without dispersion ($B_j = 0$), that is

$$\left.\begin{array}{l} u_t - \mathrm{Div}(a\nabla u) + \lambda u + f(u) = 0, \qquad \text{on} \quad \Omega, \\ \dfrac{\partial u}{\partial \vec{n}_a} + g(u) = 0, \qquad \text{on} \quad \Gamma, \end{array}\right\} \tag{4.7}$$

with initial values in $H^1(\Omega, \mathbb{R}^m)$, where $f$ and $g$ satisfy (1.2)–(1.5) and the dissipativeness assumptions (1.8), (1.9). Note that now the eigenvalue problem (1.9) reduces to

$$\begin{array}{l} -\mathrm{Div}(a\nabla v) + c^0 v = \mu v, \quad \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \vec{n}_a} + d^0 v = 0, \quad \text{on} \quad \partial\Omega, \end{array} \tag{4.8}$$

and from the diagonal and selfadjoint structure of this problem we have that the first eigenvalue, $\mu_1^0$, is given by the infimum of the first eigenvalues of each scalar eigenvalue problem, i.e. the infimum of

$$\mu_i = \inf_{\phi \in H^1(\Omega)} \frac{\int_\Omega a_i |\nabla \phi|^2 + c_i^0 \int_\Omega |\phi|^2 + d_i^0 \int_\Gamma |\phi|^2}{\int_\Omega |\phi|^2}$$

which are assumed to be positive for $i = 1, \ldots, m$.

Note that the dissipativeness assumption is met if

$$\left\{\begin{array}{l} \liminf_{|s_i| \to \infty} \dfrac{f_i(s) + \lambda s_i}{s_i} \geq 0 \\ \liminf_{|s_i| \to \infty} \dfrac{g_i(s)}{s_i} \geq 0 \end{array}\right. \tag{4.9}$$

and for each $i = 1, \ldots, m$ one of the inequalities is strict.

Moreover, we assume that there are scalar potentials $F : \mathbb{R}^m \to \mathbb{R}$ and $G : \mathbb{R}^m \to \mathbb{R}$ such that $\nabla F(s) = f(s)$ and $\nabla G(s) = g(s)$, for every $s \in \mathbb{R}^m$, a condition that is always satisfied for $m = 1$.

Consider the energy functional $V : H^1(\Omega, \mathbb{R}^m) \to \mathbb{R}$ defined by

$$V(\phi) = \frac{1}{2} \sum_{i=1}^m \int_\Omega a_i |\nabla \phi_i|^2 + \frac{\lambda}{2} \int_\Omega |\phi|^2 + \int_\Omega F(\phi) + \int_\Gamma G(\gamma(\phi)). \tag{4.10}$$

The next result ensures that $V$ is well defined

**Lemma 4.4** *Assume and $f : \mathbb{R}^m \to \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}$ is such that $\nabla F(s) = f(s)$.*
*i) If $f$ satisfies (1.2) or (1.3) then for every $\phi \in H^1(\Omega, \mathbb{R}^m)$, one has $F(\phi) \in L^1(\Omega, \mathbb{R}^m)$ and*

$$\|F(\phi)\|_{L^1} \leq c(r), \quad \text{if } \|\phi\|_{H^1} \leq r$$

*for some continuous increasing function $c(r)$.*
*ii) If $f$ satisfies (1.4) or (1.5) then for every $\phi \in H^1(\Omega, \mathbb{R}^m)$, one has $F(\phi) \in L^1(\Gamma, \mathbb{R}^m)$ and*

$$\|F(\phi)\|_{L^1} \leq c(r), \quad \text{if } \|\phi\|_{H^1} \leq r$$

*for some continuous increasing function $c(r)$.*

**Proof:** Since $f$ is a gradient, we have $F(s) = \int_0^1 f(rs)s\,dr$, for all $s \in \mathbb{R}^m$ and then $|F(s)| \leq \int_0^1 |f(rs)||s|\,dr$.

If (1.2) holds true, then for every $\eta > 0$, $|f(s)| \leq c_\eta e^{\eta|s|^2}|s| + |f(0)|$ and therefore $|F(s)| \leq \frac{c_\eta}{2\eta} e^{\eta|s|^2} + |f(0)||s|$. Since $|s| \leq e^{\eta|s|^2}$ we get $|F(s)| \leq (\frac{c_\eta}{2\eta} + |f(0)|)e^{\eta|s|^2}$ and we conclude using Lemma 3.1, as in Lemma 3.2. If (1.4) holds true, then as before and as in Lemmas 3.3 and 3.4, we get $\|F(\phi)\|_{L^1(\Gamma)} \leq c\|F(\phi)\|_{W^{1,1}(\Omega)} \leq c(r)$. If (1.3) holds true, then $|f(s)| \leq c(1 + |s|^3)$ and therefore $|F(s)| \leq c(1 + |s|^4)$ and again we get the result as in Lemma 3.2. Finally, if (1.5) holds true, then proceeding as before, we get $|F(s)| \leq c(1 + |s|^p)$ for some $p < 4$ and, at the same time, if $\phi \in H^1(\Omega, \mathbb{R}^m)$, then the trace is in $H^{1/2}(\Gamma, \mathbb{R}^m) \subset L^q(\Gamma, \mathbb{R}^m)$, for $q \leq 4$ and we get the result.□

**Theorem 4.2** *Under the above assumptions, we have*
*i) $V$ is a Liapunov function for (4.7).*
*ii) All solutions of (4.7) with initial data in $H^1(\Omega, \mathbb{R}^m)$ are globally defined.*
*iii) The problem (4.7) has a global attractor $\mathcal{A}$ in $H^1(\Omega, \mathbb{R}^m)$. Furthermore (4.7) is a gradient system and therefore, $\mathcal{A} = W^u(E)$, where $E$ is the set of equilibria of (4.7), i.e.*

$$E = \{\phi \in H^1(\Omega, \mathbb{R}^m) : L\phi + h(\phi) = 0\}$$

*and $W^u(E)$ denotes the unstable set of $E$. Moreover, if each element of $E$ is hyperbolic, then $E$ is finite and*

$$\mathcal{A} = \cup_{\phi \in E} W^u(\phi).$$

*where $W^u(\phi)$ denotes the unstable manifold of the equilibrium point $\phi$.*

**Proof:** From the smoothing effect in Theorem 3.2, we have $u_t \in H^1(\Omega, \mathbb{R}^m)$ for $t > 0$, and then taking $\phi = u_t$ as a test function in (3.3), we get

$$\frac{d}{dt} V(u(t)) = -\|u_t\|^2 \leq 0$$

for any solution $u(t)$ of (4.7) and therefore $V$ decreases along solutions of (4.7).

Observe that from the dissipativeness assumption, we have

$$F(s) + \frac{\lambda}{2}|s|^2 \geq \sum_{i=1}^{m} \frac{c_i^0}{4}|s_i|^2 - c_1, \quad G(s) \geq \sum_{i=1}^{m} \frac{d_i^0}{4}|s_i|^2 - c_1$$

for all $s \in \mathbb{R}^m$ and some constant $c_1$. Therefore,

$$V(\phi) \geq \frac{1}{2}\sum_{i=1}^{m} \int_\Omega a_i|\nabla\phi_i|^2 + \sum_{i=1}^{m} \frac{c_i^0}{4} \int_\Omega |\phi_i|^2 + \sum_{i=1}^{m} \frac{d_i^0}{4} \int_\Gamma |\phi_i|^2 - c_2$$

and then

$$V(\phi) \geq \frac{1}{4}\sum_{i=1}^{m} \int_\Omega a_i|\nabla\phi_i|^2 + \frac{1}{4}\sum_{i=1}^{m} \mu_i \int_\Omega |\phi_i|^2 - c_2$$

This inequality implies that solutions are globally defined, since $V(u(t)) \leq V(u_0)$ and then the $H^1(\Omega, \mathbb{R}^m)$ norm of the solution remains bounded since $\mu_i > 0$.

From Lemma 4.4 we get

$$V(\phi) \leq C(r),$$

if $\|\phi\|_{H^1(\Omega, \mathbb{R}^m)} \leq r$, where $C(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function. Therefore, orbits of bounded sets are bounded in $H^1(\Omega, \mathbb{R}^m)$. Also, for each $t > 0$, $S(t)$ is compact, since $A$ has compact resolvent, [16].

Since, $V$ decreases along trajectories, from La Salle's invariance principle, the $\omega$-limit set of any trajectory lies inside the set $\{\phi \in H^1(\Omega, \mathbb{R}^m), \dot{V} = 0\}$, but this is the set of equilibria of (4.7), i.e. $E = \{\phi \in H^1(\Omega, \mathbb{R}^m) : L\phi + h(\phi) = 0\}$. Therefore, $E$ attracts unitary sets in $H^1(\Omega, \mathbb{R}^m)$ under the semigroup $\{S(t),\ t \geq 0\}$. Next, we prove that $E$ is bounded in $H^1(\Omega, \mathbb{R}^m)$ and then the semigroup $\{S(t),\ t \geq 0\}$ is point dissipative, [16].

Since the equilibrium points of (4.7) satisfy $Lu + h(u) = 0$, i.e.

$$\begin{aligned} &-\mathrm{Div}(a\nabla u) + \lambda u + f(u) = 0, \quad \text{on } \Omega \\ &\frac{\partial u}{\partial \vec{n}_a} + g(u) = 0, \ \text{on } \Gamma \end{aligned} \tag{4.11}$$

taking $u$ as a test function in (4.11), we get

$$\sum_{i=1}^{m} \int_\Omega a_i|\nabla u_i|^2 + \lambda \int_\Omega |u|^2 + \int_\Omega f(u)\cdot u + \int_\Gamma g(u)\cdot u = 0$$

From the dissipative condition we get $f(s)\cdot s+\lambda|s|^2\geq\sum_{i=1}^m\frac{c_i^0}{2}|s_i|^2-c_3$ and $g(s)\cdot s\geq\sum_{i=1}^m\frac{d_i^0}{2}|s_i|^2-c_3$ for all $s\in\mathbb{R}^m$ and some constant $c_3$. Therefore,

$$\sum_{i=1}^m\int_\Omega a_i|\nabla u_i|^2+\sum_{i=1}^m\frac{c_i^0}{2}\int_\Omega|u_i|^2+\sum_{i=1}^m\frac{d_i^0}{2}\int_\Gamma|u_i|^2\leq c_4$$

thus

$$\sum_{i=1}^m\int_\Omega a_i|\nabla u_i|^2+\sum_{i=1}^m\mu_i\int_\Omega|u_i|^2\leq 2c_4$$

and then the set $E$ of equilibria is bounded in $H^1(\Omega,\mathbb{R}^m)$. The rest follows from the results in [16].□

Our next goal is proving that the attractor constructed above is a bounded set in $C(\overline{\Omega},\mathbb{R}^m)$ and moreover that the estimates (4.3), (4.4), of the previous section still hold true. For this, we first prove the following lemma

**Lemma 4.5** *Assume $N=3$ and $g$ verifies (1.5), with $\rho<1$. Then, if $u\in H^\sigma(\Omega,\mathbb{R}^m)$, with $3/2>\sigma>1$ verifies*

$$\begin{cases}-Div(a\nabla u)+\lambda u=F\in L^2(\Omega,\mathbb{R}^m)\\ \frac{\partial u}{\partial\vec{n}_a}+g(u)=0 & on\ \Gamma\end{cases}$$

*then $u\in H^{s(\sigma)}(\Omega,\mathbb{R}^m)$, where $s(\sigma)=\min\{2,A\sigma-B\}$, with $A=\rho+2$, $B=\frac{3\rho+1}{2}$. Moreover, $\|u\|_{H^{s(\sigma)}}\leq C(\|F\|_{L^2},\|u\|_{H^\sigma})$. In particular $u\in C(\overline{\Omega})$.*

**Proof:** Note that we can restrict ourselves to the case $m=1$, since we only use the smoothness and growth assumptions on $g$. Also, note that from elliptic regularity results, since $F\in L^2(\Omega)$ we get $u\in H^2_{loc}(\Omega)$, but the regularity of $g(u)$ on $\Gamma$ determines the overall regularity of $u$ on $\Omega$. Therefore, we prove that $g(u)$ has certain degree of regularity on $\Gamma$ and then use elliptic regularity theory to get the result.

Note that from (1.5) and the Sobolev inclusion for $H^\sigma(\Omega)$ we get that $Dg(u)\in L^p(\Omega)$ for $p\leq\frac{6}{(3-2\sigma)(\rho+1)}$. On the other hand, from the Sobolev inclusions for $H^{\sigma-1}(\Omega)$ we get $\nabla u\in L^q(\Omega,\mathbb{R}^3)$ for $q\leq\frac{6}{5-2\sigma}$. From this we get $Dg(u)\nabla u\in L^r(\Omega,\mathbb{R}^3)$ for $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ and this leads to $r\leq r_0=\frac{6}{(3-2\sigma)(\rho+2)+2}$.

Also, from (1.5) and the Sobolev inclusion for $H^\sigma(\Omega)$ we get $g(u)\in L^p(\Omega)$ for $p\leq\frac{6}{(3-2\sigma)(\rho+2)}$. Therefore, we get $g(u)\in W^{1,r_0}(\Omega)$. Now, from the trace theorem, $g_\Gamma(u)\in W^{1-\frac{1}{r_0},r_0}(\Gamma)\subset H^s(\Gamma)$ for $s=2-\frac{3}{r_0}$. From elliptic regularity theory, we get $u\in H^{s(\sigma)}(\Omega)$ with $s(\sigma)=\min\{2,7/2-\frac{3}{r_0}\}$ and we get the result.

Now, note that $H^s(\Omega)\subset C(\overline{\Omega})$, if $s>3/2$. Hence, if $j(\sigma)=A\sigma-B$, with $A=\rho+2$, $B=\frac{3\rho+1}{2}$ is greater than 3/2 the conclusion follows. If not, we repeat the previous argument to get $u\in H^{s^n(\sigma)}(\Omega)$ for $n\in N$. But note that the function $j(\sigma)$ is monotonic and has a unique fixed point at $\sigma_f=\frac{3\rho+1}{2(\rho+1)}<1$, since $\rho<1$. Therefore, there exist $n$ such that $s^n(\sigma)>3/2$ for any $3/2>\sigma>1$ and the conclusion follows.□

**Proposition 4.1** *Let $u(t)$ be a solution of (4.7), with initial data in $H^1(\Omega,\mathbb{R}^m)$. Then*

$$u\in C(0,\infty,C(\overline{\Omega},\mathbb{R}^m))$$

*Moreover, if $N=1,2$ or if $N=3$ and $\{u_t,\ t\geq\epsilon\}$ is bounded in $L^2(\Omega,\mathbb{R}^m)$ for $\epsilon\geq0$, then*

$$\sup_{t\geq\epsilon}\{\|u(t)\|_{C(\overline{\Omega},\mathbb{R}^m)}\}<\infty$$

*and is bounded above by a constant depending on the bound on $u(t)$ in $H^1(\Omega,\mathbb{R}^m)$ and the bound on $u_t$ in $L^2(\Omega,\mathbb{R}^m)$, if $N=3$.*

**Proof:** Note that without loss of generality, we can assume that (2.7) holds true with $\delta > 0$. From Theorem 4.2, $u(t)$ is continuous and uniformly bounded in $H^1(\Omega, \mathbb{R}^m)$ for $t \geq 0$. Hence, if $N = 1$ there is nothing to prove since $H^1(\Omega, \mathbb{R}^m) \subset C(\overline{\Omega}, \mathbb{R}^m)$. If $N \geq 2$, we get from Theorem 3.2, $u \in C(0, \infty, X^{\beta+1})$ and moreover, from the variation of constants formula, for any $\gamma < \beta + 1$

$$\|u(t)\|_{X^\gamma} \leq Mt^{-(\gamma-1/2)}e^{-\delta t}\|u(0)\|_{X^{1/2}} + \int_0^t Me^{-\delta(t-s)}(t-s)^{-(\gamma-\beta)}\|h(u(s))\|_{X^\beta}\, ds$$

and the right hand side is uniformly bounded for $t \geq \epsilon > 0$. But since $\beta > -1/2$, $\gamma$ can be chosen such that $3/4 > \gamma > 1/2$ and then we get the result if $N = 2$, since $X^\gamma \subset C(\overline{\Omega}, \mathbb{R}^m)$.

Finally, if $N = 3$, as seen in Lemma 3.4 and Theorem 3.3, in the argument above we can take $\beta = -r/2 - 1/4$ for $1/2 > r > \rho/2$ and then we get the bound in $X^\gamma$ for $\gamma < \frac{3-\rho}{4}$ and then in $H^\sigma(\Omega, \mathbb{R}^m)$ for $1 < \sigma < \frac{3-\rho}{2}$. Also, from Theorem 3.2, $u_t \in C(0, \infty, L^2(\Omega, \mathbb{R}^m))$ and reading the equation as

$$\begin{cases} -\text{Div}(a\nabla u) + \lambda u = u_t - f(u) \\ \dfrac{\partial u}{\partial \vec{n}_a} + g(u) = 0 \end{cases}$$

and working on finite time intervals, we get, from Lemma 4.5, $u \in C(0, \infty, C(\overline{\Omega}, \mathbb{R}^m))$. On the other hand, if $\{u_t,\ t \geq \epsilon\}$ is bounded in $L^2(\Omega, \mathbb{R}^m)$ for $\epsilon \geq 0$, again Lemma 4.5 gives the result.□

We now prove de following abstract result

**Proposition 4.2** *Assume $A$ is a sectorial operator with compact resolvent and assume the problem*

$$u_t + Au + h(u) = 0$$

*where $h : X^\alpha \to X^\beta$ is Lipschitz on bounded sets, with $0 \leq \alpha - \beta < 1$, defines a globally defined semigroup, $S(t)$, in $X^\alpha$ that has a global attractor $\mathcal{A}$.*

*Then, $\mathcal{A}$ is a bounded subset of $X^\gamma$ for any $\gamma \leq \beta + 1$ and moreover, there exist a constant $c$, only depending on the bound of $\mathcal{A}$ in $X^\alpha$, such that for any solution $u(t)$ lying on the attractor, one has for any $\gamma < \beta + 1$*

$$\|u_t(t)\|_{X^\gamma} \leq c, \text{ for any } t \in \mathbb{R} \tag{4.12}$$

**Proof:** Note that $\mathcal{A}$ is a bounded subset of $X^\alpha$ and is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$. Let $u(t)$, with $t \in \mathbb{R}$, be a solution lying on the attractor. From the variation of constants formula, for any $\gamma < \beta + 1$ we get $\|u(t)\|_{X^\gamma} \leq Mt^{-(\gamma-\alpha)}e^{-\delta t}\|u(0)\|_{X^\alpha} + \int_0^t Me^{-\delta(t-s)}(t-s)^{-(\gamma-\beta)}\|h(u(s))\|_{X^\beta}\, ds$. Taking $t = 1$, the right hand side above is bounded by a constant independent of $u$ and then $S(1)\mathcal{A} = \mathcal{A}$ is a bounded set of $X^\gamma$. Note that if (4.12) is proved, then we can get the bound on the attractor for the case $\gamma = \beta + 1$, since we have $Au = -u_t - h(u)$ and the right hand side is bounded in $X^\beta$.

The proof of (4.12) is based on the proof of [21, Theorem 3.5.2], [21, Lemma 3.5.1] and the invariance of the attractor. Take $u(t)$, with $t \in \mathbb{R}$, a solution lying on the attractor and consider the interval $[t_0, t_1] = [t-1, t]$ for $t \in \mathbb{R}$. Proceeding as in [21, Theorem 3.5.2], using the bound on $X^\gamma$, $\alpha < \gamma < \beta + 1$, and that the non-linearity is Lipschitz on bounded sets, we get that $j(s) = h(u(s))$ is bounded in $X^\beta$ and satisfies, for $t_0 < s < s + h \leq t_1$,

$$\|j(s+h) - j(s)\|_{X^\beta} \leq hc_1(s-t_0)^{-1+\gamma-\alpha} + c_2\int_{t_0}^s (s-r)^{-(\alpha-\beta)}\|j(r+h) - j(r)\|_{X^\beta}\, dr$$

and from here using Gronwall's lemma as in [21, pg 6], one gets $\|j(s+h) - j(s)\|_{X^\beta} \leq c_3 h(s-t_0)^{-1+\gamma-\alpha} = hK(s)$ where $c_1, c_2, c_3$ are independent of $u$ and $t$. Note that $\int_{t_0}^{t_1} K(s)\, ds$ is bounded by an absolute constant, and then, [21, Lemma 3.5.1] gives that on $[t_0, t_1]$, for any $\eta < \beta + 1$

$$\|u_t(s)\|_\eta \leq c_4(s-t_0)^{-(\eta-\beta)} + c_5\int_{t_0}^s (s-r)^{-(\eta-\beta)}K(r)\, dr$$

Taking $s = t_1 = t$ and changing variables $r = t_0 + z$, and using $t_1 - t_0 = 1$ one gets $\|u_t(t)\|_\eta \leq c_6$, for some absolute constant $c_6$, for all $t$ and $u$ on the attractor.□

Therefore, we get

**Theorem 4.3** *The attractor of (4.7) in $H^1(\Omega, \mathbb{R}^m)$ is a bounded set in $C(\overline{\Omega}, \mathbb{R}^m)$. Furthermore, for any $u \in \mathcal{A}$ one has*

$$|u_i(x)| \leq \theta_i \varphi_i(x), \text{ for all } x \in \bar{\Omega}, \ i = 1, \ldots, m$$

*with $\theta$, $\varphi(x)$ as in (4.3) and (4.4).*

**Proof:** We first prove that the attractor is bounded in $C(\overline{\Omega}, \mathbb{R}^m)$. For this, note that the attractor, $\mathcal{A}$, is bounded in $H^1(\Omega, \mathbb{R}^m)$ and satisfies $S(t)\mathcal{A} = \mathcal{A}$ for every $t \in \mathbb{R}$. Therefore, if $N = 1$ again $H^1(\Omega, \mathbb{R}^m) \subset C(\overline{\Omega}, \mathbb{R}^m)$ gives the result. If $N = 2$, and choosing $\gamma > 1/2$ in Proposition 4.2, we get the result.

On the other hand, if $N = 3$, for any solution $u(t)$, $t \in \mathbb{R}$ lying on the attractor of (4.7) one has that $u_t$ is uniformly bounded in $L^2(\Omega, \mathbb{R}^m)$ for $t \in \mathbb{R}$ and then Proposition 4.1 and the fact that $\mathcal{A}$ is bounded in $H^1(\Omega, \mathbb{R}^m)$ and invariant concludes. Now, from Corollary 4.1, $\Sigma_\theta$ is an absorbing set in $Y = X^{1/2} \cap C(\overline{\Omega}, \mathbb{R}^m)$ and, again the invariance of $\mathcal{A}$ gives the result.□

### 4.2.2 Non-gradient systems in the non-critical growth case

As can be seen from the proofs in the previous subsections we are essentially using [16, Theorems 3.4.6, 4.2.2] that amount to proving, in suitable spaces, that orbits of bounded sets are bounded and point dissipativeness. However, proving these properties for (1.1) in $H^1(\Omega, \mathbb{R}^m)$ when the system has not a gradient structure, turns out to be a difficult exercise, specially if one does not want to impose very restrictive structure conditions on the non-linearities. In general, the required a priori estimates when $m > 1$ are very involved, except for the case of the previous subsection.

Therefore to skip this technical difficulty, we adopt an indirect argument based again on the variation of constants formula and on the idea of dissipation in two spaces, [16]. The method relies on the following abstract result.

Assume $A$ is a sectorial operator with compact resolvent and consider the problem

$$u_t + Au + h(u) = 0 \tag{4.13}$$

and assume that the map $h$ is such that

$$h : X^{\alpha_0} \to X^{\beta}$$

is Lipschitz continuous on bounded subsets of $X^{\alpha_0}$ with $0 < \alpha_0 - \beta < 1$. Therefore, if $\alpha > \alpha_0$ still satisfies $0 \leq \alpha - \beta < 1$, then $h : X^{\alpha} \to X^{\beta}$ is also Lipschitz continuous on bounded subsets of $X^{\alpha}$ and from Theorem 3.1, there exists locally defined solutions of the equation above, with initial data in $X^{\alpha_0}$ and in $X^{\alpha}$.

Then the following result holds:

**Theorem 4.4** *Assume that solutions with initial data in $X^{\alpha_0}$ are globally defined and let $\{S_{\alpha_0}(t)\}$ denote the associated semigroup in $X^{\alpha_0}$. Moreover, assume that $\{S_{\alpha_0}(t)\}$ is point dissipative.*

*Then the semigroup $\{S_{\alpha}(t)\}$ in $X^{\alpha}$ is globally defined and bounded dissipative. Furthermore, there is a global attractor for $\{S_{\alpha}(t)\}$ in $X^{\alpha}$.*

**Proof:** First, note again that without loss of generality, we can assume that (2.7) holds for $\delta > 0$. We now prove that the semigroup $\{S_\alpha(t)\}$ is globally defined. If $u_0 \in X^\alpha \hookrightarrow X^{\alpha_0}$ then, from the point dissipativeness, $\{S_{\alpha_0}(t)u_0, t \geq 0\}$ is a bounded subset of $X^{\alpha_0}$ and therefore $\{h(S_{\alpha_0}(t)u_0)\}$ is a bounded subset of $X^\beta$. Thus, omitting subscripts $\alpha$, $\alpha_0$ wherever they are inessential, we have:

$$\|S(t)u_0\|_{X^\alpha} \leq Me^{-\delta t}\|u_0\|_{X^\alpha} + \int_0^t Me^{-\delta(t-s)}(t-s)^{-(\alpha-\beta)}\|h(S(s)u_0)\|_{X^\beta} ds$$

which proves that the semigroup is globally defined on $X^\alpha$ since the right hand side is bounded for $t \geq 0$.

Since $N(u_0) := \sup_{t\geq 0} \|h(S(t)u_0)\|_{X^\beta}$ is not necessarily a bounded function of $u_0$ we may not yet conclude that the semigroup $\{S_\alpha(t)\}$ is locally bounded; that is, for any bounded set $B \subset X^\alpha$ and $T > 0$ the set $\{S_\alpha(t)u_0 : 0 \leq t \leq T, u_0 \in B\}$ is a bounded subset of $X^\alpha$. In order to show this we use the following argument.

Let $B$ be a bounded subset of $X^\alpha$. Note that $B$ viewed as a subset of $X^{\alpha_0}$ is a precompact set and from the fact that $(t, u_0) \to S_{\alpha_0}(t)u_0$ from $\mathbb{R}^+ \times X^{\alpha_0}$ to $X^{\alpha_0}$ is continuous it follows that $\{S(t)u_0 : 0 \leq t \leq T,\ u_0 \in B\}$ is a bounded subset of $X^{\alpha_0}$. Using the variation of constants formula as before, gives that the right hand side is uniformly bounded on $0 \leq t \leq T$ for $u_0 \in B$ and then we conclude that $\{S_\alpha(t)\}$ is locally bounded and therefore compact.

To see that it is point dissipative we proceed as follows. Let $B_0 \subset X^{\alpha_0}$ be such that $B_0$ attracts points of $X^{\alpha_0}$ under the semigroup $\{S_{\alpha_0}(t), t \geq 0\}$. Thus, for any $u_0 \in X^\alpha$ there is a $t_0 = t_0(u_0)$ such that $dist_{X^{\alpha_0}}(S_{\alpha_0}(t)u_0, B_0) < 1$ for $t \geq t_0$. Thus, for $t \geq t_0$, we have

$$S(t)u_0 = e^{-A(t-t_0)}S(t_0)u_0 + \int_{t_0}^{t} e^{-A(t-s)}h(S(s)u_0)ds$$

and then

$$\|S(t)u_0\|_{X^\alpha} \leq M(t-t_0)^{\alpha_0-\alpha}e^{-\delta(t-t_0)}\|S(t_0)u_0\|_{X^{\alpha_0}} + \int_{t_0}^{t} Me^{-\delta(t-s)}(t-s)^{-(\alpha-\beta)}\|h(S(s)u_0)\|_{X^\beta}\, ds$$

But since $\|h(S(s)u_0)\|_{X^\beta} \leq sup_{dist_{X^{\alpha_0}}(v,B_0)<1}\|h(v)\|_{X^\beta}$ for all $s \geq t_0$, we get

$$\limsup_{t\to\infty} \|S(t)u_0\|_{X^\alpha} \leq \sup_{dist_{X^{\alpha_0}}(v,B_0)<1} \|h(v)\|_{X^\beta} \int_0^\infty Me^{-\delta s}s^{-(\alpha-\beta)}\, ds$$

and point dissipativeness follows. The rest follows from the results in [16].□

With this, we get

**Theorem 4.5** *Assuming (3.7), (3.8) and (1.8), (1.9), (1.1) defines a global semigroup in $H^1(\Omega, \mathbb{R}^m)$, which has a global compact attractor. Moreover, the attractor is a bounded set in $C(\bar{\Omega}, \mathbb{R}^m)$ and for any $u \in \mathcal{A}$ one has*

$$|u_i(x)| \leq \theta_i\varphi_i(x), \text{ for all } x \in \bar{\Omega},\ i = 1, \ldots, m$$

*with $\theta$, $\varphi(x)$ as in (4.3) and (4.4).*

**Proof:** From Theorem 3.4 we can take $\alpha_0 = s/2$, $\beta = -r/2 - 1/4$, with $r$, $s$ as in Lemma 3.7, and $\alpha = 1/2$.

Next, we prove the semigroup is globally defined and point dissipative. Let $u(t)$ be a solution in $X^{\alpha_0}$ defined on a maximal interval $[0, t_{max})$. Arguing as in Proposition 4.1, we get that for $N = 1, 2$, $u(t) \in C(\bar{\Omega}, \mathbb{R}^m)$ for any $0 < t < t_{max}$. For $N = 3$ and again as in Proposition 4.1, we get that $u(t) \in X^\gamma$ for $\gamma < \frac{5-(3-2s)(\rho+2)}{4}$ and hence $u(t) \in H^\sigma(\Omega, \mathbb{R}^m)$ for $\sigma < \frac{5-(3-2s)(\rho+2)}{2}$. Using now Lemma 4.5, we get $u(t) \in C(\bar{\Omega}, \mathbb{R}^m)$ for any $0 < t < t_{max}$. To see this, just note that we can choose $\sigma$ such that the fixed point of the function $j(\sigma) = A\sigma - B$, with $A = \rho + 2$, $B = \frac{3\rho+1}{2}$, i.e. $\sigma_f = \frac{3\rho+1}{2(\rho+1)}$, verifies $\sigma_f < \sigma$, since $s > \frac{3}{2}\frac{\rho+1}{\rho+2} > \frac{3\rho+1}{2(\rho+1)}$.

Then if $t_0 \in (0, t_{max})$, we have that $u(t_0) \in \Sigma_\theta$ for some $\theta > 0$, where $\Sigma_\theta$ is as in (4.3). Thus, from Lemma 4.3 we get that the sup norm of the solution remain bounded on $t_0 \leq t < t_{max}$ and then from Lemma 4.1 the solution exist for all $t \geq 0$ and the $X^{\alpha_0}$ norm also remains bounded for all $t \geq 0$. Furthermore, as in Corollary 4.1, we get that the semigroup is point dissipative. Consequently, Theorem 4.4, gives the existence of the attractor in $H^1(\Omega, \mathbb{R}^m)$. The rest follows as in Theorem 4.3.□

# 5 Alternative dissipativeness conditions

In this section we give alternative dissipativeness conditions to (1.8), and under these new conditions, we prove the existence of a global attractor for (1.1) and give $C(\bar{\Omega})$−diffusion independent estimates for the functions in the attractor. For this we will construct suitable "contracting regions", similar to the regions $\Sigma_\theta$ of the previous section.

## 5.1 Contracting rectangles

We will assume that $f$ and $g$ now verify the following

$$\frac{\lambda s_i + f_i(s)}{s_i} > 0, \quad \text{and} \quad \frac{g_i(s)}{s_i} > 0, \text{ for all } |s_i| > \xi_i^0 \tag{5.1}$$

for every $1 \le i \le m$. In other words, at the boundary of rectangles with faces parallel to $\Sigma^0 := \Pi_{i=1}^m \Sigma_i^0$, with $\Sigma_i^0 =: [-\bar{\xi}_i^0, \xi_i^0]$, and containing $\Sigma^0$, the vector fields $\lambda s + f(s)$ and $g(s)$ point outwards.

Note that this assumption is like (1.8), with $c_i^0, d_i^0 = 0$, but then the assumption on the eigenvalue problem (1.9) is not satisfied, since $\mu = 0$ is an eigenvalue.

We now introduce the notion of invariant regions as in [10, 37].

**Definition 5.1** *A set $\Sigma \subset \boldsymbol{R}^m$ is called a positively invariant region for the local solution of (1.1) if any solution $S(t)u_0$ satisfying $u_0(x) \in \Sigma$, for every $x \in \bar{\Omega}$ is such that $(S(t)u_0)(x) \in \Sigma$, for every $x \in \bar{\Omega}$ and for all $t$ in the maximal interval of existence of the solution.*

Our next result characterizes some of the invariant regions of the problem (1.1) (see also [4] for the case $g = 0$).

**Theorem 5.1** *Let $\bar{\xi}_j$, $\xi_j > 0$, $1 \le j \le m$ be such that $\frac{\lambda s_j + f_j(s)}{s_j} > 0$ and $\frac{g_j(s)}{s_j} > 0$ for all $s \in \boldsymbol{R}^m$ with $s_j \notin [-\bar{\xi}_j, \xi_j] =: \Sigma_j$. Then, the rectangle $\Sigma = \Pi_{j=1}^m \Sigma_j$ is an invariant region for the local solution of (1.1).*

**Proof:** If there is a solution $v(x,t) = (v_1(x,t), \cdots, v_m(x,t))$ of (1.1) such that $v(x,0) \in \Sigma$ for all $x \in \bar{\Omega}$, that does not stay in $\Sigma_1^* = (-\infty, \xi_1] \times \boldsymbol{R}^{m-1}$ for all $t \in [0, t_{max})$, then there is a $t_0$ and $x_0 \in \overline{\Omega}$ such that

$$v_1(x,t) < \xi_1, \text{ for every } \quad 0 \le t < t_0, \quad x \in \bar{\Omega}, \quad \text{and} \quad v_1(x_0, t_0) = \xi_1.$$

Therefore, if we prove that at such a point, $(v_1)_t(x_0, t_0) < 0$, then, $\Sigma_1^*$ is invariant.

Observe that $x_0 \notin \Gamma$ since in this case it would be a maximum point for $v_1(\cdot, t_0)$ with negative normal derivative, since $g_1(v) > 0$ and this leads to a contradiction. Hence, $x_0 \in \Omega$ and in particular, at $(x_0, t_0)$, $\nabla_x v_1 = 0$, $\Delta_x v_1 \le 0$ and then $\mathrm{Div}(a_1 \nabla v_1) = a_1 \Delta v_1 + \nabla a_1 \nabla v_1 \le 0$.

Thus, at $(x_0, t_0)$,

$$(v_1)_t = \mathrm{Div}(a_1 \nabla v_1) - \sum_{j=1}^{N} b_j^1 \frac{\partial v_1}{\partial x_j} - \lambda v_1 - f_1(v) \le -\lambda v_1 - f_1(v) < 0.$$

Thus, $(-\infty, \xi_1] \times \boldsymbol{R}^{m-1}$ is an invariant region. In the same way we obtain that $[-\bar{\xi}_1, \infty) \times \boldsymbol{R}^{m-1}$ is also an invariant region. Repeating the argument for $v_i$, $i = 2, \ldots, m$, and from the fact that the intersection of invariant regions is still invariant, the proof is completed.□

Let $f : \boldsymbol{R}^m \to \boldsymbol{R}^m$ and $g : \boldsymbol{R}^m \to \boldsymbol{R}^m$ be $C^1$ and $C^2$ functions, respectively, satisfying (5.1). We take the rectangle $\Sigma^0 = \Pi_{i=1}^m \Sigma_i^0 \subset \boldsymbol{R}^m$ and define $\Sigma_\tau^0 = \Pi_{i=1}^m [-\xi_i^0 - \tau, \xi_i^0 + \tau]$, $\tau \ge 0$, then $\Sigma_0^0 = \Sigma^0$ and $\{\Sigma_\tau^0, \tau \ge 0\}$ covers $\boldsymbol{R}^m$. To simplify the notation, when dealing with these rectangles, we denote $l_i := -\xi_i^0 - \tau < 0$ and $r_i := \xi_i^0 + \tau > 0$.

¿From (5.1), Theorem 5.1, and Lemma 4.1, we get, analogously to Corollary 4.1,

**Corollary 5.1** *With the above assumptions, all solutions of (1.1), exist for $t \ge 0$, and the semigroup $S(t)$ is well defined for $t \ge 0$. Moreover, for every bounded set $B \subset Y$, $\{S(t)B, t \ge 0\}$ is bounded in $Y$ and $\{S(t)B, t \ge 1\}$ is compact in $Y$.*

We will show now that the set of functions in $Y$, with values in $\Sigma^0$ is attracting, which, in turn, will show the dissipativeness of $S(t)$. For this, we will construct a suitable Liapunov functional, described below.

Let $p_{\Sigma^0} : \boldsymbol{R}^m \to \boldsymbol{R}^+$ be defined by

$$p_{\Sigma^0}(s) = \inf\{\tau \ge 0 : s \in \Sigma_\tau^0\} \tag{5.2}$$

and define $F_{\Sigma^0} : Y \to \boldsymbol{R}^+$ by

$$F_{\Sigma^0}(w) = \sup\{p_{\Sigma^0}(w(x)) : x \in \bar{\Omega}\}. \tag{5.3}$$

It is easy to check that $F_{\Sigma^0}$ is continuous on $Y$ and $F_{\Sigma^0}(w) = 0$ if and only if $w(x) \in \Sigma^0$ for all $x \in \bar{\Omega}$.

**Theorem 5.2** *Let $f$, $g$ and $\Sigma^0_\tau$, $\tau > 0$, be as before. Suppose that $u(t)$ is a solution of (1.1). Then, for any $T > 0$ for which $F_{\Sigma^0}(u(\cdot,T)) = \tau > 0$, there exists $\eta > 0$ such that*

$$\limsup_{h\to 0} \frac{F_{\Sigma^0}(u(\cdot,T+h)) - F_{\Sigma^0}(u(\cdot,T))}{h} \leq -\eta. \tag{5.4}$$

**Proof:** If $F_{\Sigma^0}(u(\cdot,T)) = \tau > 0$ and $\Sigma^0_\tau = \Pi_{i=1}^m [l_i, r_i]$, we have that $l_i \leq u_i(x,T) \leq r_i$, for all $x \in \bar{\Omega}$ and all $1 \leq i \leq m$. If $\bar{x} \in \overline{\Omega}$ is such that $u(\bar{x},T) \in \partial\Sigma^0_\tau$, then $u(\bar{x},T)$ is in one of the faces of $\Sigma^0_\tau$, say the right-hand $j^{\text{th}}$ face $\partial(\Sigma^0_\tau)^r_j$ of $\Sigma^0_\tau$, that is $u_j(\bar{x},T) = r_j$ for some $j$. Therefore, for any of these j's, there exists $\eta = \eta(\tau) > 0$ such that

$$-\lambda u_j(\bar{x},T) - f_j(u(\bar{x},T)) < -\eta.$$

Note that, $\bar{x}$ cannot be on $\Gamma$ since in this case $\bar{x}$ would be a maximum point for $u_j(\cdot,T)$ and the normal derivative at this point would be negative, since $g_j(u(\bar{x},T)) > 0$. Thus, at $(\bar{x},T)$, $\nabla u_j = 0$, $\Delta u_j \leq 0$ and $\mathrm{Div}(a_j \nabla u_j) = a_j \Delta u_j + \nabla a_j \nabla u_j \leq 0$ and then

$$\frac{\partial u_j}{\partial t} = \mathrm{Div}(a_j \nabla u_j) - \sum_{k=1}^N b^j_k \frac{\partial u_j}{\partial x_k} - \lambda u_j - f_j(u) < -\eta,$$

so that $u_j(\bar{x}, T+h) < r_j - \eta h$ for small $h$. Also, by continuity, for the remaining set of indexes, we also have $u_i(\bar{x}, T+h) < r_i - \eta h$ for small $h$. By reversing the argument we also have $u_i(\bar{x}, T+h) > l_i + \eta h$ for small $h$ and for all $i = 1, \ldots, m$.

By continuity, this holds for all $x$ in a neighborhood of $\bar{x}$. If $K_T = \{x \in \bar{\Omega}, u(x,T) \in \partial\Sigma^0_\tau\}$, then $K_T$ is compact, and by what we have just shown, there is an open set $\Omega \supset U \supset K_T$ such that if $x \in U$, and $h$ is small, $u(x, T+h) \in \Sigma^0_{(\tau-\eta h)}$.

If $x \in \overline{\Omega}\backslash U$, then $u(x,T)$ is interior to $\Sigma^0_\tau$, so there is a compact set $Q$ contained in the interior of $\Sigma^0_\tau$, and an $h_0 > 0$ such that $u(x, T+h) \in Q$ if $|h| < h_0$ for all $x \in \overline{\Omega}\backslash U$.

Thus, for sufficiently small $h$, $u(x, T+h)$ is in $\Sigma^0_{(\tau-\eta h)}$ for all $x \in \bar{\Omega}$, and so

$$F_{\Sigma^0}(u(\cdot, T+h)) \leq \tau - \eta h.$$

Therefore,

$$\frac{F_{\Sigma^0}(u(\cdot,T+h)) - F_{\Sigma^0}(u(\cdot,T))}{h} \leq -\eta.$$

and the result is proved.□

As an immediate consequence, we have

**Corollary 5.2** *The map $F_{\Sigma^0} : Y \to \mathbb{R}^+$ is a Liapunov functional for (1.1) and*

$$\{\phi \in Y : \dot{F}_{\Sigma^0}(\phi) = 0\} \subset \{\phi \in Y : F_{\Sigma^0}(\phi) = 0\} = \{\phi \in Y : \phi(x) \in \Sigma^0, \forall x \in \bar{\Omega}\} = \mathcal{M}_\infty$$

**Proof:** The proof is obvious, since from the theorem, if $F_{\Sigma^0} > 0$, then it decreases strictly.□

Note that $\mathcal{M}_\infty$ is not bounded in $Y$ and therefore point dissipativeness does not follow from the corollary. However, as in Theorem 4.1, we have

**Theorem 5.3** *Suppose that $f, g : \mathbb{R}^m \to \mathbb{R}^m$ are $C^1$ and $C^2$, respectively, and satisfy (5.1). Then, the problem (1.1) has a global attractor $\mathcal{A}$ in $Y$. Furthermore,*

$$u(x) \in \Sigma^0, \text{ for all } x \in \bar{\Omega}, \text{ and } u \in \mathcal{A}. \tag{5.5}$$

**Proof:** It follows from La Salle's invariance principle and the previous corollary that for any $u_0 \in Y$, the $\omega$-limit set $\omega(u_0)$ is contained in $\mathcal{M}_\infty$. Thus, given $\tau > 0$ there exists $t_\tau > 0$ such that $S(t)u_0(x) \in \Sigma^0_\tau$, for all $t \geq t_\tau$ and $x \in \bar{\Omega}$. As in Theorem 4.1, we obtain

$$\|S(t)u_0\|_{X^\alpha} \leq M\, e^{-\delta(t-t_\tau)} L(\tau) + M\, P(\tau)\, |\Omega|^{\frac{1}{2}} \int_{t_\tau}^t e^{-\delta(t-s)} (t-s)^{-(\alpha-\beta)} ds, \tag{5.6}$$

where $\|S(s)u_0\|_Y \leq L(\tau)$, $\forall\, s \geq t_\tau$ and $P(\tau) = \sup_{s\in\Sigma_\tau^0}\left\{|\Omega|^{\frac{1}{2}}|f(s)| + |\Gamma|^{\frac{1}{2}}|g(s)|\right\}$. Letting first $t \to \infty$ and then $\tau \to 0$, we obtain

$$\limsup_{t\to\infty} \|S(t)u_0\|_{X^\alpha} \leq M\, P(0)\, |\Omega|^{\frac{1}{2}} \int_0^\infty e^{-\delta s} s^{-(\alpha-\beta)} ds, \tag{5.7}$$

and $S(t)$ is point dissipative in $Y$. Again [16, Theorem 3.4.6], gives the existence of the attractor in $Y$.

Now, if $u \in \mathcal{A}$, there is a complete precompact orbit through $u$ and $\sup_{t\in R}\|S(t)u\|_Y < \infty$, which implies

$$\sup_{t\in R} F_{\Sigma^0}(S(t)u) < \infty.$$

But, $F_{\Sigma^0}(S(-t)u)$ is increasing for $t \geq 0$ and is bounded above, thus the limit $\ell = \lim_{t\to-\infty} F_{\Sigma^0}(S(t)u)$ exists. If $z \in \alpha(u)$, then $F_{\Sigma^0}(z) = \ell$, so also $F_{\Sigma^0}(S(t)z) = \ell$, $t \in \mathbb{R}$, and so $\dot{F}_{\Sigma^0}(z) = 0$. Thus $\alpha(y) \subset \mathcal{M}_\infty$ and from Theorem 6.4 it attracts $S(t)u$. But then for any $\tau > 0$, and for sufficiently negative $t_\tau$, $S(t_\tau)u(x) \in \Sigma_\tau^0$, for all $x \in \bar{\Omega}$, and from Theorem 5.1, $S(t)u(x) \in \Sigma_\tau^0$, for all $x \in \bar{\Omega}$, for $t \geq t_\tau$. In particular, with $t = 0$, $u(x) \in \Sigma_\tau^0$, for all $x \in \bar{\Omega}$. Since $\tau > 0$ is arbitrary, we get the result.□

## 5.2 Contracting piecewise smooth convex regions

Now we construct different, smooth, contracting regions for (1.1). For this, we define, as in [37]

**Definition 5.2** *A smooth function $V : \mathbb{R}^m \to \mathbb{R}$ is said non-degenerate eventually convex if there exists $\tau_0$ (and without loss of generality we can always assume that $\tau_0 = 0$) such that for every $s \in \mathbb{R}^m$, with $V(s) > \tau_0$, $\nabla_s V(s) \neq 0$ and $d^2V(s)(\eta,\eta) \geq 0$ for every $\eta \in \mathbb{R}^m$.*

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a non-degenerate eventually convex function such that $V_\tau = \{s \in \mathbb{R}^m : V(s) \leq \tau\}$, $\tau \geq 0$ are bounded sets and verify

$$(\lambda s + f(s)) \cdot \vec{\nu}(s) > 0, \quad \text{and} \quad g(s)\cdot\vec{\nu}(s) > 0 \tag{5.8}$$

for every $s \in \partial V_\tau = \{s \in \mathbb{R}^m : V(s) = \tau\}$ and $\tau > 0$, where $\vec{\nu}(s)$ is the outward normal to $\partial V_\tau$ at $s$. Note that $\{V_\tau, \tau \geq 0\}$ is an increasing family of sets that covers $\mathbb{R}^m$ and $\nabla_s V(s)$ is an outward normal vector to $\partial V_\tau$.

First we prove the following theorem

**Theorem 5.4** *The convex set $V_\tau$, $\tau > 0$, is an invariant region for the local solution of (1.1).*

**Proof:** If there is a solution $v(x,t) = (v_1(x,t), \cdots, v_m(x,t))$ of (1.1) with initial data $v(x,0) \in V_\tau$ for all $x \in \bar{\Omega}$, that does not stay in $V_\tau$ for all $t \in [0, t_{max})$, then there is a $t_0$ and $x_0 \in \overline{\Omega}$ such that

$$V(v(x,t)) < \tau,\ 0 \leq t < t_0,\ x \in \overline{\Omega}, \text{ and } V(v(x_0,t_0)) = \tau.$$

Therefore, if we prove that at such a point, $\frac{d}{dt}V(v(x_0,t_0)) < 0$, then, $V_\tau$ is invariant.

Observe that $x_0 \notin \Gamma$ since in this case, if $\vec{n}$ denotes the unit outward normal to $\partial\Omega$ at $x_0$

$$0 \leq \nabla_x V(v(x_0,t_0))\cdot\vec{n} = \nabla_s V(v(x_0,t_0))\cdot D_x v(x_0,t_0)\cdot\vec{n} = -\nabla_s V(v(x_0,t_0)) g(v(x_0,t_0)) < 0$$

where for the last inequality we used that $\nabla_s V(s)$ is an outward normal to $\partial V_\tau$ and (5.8). But then $x_0 \in \Omega$ is a point of maximum for $V(v(\cdot,t_0))$, and then, at $(x_0,t_0)$, $\nabla_x V(v) = \nabla_s V(v)\cdot D_x v = 0$ and $\Delta_x V(v) \leq 0$, but since $V$ is convex outside $V_0$ we have

$$0 \geq \Delta_x V(v) = d^2V(v)(\nabla v, \nabla v) + \nabla V_s(v)\cdot\Delta v \geq \nabla V_s(v)\cdot\Delta v.$$

which, in turn, implies $\nabla_s V(v)\cdot \mathrm{Div}\,(a\nabla v) \leq 0$. Thus, at $(x_0,t_0)$

$$\frac{d}{dt}V(v) = \nabla_s V(v)\frac{\partial v}{\partial t} = \nabla_s V(v)\mathrm{Div}(a\nabla v) - \sum_{j=1}^{N} B_j \nabla_s V(v)\cdot\frac{\partial v}{\partial x_j} - \nabla_s V(v)\,(\lambda v + f(v)) \leq$$

$$\leq -\nabla_s V(v)\,(\lambda v + f(v)) < 0$$

where for the last inequality we used again (5.8). Therefore, $V_\tau$ is invariant and the proof is completed.□

As in previous sections, we obtain

**Corollary 5.3** *With the above assumptions, all solutions of (1.1), exist for $t \geq 0$, and the semigroup $S(t)$ is well defined for $t \geq 0$. Moreover, for every bounded set $B \subset Y$, $\{S(t)B, t \geq 0\}$ is bounded in $Y$ and $\{S(t)B, t \geq 1\}$ is compact in $Y$.*

We now show that, under the above conditions, the semigroup $\{S(t),\, t \geq 0\}$ is point dissipative. For this, as in the previous section, we will construct a suitable Liapunov function.

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ be $C^1$ and $C^2$ functions, respectively, satisfying (5.8) and $F_{V_0} : Y \to \mathbb{R}^+$ be defined by

$$F_{V_0}(w) = \sup\{V(w(x)) :\ x \in \bar{\Omega}\}. \tag{5.9}$$

It is easy to check that $F_{V_0}$ is a continuous on Y and $F_{V_0}(w) = 0$ if and only if $w(x) \in V_0$ for all $x \in \bar{\Omega}$. Then, we prove

**Theorem 5.5** *Let $f$, $g$ and $V_\tau$, $\tau > 0$, be as before. Suppose that $u(t)$ is a solution of (1.1). Then, for any $T > 0$ for which $F_{V_0}(u(\cdot, T)) = \tau > 0$, there exists $\eta > 0$ such that*

$$\limsup_{h \to 0} \frac{F_{V_0}(u(\cdot, T+h)) - F_{V_0}(u(\cdot, T))}{h} \leq -\eta. \tag{5.10}$$

**Proof:** If $F_{V_0}(u(\cdot, T)) = \tau > 0$ and if $\bar{x} \in \overline{\Omega}$ is such that $u(\bar{x}, T) \in \partial V_\tau$, then from (5.8), there exists $\eta = \eta(\tau) > 0$ such that

$$-(\lambda u(\bar{x}, T) + f(u(\bar{x}, T))) \cdot \nabla_s V(u(\bar{x}, T)) < -\eta.$$

Again, $\bar{x}$ cannot be on $\Gamma$ since in this case, $\bar{x}$ is a maximum point for $V(u(\cdot, T))$ and if $\vec{n}$ is the unit outward normal to $\partial\Omega$ at $\bar{x}$, as before,

$$0 \leq \nabla_x V(v(\bar{x}, T)) \cdot \vec{n} = -\nabla_s V(v(\bar{x}, T)) \cdot g(v(\bar{x}, T)) < 0$$

which is a contradiction. Thus, $\bar{x} \in \Omega$ and $\nabla_x V(u(\bar{x}, T)) = 0$ and $\Delta_x V(u(\bar{x}, T)) \leq 0$. Consequently, $\nabla_s V(u) \cdot \mathrm{Div}\ (a\nabla u) \leq 0$ at $(\bar{x}, T)$ and, as before,

$$\frac{\partial V(u)}{\partial t} = \nabla_s V(u) \cdot \mathrm{Div}(a\nabla u) - \sum_{k=1}^{N} B_k \nabla_s V(u) \cdot \frac{\partial u}{\partial x_k} - \nabla_s V(u) \cdot (\lambda u - f(u)) < -\eta,$$

so that $V(u(\bar{x}, T+h)) < \tau - \eta h$ for small $h$. By continuity, this holds for all $x$ in a neighborhood of $\bar{x}$. If $K_T = \{x : u(x, T) \in \partial V_\tau\}$, then $K_T$ is compact, and by what we have just shown, there is an open set $\Omega \supset U \supset K_T$ such that if $x \in U$, and $h$ is small, $u(x, T+h) \in V_{(\tau - \eta h)}$. If $x \in \overline{\Omega} \backslash U$, then $u(x, T)$ is interior to $V_\tau$, so there exists a compact set $Q$ contained in the interior of $V_\tau$, and $h_0 > 0$ such that $u(x, T+h) \in Q$ if $|h| < h_0$ for all $x \in \bar{\Omega} \setminus U$.

Thus, for sufficiently small $h$, $u(x, T+h)$ is in $V_{(\tau-\eta h)}$ for all $x \in \bar{\Omega}$, and so

$$F_{V_0}(u(\cdot, T+h)) \leq \tau - \eta h.$$

Therefore,

$$\frac{F_{V_0}(u(\cdot, T+h)) - F_{V_0}(u(\cdot, T))}{h} \leq -\eta$$

and the result is proved.□

As an immediate consequence, we have

**Corollary 5.4** *The map $F_{V_0} : Y \to \mathbb{R}^+$ is a Liapunov functional for (1.1) and*

$$\{\phi \in Y :\ \dot{F}_{V_0}(\phi) = 0\} \subset \{\phi \in Y :\ F_{V_0}(\phi) = 0\} = \{\phi \in Y :\ \phi(x) \in V_0,\ \forall x \in \bar{\Omega}\} = \mathcal{V}_\infty$$

Note that $\mathcal{V}_\infty$ is not bounded in $Y$ and therefore point dissipativeness does not follow from the corollary. However, as before, we have

**Theorem 5.6** *Suppose that $f, g : \mathbb{R}^m \to \mathbb{R}^m$ are $C^1$ and $C^2$, respectively, and satisfy (5.8). Then, the problem (1.1) has a global attractor $\mathcal{A}$ in $X^\alpha$. Furthermore,*

$$u(x) \in V_0, \ \textit{for all } x \in \bar{\Omega}, \ \textit{and } u \in \mathcal{A}. \tag{5.11}$$

The proof is as in the previous section, with $V_0$, $V_\tau$ and $\mathcal{V}_\infty$ replacing $\Sigma^0$, $\Sigma^0_\tau$ and $\mathcal{M}_\infty$, respectively.

# 6 Appendix

## 6.1 The first eigenvalue of differential operators

We now collect some results on the spectrum of differential operators which are not necessarily selfadjoint, see [32, 33, 23]. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, and consider the elliptic operator $L^c$, defined by

$$L^c[u] = -\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + cu \tag{6.1}$$

together with the boundary operator

$$B^d[u] \equiv \sum_{i=1}^{N} e^i(x)\frac{\partial u}{\partial x_i} + du, \text{ on } \Gamma = \partial\Omega. \tag{6.2}$$

where $a^{ij}$ is symmetric and positively definite, $e = (e^1, \ldots, e^N)$ is a vector that points outwards to $\Gamma = \partial\Omega$, and all coefficients are real and bounded in $\bar{\Omega}$.

For studying the spectrum of $(L^c, B^d)$ we are lead to the eigenvalue problem

$$(P)_{c,d} \begin{cases} L^c[u] = \mu u, \text{ on } \Omega \\ B^d[u] = 0, \text{ on } \Gamma \end{cases}$$

**Theorem 6.1 [32, 33]**
*Suppose that there exists $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $w > 0$ in $\bar{\Omega}$, such that, $B^d[w] \geq 0$ on $\Gamma$. Then the continuous and discrete spectrum of $(L^c, B^d)$ are contained in the half-plane*

$$Re(\mu) \geq \inf\left(\frac{L^c[w]}{w}\right). \square \tag{6.3}$$

If $(L^c, B^d)$ does not have residual spectrum, then the following theorem holds.

**Theorem 6.2 [32, 33]**
*Suppose that there exists a function $w$ satisfying the conditions of the preceding theorem. Then, if the spectrum of $(L^c, B^d)$ is non-empty, there exists a real number in the spectrum $\mu_1$, such that the whole spectrum is contained in the half-plane*

$$Re(\mu) \geq \mu_1 \geq \inf\left(\frac{L^c[w]}{w}\right). \square \tag{6.4}$$

**Theorem 6.3 [23]**
*If $\Omega$ and all the coefficients of $(L^c, B^d)$ are smooth enough, for $\mu$ sufficiently small, the resolvent $R_\mu$ is completely continuous in the sup norm, and all the conditions of the preceding Theorem are satisfied. Therefore the spectrum is discrete, and thus $\mu_1$ is an eigenvalue. Moreover, there exists a positive eigenfunction $\phi_1$, associated to $\mu_1$.* $\square$

Let us suppose that all the coefficients of $(P)_{c,d}$ are smooth enough. According to the results above, we define $\mu(c,d) = \mu_1$, which is an eigenvalue associated to a positive eigenfunction. We will say then that $\mu(c,d)$ is the first eigenvalue of $(L^c, B^d)$.

We will show below some properties of the function $\mu(c,d)$.

**Lemma 6.1** *If there exists an eigenpair $(\tilde{\mu}, \tilde{\phi})$ of $(P)_{c,d}$, such that, $\tilde{\phi} \geq 0$ then $\tilde{\mu} = \mu(c,d)$.*

**Proof:** Taking $w = \tilde{\phi}$ in the Theorem above, we have $Re(\mu) \geq \inf \left( \frac{L^c[\tilde{\phi}]}{\tilde{\phi}} \right) = \inf \left( \frac{\tilde{\mu}\tilde{\phi}}{\tilde{\phi}} \right) = \tilde{\mu}$.□

**Lemma 6.2** *With the above notations, we have*
*i) For every $c, d \in \mathbb{R}$, $\mu(c,d) = \mu(0,d) + c$, and $\mu(0,0) = 0$.*
*ii) Given $d_1 > d_2$, then $\mu(0,d_1) > \mu(0,d_2)$.*

**Proof:**
i) The first claim is obvious. If $c, d = 0$ then we take $w = 1$ which is clearly a positive eigenfunction for the eigenvalue $\mu = 0$ in $(P)_{0,0}$ and the result follows.
ii) Let $\phi_2$ be a positive eigenfunction associated with $\mu(0,d_2)$. We take $w = \phi_2$, in $(P)_{0,d_1}$, and note that $B^{d_1}[w] = (d_1 - d_2)w \geq 0$. Hence, $\mu(0,d_1) \geq \inf \left( \frac{-L^0[w]}{w} \right) = \mu(0,d_2)$. But $\mu(0,d)$ is analytic function of $d$, thus we get the result.□

For the particular operator in former sections of this paper, we have

**Lemma 6.3** *If $L^c$ is of the form $L^c[u] = -Div(a(x)\nabla u) + cu$ and $B^d[u] = \frac{\partial u}{\partial \vec{n}_a} + du$, we have*

$$\mu(c,d) = \inf_{\phi \in H^1(\Omega)} \frac{\int_\Omega a|\nabla \phi|^2 + c\int_\Omega |\phi|^2 + d\int_\Gamma |\phi|^2}{\int_\Omega |\phi|^2}$$

*and*

$$\mu(0,d) \leq \mu^*, \quad \lim_{d \to \infty} \mu(0,d) = \mu^*$$

*where $\mu^*$ is the first eigenvalue of $L^0$ subjected to Dirichlet boundary conditions.*

**Proof:** The first part is rather well known, [11]. Also, note that

$$\mu(0,d) = \inf_{\phi \in H^1(\Omega)} \frac{\int_\Omega a|\nabla \phi|^2 + d\int_\Gamma |\phi|^2}{\int_\Omega |\phi|^2} \leq \inf_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega a|\nabla \phi|^2}{\int_\Omega |\phi|^2} = \mu^*$$

Since $\mu(0,d)$ is increasing and bounded, then the limit exists and $\lim_{d\to\infty} \mu(0,d) \leq \mu^*$. Let $d_n$ converging to $\infty$ and $\phi_n \in H^1(\Omega)$, such that $\|\phi_n\| = 1$, norm in $L^2(\Omega)$, and $\mu(0,d_n) = \int_\Omega a|\nabla\phi_n|^2 + d_n \int_\Gamma |\phi_n|^2 \leq \mu^*$. Thus, $\{\phi_n\}$ is bounded in $H^1(\Omega)$ and, taking sub-sequences, if necessary, we can assume $\phi_n \to \phi$, weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and $\gamma(\phi_n) \to \gamma(\phi)$ weakly in $L^2(\Gamma)$. Therefore, $\|\phi\| = 1$ and since $\mu(0,d_n)$ is bounded we get $\int_\Gamma |\phi_n|^2 \to 0$. Hence, $\phi \in H_0^1(\Omega)$. From lower semi-continuity, $\int_\Omega a|\nabla\phi|^2 \leq \liminf_n \int_\Omega a|\nabla\phi_n|^2$ and then $\lim_{n\to\infty} \mu(0,d_n) \geq \int_\Omega a|\nabla\phi|^2 \geq \mu^*$. Since the argument is independent of the sequence we take, we get the result.□

Note that, in terms of hypothesis (1.8) and (1.9), these results imply that for a given $d^0 \in \mathbb{R}^m$ as before, if $c^0 \in \mathbb{R}^m$ is (component-wise) large enough, the hypothesis is met, since one is pushing the spectrum of the operator to the right in the complex plane. However, the converse is not true. In fact if $c^0 \in \mathbb{R}^m$ is given and some component is very negative, no matter how big $d^0 \in \mathbb{R}^m$ is taken, the first eigenvalue of (1.9) can not be made positive. This means that one can compensate possible wrong signs in the flux terms with a suitable reaction term, but the converse is only true if the reaction term is not too big.

## 6.2 La Salle's invariance principle

In this subsection we state classical results in invariance theory, that have been used in previous sections. The reader is referred to [21, Chap. 4, sec 4.3] and [16] for a more complete presentation.

**Theorem 6.4** *Assume $S(t)$ is a semigroup in a complete metric space $Y$ and suppose that $u_0 \in Y$ is such that $\{S(t)u_0,\ t \geq 0\}$ lies in a compact set in $Y$. Then, the $\omega$-limit set $\omega(u_0)$ is nonempty, compact, invariant, connected, and $dist(S(t)u_0, \omega(u_0)) \to 0$ as $t \to +\infty$.*

*Similarly, if there is a negative orbit through $u_0$ and $\{S(t)u_0,\ t \leq 0\}$ lies in a compact set in $Y$ then, the $\alpha$-limit set, $\alpha(u_0)$ is nonempty, compact, invariant, connected, and $dist(S(t)u_0, \alpha(u_0)) \to 0$ as $t \to -\infty$.* □

Next, we recall the definition of Liapunov function.

**Definition 6.1** *With the above notations, a Liapunov function for $S(t)$ is a continuous real valued function $V : Y \to \mathbb{R}$ such that*

$$\dot{V}(\phi) = \limsup_{t \to 0^+} \frac{V(S(t)\phi) - V(\phi)}{t} \leq 0$$

*for all $\phi \in Y$.*

Then, we have

**Theorem 6.5** *Let $V$ be a Liapunov function for $S(t)$ on $Y$ and define $E = \{\phi \in Y : \dot{V}(\phi) = 0\}$ and $\mathcal{M}$ as the maximal invariant subset of $E$. If $\{S(t)u_0,\ t \geq 0\}$ lies in a compact set in $Y$, then $S(t)u_0 \to \mathcal{M}$ as $t \to +\infty$.*

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# NOTAS DO ICMSC

## SÉRIE MATEMÁTICA


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