UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

CONFIGURATIONS OF QUADRATIC SYSTEMS POSSESSING THREE DISTINCT INFINITE SINGULARITIES AND INVARIANT PARABOLAS

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Nº 448

NOTAS DO ICMC

SÉRIE MATEMÁTICA

São Carlos – SP
Abr./2024

Configurations of quadratic systems possessing three distinct infinite singularities and invariant parabolas

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Abstract

Denote by **QS** the class of all non-degenerate planar quadratic differential systems and by **QSP** the subclass of **QS** of all systems possessing at least one invariant parabola. In this paper we consider the subfamily of **QSP** defined by the condition $\eta \neq 0$, which we denote by $\mathbf{QSP}_{(\eta\neq 0)}$. We investigate all possible configurations of invariant parabolas and invariant straight lines which systems in $\mathbf{QSP}_{(\eta\neq 0)}$ could possess and their geometric properties encoded in such configurations. The classification presented here is taken modulo the action of the group of real affine transformations and time rescaling and it is given in terms of affine invariant polynomials. It yields a total of 146 distinct configurations. The obtained classification is an algorithm which makes possible for any given real quadratic differential system in $\mathbf{QSP}_{(\eta\neq 0)}$ to specify its configuration of invariant parabolas and straight lines. This work will prove helpful in studying the integrability of the systems in in $\mathbf{QSP}_{(\eta\neq 0)}$.

1 Introduction and statement of main results

To every planar differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P, $Q \in \mathbb{R}[x,y]$, i.e. P, Q are polynomials in x, y over $\mathbb{R}$, is associated the vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

The *degree* of such a system is the integer $m = \max(\deg P, \deg Q)$. In particular we say that a system (1) is a *quadratic* differential system when $m = 2$ and here **QS** denotes the whole class of real quadratic differential systems. From now on we are assuming that P and Q are coprime polynomials. Otherwise doing a rescaling of the time, systems (1) can be reduced to linear or constant systems. Quadratic differential systems under such assumptions are called *non–degenerate quadratic systems*.

Quadratic systems emerge in various research fields including models of population dynamics [6], fluid dynamics [9], control systems [11] and even quantum dynamics [3]. As a consequence, **QS** are subject of great interest for mathematicians and researchers from other areas of science and, many papers have been published on such systems, see for example [1] for a bibliographical survey.

Given $f \in \mathbb{C}[x, y]$, we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1) if there exists $K \in \mathbb{C}[x, y]$ (it is called cofator of the invariant curve $f = 0$) such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

Quadratic systems with an invariant algebraic curve have been studied by many authors, for example Schlomiuk and Vulpe in [19, 21] have studied quadratic systems with invariant straight lines; Qin Yuan-xum [14] have investigated the quadratic systems having an ellipse as limit cycle; Druzhkova [10] presented the necessary and sufficient conditions on the coefficients of a quadratic system and also on the coefficients of a conic so as to have the conic as an invariant curve of the system; Christopher [7] presented a normal form for quadratic systems possessing invariant parabolas; Cairó and Llibre in [5] studied the quadratic systems having invariant algebraic conics in order to investigate the Darboux integrability of such systems.

The main goal of this research is to investigate non–degenerate quadratic systems having invariant conics. The irreducible affine conics over the field $\mathbb{R}$ are the hyperbolas, ellipses and parabolas. One way to distinguish them is to consider their points at infinity. The term hyperbola is used for a real irreducible affine conic which has two real points at infinity. This distinguishes it from the other two irreducible real conics: the parabola has just one real point at infinity at which the multiplicity of intersection of the conic with the line at infinity is two, and the ellipse which has two complex points at infinity.

Inside this proposal, the classification of **QS** with invariant hyperbolas [17, 16] and with invariant ellipses [15, 13] are obtained in previous works. In this work we study the class **QSP** of non-degenerate quadratic differential systems having an invariant parabolas. The investigation of such a class of systems is done applying the invariant theory.

The group of real affine transformations and time rescaling acts on the class **QS** and due to this, modulo this group action the quadratic systems depend on five parameters. The same group acts on the **QSP** and modulo this action systems in this class depend on at most three parameters. As we want this study to be intrinsic, independent of the normal form given to the systems, we use here invariant polynomials and geometric invariants for the desired classification.

In the paper [23] the necessary and sufficient conditions for a non-degenerate quadratic system in **QS** to have invariant parabolas are provided. Moreover in that paper the invariant criteria which provide the number, position and multiplicity of such parabolas are determined.

The present paper is a continuation of [23]. More precisely using the conditions from that paper we present the classification of all configurations of invariant parabolas and invariant lines which a system in $\mathbf{QSP}_{(\eta\neq 0)}$ could possess. The investigation of the configurations of the family of systems in $\mathbf{QSP}_{(\eta=0)}$ is in progress.

An important ingredient in this work is the notion of *configuration of algebraic solutions* of a polynomial differential system. This notion appeared for the first time in [19].

Definition 1. Consider a planar polynomial system which has a finite number of algebraic solutions and a finite number of singularities, finite or infinite. By *configuration of algebraic solutions* of this system we mean the set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singularities of this system located on these curves, each one of these singularities endowed with its own multiplicity.

We point out that in [8] the notions of multiplicities (infinitesimal; integrable; algebraic; geometric; holonomic) of an algebraic invariant curve are given. Here we use the definition of geometric multiplicity based on perturbations in the family **QS**.

Definition 2. We say that an invariant conic $\Phi(x,y) = p + qx + ry + sx^2 + 2txy + uy^2 = 0$, $(s,t,u) \neq (0,0,0)$, $(p,q,r,s,t,u) \in \mathbb{C}^6$ for a quadratic vector field X has *multiplicity* m if there exists a sequence of real quadratic vector fields X_k converging to X, such that each X_k has m distinct (complex) invariant conics $\Phi_k^1 = 0, \ldots, \Phi_k^m = 0$, converging to $\Phi = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$. In the case when an invariant conic $\Phi(x,y) = 0$ has multiplicity one we call it *simple*.

Our main results are stated in the following theorem.

Main Theorem. **(*A*)** *The conditions $\eta \neq 0$, and $\chi_1 = \chi_2 = 0$ are necessary for a quadratic system in the class $\boldsymbol{QSP}_{(\eta\neq 0)}$ to possess at least one invariant parabola.*

(*B*) *Assume that for a system (S) in the class $\boldsymbol{QSP}_{(\eta\neq 0)}$ the condition $\chi_1 = \chi_2 = 0$ is satisfied.*

- **(B_1)** *If $\eta > 0$ then the system (S) could possess only one of the configurations Config. $\mathcal{P}$.1–Config. $\mathcal{P}$.114 presented in Figure 1. Moreover for each one of these configurations the corresponding conditions for its realization could be collected from Diagrams 1 and 2.*
- **(B_2)** *If $\eta < 0$ then the system (S) could possess only one of the configurations Config. $\mathcal{P}$.115–Config. $\mathcal{P}$.146 presented in Figure 2. Moreover for each one of these configurations the corresponding conditions for its realization could be collected from Diagram 3.*

(*C*) *The Diagrams 1, 2 and 3 actually contain the global bifurcation diagram in the 12-dimensional space of parameters of the systems belonging to family $\boldsymbol{QSP}_{(\eta\neq 0)}$, which possess at least one invariant parabola. The corresponding conditions are given in terms of invariant polynomials with respect to the group of affine transformations and time rescaling.*

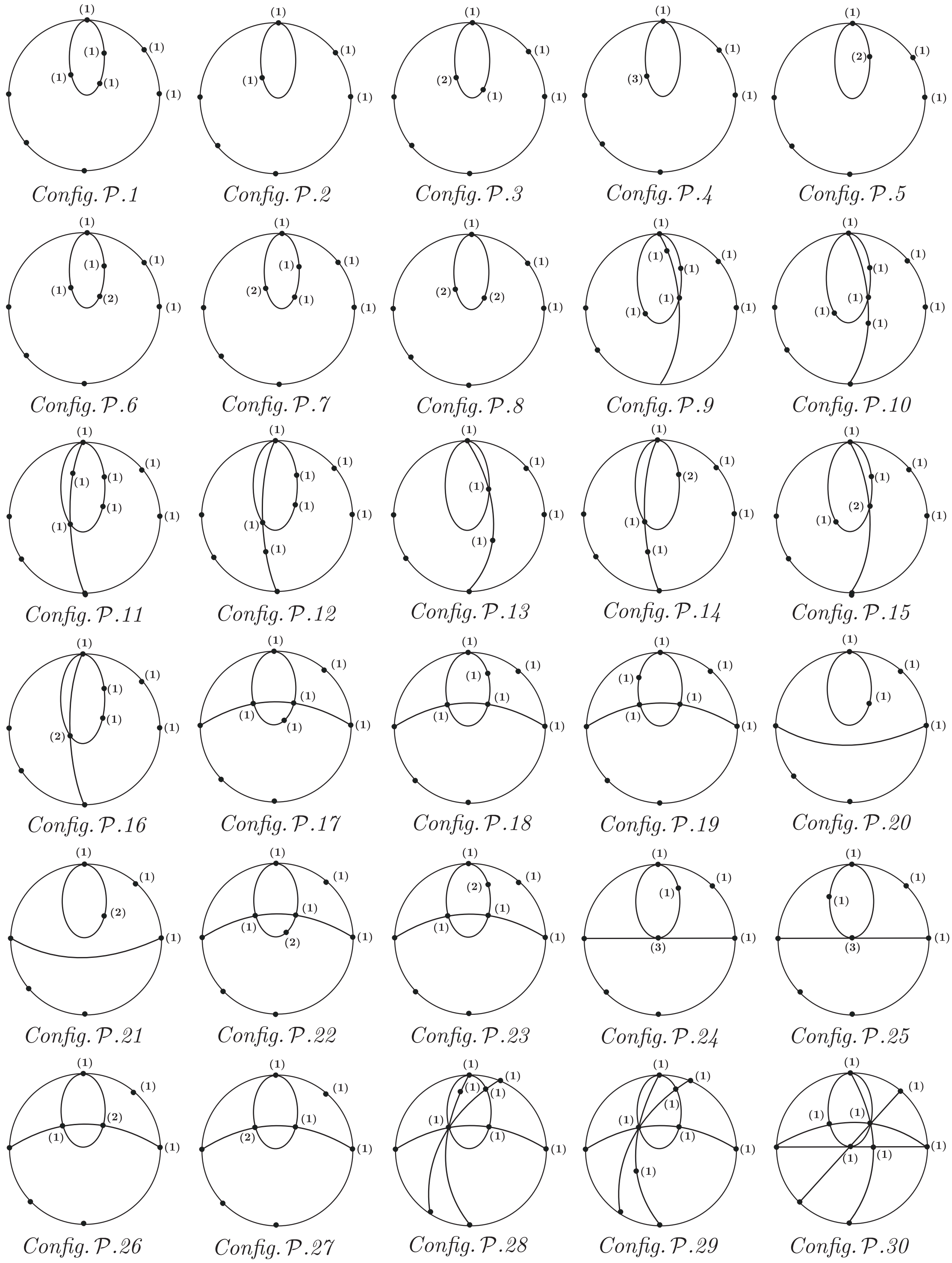

Figure 1: Configurations of systems in **QSP** in the case $\eta > 0$

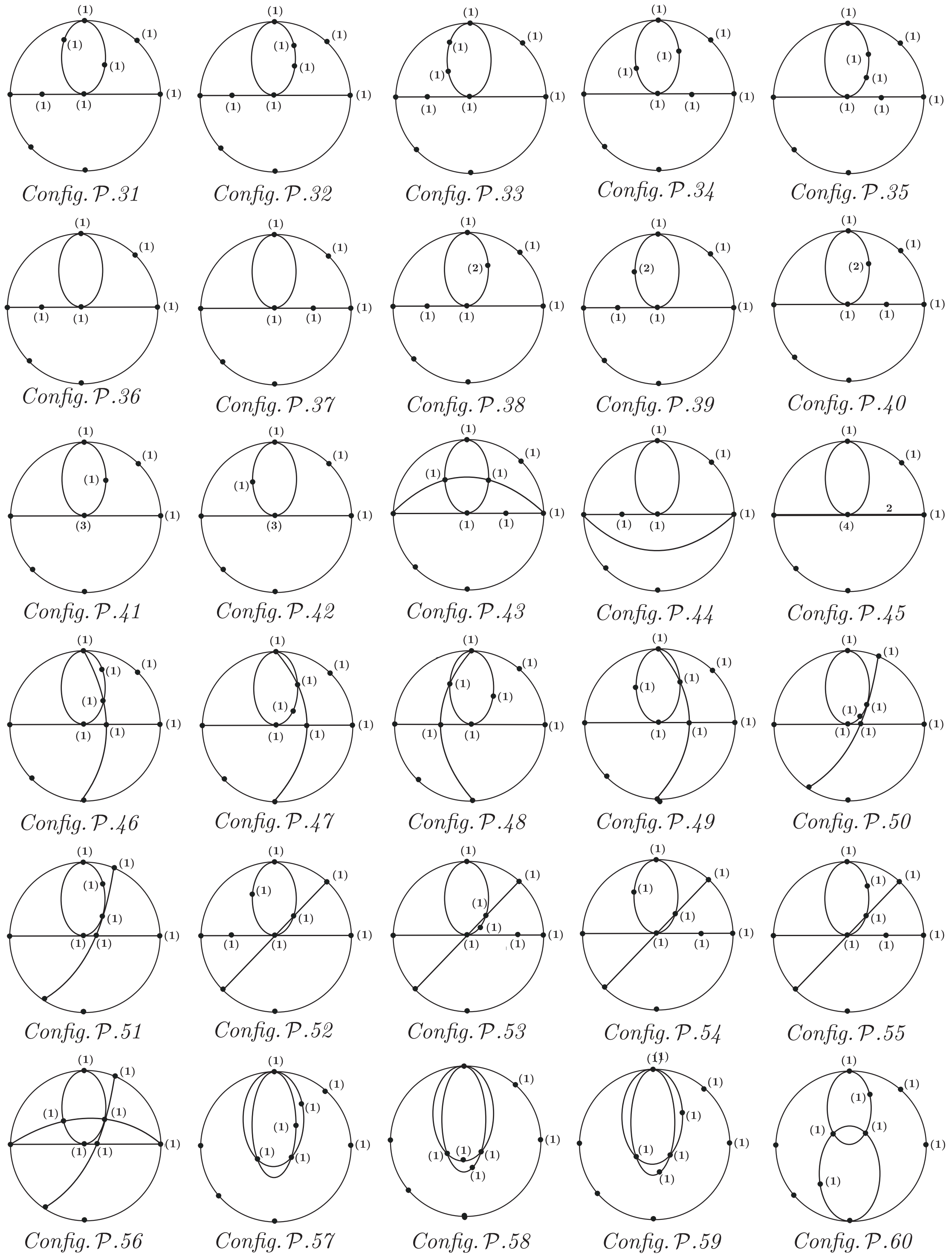

Figure 1 (cont.): Configurations of systems in **QSP** in the case $\eta > 0$

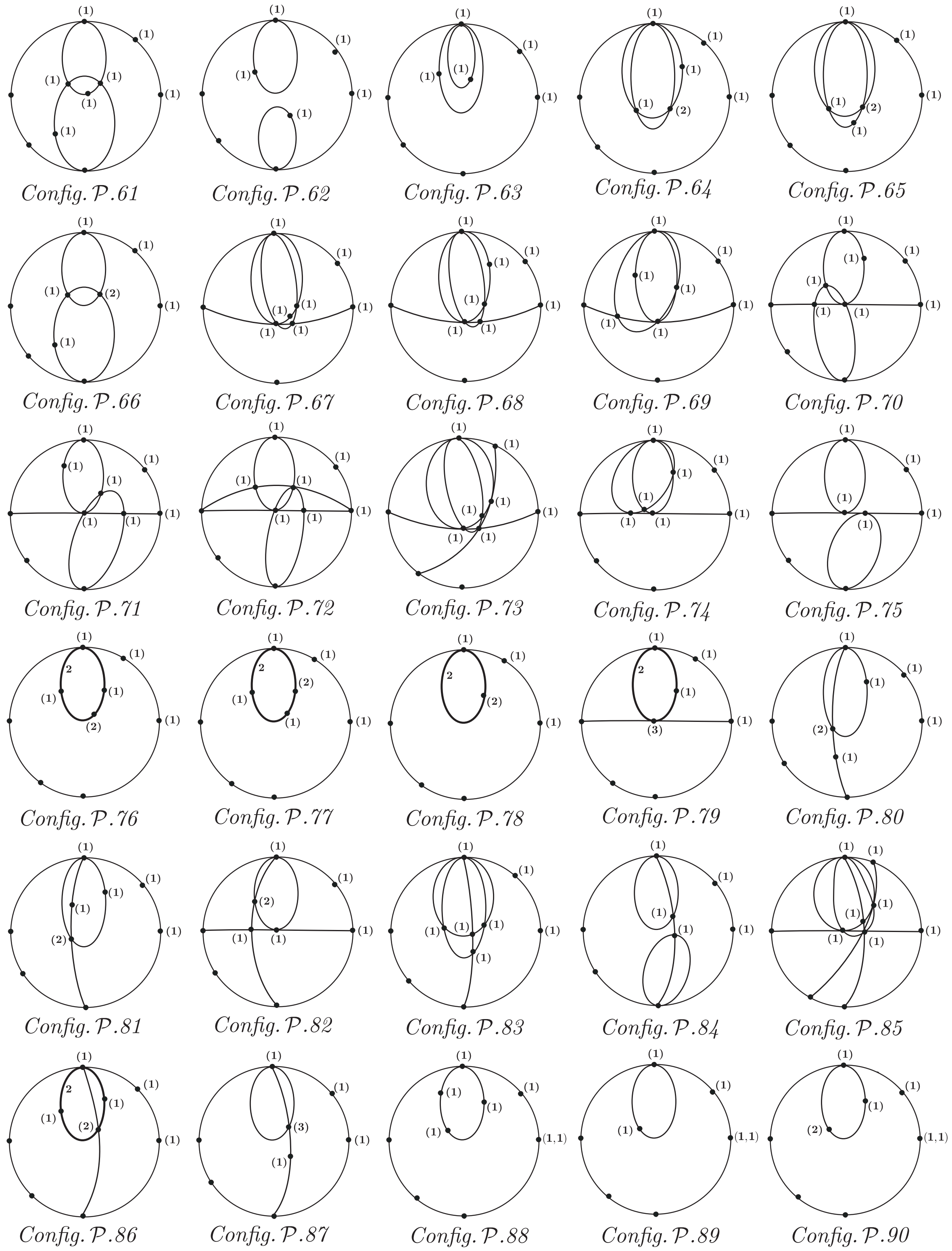

Figure 1 (cont.): Configurations of systems in **QSP** in the case $\eta > 0$

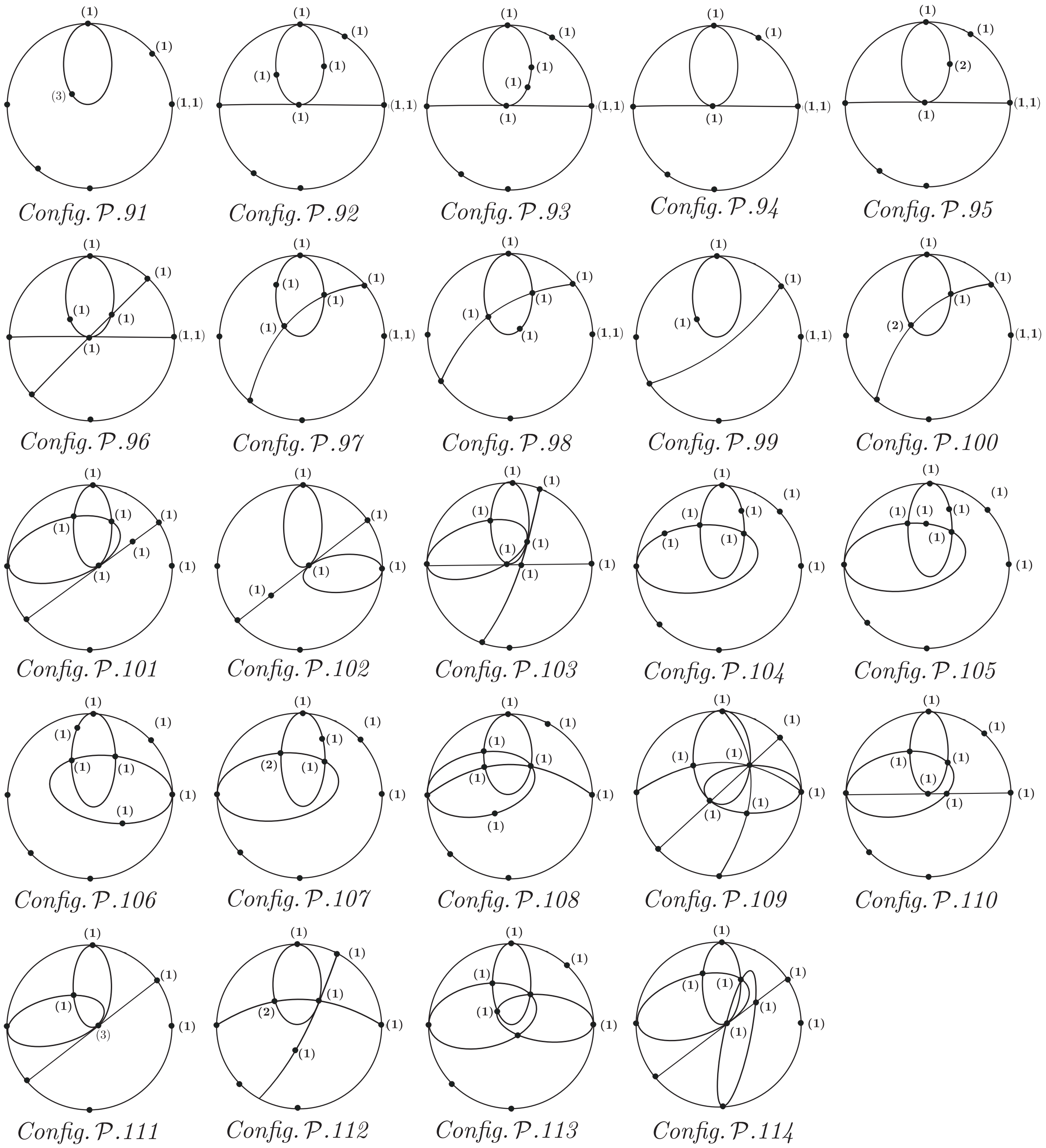

Figure 1 (cont.): Configurations of systems in **QSP** in the case $\eta > 0$

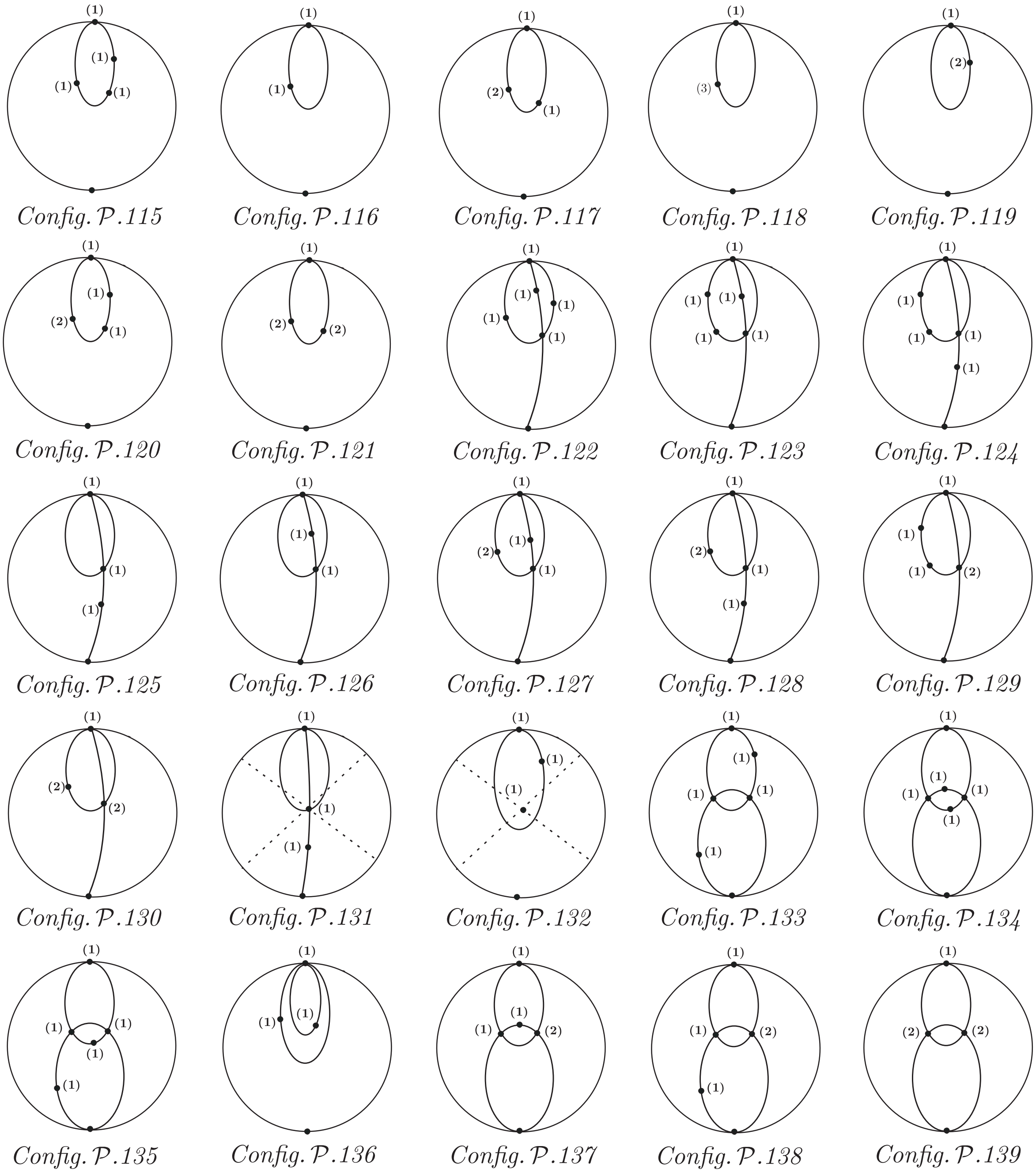

Figure 2: Configurations of systems in **QSP** in the case $\eta < 0$

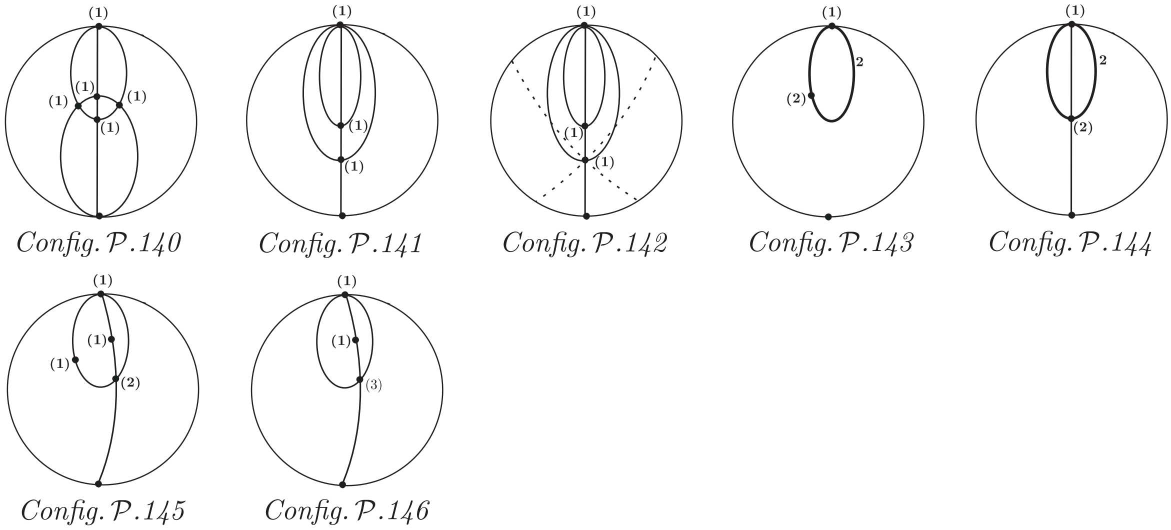

Figure 2 (cont.): Configurations of systems in **QSP** in the case $\eta < 0$

2 Preliminaries

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Such a system (2) can be identified with a point in $\mathbb{R}^{12}$. Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ and consider the ring $\mathbb{R}[a_{00}, a_{10}, \ldots, a_{02}, b_{00}, b_{10}, \ldots, b_{02}, x, y]$ which we shall denote $\mathbb{R}[\tilde{a}, x, y]$.

It is known that on the set **QS** of all quadratic differential systems (2) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [20]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS** . We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map **QS** $\longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a} = (a_{00}, \ldots, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants, the T–comitants and the CT–comitants. For their detailed definitions as well as their constructions we refer the reader to the paper [20] (see also [1]).

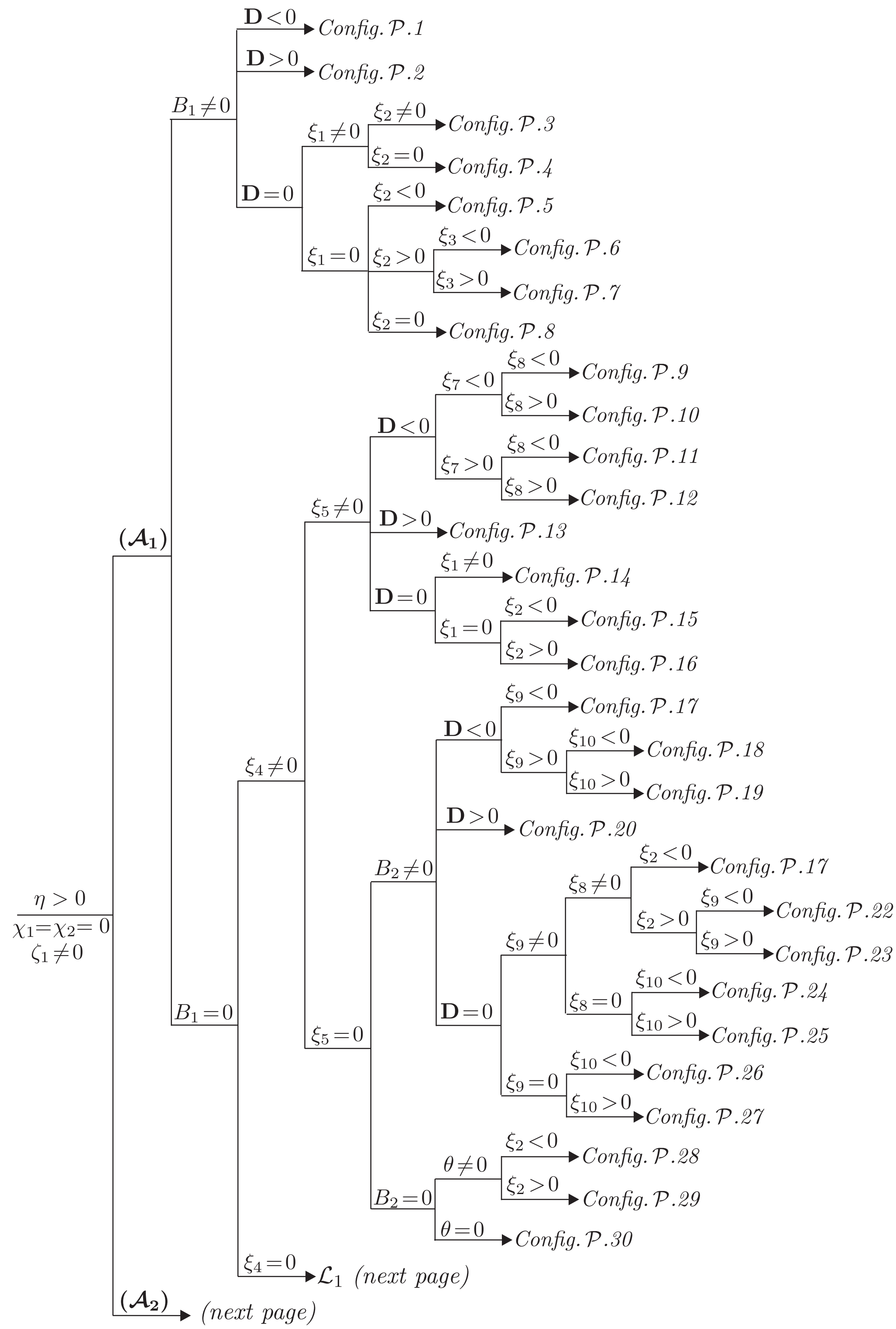

Diagram 1: Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$

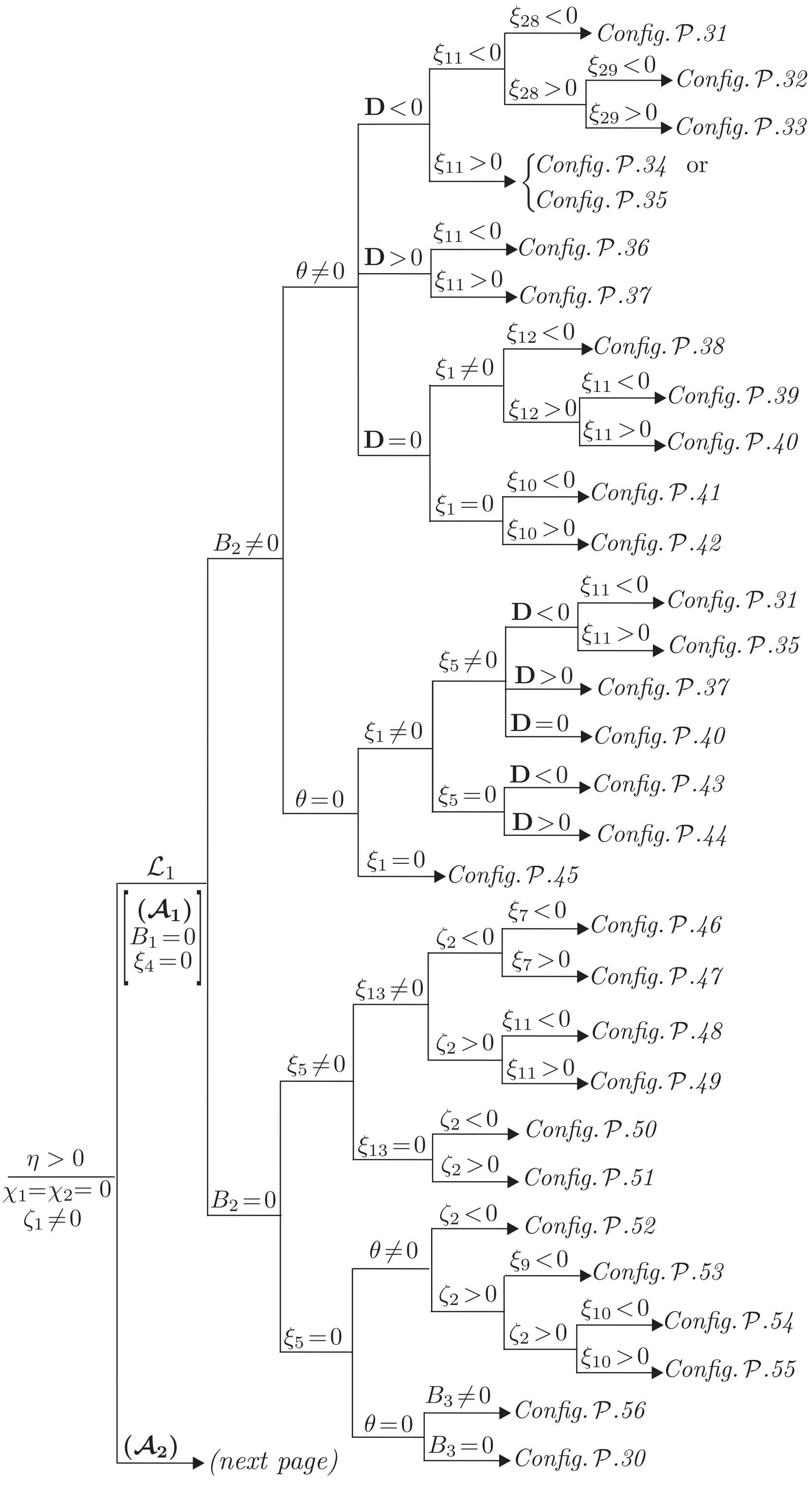

Diagram 1 (cont.): Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$

2.1 The main invariant polynomials associated to invariant parabolas

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2):

$$
\begin{aligned}
C_i(\tilde{a}, x, y) &= y p_i(x, y) - x q_i(x, y), \;\; (i = 0, 1, 2) \\
D_i(\tilde{a}, x, y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \;\; (i = 1, 2).
\end{aligned}
\tag{3}
$$

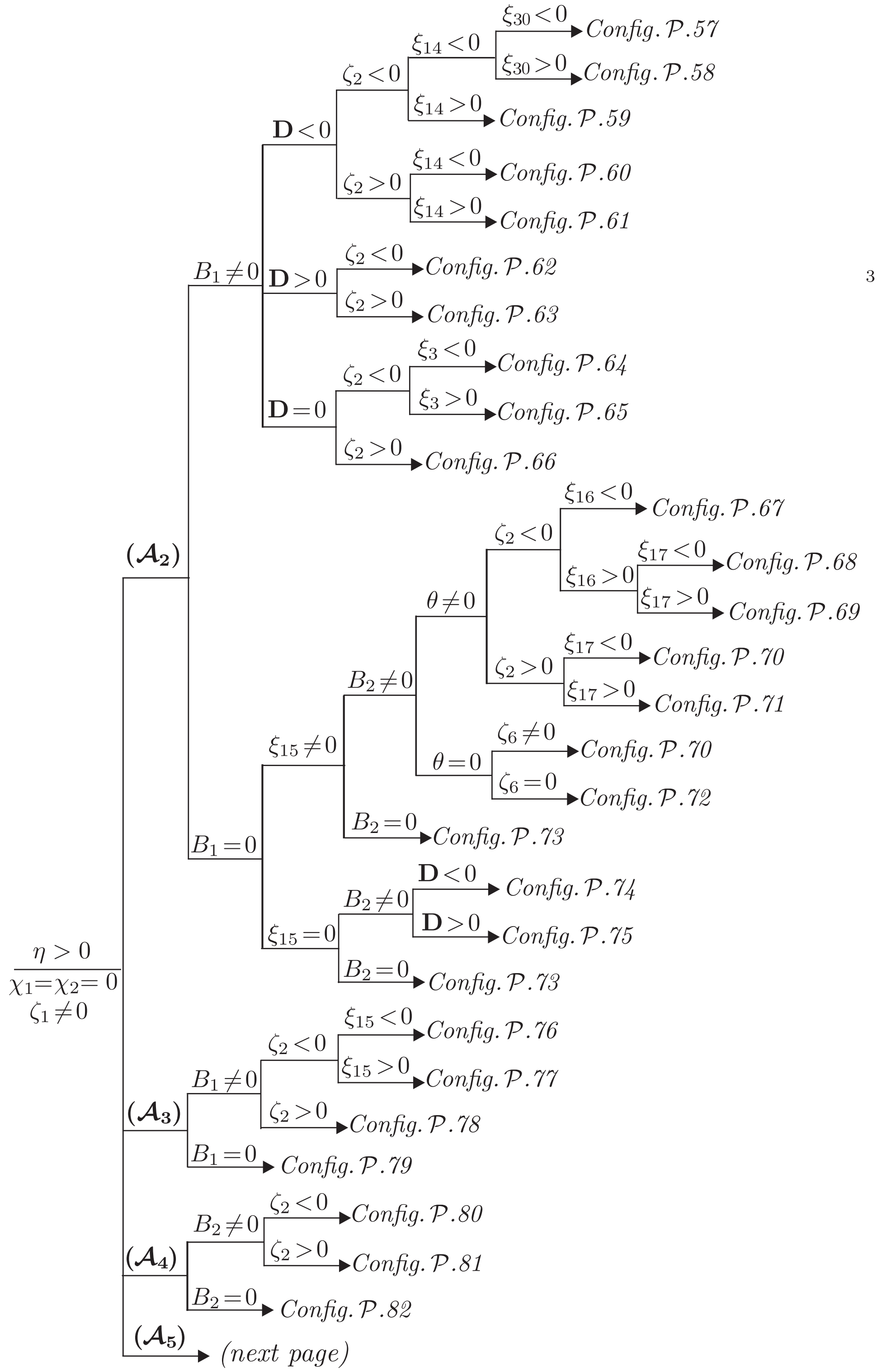

Diagram 1 (cont.): Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$

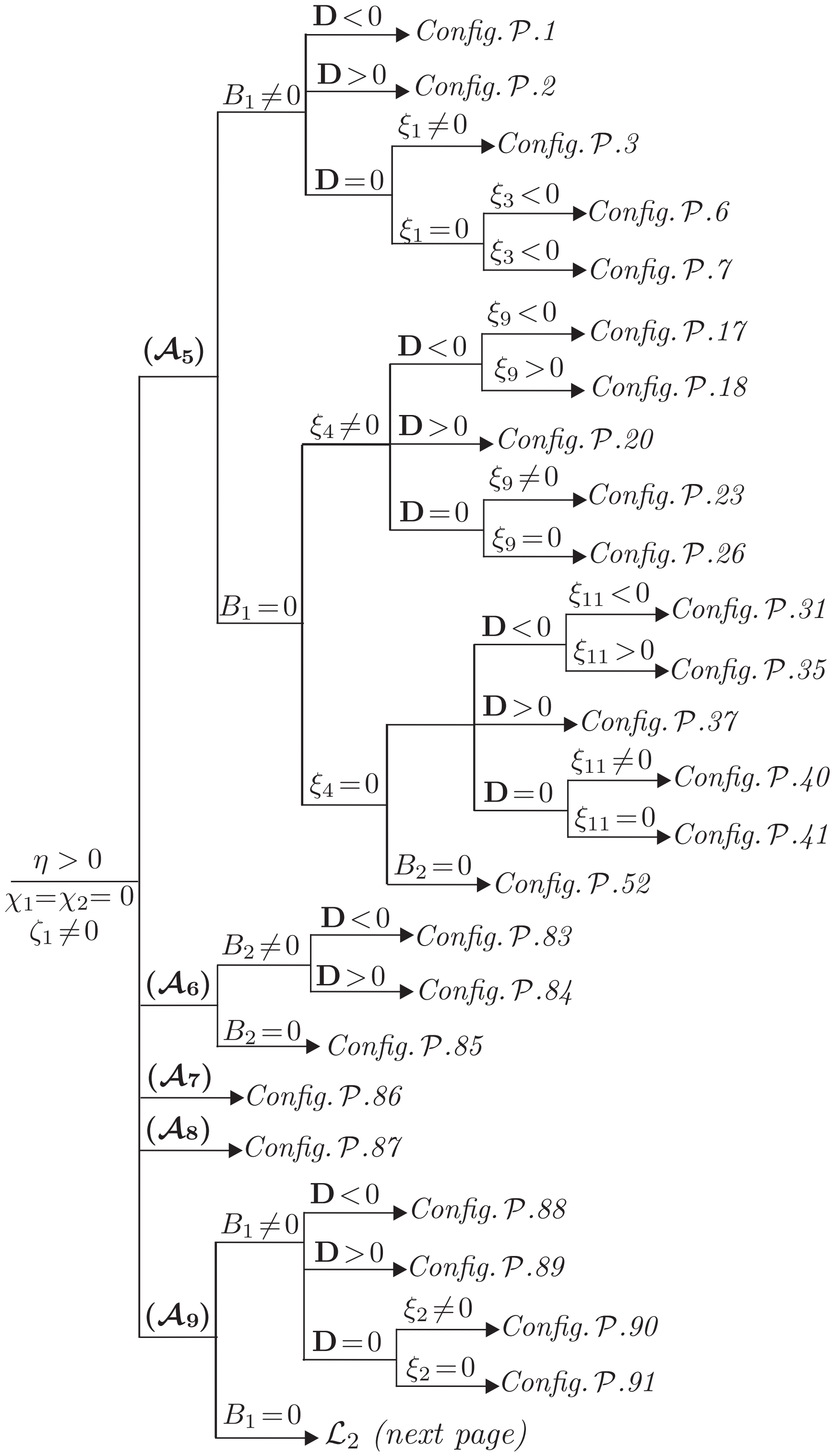

Diagram 1 (cont.): Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$

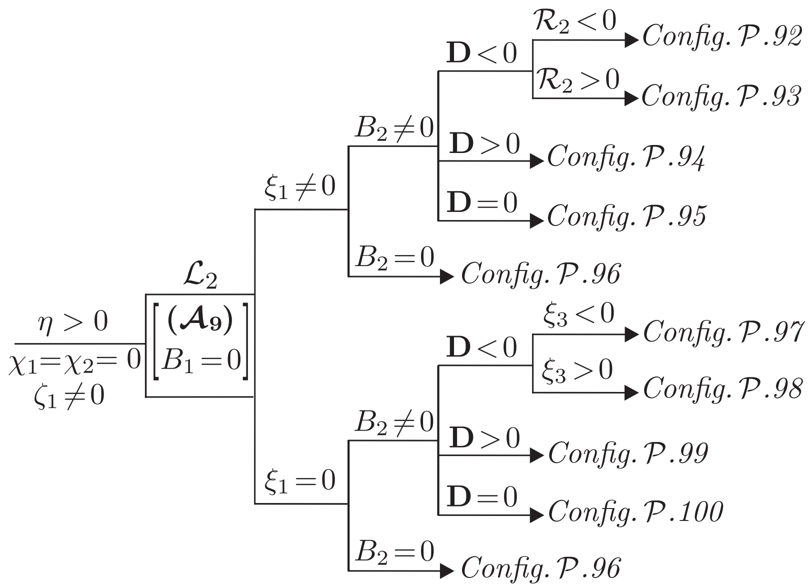

Diagram 1 (cont.): Conditions for the configurations of systems in **QSP** in the case $\eta>0$, $\zeta_1\neq0$

As it was shown in [22] these polynomials of degree one in the coefficients of systems (2) are GL–comitants of these systems. Let $f,\ g\in\mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)}\in\mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index k of (f,g)* (cf. [12], [18])).

Proposition 1 (see [24])**.** *Any GL–comitant of systems* (2) *can be constructed from the elements* (3) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 1. *We point out that the elements* (3) *generate the whole set of GL–comitants and hence also the set of affine comitants as well as the set of T-comitants.*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1=(C_0,C_1)^{(1)}, & T_2=(C_0,C_2)^{(1)}, & T_3=(C_0,D_2)^{(1)},\\ T_4=(C_1,C_1)^{(2)}, & T_5=(C_1,C_2)^{(1)}, & T_6=(C_1,C_2)^{(2)},\\ T_7=(C_1,D_2)^{(1)}, & T_8=(C_2,C_2)^{(2)}, & T_9=(C_2,D_2)^{(1)}. \end{array}\tag{4}$$

Using these GL–comitants as well as the polynomials (3) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly

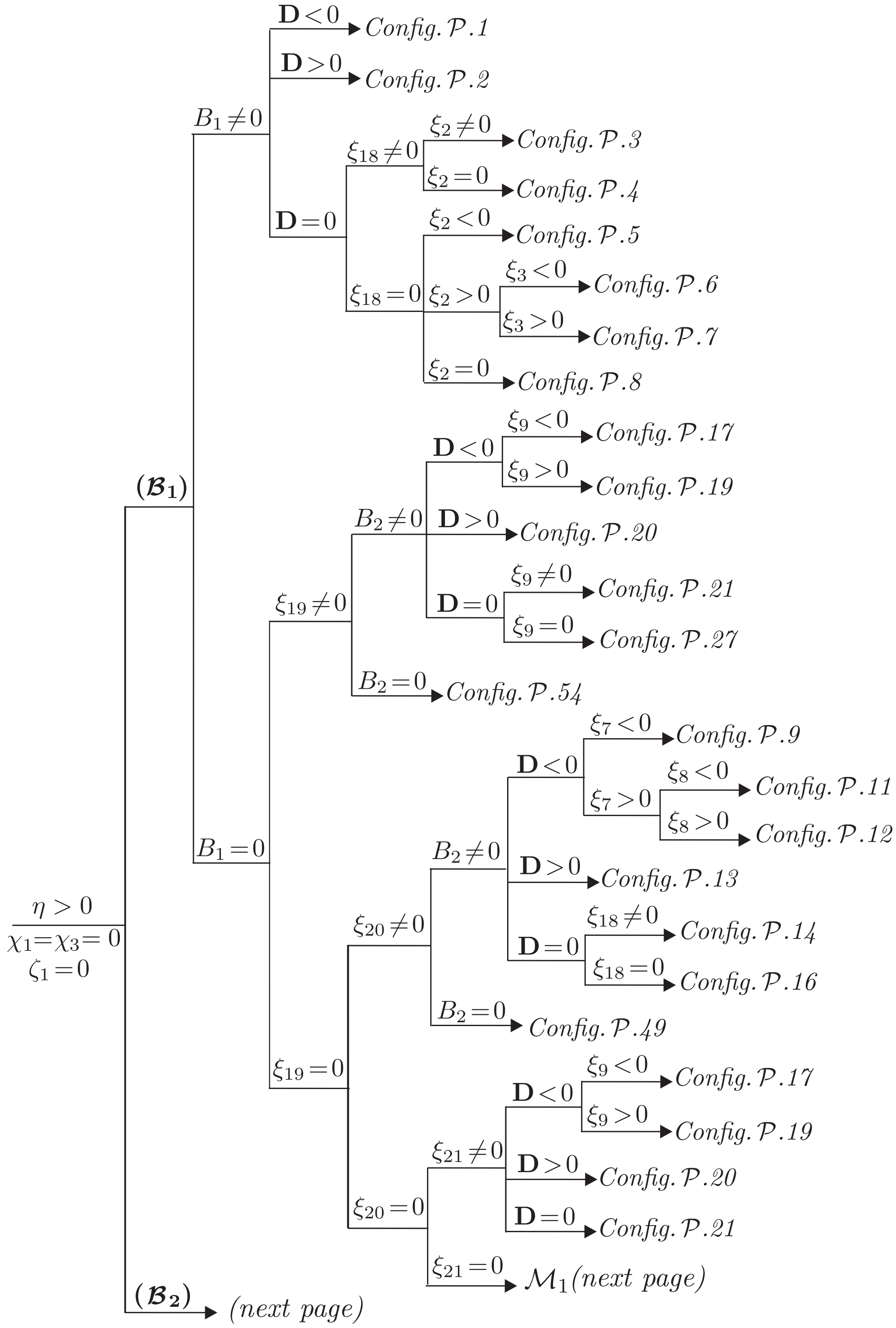

Diagram 2: Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 = 0$

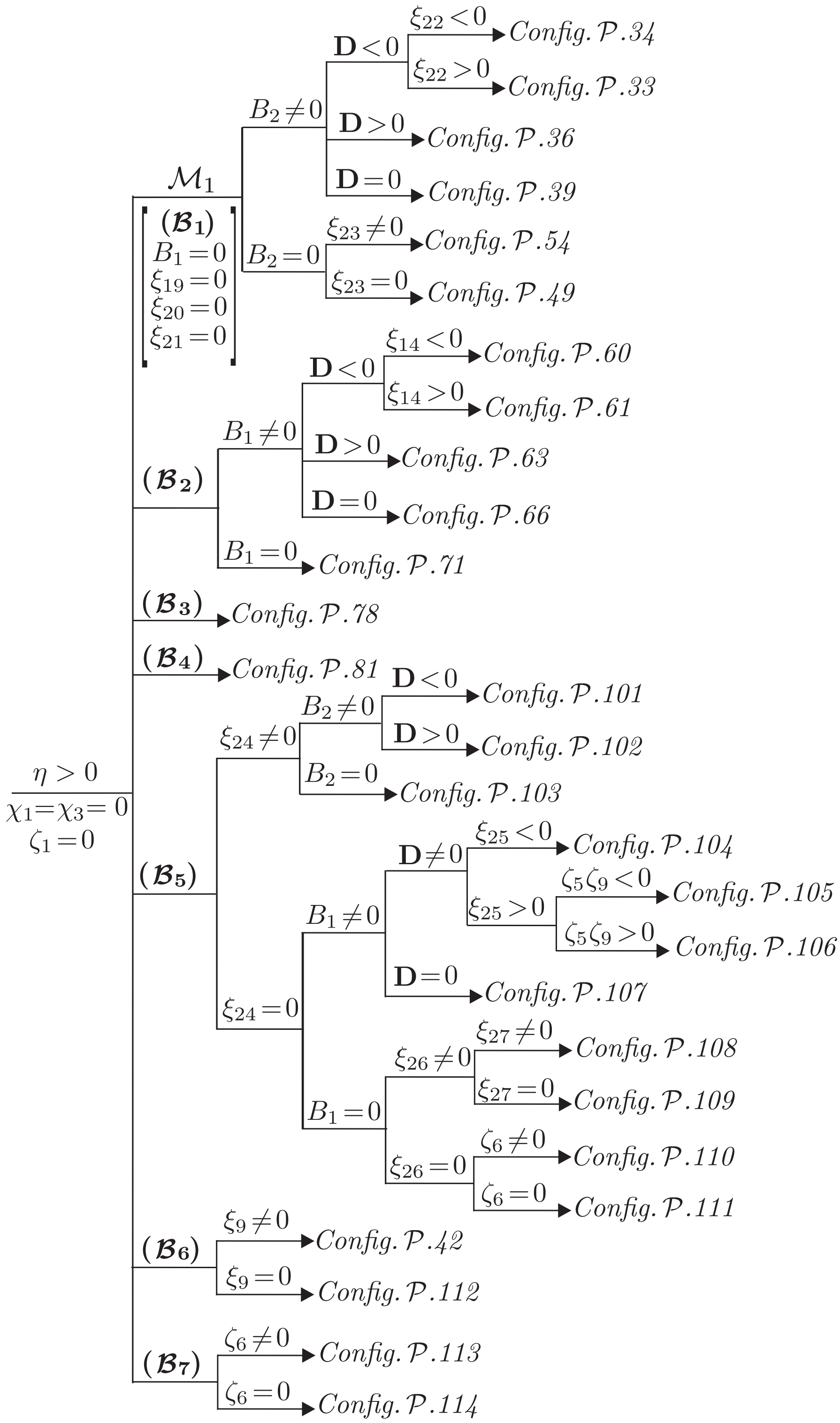

Diagram 2 (cont.): Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 = 0$

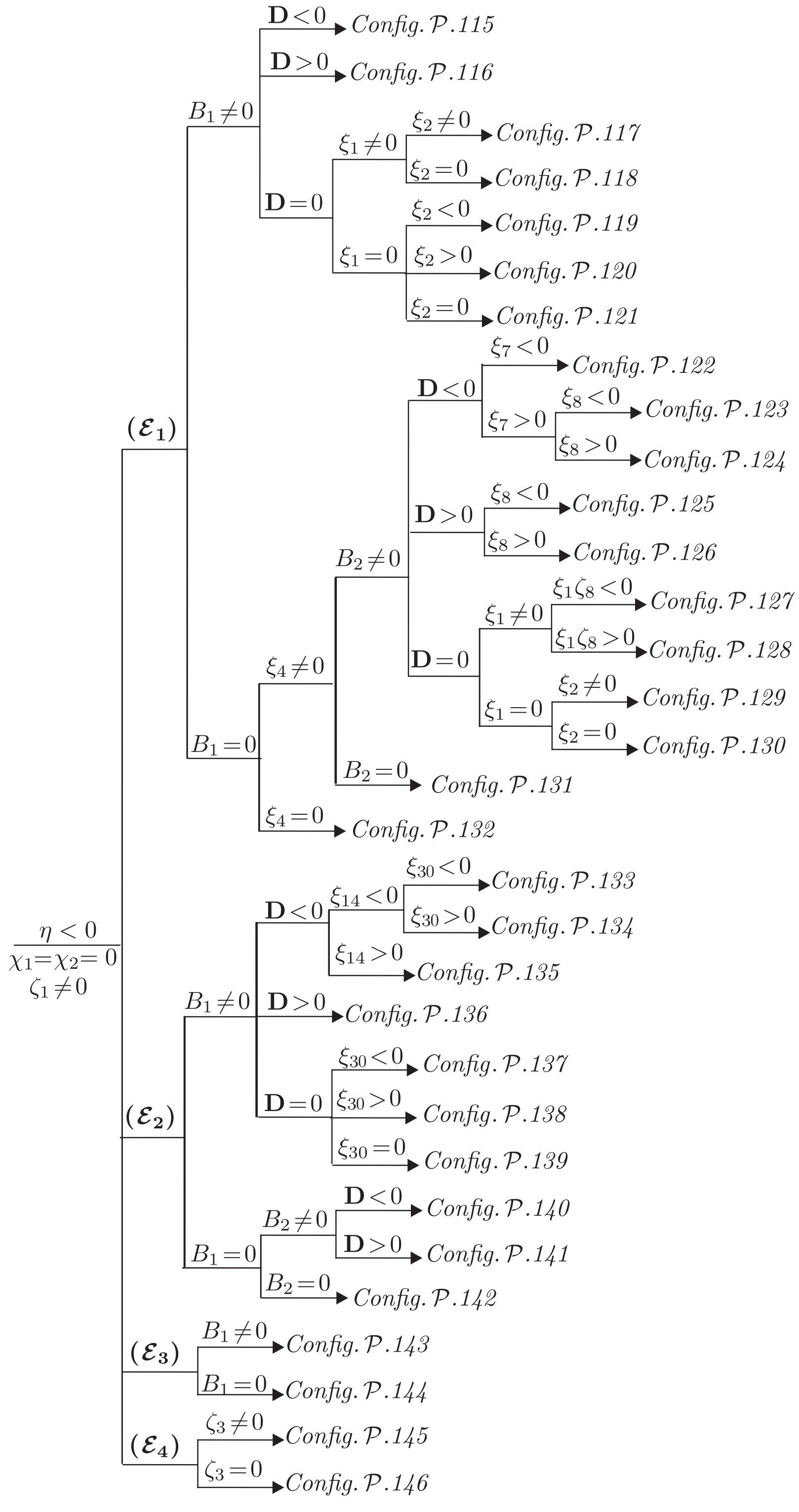

Diagram 3: Conditions for the configurations of systems in **QSP** in the case $\eta < 0$

for every canonical system we shall define here a family of T–comitants expressed through C_i $(i = 0, 1, 2)$ and D_j $(j = 1, 2)$:

$$\hat{A} = \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)} / 144,$$
$$\widehat{D} = \left[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\right] / 36,$$
$$\widehat{E} = \left[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\right] / 72,$$
$$\widehat{F} = \Big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widehat{E}$$
$$- 24\left(C_2, \widehat{D}\right)^{(2)} + 120\left(D_2, \widehat{D}\right)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\Big] / 144,$$
$$\widehat{B} = \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)$$
$$+ 2\,(D_2, T_9)^{(1)}\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\right)$$
$$+ 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)]$$
$$+ 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\left(2D_2^2 - T_8\right)$$
$$- 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\left(D_1^2 + 4T_3\right)$$
$$+ 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\left(T_8 - 7D_2^2 - 42T_9\right)$$
$$+ 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\left[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\right] -$$
$$- 16D_1D_2T_3\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\,(7T_6 + 2T_7)$$
$$-252D_1D_2T_4T_9\Big\} / (2^8 3^3),$$
$$\widehat{K} = (T_8 + 4T_9 + 4D_2^2)/72, \quad \widehat{H} = (8T_9 - T_8 + 2D_2^2)/72.$$

These polynomials in addition to (3) and (4) will serve as bricks in constructing affine invariant polynomials for systems (2).

The following 42 affine invariants $A_1, \ldots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [4] by constructing $A_1, \ldots, A_{42}$ using the above bricks.

$$\begin{aligned}
A_1 &= \hat{A},\\
A_2 &= (C_2, \widehat{D})^{(3)}/12,\\
A_3 &= \left[C_2, D_2)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}/48,\\
A_4 &= (\widehat{H}, \widehat{H})^{(2)},\\
A_5 &= (\widehat{H}, \widehat{K})^{(2)}/2,\\
A_6 &= (\widehat{E}, \widehat{H})^{(2)}/2,\\
A_7 &= \left[C_2, \widehat{E})^{(2)}, D_2\right)^{(1)}/8,\\
A_8 &= \left[\widehat{D}, \widehat{H})^{(2)}, D_2\right)^{(1)}/8,\\
A_9 &= \left[\widehat{D}, D_2)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}/48,\\
A_{10} &= \left[\widehat{D}, \widehat{K})^{(2)}, D_2\right)^{(1)}/8,\\
A_{11} &= (\widehat{F}, \widehat{K})^{(2)}/4,\\
A_{12} &= (\widehat{F}, \widehat{H})^{(2)}/4,\\
A_{13} &= \left[C_2, \widehat{H})^{(1)}, \widehat{H}\right)^{(2)}, D_2\big)^{(1)}/24,
\end{aligned}$$

$$
\begin{aligned}
A_{14} &= (\widehat{B}, C_2)^{(3)}/36,\\
A_{15} &= (\widehat{E}, \widehat{F})^{(2)}/4,\\
A_{16} &= \big[\widehat{E}, D_2)^{(1)}, C_2\big)^{(1)}, \widehat{K}\big)^{(2)}/16,\\
A_{17} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64,\\
A_{18} &= \big[\widehat{D}, \widehat{F})^{(2)}, D_2\big)^{(1)}/16,\\
A_{19} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{H}\big)^{(2)}/16,\\
A_{20} &= \big[C_2, \widehat{D})^{(2)}, \widehat{F}\big)^{(2)}/16,\\
A_{21} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(2)}/16,\\
A_{22} &= \tfrac{1}{1152}\big[C_2, \widehat{D})^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)},\\
A_{23} &= \big[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\big)^{(2)}/8,\\
A_{24} &= \big[C_2, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/32,\\
A_{25} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(2)}/16,\\
A_{26} &= (\widehat{B}, \widehat{D})^{(3)}/36,\\
A_{27} &= \big[\widehat{B}, D_2)^{(1)}, \widehat{H}\big)^{(2)}/24,\\
A_{28} &= \big[C_2, \widehat{K})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{E}\big)^{(2)}/16,\\
A_{29} &= \big[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\big)^{(3)}/96,\\
A_{30} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/288,\\
A_{31} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/64,\\
A_{32} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{H}\big)^{(1)}, D_2\big)^{(1)}/64,\\
A_{33} &= \big[\widehat{D}, D_2)^{(1)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128,\\
A_{34} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64,\\
A_{35} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128,\\
A_{36} &= \big[\widehat{D}, \widehat{E})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/16,\\
A_{37} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/576,\\
A_{38} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/64,\\
A_{39} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{H}\big)^{(2)}/64,\\
A_{40} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{K}\big)^{(2)}/64,\\
A_{41} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/64,\\
A_{42} &= \big[\widehat{D}, \widehat{F})^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above we construct two groups of affine invariant polynomials. The first group contains invariant polynomials related to the existence of an invariant parabola for a quadratic system and they are:

$$
\begin{aligned}
\chi_1 &= 32A_3 + 45A_4 - 160A_5;\\
\chi_2 &= 32A_8(14A_8 - 48A_9 + 37A_{10} + 24A_{11}) + 16A_5(76A_{17} + 74A_{18} + 313A_{19} - 80A_{20} - 167A_{21})\\
&\quad + A_4(160A_2^2 + 368A_{18} - 3363A_{19} + 736A_{20} + 2109A_{21}) + 32(17A_{10}^2 + 27A_{10}A_{11} + 24A_{11}^2\\
&\quad - 48A_9A_{12} + 51A_{10}A_{12} + 24A_{11}A_{12} + 288A_6A_{14} - 96A_7A_{14});
\end{aligned}
$$

$$
\begin{aligned}
\chi_3 =\ & 6520480A_{20}(407A_{18} - 2253A_{21}) + 24A_{18}(1057715458A_{19} + 5944853225A_{21}) \\
& + 28800A_{14}(1872476A_{25} - 122259A_{26}) + 144A_{12}(3620283092A_{29} - 1554910481A_{30}) \\
& + 1440A_{15}(107225339A_{25} - 19561440A_{26}) - 72A_{11}(8198511476A_{29} - 2965514443A_{30}) \\
& + 652048(4544A_{18}^2 + 125A_{20}^2 - 8955A_2A_{42}) - 9(264364688A_{19}^2 + 39417454842A_{19}A_{21} \\
& - 54474141921A_{21}^2) + 3448898760A_{19}A_{20}; \\
\chi_4 =\ & 62713A_{10}^2 + 45787A_{10}A_{11} - 157928A_{11}^2 + 81202A_{10}A_{12} + A_{19}353474A_{11}A_{12} - 145848A_{12}^2 \\
& + 64320A_7A_{15} + 28600A_5A_{17}; \\
\zeta_1 =\ & 13A_4 - 24A_5; \\
\zeta_2 =\ & -A_4; \\
\zeta_3 =\ & 16A_5 - 17A_4; \\
\zeta_4 =\ & 9A_1A_4 - 7A_1A_5 - 2A_{16}; \\
\zeta_5 =\ & 166A_8 + 384A_9 - 1120A_{10} - 512A_{11} - 62A_{12}; \\
\zeta_6 =\ & A_6; \\
\zeta_7 =\ & 40(71436A_7A_{20} - 640883A_7A_{21} + 1008622A_1A_{32}) + 12A_{12}(3585035A_{14} + 14919259A_{15}) \\
& - 5(8092193A_{10} + 15970731A_{11})A_{14} - (129780821A_{10} + 269944167A_{11})A_{15}; \\
\zeta_8 =\ & A_2; \\
\zeta_9 =\ & 1040(2256A_7A_{15} + 143A_3A_{21}) - 264(162941A_{10} + 315202A_{11})A_{12} \\
& + 3A_{11}(25887132A_{10} + 24385177A_{11}) + 20603609A_{10}^2 + 24896016A_{12}^2; \\
\zeta_{10} =\ & 250A_1^2 + 34A_{11} - 41A_{12}; \\
\mathcal{R}_1 =\ & 531A_2A_4 - 1472A_2A_5 - 8352A_1A_6 + 320A_{22} - 3216A_{23} + 1488A_{24}; \\
\mathcal{R}_2 =\ & 15A_{10} - 10A_8 - 6A_9; \\
\mathcal{R}_3 =\ & 4800(6650951968A_{14}A_{15} - 2382132830A_{14}^2 - 9860550485A_{15}^2) + 1600(4765089473A_{11} \\
& - 7838161089A_{12})A_{20} + 640(15664652914A_{11} - 50944340271A_{12})A_{18} \\
& - 6(20392663986679A_{10} + 34357804389813A_{11} - 739275727012A_{12})A_{21} \\
& + 3(46944212550227A_{10} + 83455057317969A_{11} - 22899810934956A_{12})A_{19}; \\
\mathcal{R}_4 =\ & 251A_1^2 + 25A_{12}; \\
\mathcal{R}_7 =\ & 62250A_1^2 + 8956A_9 - 46223A_{10} - 50129A_{11} + 14766A_{12}.
\end{aligned}
$$

The invariant polynomials from the second group are responsible for the classification of the configurations of invariant parabolas and lines. They are:

$$
\begin{aligned}
\xi_1 =\ & 342A_1^2A_2 + A_2(35A_{10} - 15A_8 - 16A_9 + 97A_{11} - 83A_{12}) - 48A_1(4A_{14} + 3A_{15}) \\
& + 16(2A_{32} + A_{33} - 3A_{34}) + 90A_{31}; \\
\xi_2 =\ & -A_{19};
\end{aligned}
$$

$$
\begin{aligned}
\xi_3 =\ & 12(49836514A_8^2 - 40804544A_8A_9 - 63384469A_8A_{10} - 4515985A_{10}^2 + 93824435A_8A_{11} \\
& - 23552547A_{10}A_{11} + 51595312A_{11}^2 + 202411827A_1^2A_{12}) - 763176315A_4A_{21} \\
& - 16(30603408A_9A_{12} + 10917387A_7A_{14} + 14011860A_7A_{15} - 75865539A_5A_{17} \\
& - 115398446A_5A_{18} - 54568383A_5A_{21}) - 4(86656770A_6A_{14} + 404823654A_6A_{15} \\
& - 68396637A_5A_{19} + 25391678A_5A_{20}) - 6A_{12}(154041735A_8 + 47473233A_{10} \\
& - 170661233A_{11} + 202411827A_{12}); \\
\xi_4 =\ & 800(175A_2A_5A_7 - 336A_1A_3A_8 - 16500A_{13}A_{14} - 9300A_{13}A_{15} - 47001A_6A_{22} + 39861A_7A_{23} \\
& - 3150A_6A_{24} - 10242A_7A_{24} + 168792A_5A_{28}) + 240(173478A_8A_{16} + 128774A_{10}A_{16} \\
& + 151602A_{11}A_{16} + 134102A_{12}A_{16} + 8799A_4A_{27} - 134102A_5A_{27}) - 1879552(3A_9A_{16} - A_7A_{22}) \\
& + 75(50400A_6A_{23} - 646151A_4A_{28}); \\
\xi_5 =\ & 2000(802A_{13}A_{14} + 315A_6A_{23} - 210A_6A_{24}) + 320(28A_1A_3A_{11} - 13757A_8A_{16} - 11282A_{12}A_{16} \\
& + 3336A_7A_{24} + 11282A_5A_{27}) + 80(16038A_{13}A_{15} - 30398A_{10}A_{16} - 36154A_{11}A_{16} + 46738A_6A_{22} \\
& - 45142A_7A_{23} - 162339A_5A_{28}) + 151552(3A_9A_{16} - A_7A_{22}) - 15A_4(28392A_{27} - 313721A_{28}); \\
\xi_6 =\ & 1536(16671538A_7A_{14} - 5655800A_{11}^2 - 5655800A_{11}A_{12} - 134975925A_6A_{15} + 14236220A_7A_{15}) \\
& + 128(42330182A_8A_9 + 279065017A_8A_{11} - 857954A_8A_{12} + 138313062A_9A_{12} \\
& - 633595086A_6A_{14} - 35417298A_5A_{20}) + 64(171565045A_8^2 + 343921603A_5A_{17}) \\
& - 32(1111806317A_8A_{10} + 256225409A_{10}^2 + 874265715A_{10}A_{11} + 2536914399A_{10}A_{12} \\
& - 936841383A_5A_{18}) - 16A_5(2168875001A_{19} + 1048355233A_{21}) + A_4(26458433203A_{19} \\
& - 4734012269A_{21}); \\
\xi_7 =\ & -A_4\big[3200A_{12}(14657A_8 - 1615148A_{10} + 318175A_{11}) - 640(388968A_9^2 - 7748782A_{10}^2 \\
& - 592379A_9A_{12}) - 160(13079737A_8A_{10} - 27509045A_8A_{11} - 63353923A_9A_{11} - 16215395A_{10}A_{11} \\
& - 36662125A_{11}^2) + 4121433952A_9A_{10}\big]; \\
\xi_8 =\ & -A_4\big[512A_9(1275434A_{10} + 2193137A_{11} - 170333A_{12}) - 1280(30087A_9^2 + 424036A_{10}^2 \\
& + 1052798A_{10}A_{11} + 48550A_{11}^2 + 61603A_8A_{12}) - 640(608587A_8A_{10} + 248041A_8A_{11} \\
& + 430261A_{10}A_{12} + 525475A_{11}A_{12})\big]; \\
\xi_9 =\ & -A_4\big[48(675908847A_8A_9 + 1141726617A_9A_{12} + 7216376855A_{10}A_{12} - 4015621128A_6A_{14} \\
& + 3915909450A_7A_{15}) - 12(16745223889A_8^2 + 5997051735A_8A_{11} - 26372062499A_{10}A_{11} \\
& + 2601951027A_8A_{12} - 7916516650A_7A_{14} - 30105649725A_6A_{15} + 20512413539A_5A_{17} \\
& - 1497206278A_4A_{19} - 4791714129A_4A_{21}) + 2(220220676003A_8A_{10} + 58687175103A_{10}^2 \\
& + 14685562719A_{11}^2 + 9716839839A_{11}A_{12} - 219193688911A_5A_{18} - 4467110471A_5A_{20}) \\
& + 3A_5(36033875127A_{19} - 37652431103A_{21})\big]; \\
\xi_{10} =\ & A_4\big[48(568199091031A_8A_9 - 248186616391A_9A_{10} + 314207594667A_9A_{11} + 5804879973A_9A_{12} \\
& - 3905825755777A_{10}A_{12} - 2095407390920A_6A_{14} - 546799764750A_7A_{15}) + 12(6550908482493A_8^2 \\
& - 3402501855145A_8A_{11} - 3448022811579A_{10}A_{11} + 2284925158471A_8A_{12} + 2482932379806A_7A_{14} \\
& - 11017448610465A_6A_{15} + 5894909506479A_5A_{17}) - 2(131290745988327A_8A_{10} \\
& - 17334476527245A_{10}^2 - 11980168965A_{11}^2 + 21428060568795A_{11}A_{12} - 62352140313275A_5A_{18} \\
& + 3924064256285A_5A_{20}) - 3(2258722903315A_5A_{19} + 9533558573843A_4A_{21} - 10218122423819A_5A_{21})\big];
\end{aligned}
$$

$\xi_{11} = \zeta_1\zeta_2\xi_6;$

$\xi_{12} = 1288A_1^2 + 117A_{10} + 351A_{11} - 352A_{12};$

$\xi_{13} = 61A_2^2 - 20A_{17} - 8A_{18} + 24A_{19} - 28A_{20} + 12A_{21};$

$\xi_{14} = 9854A_{11} - 3005A_8 - 3296A_9 + 13578A_{10} - 991A_{12};$

$\xi_{15} = 8A_5 - 9A_4;$

$\xi_{16} = (525A_8 - 4448A_9 + 10554A_{10} - 1378A_{11} + 8087A_{12});$

$\xi_{17} = 10005A_8 + 9856A_9 - 38348A_{10} - 27404A_{11} + 8371A_{12};$

$$\begin{aligned}\xi_{18} = {} & 2240(15452233775A_{14}^2 + 742923092360A_{14}A_{15} - 145263086200A_{15}^2 + 10151798384A_{11}A_{18} \\ & - 68919094926A_{12}A_{18} - 14663220305A_{11}A_{20} + 7194838365A_{12}A_{20}) + 16A_{19}(88266907919051A_8 \\ & + 12824946044853A_{11} + 119819326860153A_{12}) - 7A_{21}(138073671324637A_{10} + 258358507987439A_{11} \\ & - 32813284182036A_{12});\end{aligned}$$

$$\begin{aligned}\xi_{19} = {} & 429A_9(629A_{10} + 1275A_{11} - 900A_{12}) + 100(2145A_8A_{11} - 1595A_5A_{17} - 2970A_5A_{18} + 2886A_2A_{23} \\ & - 559A_2A_{24});\end{aligned}$$

$$\begin{aligned}\xi_{20} = {} & 4A_2(47A_2^2 - 468A_{18} + 3478A_{19} + 9A_{20}) - 9189A_2A_{21} + 12(-682A_1A_{25} + 2592A_1A_{26} + 395A_{38} \\ & + 35A_{39});\end{aligned}$$

$$\begin{aligned}\xi_{21} = {} & 24(675906A_{40} - 672409A_{39} + 6578A_{41} + 110106A_{42}) - 73404A_2(74A_{18} + A_{20}) + 4(99911A_2^3 \\ & - 2048846A_{38}) - 15133791A_2A_{21};\end{aligned}$$

$\xi_{22} = 84A_{12} - 68A_{10} - 141A_{11};$

$\xi_{23} = 5A_8 - 3A_9;$

$$\begin{aligned}\xi_{24} = {} & 625A_2(12A_2A_3 - 775A_1A_6) - 62(13500A_1^2 + 275A_8 - 276A_9)A_9 + 10A_3(2561A_{17} + 3240A_{18} \\ & + 2550A_{19});\end{aligned}$$

$\xi_{25} = -(46A_{18} + 537A_{19} + 134A_{20});$

$\xi_{26} = 41A_1A_2 + 16A_{14} - 18A_{15};$

$\xi_{27} = A_1;$

$$\begin{aligned}\xi_{28} = {} & 64(72137434664A_8^2 + 3322490880A_9^2 - 58216412276A_{10}^2 - 217656099219A_{10}A_{11} - 63098236389A_{11}^2 \\ & - 250756327503A_{10}A_{12} - 71858710389A_{11}A_{12} + 96A_9(449920640A_{11} + 1009660963A_{12}) \\ & + 6A_8(21795888048A_9 - 66020231422A_{10} - 21118997424A_{11} + 2573485725A_{12})) \\ & - 384(62739943233A_6A_{14} - 27065693406A_7A_{14} + 7592410800A_6A_{15} - 10442342780A_7A_{15}) \\ & + A_4(2998959134256A_{17} + 4635359414448A_{18} + 1132776129074A_{19} - 1187818900002A_{20} \\ & - 5542617623395A_{21}) + 32A_3(19078937382A_{20} + 81853956367A_{21});\end{aligned}$$

$$\begin{aligned}\xi_{29} = {} & 497213324620A_8^2 - 1001736600522A_{10}^2 - 870653569536A_9A_{11} + 337754949134A_{11}^2 \\ & + A_8(2170429037822A_{10} - 1858453397512A_9 + 2112595332132A_{11} - 304022217484A_{12}) \\ & - 987799827976A_9A_{12} + 949933240214A_{11}A_{12} + A_{10}(-648979472052A_{11} + 956487534504A_{12}) \\ & - 4(125652578829A_6A_{14} + 240347919318A_7A_{14} - 775425835368A_6A_{15} + 405563103412A_7A_{15}) \\ & - A_4(197626785161A_{20} + 1540932760870A_{21}) + A_5(1910970964424A_{17} + 2668708281714A_{18} \\ & + 182967974851A_{19} + 280452031438A_{20} + 2136843181298A_{21});\end{aligned}$$

$\xi_{30} = 3512A_{10} - 1695A_8 - 544A_9 + 4576A_{11} - 3329A_{12}.$

2.2 Preliminary results involving the use of polynomial invariants

A few more definitions and results which play an important role in the proof of the part (B) of the Main Theorem are needed. We do not prove these results here but we indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$ constructed in [2] and acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$L_1 = 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},$$
$$L_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.$$

Using this operator and the affine invariant $\mu_0 = \text{Res}_x\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1, \ldots, 4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2, \mathbb{R})$ (see [2]). Their geometrical meaning is revealed in Lemma 5.2 of [1]. Using these invariant polynomials we construct the invariant polynomials $\mathbf{D}$ and $\mathbf{R}$ which are responsible for the existence of multiple finite singularity of a quadratic system:

$$\mathbf{D} = \Big[3\big((\mu_3, \mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48, \quad \mathbf{R} = 3\mu_1^2 - 8\mu_0\mu_2,$$

Next we construct the following T-comitants (for the definition of T-comitants see [20]) which are responsible for the existence of invariant straight lines of systems (2):

$$B_3(\tilde{a}, x, y) = (C_2, \widehat{D})^{(1)} = \text{Jacob}(C_2, \widehat{D}),$$
$$B_2(\tilde{a}, x, y) = (B_3, B_3)^{(2)} - 6B_3(C_2, \widehat{D})^{(3)},$$
$$B_1(\tilde{a}) = \text{Res}_x\left(C_2, \widehat{D}\right)/y^9 = -2^{-9}3^{-8}\left(B_2, B_3\right)^{(4)}.$$

Lemma 1 (see [19])**.** *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

To detect the parallel invariant lines we need the following invariant polynomials:

$$N(\tilde{a}, x, y) = D_2^2 + T_8 - 2T_9 = 9\widehat{N},$$
$$\theta(\tilde{a}) = 2A_5 - A_4\ (\equiv \text{Discriminant}\big(N(a, x, y)\big)/1296).$$

Lemma 2 (see [19])**.** *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (2) *corresponding to $\tilde{a} \in \mathbb{R}^{12}$ is the condition $\theta(\tilde{a}) = 0$ (respectively $N(\tilde{a}, x, y) = 0$).*

Now we introduce some important GL-comitant in the study of the invariant conics. Considering $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2/x^3, \xi], \quad M(\tilde{a}, x, y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. We point out (see [22]) that the invariant polynomials C_2, η and M are responsible for the number of infinite singularities and their kind (real or complex).

In this paper we consider only the case $\eta \neq 0$, i.e. $\eta > 0$ and $\eta < 0$. In the first case by [22] a quadratic system possesses at infinity three real distinct singularities whereas in the second case it possesses one real and two complex singularities.

In [23] the classification of the class **QSP** of quadratic systems possessing at least one invariant parabola is performed. More exactly in this paper necessary and sufficient conditions are determined for a quadratic system to belong to **QSP**.

We extract from [23] only the information related to the case $\eta \neq 0$ and for this we need some notations.

Definition 3. *By the direction of an invariant parabola of a quadratic system* (S) *we mean the direction of its axis of symmetry which intersects the invariant line* $Z = 0$ *at an infinite singular point of* (S).

In order to distinguish the invariant parabolas that a quadratic system could have we use the following notations:

- $\cup$ for a simple invariant parabola;
- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect);
- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\bigcup^2$ for one double invariant parabola;
- $\Cup\subset$ for three simple invariant parabolas: two in one direction and one in another direction.

The proof of the next three propositions could be found in [23].

Proposition 2. *Assume that for a non-degenerate arbitrary quadratic system the conditions* $\eta > 0$, $\chi_1 = 0$ *and* $\zeta_1 \neq 0$ *are satisfied. Then this system could possess invariant parabolas only in one direction. More exactly it could only possess one of the following sets of invariant parabolas:* $\cup$, $\Cup$ *and* $\bigcup^2$. *Moreover this system has one of the above sets of parabolas if and only if* $\chi_2 = 0$ *and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{A}_1)$ $\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 \quad \Rightarrow \cup$;
$(\mathcal{A}_2)$ $\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0 \quad \Rightarrow \Cup$;
$(\mathcal{A}_3)$ $\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0 \quad \Rightarrow \bigcup^2$;
$(\mathcal{A}_4)$ $\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0 \quad \Rightarrow \cup$;
$(\mathcal{A}_5)$ $\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 \quad \Rightarrow \cup$;
$(\mathcal{A}_6)$ $\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0\ \zeta_5 \neq 0 \quad \Rightarrow \Cup$;
$(\mathcal{A}_7)$ $\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0 \quad \Rightarrow \bigcup^2$;
$(\mathcal{A}_8)$ $\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0 \quad \Rightarrow \cup$;
$(\mathcal{A}_9)$ $\zeta_2 = 0,\ \zeta_6 \neq 0,\ \mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0 \quad \Rightarrow \cup$.

Moreover in the case of the existence of an invariant parabola a system with $\eta > 0$ and $\zeta_1 \neq 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \tag{5}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

Proposition 3. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\chi_1 = \zeta_1 = 0$ are satisfied. Then this system could possess invariant parabolas in one or two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\boldsymbol{\cup}^2$, $\cup\subset$ and $\Cup\subset$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

($\mathcal{B}_1$) $\chi_4 \neq 0,\ \zeta_7 \neq 0,\ \mathcal{R}_3 \neq 0 \quad \Rightarrow \cup$;
($\mathcal{B}_2$) $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 \neq 0 \quad \Rightarrow \Cup$;
($\mathcal{B}_3$) $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 = 0 \quad \Rightarrow \boldsymbol{\cup}^2$;
($\mathcal{B}_4$) $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 = 0 \quad \Rightarrow \cup$;
($\mathcal{B}_5$) $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 \neq 0 \quad \Rightarrow \cup\subset$;
($\mathcal{B}_6$) $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 = 0,\ \zeta_{10} \neq 0 \quad \Rightarrow \cup$;
($\mathcal{B}_7$) $\chi_4 = 0,\ \zeta_5 = 0,\ \zeta_6 \neq 0 \quad \Rightarrow \Cup\subset$.

Moreover in the case of the existence of an invariant parabola a system with $\eta > 0$ and $\zeta_1 = 0$ could be brought via an affine transformation and time rescaling to the systems (5) *with $g = 2$.*

Proposition 4. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta < 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (real) direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\boldsymbol{\cup}^2$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

($\mathcal{E}_1$) $\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 \quad \Rightarrow \cup$;
($\mathcal{E}_2$) $\zeta_4 = 0,\ \mathcal{R}_7 \neq 0,\ \zeta_5 \neq 0 \quad \Rightarrow \Cup$;
($\mathcal{E}_3$) $\zeta_4 = 0,\ \mathcal{R}_7 \neq 0,\ \zeta_5 = 0 \quad \Rightarrow \boldsymbol{\cup}^2$;
($\mathcal{E}_4$) $\zeta_4 = 0,\ \mathcal{R}_7 = 0,\ \zeta_5 \neq 0 \quad \Rightarrow \cup$.

Moreover in the case of the existence of an invariant parabola a system with $\eta < 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + (2n-1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \tag{6}$$

with $C_2 = x(x^2 + y^2)$, possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

3 The proof of the Main Theorem

The statement $\boldsymbol{(A)}$ of Main Theorem follows from Lemma 2.4 of [23]. The statement $\boldsymbol{(C)}$ follows directly from the form of the conditions given in Diagrams 1, 2 and 3. These conditions could be evaluated for any point $a \in \mathbb{R}^{12}$ corresponding to a quadratic system with the condition $\eta \neq 0$.

In order to prove the statement $\boldsymbol{(B)}$ of Main Theorem we have to examine the sets of conditions provided by each one of Propositions 2, 3 and 4.

3.1 Systems in $\mathbf{QSP}_{(\eta>0)}$ with the condition $\zeta_1 \neq 0$

In what follows we examine the configurations of the systems in $\mathbf{QSP}_{(\eta>0)}$ in each one of the cases provided by Proposition 2. According to this proposition we consider the canonical form (5), i.e. the systems

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \tag{7}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

3.1.1 The statement $(\mathcal{A}_1)$

For systems (7) we calculate

$$\begin{aligned}
\zeta_1 =& 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g), \quad \zeta_3 = 8(1+2g)^2,\\
\zeta_4 =& (g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/16,\\
\mathcal{R}_1 =& -15g(1+g)(g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/2,\\
B_1 =& m(g+8m+4n)(gn-2m-n)(1+2g+g^2-4m+2n+2gn)\\
&\times(g+2g^2+g^3+4m+2n+2gn)/4.
\end{aligned} \tag{8}$$

3.1.1.1 The case $B_1 \neq 0$. The according to Lemma 1 systems (7) could not possess any invariant line.

Let us examine the finite singularities of these systems. Following [1, Proposition 5.1] we calculate the invariant polynomial $\mathbf{D} = 48\mathcal{F}_1^2\mathcal{F}_2$, where

$$\begin{aligned}
\mathcal{F}_1 =& -4m^2 + 2(g+1)m(g^2-2n) - (g+1)^2n^2;\\
\mathcal{F}_2 =& 108m^2 + 2(g-1)m(1-2g+g^2-18n) + n^2(16n-1+2g-g^2).
\end{aligned} \tag{9}$$

So we discuss these two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

3.1.1.1.1 The subcase $\mathbf{D} \neq 0$. We determine that systems (7) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned}
x_1 =& -\frac{2m+n+gn}{g(1+g)}, \; y_1 = \frac{2m}{1+g}; \quad x_2 = \frac{1}{6\mathcal{Z}^{1/3}}\big[\mathcal{Y} + (1-g)\mathcal{Z}^{1/3} + \mathcal{Z}^{2/3}\big],\\
y_2 =& \frac{1}{36\mathcal{Z}}\big[3(\mathcal{Y}+4n)\mathcal{Z} + \mathcal{Y}^2\mathcal{Z}^{1/3} - 2(g-1)\mathcal{Y}\mathcal{Z}^{2/3} - 2(g-1)\mathcal{Z}^{4/3} + \mathcal{Z}^{5/3};\\
x_3 =& \frac{1}{12\mathcal{Z}^{1/3}}\big[-(1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1-i\sqrt{3})\mathcal{Z}^{2/3}\big],\\
y_3 =& -\frac{1}{72\mathcal{Z}}\big[-6(\mathcal{Y}+4n)\mathcal{Z} + (1-i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1+i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3}\\
&-2(1-i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1+i\sqrt{3})\mathcal{Z}^{5/3}\big];\\
x_4 =& \frac{1}{12\mathcal{Z}^{1/3}}\big[(-1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1+i\sqrt{3})\mathcal{Z}^{2/3}\big],\\
y_4 =& -\frac{1}{72\mathcal{Z}}\big[-6(\mathcal{Y}+4n)\mathcal{Z} + (1+i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1-i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3}\\
&-2(1+i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1-i\sqrt{3})\mathcal{Z}^{5/3}\big],
\end{aligned} \tag{10}$$

where

$$\mathcal{Z} = 1 - 3g + 3g^2 - g^3 - 108m - 18n + 18gn + 6\sqrt{3}\sqrt{\mathcal{F}_2}, \quad \mathcal{Y} = (1-g)^2 - 12n.$$

Calculations yield:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = \frac{\mathcal{F}_1}{g^2(1+g)^2}$$

and therefore we deduce that three singularities M_2, M_3 and M_4 of systems (7) are located on the invariant parabola. Moreover M_1 is located outside the parabola and could belong to it if and only if the condition $\mathcal{F}_1 = 0$ holds, where $\mathcal{F}_1$ is given in (9). However we have $\mathbf{D} = 48\mathcal{F}_1^2\mathcal{F}_2 \neq 0$ and hence on the parabola we always have three distinct singularities.

On the other hand according to [1, Proposition 5.1] if $\mathbf{D} > 0$ systems (7) possess two real and two complex finite singularities. For $\mathbf{D} < 0$ we could have either four real or four complex finite singularities. However since M_1 is a real singular point for these systems we conclude that in the case $\mathbf{D} < 0$ we have four real finite distinct singularities.

Thus since the real singularity M_1 is outside the invariant parabola and all three finite singularities on the parabola (real or complex) are distinct and furthermore we could not have any invariant line we arrive at the configuration *Config.* $\mathcal{P}$*.1* if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}$*.2* if $\mathbf{D} > 0$.

3.1.1.1.2 The subcase $\mathbf{D} = 0$. This implies $\mathcal{F}_1\mathcal{F}_2 = 0$ and for systems (7) we calculate:

$$\xi_1 = -6\zeta_4\mathcal{F}_1 \ \Rightarrow \ \mathcal{F}_1 = 0 \ \Leftrightarrow \ \xi_1 = 0.$$

So we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: The possibility $\xi_1 \neq 0$. Then $\mathcal{F}_1 \neq 0$ and therefore the condition $\mathbf{D} = 0$ implies $\mathcal{F}_2 = 0$.

We observe that the polynomial $\mathcal{F}_2$ is quadratic with respect to the parameter m and we calculate

$$\text{Discrim}\,[\mathcal{F}_2, m] = 4(1 - 2g + g^2 - 12n)^3.$$

Therefore since the parameters m, n and g of systems (7) must be real we conclude that the condition $1 - 2g + g^2 - 12n \geq 0$ has to be fulfilled. So setting a new parameter v: $1 - 2g + g^2 - 12n = v^2 \geq 0$ we get $n = \big[(g-1)^2 - v^2\big]/12$ and then we calculate

$$\mathcal{F}_2 = \frac{1}{432}\big[216m - (1-g+v)^2(g-1+2v)\big]\big[216m - (1-g-v)^2(g-1-2v)\big] = 0$$

and due to the change $v \to -v$ we could force the first factor to vanish. Then we obtain

$$m = (1-g+v)^2(g-1+2v)/216$$

and considering the expression for the parameters m and n we arrive at the 2-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{(1-g+v)^2(g-1+2v)}{216} + \frac{(g-1)^2 - v^2}{12}x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} =& \frac{(1-g+v)^2(g-1+2v)}{108}x + \frac{(g-1)^2 - v^2}{6}, y + (g-1)xy + 2y^2, \end{aligned} \tag{11}$$

possessing the invariant parabola $\Phi(x,y)=x^2-y=0$. We observe that for the above systems we have the following conditions on the parameters g and v:

$$\begin{aligned}
&\zeta_1\zeta_2\zeta_3\zeta_4\mathcal{R}_1\neq 0 \;\Leftrightarrow\; g(g-2)(1+g)(3+g)(1+2g)(2+4g-v)(4+8g+v)\neq 0;\\
&\xi_1\neq 0 \;\Leftrightarrow\; (g-2)(3+g)(g-1-v)(2+g-v)(2+4g-v)(4+8g+v)^2\\
&\qquad\qquad \times(4-2g-2g^2-4v-8gv+v^2)\neq 0;\\
&B_1\neq 0 \;\Leftrightarrow\; (g-1-v)(2+g-v)(2+4g-v)(2g-2+v)(1+2g+v)(4+2g+v)\\
&\qquad\qquad \times(g-1+2v)(2+g+2v)\neq 0.
\end{aligned}\tag{12}$$

We determine that systems (14) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates

$$\begin{aligned}
&x_1=\frac{1-g+v}{6},\;\; y_1=\frac{(1-g+v)^2}{36};\quad x_2=\frac{1-g-2v}{6},\;\; y_2=\frac{(1-g-2v)^2}{36};\\
&x_3=\frac{(1-g+v)(5g^2-4-g+4v+5gv-v^2)}{54g(1+g)},\;\; y_3=\frac{(1-g+v)^2(g-1+2v)}{108(1+g)}.
\end{aligned}\tag{13}$$

We calculate

$$\Phi(x_1,y_1)=\Phi(x_2,y_2)=0,\quad \Phi(x_3,y_3)=\frac{(g-1-v)^2(2+g-v)^2(4-2g-2g^2-4v-8gv+v^2)}{2916g^2(1+g)^2}$$

and we conclude that the singular points M_1 and M_2 are located on the invariant parabola.

On the other hand considering the conditions (12) we obtain that M_3 will be located on $\Phi(x,y)=0$ if and only if

$$\alpha=4-2g(1+g)-4v-8gv+v^2=0.$$

However considering (12) we conclude that $\alpha\neq 0$ (due to $\xi_1\neq 0$) and hence the singularity M_3 is not located on the invariant parabola in the considered case.

We claim that M_2 is a multiple singularity of systems (11). Indeed, applying the corresponding translation, we could place M_2 at the origin of coordinates and we arrive at the systems

$$\begin{aligned}
\dot x=&-\frac{(g-v-1)\left(5g^2+5gv-g-v^2+4v-4\right)}{54(g+1)}x-\frac{(2g+v+1)(4g-v+2)^2}{54g(g+1)}y+gx^2+xy,\\
\dot y=&\frac{g(g+2v-1)(g-v-1)^2}{54(g+1)}x+(g-1)xy+2y^2+\frac{g-v-1}{54g(g+1)}\big[6g^3-(v-2)^2\\
&-g(v-2)(3v-2)+2g^2(1+3v)\big]\,y,
\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials: $\mu_4=\mu_3=0$ and

$$\mu_2=-\frac{1}{324}v(g-v-1)(g-v+2)\big[(2g+v-2)x+6y\big]\big[g(g-v-1)x+(2+4g-v)y\big],$$

and by [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity at least 2. We observe that due to the condition $\xi_1\neq 0$ we have $\mu_2=0$ if and only if $v=0$. In this case we calculate

$$\mu_2=0,\quad \mu_1=-\frac{1}{27}(g-1)\big[g(g-1)(4+5g)x+2(13g+16g^2-2)y\big]\neq 0,$$

due to $\xi_1 \neq 0$. According to [1, Lemma 5.2, statement (ii)] we have a double point if $v \neq 0$ and a triple one if $v = 0$.

On the other hand for systems (11) we have

$$\xi_2 = \frac{1}{209952}(g-1-v)^2(2+g-v)^2 v^2 \alpha^2$$

and due to the conditions (12) we conclude that the condition $v = 0$ is equivalent to $\xi_2 = 0$.

Thus for systems (11) we have the configuration *Config.* $\mathcal{P}$*.3* if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}$*.4* if $\xi_2 = 0$.

2: The possibility $\xi_1 = 0$. This implies $\mathcal{F}_1 = 0$ and we observe that the polynomial $\mathcal{F}_1$ is quadratic with respect to the parameter m and we calculate

$$\text{Discrim}\,[\mathcal{F}_1, m] = 4g^2(1+g)^2(g^2-4n).$$

Since $g(g+1) \neq 0$ (due to $\zeta_2 \neq 0$) we must have $g^2 - 4n \geq 0$. So we set a new parameter u as follows: $g^2 - 4n = u^2 \geq 0$ and we get $n = (g^2 - u^2)/4$. Then calculation yields

$$\mathcal{F}_1 = -\frac{1}{16}\left[8m - (1+g)(g+u)^2\right]\left[8m - (1+g)(g-u)^2\right] = 0$$

and due to the change $u \to -u$ we could force the second factor to vanish. In this case we obtain

$$m = (1+g)(g-u)^2/8$$

and considering the expression for the parameters m and n we arrive at the 2-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{(1+g)(g-u)^2}{8} + \frac{g^2-u^2}{4}x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} =& \frac{(1+g)(g-u)^2}{4}x + \frac{g^2-u^2}{2}y + (g-1)xy + 2y^2 \end{aligned} \tag{14}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$. We observe that for the above systems we have the following condition on the parameters g and u:

$$\begin{aligned} \zeta_1\zeta_2\zeta_3\zeta_4\mathcal{R}_1 \neq 0 \ &\Leftrightarrow\ (g-2)g(1+g)(3+g)(1+2g)(1+2g+u)(1+5g+5g^2-u-2gu) \neq 0; \\ B_1 \neq 0 \ &\Leftrightarrow\ g(1+g)(g-u)(1+g-u)(1+2g-u)(-1+u)(1+u) \neq 0. \end{aligned} \tag{15}$$

We determine that systems (14) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with the coordinates

$$\begin{aligned} x_1 = \frac{u-g}{2},\ y_1 = \frac{(u-g)^2}{4};\ \ x_{2,3} = \frac{1}{4}\left(1-u \pm \sqrt{Z_1}\right), \\ y_{2,3} = \frac{1}{8}\left[1-2g-2g^2+2gu+u^2 \mp (u-1)\sqrt{Z_1}\right],\ \ Z_1 = -4g^2+4g(-1+u)+(1+u)^2. \end{aligned} \tag{16}$$

We calculate

$$\Phi(x_1,y_1) = \Phi(x_2,y_2) = \Phi(x_3,y_3) = 0$$

and therefore all three singularities are located on the invariant parabola.

We point out that M_1 is a multiple singularity of systems (14). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\begin{aligned}\dot{x} =& -\frac{1}{2}g(g-u)x + \frac{1}{2}(u-2g-1)y + gx^2 + xy,\\ \dot{y} =& \frac{1}{2}g(g-u)^2x + \frac{1}{2}(2g-u+1)(g-u)y + (g-1)xy + 2y^2,\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$ and

$$\mu_2 = \frac{1}{2}g(g+1)(g-u)(g-u+1)\big[g(g-u)x^2 + (2g-1+-u)xy + 2y^2\big] \neq 0,$$

due to the conditions (15). By [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity exactly 2.

On the other hand it is clear that the singularities M_2 and M_3 could be complex (respectively real; coinciding) if $Z_1 < 0$ (respectively $Z_1 > 0$; $Z_1 = 0$). We observe that for systems (14) we have:

$$\xi_2 = g^2(1+g)^2(g-u)^2(1+g-u)^2 Z_1$$

and due to the conditions (15) we conclude that the sign of Z_1 is governed by the invariant polynomial ξ_2. So we discuss three cases: $\xi_2 < 0$, $\xi_2 > 0$ and $\xi_2 = 0$.

1.1: *The case* $\xi_2 < 0$. This implies $Z_1 < 0$ and then systems (14) possess only one real singular point M_1 (which is double) and evidently we get the configuration *Config.* $\mathcal{P}$*.5*.

1.2: *The case* $\xi_2 > 0$. Then $Z_1 > 0$ and this implies the existence of three real singularities and we have to determine the position of the double point with respect to the simple ones. So we calculate

$$(x_3 - x_1)(x_2 - x_1) = (g-u)(1+g-u)/2 \equiv \alpha_1/2, \quad \mathrm{sign}\big((x_3-x_1)(x_2-x_1)\big) = \mathrm{sign}\,(\alpha_1), \qquad (17)$$

where $\alpha_1 \neq 0$ due to $B_1 \neq 0$. This means that the singularity M_1 could not coalesce with one of the singularities M_2 or M_3.

On the other hand for systems (14) calculations yield:

$$\xi_3 = \frac{27249129}{2} g^2(1+g)^2\alpha_1^3 Z_1.$$

So due to the conditions (15) we deduce that $\mathrm{sign}\,(\xi_3) = \mathrm{sign}\,(\alpha_1 Z_1)$.

Therefore in the case $\xi_3 < 0$ the double singular point M_1 is located on the parabola between M_2 and M_3 and we arrive at the configuration *Config.* $\mathcal{P}$*.6*.

If $\xi_3 > 0$ we evidently get the configuration *Config.* $\mathcal{P}$*.7*.

1.3: *The case* $\xi_2 = 0$. Then $Z_1 = 0$ which implies the coalescence of the singularities M_2 and M_3. Therefore systems (14) possess two double singularities located on the invariant parabola. So we obtain the configuration *Config.* $\mathcal{P}$*.8*.

It remains to mention that the case $u = 0$ (i.e. when the discriminant of $\mathcal{F}_1$ vanishes) is included in the previous examination because the condition $u \neq 0$ was not necessary. So in this case we obtain the same configurations for the provided conditions, respectively.

3.1.1.2 The case $B_1 = 0$. Considering (8) we observe that the condition $B_1 = 0$ splits into five conditions at the coefficient level. However due to an affine transformation we could decrease this number. More exactly we have the following lemma.

Lemma 3. *The condition $(g + 8m + 4n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0$ for systems (7) could be transferred to the condition $m(gn - 2m - n) = 0$ via an affine transformation.*

Proof: Applying to systems (7) the transformation

$$x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4,$$

we obtain the systems

$$\begin{aligned}\dot{x}_1 =& -\frac{1}{8}(g + 8m + 4n) + \frac{1}{4}(1 + 2g + 4n)x_1 + \frac{g}{2}y_1 - (1 + g)x_1^2 + x_1y_1,\\ \dot{y}_1 =& -\frac{1}{4}(g + 8m + 4n)x_1 + \frac{1}{2}(1 + 2g + 4n)y_1 - (g + 2)x_1y_1 + 2y_1^2.\end{aligned}$$

So setting the new parameters

$$\begin{aligned}m_1 =& -\frac{1}{8}(g + 8m + 4n), \quad n_1 = \frac{1}{4}(1 + 2g + 4n), \quad g_1 = -(1 + g) \ \Rightarrow\\ m =& -\frac{1}{8}(g_1 + 8m_1 + 4n_1), \quad n = \frac{1}{4}(1 + 2g_1 + 4n_1) \ \ g = -(1 + g_1),\end{aligned} \tag{18}$$

we obtain the family of systems

$$\dot{x}_1 = m_1 + nx_1 - \frac{1 + g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \quad \dot{y} = 2m_1x_1 + 2n_1y_1 + (g_1 - 1)x_1y_1 + 2y_1^2$$

which coincide with family (7) (up to notations). Then considering (18) calculations yield:

$$g + 8m + 4n = -8m_1, \quad 1 + 2g + g^2 - 4m + 2n + 2gn = 2(2m_1 + n_1 - g_1n_1)$$

and this completes the proof of the lemma. ■

Thus by Lemma 3 in order to examine the condition $B_1 = 0$ it is sufficient to consider the condition

$$m(gn - 2m - n)(g + 2g^2 + g^3 + 4m + 2n + 2gn) = 0.$$

In order to determine the invariant conditions which distinguish the three possibilities provided by the above equality, for systems (7) we calculate:

$$\begin{aligned}\xi_4 =& 21 \cdot 2^6 5^4 m(g + 8m + 4n)\zeta_4,\\ \xi_5 =& -14 \cdot 5^5(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn)\zeta_4.\end{aligned} \tag{19}$$

Hence due to $\zeta_4 \neq 0$ the condition $\xi_4 = 0$ is equivalent to $m(g + 8m + 4n) = 0$ (this implies $B_1 = 0$), whereas the condition $\xi_5 = 0$ is equivalent to $(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn = 0)$ (this also implies $B_1 = 0$).

3.1.1.2.1 The subcase $\xi_4 \neq 0$. Then $m(g+8m+4n) \neq 0$ and we consider two possibilities: $\xi_5 \neq 0$ and $\xi_5 = 0$.

1: The possibility $\xi_5 \neq 0$. In this case we have $(gn-2m-n)(1+2g+g^2-4m+2n+2gn) \neq 0$ and therefore the condition $B_1 = 0$ implies $g+2g^2+g^3+4m+2n+2gn = 0$. This yields $m = -(1+g)(g+g^2+2n)/4$ and we get the family of systems

$$\begin{aligned} \dot{x} =& -\frac{1}{4}(1+g-2x)(g+g^2+2n+2gx+2y),\\ \dot{y} =& -\frac{(1+g)(g+g^2+2n)}{2}x+2ny+(g-1)xy+2y^2 \end{aligned} \tag{20}$$

possessing the invariant line $x = (g+1)/2$. For these systems we calculate

$$\begin{aligned} B_2 =& -81g^2(1+g)^2(g+g^2+2n)(1+4g+2g^2+4n)(1+2g+g^2+4n)^2x^4,\\ \xi_4 =& 13125g(1+g)(g-2)(3+g)(1+2g)(g+g^2+2n)(1+4g+2g^2+4n)(1+6g+5g^2+4n),\\ \xi_5 =& -(21875/16)(g-2)g(1+g)(3+g)(1+2g)(1+2g+g^2+4n)^2(1+6g+5g^2+4n), \end{aligned}$$

and we observe that the condition $\xi_4\xi_5 \neq 0$ implies $B_2 \neq 0$. Then by Lemma 1 besides the invariant line $x = (g+1)/2$ systems (20) could not possess invariant lines in other directions. However they could have a parallel invariant line and by Lemma 2 for this it is necessary $\theta = 0$ and this condition implies $(g-1)(g+2) = 0$. A straightforward calculation shows us that none of the conditions $g = 1$ or $g = -2$ could imply the appearance of an additional parallel invariant line.

Next we determine that systems (20) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{aligned} x_1 = \frac{1+g}{2},\ y_1 = \frac{(1+g)^2}{4};\quad x_2 = \frac{1+g}{2},\ y_2 = -\frac{g+g^2+2n}{2};\quad x_{3,4} = \frac{1}{2}\big(-g \pm \sqrt{Z_2}\big),\\ y_{3,4} = \frac{1}{2}\big(-g-2n \mp g\sqrt{Z_2}\big),\quad Z_2 = -(2g+g^2+4n). \end{aligned} \tag{21}$$

We determine that the singularities M_1, M_3 and M_4 are located on the invariant parabola. At the same time M_1 and M_2 are located on the invariant line $x = (g+1)/2$ and M_1 is the point of intersection of this invariant line with the parabola.

In order to determine the reciprocal position of the singularities M_1 and M_2 on the vertical invariant line we calculate

$$y_2 - y_1 = -\frac{1+4g+3g^2+4n}{4} \equiv -\frac{\alpha_2}{4} \ \Rightarrow\ \text{sign}\,(y_2-y_1) = -\text{sign}\,(\alpha_2). \tag{22}$$

Since the singularities M_2 and M_3 are either complex or real or coinciding depending on the value of Z_2 we need to distinguish these conditions using affine invariant polynomials. For systems (20) we calculate:

$$\begin{aligned} \zeta_4 =& \frac{1}{16}(g-2)(3+g)(1+2g)(1+6g+5g^2+4n) \equiv \frac{1}{16}(g-2)(3+g)(1+2g)\beta_2,\\ \mathbf{D} =& -\frac{3}{4}g^4(1+g)^4\beta_2^2\alpha_2^2Z_2,\quad \zeta_2 = 4g(1+g), \end{aligned} \tag{23}$$

and due to $\zeta_2\zeta_4 \neq 0$ we conclude that $\mathbf{D} = 0$ is equivalent to $\alpha_2 Z_2 = 0$. Moreover if $\mathbf{D} \neq 0$ then $\text{sign}\,(\mathbf{D}) = -\text{sign}\,(Z_2)$. So we discuss three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D}<0$. This implies $Z_2>0$ and systems (20) possess four real singularities. Clearly it is necessary to know the position of the real singularities $M_{3,4}$ with respect to M_1 all located on the invariant parabola. We calculate

$$(x_3-x_1)(x_4-x_1)=\frac{\beta_2}{4},\quad (x_3-x_1)+(x_4-x_1)=-(1+2g),$$
$$\text{sign}\big((x_3-x_1)(x_4-x_1)\big)=\text{sign}\,(\beta_2),\quad \text{sign}\big((x_1-x_3)+(x_1-x_4)\big)=-\text{sign}\,(1+2g).$$

We observe that $\beta_2\neq 0$ due to the condition $\zeta_4\neq 0$ and moreover $\alpha_2\neq 0$ due to $\mathbf{D}\neq 0$.

On the other hand we need the invariant polynomials which govern the signs of β_2 and α_2. So for systems (20) we calculate:

$$\xi_7=1174627500\,g^2(1+g)^2(1+2g)^2\alpha_2^2\beta_2 Z_2,\quad \xi_8=61822500g^2(1+g)^2(1+2g)^2\alpha_2\beta_2^2 Z_2$$

and due to the condition $\mathbf{D}<0$ which implies $g(1+g)(1+2g)\alpha_2\beta_2\neq 0$ and $Z_2>0$ (this also implies $\xi_7\xi_8\neq 0$) we have the next relations:

$$\text{sign}\,(\beta_2)=\text{sign}\,(\xi_7),\quad \text{sign}\,(\alpha_2)=\text{sign}\,(\xi_8).$$

Thus considering the above relations in the case $\mathbf{D}<0$ we detect the following configurations:

$\xi_7<0,\ \xi_8<0 \ \Rightarrow\ (x_3-x_1)(x_4-x_1)<0,\ y_2>y_1 \ \Rightarrow$ *Config.* $\mathcal{P}$*.9*;
$\xi_7<0,\ \xi_8>0 \ \Rightarrow\ (x_3-x_1)(x_4-x_1)<0,\ y_2<y_1 \ \Rightarrow$ *Config.* $\mathcal{P}$*.10*;
$\xi_7>0,\ \xi_8<0 \ \Rightarrow\ (x_3-x_1)(x_4-x_1)>0,\ y_2>y_1 \ \Rightarrow$ *Config.* $\mathcal{P}$*.11*;
$\xi_7>0,\ \xi_8>0 \ \Rightarrow\ (x_3-x_1)(x_4-x_1)>0,\ y_2<y_1 \ \Rightarrow$ *Config.* $\mathcal{P}$*.12.*

1.2: *The case* $\mathbf{D}>0$. Then $Z_2<0$ and we claim that this condition implies $\alpha_2>0$. Indeed supposing the contrary (i.e. $\alpha_2<0$) we must have $Z_2+\alpha_2<0$. However calculations yield:

$$Z_2+\alpha_2=-(2g+g^2+4n)+(1+4g+3g^2+4n)=(1+g)^2+g^2>0. \tag{24}$$

The contradiction we obtained proves our claim.

Therefore since M_2 and M_3 are complex we arrive at the configuration *Config.* $\mathcal{P}$*.13*.

1.3: *The case* $\mathbf{D}=0$. Considering (23) we deduce that due to $\zeta_2\zeta_4\neq 0$ the condition $\mathbf{D}=0$ implies $\alpha_2 Z_2=0$.

On the other hand for systems (20) we calculate:

$$\xi_1=g^2(1+g)^2(g-2)(3+g)(1+2g)\alpha_2\beta_2,\quad \zeta_3=8(1+2g)^2.$$

So due to $\zeta_2\zeta_3\zeta_4\neq 0$ (i.e. $g(1+g)(g-2)(3+g)(1+2g)\beta_2\neq 0$) we obtain that the condition $\alpha_2=0$ is equivalent to $\xi_1=0$. So we discuss two subcases: $\xi_1\neq 0$ and $\xi_1=0$.

1.3.1: *The subcase* $\xi_1\neq 0$. In this case the condition $\mathbf{D}=0$ implies $Z_2=0$. Then M_3 and M_4 coalesce producing a double point located on the invariant parabola. Considering (24) we deduce that the condition $Z_2=0$ implies $\alpha_2>0$.

Thus it is not too difficult to determine that in this case we arrive at the configuration *Config.* $\mathcal{P}$*.14*.

1.3.2: *The subcase* $\xi_1 = 0$. This implies $\alpha_2 = 0$ and as we have mentioned earlier (see formulas (22)) in this case we get $y_2 = y_1$ and hence the intersection point M_1 of the invariant line $x = (g+1)/2$ with the parabola becomes a double singularity of systems (20). Moreover the position of the real singularities M_3 and M_4 with respect to M_1 depends on the sign of β_2.

So the condition $\alpha_2 = 0$ implies $n = -(1+g)(1+3g)/4$ and then we obtain

$$\beta_2 = 2g(1+g), \quad \zeta_2 = 4g(1+g) \;\Rightarrow\; \operatorname{sign}(\beta_2) = \operatorname{sign}(\zeta_2).$$

Thus in the case $\alpha_2 = 0$ (i.e. $\xi_1 = 0$) we obtain the following two configurations:

$$\begin{aligned} &\zeta_2 < 0 \quad \Rightarrow \quad (x_3 - x_1)(x_4 - x_1) < 0,\; y_2 = y_1 \quad \Rightarrow \quad \textit{Config.}\,\mathcal{P}.15;\\ &\zeta_2 > 0 \quad \Rightarrow \quad (x_3 - x_1)(x_4 - x_1) > 0,\; y_2 = y_1 \quad \Rightarrow \quad \textit{Config.}\,\mathcal{P}.16. \end{aligned}$$

2: The possibility $\xi_5 = 0$. Considering (19) and the condition $\zeta_4 \neq 0$ we obtain that the condition $\xi_5 = 0$ implies

$$(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0.$$

On the other hand according to Lemma 3 it is sufficient to examine the condition given by the first factor because the condition defined by the second factor could be brought to the first one via an affine transformation.

So in what follows we assume that for systems (7) the condition $gn - 2m - n = 0$ holds. Then $m = n(g-1)/2$ and we arrive at the family of systems

$$\dot{x} = \frac{n(g-1)}{2} + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = (n+y)(gx - x + 2y) \tag{25}$$

which possess the invariant line $y = -n$ and the invariant parabola $\Phi(x,y) = x^2 - y = 0$. For these systems we calculate

$$\begin{aligned} B_2 =& -81g^2(1+4n)\big[(1+g)^2 + 4n\big]^2 y^4/2,\\ \xi_4 =& 26250(g-1)g(1+g)(g-2)(3+g)n(1+4n)(1+6g+9g^2+4n) \end{aligned} \tag{26}$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

2.1: *The case* $B_2 \neq 0$. In this case by Lemma 1 systems (25) could not possess invariant lines in other directions than the invariant line $y = -n$. But by Lemma 2 these systems could possess an invariant line parallel to the existent one if $\theta = -8(g-1)(2+g) = 0$. So due to $\xi_4 \neq 0$ the condition implies $g = -2$. However in this case systems (25) do not have any invariant line parallel to $y = -n$.

Next we determine that systems (25) possess the finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{aligned} x_1 = \sqrt{-n},\; y_1 = -n; \quad x_2 = -\sqrt{-n},\; y_2 = -n; \quad x_3 = \frac{1-g}{2},\; y_3 = \frac{(1-g)^2}{4};\\ x_4 = -\frac{2n}{1+g},\; y_4 = \frac{n(g-1)}{1+g}. \end{aligned} \tag{27}$$

We observe that singular points M_1, M_2 and M_3 are located on the invariant parabola $\Phi(x,y) = x^2 - y = 0$. Moreover M_1 and M_2 are the points of intersection of the invariant line $y = -n$ and they are either complex for $n > 0$ or real for $n < 0$ or they coincide if $n = 0$.

On the other hand for systems (25) calculations yield:

$$\mathbf{D} = 48g^4n^3(1-g^2+4n)^2(1-2g+g^2+4n)^2 \equiv 48g^4n^3\,\alpha_3^2\,\beta_3^2$$

and it is clear that in the case $\mathbf{D} \neq 0$ we have $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(n)$.

To determine the position of the singular point M_4 we calculate

$$\Phi(x_4, y_4) = \frac{n\alpha_3}{(1+g)^2}$$

and since $n \neq 0$ (due to $\xi_4 \neq 0$) we deduce that the singular point M_4 lies on the invariant parabola if and only if $\alpha_3 = 0$.

To examine the configurations of systems we consider three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

2.1.1: *The subcase* $\mathbf{D} < 0$. Then $n < 0$ and the singular point M_1 and M_2 are real and in order to determine the position of the singularity M_3 with respect to the real singularities M_1 and M_2 we calculate

$$(x_3-x_1)(x_3-x_2) = (1-g)^2+4n \equiv \beta_3,\ \ (x_3-x_1)+(x_3-x_2) = 1-g;$$
$$\operatorname{sign}\big((x_3-x_1)(x_3-x_2)\big) = \operatorname{sign}(\beta_3),\quad \operatorname{sign}\big((x_3-x_1)+(x_3-x_2)\big) = \operatorname{sign}(1-g).$$

We observe that $\alpha_3\beta_3 \neq 0$ due to $\mathbf{D} \neq 0$ and we have to determine invariant polynomials which are responsible for the signs of β_3 and $g-1$. Calculations yield:

$$\begin{aligned}\xi_9 =&5589813240\,g^6(1+g)^2(1+2g+g^2+4n)^2\beta_3,\\ \xi_{10} =&24814861965\,(g-1)g^2(1+g)^2(1+2g+g^2+4n)^2(1+6g+9g^2+4n)^2/2,\end{aligned} \tag{28}$$

and taking into account the condition $\xi_4 B_2 \neq 0$ and (26) we deduce that $\operatorname{sign}(\beta_3) = \operatorname{sign}(\xi_9)$ and $\operatorname{sign}(g-1) = \operatorname{sign}(\xi_{10})$.

Thus considering the above relations in the case $\mathbf{D} < 0$ we arrive at the following configurations:

$$\begin{array}{lcl}\xi_9 < 0 & \Rightarrow\ (x_3-x_1)(x_3-x_2) < 0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.17;\\ \xi_9 > 0,\ \xi_{10} < 0 & \Rightarrow\ (x_3-x_1) > 0, (x_3-x_2) > 0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.18;\\ \xi_9 > 0,\ \xi_{10} > 0 & \Rightarrow\ (x_3-x_1) < 0, (x_3-x_2) < 0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.19.\end{array}$$

2.1.2: *The subcase* $\mathbf{D} > 0$. Then $n > 0$ and the singular points M_1 and M_2 are complex. So due to the condition $\alpha_3 \neq 0$ we arrive at the configuration *Config.* $\mathcal{P}.20$.

2.1.3: *The subcase* $\mathbf{D} = 0$. This implies $n\,\alpha_3\,\beta_3 = 0$ and we have to distinguish these three cases. From (28) we observe that due to $\xi_4 B_2 \neq 0$ the condition $\xi_9 = 0$ is equivalent to $\beta_3 = 0$.

On the other hand for systems (25) we have

$$\zeta_8 = gn(1+2g+g^2+4n)$$

and due to $B_2 \neq 0$ the condition $\zeta_8 = 0$ is equivalent to $n = 0$. So we discuss the above mentioned possibilities.

2.1.3.1: *The possibility* $\xi_9 \neq 0$. Then $\beta_3 \neq 0$ and the condition $\mathbf{D} = 0$ implies $n\alpha_3 = 0$. So we examine two cases: $\zeta_8 \neq 0$ and $\zeta_8 = 0$.

2.1.3.1.1: *The case* $\zeta_8 \neq 0$. Then $n \neq 0$ and this implies $\alpha_3 = 0$ and we obtain $n = (g^2-1)/4$. Considering (27) we observe that in this case the singular point M_4 coalesces with M_3 producing a double singular point on the invariant parabola. So we obtain that the finite singularities of systems (25) have the following coordinates:

$$x_1 = \frac{\sqrt{1-g^2}}{2},\ y_1 = \frac{1-g^2}{4};\quad x_2 = -\frac{\sqrt{1-g^2}}{2},\ y_2 = \frac{1-g^2}{4};$$
$$x_3 = x_4 = \frac{1-g}{2},\ y_3 = y_4 = \frac{(1-g)^2}{4}.$$

We note that in this case $\beta_3 = 2g(g-1)$ and it is clear that we need to determine in invariant way the signs of the expressions $1-g^2$ and $g(g-1)$. So for systems (25) with $n = (g^2-1)/4$ we calculate:

$$\xi_2 = (1-g^2)^3 g^4,\quad \xi_9 = 44718505920(g-1)g^9(1+g)^4$$

and we observe that $\operatorname{sign}(\xi_2) = \operatorname{sign}(1-g^2)$ and $\operatorname{sign}(\xi_9) = \operatorname{sign}(g(g-1))$.

Thus in the case $\alpha_3 = 0$ which implies to $\mathbf{D} = 0$ (then we have a double real singularity on the invariant parabola) we obtain the following configurations:

$\xi_2 < 0 \quad \Rightarrow M_1$ and M_2 are complex $\quad \Rightarrow$ *Config.* $\mathcal{P}$*.21*;
$\xi_2 > 0,\ \xi_9 < 0 \quad \Rightarrow (x_3-x_1)(x_3-x_2) < 0 \quad \Rightarrow$ *Config.* $\mathcal{P}$*.22*;
$\xi_2 > 0,\ \xi_9 > 0 \quad \Rightarrow (x_3-x_1) > 0, (x_3-x_2) > 0 \quad \Rightarrow$ *Config.* $\mathcal{P}$*.23.*

2.1.3.1.2: *The case* $\zeta_8 = 0$. This implies $n = 0$ and the three finite singular points M_1, M_2 and M_4 coalesce and we get the triple singular point $(0,0)$ located on the invariant parabola which is also the point of tangency of the line $y = 0$ with the parabola. We observe that the singular point $M_3\big((1-g)/2, (1-g)^2/4\big)$ coalesces with the triple point if and only if $g = 1$. However we have $g-1 \neq 0$ due to $\xi_4 \neq 0$.

Thus considering the relation $\operatorname{sign}(g-1) = \operatorname{sign}(\xi_{10})$ we obtain the configuration *Config.* $\mathcal{P}$*.24* if $\xi_{10} < 0$ and *Config.* $\mathcal{P}$*.25* if $\xi_{10} > 0$.

2.1.3.2: *The possibility* $\xi_9 = 0$. This implies $\beta_3 = 0$ and hence we get $n = -(g-1)^2/4$. We observe that in this case considering (27) we obtain

$$x_1 = \frac{1}{2}\sqrt{(g-1)^2},\ y_1 = \frac{(g-1)^2}{4};\quad x_2 = -\frac{1}{2}\sqrt{(g-1)^2},\ y_2 = \frac{(g-1)^2}{4};$$
$$x_3 = \frac{1-g}{2},\ y_3 = \frac{(g-1)^2}{4};\quad x_4 = \frac{(g-1)^2}{2(1+g)},\ y_4 = \frac{(g-1)^3}{4(1+g)}.$$

We observe that the singular point M_3 coincides either with M_1 or M_2. And since x_1 is positive and x_2 is negative we conclude that M_3 coalesces with M_1 if $1-g > 0$ and with M_2 if $1-g < 0$.On the other hand for systems (25) with $n = -(g-1)^2/4$ we have

$$\xi_{10} = 12705209326080(g-1)g^6(1+g)^4$$

and hence we have $\operatorname{sign}(\xi_{10}) = \operatorname{sign}(g-1)$. Therefore it is not difficult to determine that we obtain the configuration *Config.* $\mathcal{P}$*.26* if $\xi_{10} < 0$ and *Config.* $\mathcal{P}$*.27* if $\xi_{10} > 0$.

2.2: *The case* $B_2 = 0$. Since $\xi_4 \neq 0$ (i.e. $g(1+4n) \neq 0$) considering (26) this condition implies $(1+g)^2 + 4n = 0$.

Then we get $n = -(1+g)^2/4$ and this leads to the family of systems

$$\dot{x} = -\frac{1}{8}(1+g-2x)(-1+g^2+4gx+4y), \quad \dot{y} = -\frac{1}{4}(1+2g+g^2-4y)(-x+gx+2y) \tag{29}$$

which possess the following three invariant affine lines:

$$1+g-2x=0, \quad 1+2g+g^2-4y=0, \quad 1-g^2-4x+4y=0.$$

For these systems we have $B_2 = B_3 = 0$ and we see that the above systems possess invariant line in three directions. However we could have parallel invariant lines and by Lemma 2 for this it is necessary $\theta = 0$. So we discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: *The subcase* $\theta \neq 0$. We determine that the above systems possess the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates:

$$x_1 = \frac{1+g}{2},\ y_1 = \frac{(1+g)^2}{4};\ \ x_2 = -\frac{1+g}{2},\ y_2 = \frac{(1+g)^2}{4};$$
$$x_3 = \frac{1-g}{2},\ y_3 = \frac{(1-g)^2}{4};\ \ x_4 = \frac{1+g}{2},\ y_4 = \frac{1-g^2}{4}.$$

We detect that the singular point M_1 is the point of intersection of all three invariant lines together with the invariant parabola. Since this point as well as the singular point M_4 are located on the vertical invariant line $1+g-2x=0$ the position of these two points are important for determining the configurations of systems (29). So we obtain

$$y_4 - y_1 = -\frac{g(g+1)}{2} \ \Rightarrow\ \text{sign}\,(y_4 - y_1) = -\text{sign}\,\big(g(g+1)\big).$$

We point out that the position of the vertical invariant line $x = (g+1)/2$ is also important and we have to consider sign $(g+1)$.

On the other hand for systems (29) we calculate:

$$\zeta_2 = 4g(1+g),$$

and then we determine the following configurations:

$\zeta_2 < 0$ (i.e. $-1 < g < 0$) $\Rightarrow$ $x_1 > 0,\ y_4 > y_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.28*;
$\zeta_2 > 0$ and $g < -1$ $\Rightarrow$ $x_1 < 0,\ y_4 < y_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.29*;
$\zeta_2 > 0$ and $g > 0$ $\Rightarrow$ $x_1 > 0,\ y_4 < y_1$ $\Rightarrow$ $\simeq$*Config.* $\mathcal{P}$*.29.*

2.2.2: *The subcase* $\theta = 0$. This condition implies $(g-1)(g+2) = 0$. If $g = 1$ we arrive at the system

$$\dot{x} = (x-1)(x+y), \quad \dot{y} = 2(y-1)y \tag{30}$$

possessing four invariant affine lines: $x = 1$, $y = 0$, $y = 1$ and $y = x$. Therefore it is easy to determine that this system possesses the configuration *Config.* $\mathcal{P}$*.30.*

Assuming $g = -2$ we arrive at system

$$\dot{x} = (1+2x)(3-8x+4y)/8, \quad \dot{y} = -(4y-1)(3x-2y)/4$$

which via the transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$ could be brought to the system (30) having the configuration *Config.* $\mathcal{P}$*.30.*

3.1.1.2.2 The subcase $\xi_4 = 0$. Considering (19) and the condition $\zeta_4 \neq 0$ we obtain that the condition $\xi_4 = 0$ implies

$$m(g + 8m + 4n) = 0.$$

On the other hand according to Lemma 3 it is sufficient to examine the condition $m = 0$ because the condition $g + 8m + 4n = 0$ could be brought to $m = 0$ via an affine transformation.

So $m = 0$ and we arrive at the family of systems

$$\dot{x} = nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = y(2n - x + gx + 2y) \tag{31}$$

which possess the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$. It is clear that the invariant line $y = 0$ is tangent to the invariant parabola at the origin of coordinates.

We determine that for the above systems the following condition holds:

$$\zeta_1\zeta_2\zeta_3\zeta_4\mathcal{R}_1 \neq 0 \;\Rightarrow\; g(1+g)(g-2)(3+g)(1+2g)(1+3g)(1+4g+3g^2+2n) \neq 0. \tag{32}$$

For the above systems we calculate

$$B_2 = -81(1+g)^2(1+g+2n)(g+g^2+2n)(g+4n)y^4/2, \quad \theta = -8(g-1)(2+g) \tag{33}$$

and we consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. Then besides the invariant line $y = 0$ systems (31) could not possess invariant lines in other directions. However we could have a parallel invariant line to the line $y = 0$ and by Lemma 2 for this it is necessary $\theta = 0$. So we discuss two cases: $\theta \neq 0$ and $\theta = 0$.

1.1: *The case* $\theta \neq 0$. We determine that systems (31) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned} &x_1 = 0,\ y_1 = 0; \quad x_2 = -\frac{n}{g},\ y_2 = 0; \quad x_{3,4} = \frac{1}{4}\big(1 - g \pm \sqrt{Z_3}\,\big),\\ &y_{3,4} = \frac{1}{8}\big[(1-g)^2 - 8n \pm (1-g)\sqrt{Z_3}\,\big], \quad Z_3 = (1-g)^2 - 16n. \end{aligned} \tag{34}$$

We observe that $\Phi(x_3, y_3) = \Phi(x_4, y_4) = 0$ and this means that the singular points M_3 and M_4 are located on the invariant parabola. Moreover the singularity M_2 lies on the invariant line $y = 0$ and coalesces with M_1 if and only if $n = 0$. The singularities M_3 and M_4 are complex (respectively, real) if $Z_3 < 0$ (respectively, $Z_3 > 0$) and they coincide (producing a multiple singular point) if $Z_3 = 0$.

On the other hand for systems (31) we have

$$\mathbf{D} = 48(1+g)^4 n^6(-1 + 2g - g^2 + 16n) = -48(1+g)^4 n^6 Z_3$$

and we discuss three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1.1.1: *The subcase* $\mathbf{D} < 0$. Then $Z_3 > 0$ and therefore the finite singularities M_3 and M_4 are real and they are distinct because $n \neq 0$ (due to $\mathbf{D} \neq 0$). Clearly we need to determine their positions on the parabola with respect to the singularity M_1 and we calculate:

$$\begin{aligned} &(x_3 - x_1)(x_4 - x_1) = n, \quad (x_3 - x_1) + (x_4 - x_1) = (1-g)/2;\\ &\text{sign}\big((x_3 - x_1)(x_4 - x_1)\big) = \text{sign}\,(n), \quad \text{sign}\big((x_3 - x_1) + (x_4 - x_1)\big) = \text{sign}\,(1-g). \end{aligned} \tag{35}$$

We point out that $g-1 \neq 0$ (due to $\theta \neq 0$) and $\text{sign}\,(1-g)$ is important only in the case $n>0$ (i.e. when $(x_3-x_1)(x_4-x_1)>0$). On the other hand we have

$$x_2 - x_1 = -n/g \;\Rightarrow\; \text{sign}\,(x_2-x_1) = -\text{sign}\,(gn).$$

For systems (31) calculations yield:

$$\begin{aligned}
\xi_{11} =& -95982880\,gn(g-2)^2(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2,\\
\xi_{28} =& 3244620(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2(g+4n),\\
\xi_{29} =& 3244620(g-1)g(1+g)^2(3+g)(1+3g)(1+4g+3g^2+2n)^2(g+4n),
\end{aligned}$$

and we have the next remark.

Remark 2. *We observe that due to the condition* (32) *we obtain that* $\xi_{11} \neq 0$ *and* $\text{sign}\,(\xi_{11}) = -\text{sign}\,(gn)$. *If we have* $\xi_{11}<0$ *(i.e.* $gn>0$*) then* $\text{sign}\,(\xi_{28}) = \text{sign}\,(g+4n)$. *Moreover in the case* $g>0$ *and* $n>0$ *we obtain* $\text{sign}\,(\xi_{29}) = \text{sign}\,(g-1)$.

Thus considering the above relations in the case $\mathbf{D}<0$ we detect the following configurations:

$\xi_{11}<0$, $n<0$ (then $g<0$) $\Rightarrow (x_3-x_1)(x_4-x_1)<0$, $x_2<x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.31*;
$\xi_{11}<0$, $n>0$ (then $g>0$), $g<1$ $\Rightarrow x_3>x_1$, $x_4>x_1$, $x_2<x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.32*;
$\xi_{11}<0$, $n>0$ (then $g>0$), $g>1$ $\Rightarrow x_3<x_1$, $x_4<x_1$, $x_2<x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.33*;
$\xi_{11}>0$, $n<0$ (then $g>0$) $\Rightarrow (x_3-x_1)(x_4-x_1)<0$, $x_2>x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.34*;
$\xi_{11}>0$, $n>0$ (then $g<0$) $\Rightarrow x_3>x_1$, $x_4>x_1$, $x_2>x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.35*.

Taking into account Remark 2 we obtain the following invariant conditions:

$\xi_{11}<0$, $\xi_{28}<0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.31*;
$\xi_{11}<0$, $\xi_{28}>0$, $\xi_{29}<0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.32*;
$\xi_{11}<0$, $\xi_{28}>0$, $\xi_{29}>0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.33*;
$\xi_{11}>0$ $\Rightarrow \begin{cases} \textit{Config.}\,\mathcal{P}.34 \text{ or}\\ \textit{Config.}\,\mathcal{P}.35.\end{cases}$

1.1.2: *The subcase* $\mathbf{D}>0$. Then $Z_3<0$ and hence the finite singularities M_3 and M_4 are complex. On the other hand this condition implies $n>0$ and therefore the singular point $M_2(-n/g,0)$ could not coalesce with $M_1(0,0)$. Moreover its position with respect to the singular point M_1 depends on the sign of the parameter g.

It is easy to determine that the invariant line $y=0$ has a common point with the parabola $y=x^2$ and this is the point of tangency $M_1(0,0)$ and the finite singularity $M_2(-n/g,0)$ lies on the invariant line $y=0$.

On the other hand due to $n>0$ we obtain $\text{sign}\,(\xi_{11}) = -\text{sign}\,(gn) = -\text{sign}\,(g)$. Therefore we obtain the configuration *Config.* $\mathcal{P}$*.36* if $\xi_{11}<0$ and *Config.* $\mathcal{P}$*.37* if $\xi_{11}>0$.

1.1.3: *The subcase* $\mathbf{D}=0$. Considering (32) we deduce that the condition $\mathbf{D}=0$ implies $nZ_3=0$. For systems (31) we calculate

$$\xi_1 = 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8$$

and due to (32) we obtain that the condition $n=0$ is equivalent to $\xi_1=0$. So we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1=0$.

1.1.3.1: *The possibility* $\xi_1 \neq 0$. Then the condition $\mathbf{D} = 0$ implies $Z_3 = 0$. Considering (34) we obtain $n = (1-g)^2/16$ and then calculations yield

$$x_1 = 0,\ y_1 = 0;\quad x_2 = -\frac{(1-g)^2}{16g},\ y_2 = 0;\quad x_3 = x_4 = \frac{1-g}{4},\quad y_3 = y_4 = \frac{(1-g)^2}{16};$$
$$\mathrm{sign}\,(x_2 - x_1) = -\mathrm{sign}\,(g),\quad \mathrm{sign}\,(x_3 - x_1) = \mathrm{sign}\,(1-g).$$

Therefore we have a double singular point on the invariant parabola and for the parameter g we have the following possible bifurcation values: $g \in \{0, 1\}$.

On the other hand considering for systems (31) with $n = (1-g)^2/16$ (i.e. $Z_3 = 0$) we calculate:

$$\xi_{11} = -\frac{2999465}{32}(g-2)^2(g-1)^2 g(1+g)^2(3+g)^2(1+3g)^2(3+5g)^4,\ \theta = -8(g-1)(2+g),$$
$$\zeta_2 = 4g(g+1),\quad \zeta_4 = (g-2)(3+g)(1+3g)(3+5g)^2/128,\quad \xi_{12} = g(g-1)^3\psi_1(g)$$

where $\psi_1(g) = 1105 + 1774g + 961g^2$. We observe that $\mathrm{Discrim}\,[\psi_1(g), g] = -1100544 < 0$. Therefore taking into account the conditions $\zeta_4\theta \neq 0$ and we conclude that

$$\mathrm{sign}\,(\xi_{11}) = -\mathrm{sign}\,(g),\quad \mathrm{sign}\,(\xi_{12}) = \mathrm{sign}\,(g(g-1)).$$

So considering the above relations we determine the following configurations:

$\xi_{12} < 0 \Rightarrow x_2 < x_1,\ x_3 > x_1 \Rightarrow$ *Config.* $\mathcal{P}$*.38*;
$\xi_{12} > 0,\ \xi_{11} < 0 \Rightarrow x_2 < x_1,\ x_3 > x_1 \Rightarrow$ *Config.* $\mathcal{P}$*.39*;
$\xi_{12} > 0,\ \xi_{11} > 0 \Rightarrow x_2 > x_1,\ x_3 > x_1 \Rightarrow$ *Config.* $\mathcal{P}$*.40*.

1.1.3.2: *The possibility* $\xi_1 = 0$. In this case $n = 0$ and the singular point $M_2(-n/g, 0)$ coalesces with $M_1(0,0)$. Moreover one of the singular points either M_3 or M_4 coalesces with $M_1(0,0)$ and we obtain a triple finite singularity $M_1(0,0)$. It is clear that we could get two distinct singularities depending on the position of the simple singularity (M_3 or M_4) and this position is defined by $\mathrm{sign}\,(1-g)$ (see (35)).

Since in the case $n = 0$ for systems (31) we have

$$\xi_{10} = 24814861965(g-1)g^2(1+g)^6(1+3g)^4/2$$

we conclude that $\mathrm{sign}\,(\xi_{10}) = \mathrm{sign}\,(g-1)$. Therefore we get the configuration *Config.* $\mathcal{P}$*.41* if $\xi_{10} < 0$ and *Config.* $\mathcal{P}$*.42* if $\xi_{10} > 0$.

1.2: *The case* $\theta = 0$. This implies $(g-1)(g+2) = 0$ and for systems (31) we calculate

$$\begin{aligned} \xi_1 &= 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8,\\ \xi_5 &= -21875(g-1)(1+g)(g-2)(3+g)(1+3g)n(1+g+2n)(1+4g+3g^2+2n)/8. \end{aligned} \tag{36}$$

and we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.2.1: *The subcase* $\xi_1 \neq 0$. This implies $n \neq 0$ and considering the condition (32) and $B_2 \neq 0$ (i.e. $1 + g + 2n \neq 0$) we conclude that the condition $g = 1$ is equivalent to $\xi_5 = 0$. So we consider two possibilities: $\xi_5 \neq 0$ and $\xi_5 = 0$.

1.2.1.1: *The possibility* $\xi_5 \neq 0$. Then $g-1 \neq 0$ and the condition $\theta = 0$ implies $g = -2$. It is easy to determine that for $g = -2$ systems (31) do not have any invariant line parallel with $y = 0$.

On the other hand in the case $g = -2$ for systems (31) we have

$$\mathbf{D} = 48n^6(16n-9), \quad \xi_{11} = 76786304000n(5+2n)^2, \quad \zeta_4 = 5(5+2n)/4$$

and hence $\text{sign}\,(\xi_{11}) = \text{sign}\,(n)$. Moreover $\text{sign}\,(\mathbf{D}) = \text{sign}\,(16n-9)$ and due to $n \neq 0$ we obtain that $\mathbf{D} = 0$ is equivalent to $16n - 9 = 0$.

Therefore since $g = -2 < 0$, taking into consideration the examination of systems (31) given above we arrive at the following configurations:

$\mathbf{D} < 0$, $\xi_{11} < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.31*;
$\mathbf{D} < 0$, $\xi_{11} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.35*;
$\mathbf{D} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.37*.
$\mathbf{D} = 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.40*.

1.2.1.2: *The possibility* $\xi_5 = 0$. Then $g = 1$ and this leads to the systems

$$\dot{x} = nx - y + x^2 + xy, \quad \dot{y} = 2y(n+y)$$

possessing additionally the invariant line $y + n = 0$. Considering (34) we obtain:

$$x_1 = 0,\ y_1 = 0; \quad x_2 = -n,\ y_2 = 0; \quad x_{3,4} = \pm\sqrt{-n}, \quad y_3 = y_4 = -n.$$

We observe that the invariant line $y = -n$ intersects the invariant parabola $\Phi(x,y) = x^2 - y = 0$ at two points $M_{3,4}(\pm\sqrt{-n}, -n)$ which are distinct due to $\xi_1 \neq 0$ (i.e. $n \neq 0$). Moreover they are real if $n < 0$ and complex if $n > 0$. We calculate $\mathbf{D} = 12288n^7$ and hence $\text{sign}\,(\mathbf{D}) = \text{sign}\,(n)$. Therefore we arrive at the configuration *Config.* $\mathcal{P}$*.43* for $\mathbf{D} < 0$ and *Config.* $\mathcal{P}$*.44* for $\mathbf{D} > 0$.

1.2.2: *The subcase* $\xi_1 = 0$. This implies $n = 0$ and then the line $y = -n$ coalesces with $y = 0$ and we get one double invariant line. Moreover all finite singular point coalesce producing a singular point $M_1(0,0)$ of multiplicity four. As a result we get the configuration *Config.* $\mathcal{P}$*.45*.

2: *The possibility* $B_2 = 0$. First of all we set the next remark.

Remark 3. *The condition* $B_2 = 0$ *implies for systems* (31) $n \neq 0$.

Indeed in the case $n = 0$ for systems (31) we get

$$B_2 = -81g^2(1+g)^4y^4/2 \neq 0$$

due to the condition $\zeta_2 \neq 0$ (i.e. $g(g+1) \neq 0$).

Thus $n \neq 0$ and since $g + 1 \neq 0$ considering (33) we get the condition

$$(1+g+2n)(g+g^2+2n)(g+4n) = 0$$

and considering (36) we examine two cases: $\xi_5 \neq 0$ and $\xi_5 = 0$.

2.1: *The case* $\xi_5 \neq 0$. Then by (36) we get $1 + g + 2n \neq 0$ and hence the condition $B_2 = 0$ implies $(g + g^2 + 2n)(g + 4n) = 0$.

On the other hand for systems (31) we calculate

$$\xi_{13} = 27(1+g)^2(3+g)(1+3g)n^2(g+4n)/4$$

and considering Remark 3 and the condition (32) we conclude that the condition $\xi_{13} = 0$ is equivalent to $g + 4n = 0$.

2.1.1: *The subcase* $\xi_{13} \neq 0$. Then $g + 4n \neq 0$ and therefore $B_2 = 0$ implies $g + g^2 + 2n = 0$. In this case we get $n = -g(g+1)/2$ and we arrive at the following family of systems

$$\dot{x} = (2x - 1 - g)(gx + y)/2, \quad \dot{y} = -y(g + g^2 + x - gx - 2y) \tag{37}$$

possessing the invariant lines $y = 0$ and $x = (g+1)/2$. Considering Lemma 1 for these systems we calculate

$$B_3 = -3(g-1)g(1+g)^2(1+2g)x^2y^2/4, \quad \theta = -8(g-1)(g+2),$$
$$\xi_5 = -21875(g-2)(g-1)^2g(1+g)^4(3+g)(1+2g)(1+3g)/16.$$

We observe that $B_3 \neq 0$ due to the condition $\xi_5 \neq 0$ and hence by Lemma 1 the above systems could not have any invariant line in the third direction. However according to Lemma 2 we could have parallel invariant lines if the condition $\theta = 0$ holds. Due to $B_3 \neq 0$ (i.e. $g - 1 \neq 0$) we deduce that the condition $\theta = 0$ is equivalent to $g + 2 = 0$. It is easy to determine that for $g = -2$ systems (37) do not have any invariant line which is parallel either with $y = 0$ or $x = (g+1)/2$.

Next we determine that systems (37) possess the following finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates:

$$x_1 = 0,\ y_1 = 0;\ \ x_2 = \frac{1+g}{2},\ y_2 = 0;\ \ x_3 = -g,\ y_3 = g^2;\ \ x_4 = \frac{1+g}{2},\ y_4 = \frac{(1+g)^2}{4}.$$

We observe that the invariant line $x = (g+1)/2$ intersects the invariant parabola at the point M_4 and the invariant line $y = 0$ (which is tangent to the parabola at M_1) at the singular point M_2. So to determine the positions of the line $x = (g+1)/2$ as well as of the singularities we calculate:

$$x_2 - x_1 = \frac{1+g}{2},\ x_3 - x_1 = -g,\ x_3 - x_4 = -\frac{1+3g}{2},$$
$$\mathrm{sign}\,(x_2 - x_1) = \mathrm{sign}\,(1+g),\ \mathrm{sign}\,(x_3 - x_1) = -\mathrm{sign}\,(g),\ \mathrm{sign}\,(x_3 - x_4) = -\mathrm{sign}\,(1+3g).$$

As we can see for the parameter g we have the following possible bifurcation values: $g \in \{-1, -1/3, 0\}$.

On the other hand for systems (37) we calculate:

$$\zeta_2 = 4g(g+1), \quad \xi_7 = 1174627500g^4(1+g)^7(1+2g)^2(1+3g),$$
$$\xi_{11} = 47991440(-2+g)^2g^2(1+g)^5(3+g)^2(1+2g)^2(1+3g)^2$$

and we observe that

$$\mathrm{sign}\,(\zeta_2) = \mathrm{sign}\,(g(g+1)), \quad \mathrm{sign}\,(\xi_7) = \mathrm{sign}\,((g+1)(1+3g)), \quad \mathrm{sign}\,(\xi_{11}) = \mathrm{sign}\,(g+1).$$

Moreover in the case $\zeta_2 < 0$ we have $-1 < g < 0$ (i.e. $g + 1 > 0$) and then $\mathrm{sign}\,\xi_7 = \mathrm{sign}\,(1+3g)$.

Thus considering the above relations we detect the following configurations:

$\zeta_2 < 0,\ \xi_7 < 0$ (i.e. $-1 < g < -1/3$) $\Rightarrow$ $x_2 > x_1,\ x_3 > x_1,\ x_3 > x_4$ $\Rightarrow$ *Config.* $\mathcal{P}$*.46*;
$\zeta_2 < 0,\ \xi_7 > 0$ (i.e. $-1/3 < g < 0$) $\Rightarrow$ $x_2 > x_1,\ x1 < x_3 < x_4$ $\Rightarrow$ *Config.* $\mathcal{P}$*.47*;
$\zeta_2 > 0,\ \xi_{11} < 0$ (i.e. $g < -1$) $\Rightarrow$ $x_2 < x_1,\ x_3 > x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.48*;
$\zeta_2 > 0,\ \xi_{11} > 0$ (i.e. $g > 0$) $\Rightarrow$ $x_2 > x_1,\ x_3 < x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.49.*

2.1.2: *The subcase* $\xi_{13} = 0$. This implies $g+4n = 0$ (i.e. $n = -g/4$) and we arrive at the family of systems

$$\dot{x} = -gx/4 - (1+g)y/2 + gx^2 + xy, \quad \dot{y} = -y(g + 2x - 2gx - 4y)/2 \tag{38}$$

possessing the invariant lines $y = 0$ and $y = x - 1/4$. For these systems we have

$$\xi_5 = 21875(g-2)g(1+g)(g-1)(2+g)(3+g)(1+2g)(1+3g)(2+3g)/128,$$
$$B_3 = 3g(1+g)(1+2g)(x-y)^2y^2/8, \quad \theta = -8(g-1)(2+g)$$

and since $\xi_5 \neq 0$ we obtain $B_3\theta \neq 0$. So by Lemmas 1 and 2 we conclude that the above systems could not have a third invariant line.

Next we determine that systems (38) possess the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_2 = 1/4,\ y_2 = 0; \quad x_3 = 1/2,\ y_3 = 1/4; \quad x_4 = -\frac{g}{2},\ y_4 = \frac{g^2}{4};$$
$$\text{sign}\,(x_4 - x_1) = -\text{sign}\,(g), \quad x_4 - x_3 = -(g+1)/2 \ \Rightarrow\ \text{sign}\,(x_4 - x_3) = -\text{sign}\,(g+1).$$

It could be checked directly that the invariant line $y = x - 1/4$ is tangent to the invariant parabola at the singular point $M_3(1/2, 1/4)$. Therefore considering the above relations we detect the following configurations:

$\zeta_2 < 0$ (i.e. $-1 < g < 0$) $\Rightarrow$ $x_4 > x_1,\ x_4 < x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.50*;
$\zeta_2 > 0$ and $g < -1$ $\Rightarrow$ $x_4 > x_1,\ x_4 > x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.51*;
$\zeta_2 > 0$ and $g > 0$ $\Rightarrow$ $x_4 < x_1,\ x_4 < x_3$ $\Rightarrow$ $\simeq$*Config.* $\mathcal{P}$*.51.*

2.2: *The case* $\xi_5 = 0$. Considering (36), the conditions (32) and Remark 3 imply $(g-1)(1+g+2n) = 0$ and we examine two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: *The subcase* $\theta \neq 0$. Then $g-1 \neq 0$ and we get $1+g+2n = 0$. Therefore $n = -(1+g)/2 \neq 0$ and we arrive at the family of systems

$$\dot{x} = -(1+g)(x+y)/2 + gx^2 + xy, \quad \dot{y} = -y(1 + g + x - gx - 2y), \tag{39}$$

possessing the invariant lines $y = 0$ and $y = x$ and the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates:

$$x_1 = 0,\ y_1 = 0; \quad x_2 = \frac{1+g}{2g},\ y_2 = 0; \quad x_3 = 1,\ y_3 = 1; \quad x_4 = -\frac{1+g}{2},\ y_4 = \frac{(1+g)^2}{4}.$$

On the other hand considering Lemma 1 we calculate

$$B_3 = 3(g-1)(1+g)^2(x-y)^2y^2/4 \neq 0$$

due to the conditions (32) and $\theta \neq 0$. Then by Lemma 1 we could not have any invariant line in the third direction. Moreover by Lemma 2 we could not have parallel invariant lines due to $\theta \neq 0$.

Next considering the coordinates of the finite singularities of these systems it follows immediately:

$$\operatorname{sign}(x_4 - x_1) = -\operatorname{sign}(1+g), \quad \operatorname{sign}(x_2 - x_1) = \operatorname{sign}\big(g(1+g)\big),$$
$$x_4 - x_3 = -(g+3)/2 \ \Rightarrow \operatorname{sign}(x_4 - x_3) = -\operatorname{sign}(g+3).$$

We remark that $g(g+1)(g+3) \neq 0$ due to the condition (32) and hence for the parameter g we have the following possible bifurcation values: $g \in \{-3, -1, 0\}$.

On the other hand for systems (39) we calculate:

$$\zeta_2 = 4g(1+g), \quad \xi_9 = 5589813240(g-1)^2 g^2 (1+g)^9 (3+g),$$
$$\xi_{10} = -223333757685(g-1)^2 g^2 (1+g)^6 (2+g)(1+3g)^2/2$$

and hence we have

$$\operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(1+g)\big), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}\big((1+g)(3+g)\big), \quad \operatorname{sign}(\xi_{10}) = -\operatorname{sign}(2+g).$$

Remark 4. *We observe that the conditions $\zeta_2 > 0$ and $\xi_9 > 0$ imply either $g > 0$ or $g < -3$. In order to distinguish these two possibilities we use the invariant ξ_{10} even if this invariant does not vanish in the bifurcation values of g.*

Considering the above remark we arrive at the following configurations:

$\zeta_2 < 0$ (i.e. $-1 < g < 0$)	$\Rightarrow\ x_2 < x_1,\ x_4 < x_1$	$\Rightarrow$ *Config. $\mathcal{P}$.52*;
$\zeta_2 > 0,\ \xi_9 < 0$ (i.e. $-3 < g < -1$)	$\Rightarrow\ x_2 > x_1,\ x_1 < x_4 < x_3$	$\Rightarrow$ *Config. $\mathcal{P}$.53*;
$\zeta_2 > 0,\ \xi_9 > 0,\ \xi_{10} < 0$ (i.e. $g > 0$)	$\Rightarrow\ x_2 > x_1,\ x_3 < x_1$	$\Rightarrow$ *Config. $\mathcal{P}$.54*;
$\zeta_2 > 0,\ \xi_9 > 0,\ \xi_{10} > 0$ (i.e. $g < -3$)	$\Rightarrow\ x_2 > x_1,\ x_4 > x_3$	$\Rightarrow$ *Config. $\mathcal{P}$.55.*

2.2.2: *The subcase $\theta = 0$.* This implies $(g-1)(g+2) = 0$ and we discuss two possibilities: $B_3 \neq 0$ and $B_3 = 0$.

2.2.2.1: *The possibility $B_3 \neq 0$.* We claim that in this case we get the same configuration either if $g = 1$ or $g = -2$.

Indeed, assume first $g = -2$. Then calculations yield

$$\begin{gathered}\xi_5 = -328125n(2n-1)(5+2n)/2, \quad B_2 = -162(1+n)(2n-1)^2 y^4,\\ B_3 = 3y^2\big[n(-5+4n)x^2 + 2(1+n)xy - (1+n)y^2\big]/2,\end{gathered} \tag{40}$$

and evidently the condition $\xi_5 = B_2 = 0$ gives us $n = 1/2$. This leads to the system

$$\dot{x} = (x+y)/2 - 2x^2 + xy, \quad \dot{y} = y(1 - 3x + 2y) \tag{41}$$

possessing three invariant affine lines: $y = 0$, $y = x$ and $y = x - 1/4$. It is not difficult to determine that this system has the configuration equivalent to *Config. $\mathcal{P}$.56.*

Suppose now $g = 1$. Then we have

$$\xi_5 = 0, \quad B_2 = -648(1+n)^2(1+4n)y^4, \quad B_3 = -3(1+n)y^2(4nx^2 + 2xy - y^2)$$

and due to $B_3 \neq 0$ the condition $B_2 = 0$ implies $n = -1/4$. In this case we arrive at the system

$$\dot{x} = -x/4 - y + x^2 + xy, \quad \dot{y} = y(4y-1)/2$$

which via the affine transformation $x_1 = -x + 1/2$, $\;y_1 = -x + y + 1/4$ we could be brought to system (41). Thus our claim is proved and we get the configuration *Config.* $\mathcal{P}$*.56*.

2.2.2.2: *The possibility* $B_3 = 0$. Considering (40) we conclude that the condition $g = -2$ implies $B_3 \neq 0$ and hence the condition $\theta = 0$ gives us $g = 1$. In this case we arrive at the system

$$\dot{x} = (x-1)(x+y), \quad \dot{y} = 2(y-1)y$$

possessing four invariant affine lines: $x = 1$, $y = 0$, $y = 1$ and $y = x$. Therefore it is easy to determine that this system possesses the configuration equivalent to *Config.* $\mathcal{P}$*.30*.

3.1.2 The statement ($\boldsymbol{\mathcal{A}_2}$)

According to this statement of Proposition 2 for systems (7) the condition $\zeta_4 = 0$ must hold. Considering (8) we obtain

$$(g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn) = 0$$

and since $(g-2)(3+g) \neq 0$ (due to $\zeta_1 \neq 0$) we get

$$m = \frac{1}{4}(1+3g)(1+4g+3g^2+2n).$$

Then we arrive at the 2-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{1}{4}(1+3g)(1+4g+3g^2+2n) + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} =& \frac{1}{2}(1+3g)(1+4g+3g^2+2n)\,x + 2ny + (g-1)xy + 2y^2 \end{aligned} \tag{42}$$

possessing the following two invariant parabolas: $\Phi_1(x,y) = x^2 - y = 0$ and

$$\begin{aligned} \Phi_2 =& (1+4g+3g^2+2n)(1+4g+3g^2+4n)) + 2(1+g)(1+4g+3g^2+4n)x \\ &+ 4g(1+g)x^2 - 2(1+6g+5g^2+4n)y = 0. \end{aligned} \tag{43}$$

Following the statement ($\boldsymbol{\mathcal{A}_2}$) for the above systems we calculate

$$\begin{aligned} \zeta_1 =& 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g), \quad \zeta_3 = 8(1+2g)^2, \\ \zeta_4 =& 0, \quad \zeta_5 = 19(g-2)(3+g)(1+4g+3g^2+4n)^2/4, \\ \mathcal{R}_2 =& -(g-2)(3+g)(8+27g+27g^2)(1+6g+5g^2+4n)/16, \\ B_1 =& g(1+g)(1+2g)(1+3g)(2+3g)(1+4g+3g^2+2n)(1+6g+5g^2+4n) \\ &\times (1+6g+6g^2+4n)(1+6g+9g^2+4n)(5+14g+9g^2+4n)/32. \end{aligned} \tag{44}$$

According to Lemma 1 for the existence of an invariant line of systems (42) the condition $B_1 = 0$ is necessary.

3.1.2.1 The case $B_1 \neq 0$. Then we could not have any invariant line. We determine that systems (42) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned} x_1 &= -\frac{1+3g}{2},\ y_1 = \frac{(1+3g)^2}{4};\quad x_2 = -\frac{(1+g)(1+3g)^2 + 4(1+2g)n}{2g(1+g)},\\ y_2 &= \frac{(1+3g)(1+4g+3g^2+2n)}{2(1+g)};\quad x_{3,4} = \frac{1}{2}\big(1+g \pm \sqrt{Z_4}\,\big),\\ y_{3,4} &= -\frac{1}{2}\big(2g+2g^2+2n \mp (g+1)\sqrt{Z_4}\,\big),\quad Z_4 = -(1+6g+5g^2+4n). \end{aligned} \tag{45}$$

In order to determine the position of the finite singularities with respect to the parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ we calculate

$$\Phi_1(x_1, y_1) = \Phi_1(x_3, y_3) = \Phi_1(x_4, y_4) = 0;\quad \Phi_2(x_2, y_2) = \Phi_2(x_3, y_3) = \Phi_2(x_4, y_4) = 0.$$

Therefore we deduce that the finite singularities M_3 and M_4 are the points of intersection of these two invariant parabolas. We observe that the points of intersection of the invariant parabolas are complex if $Z_4 < 0$ and they are real if $Z_4 > 0$.

On the other hand for systems (42) we calculate:

$$\mathbf{D} = -3Z_4(1+4g+3g^2+4n)^2\,\alpha_4^2\,\beta_4^2/4, \tag{46}$$

where

$$\alpha_4 = 5+22g+21g^2+4n,\quad \beta_4 = (1+g)(1+3g)(1+6g+7g^2) + 4(1+2g)^2 n. \tag{47}$$

So if $\mathbf{D} \neq 0$ then $\text{sign}\,(\mathbf{D}) = -\text{sign}\,(Z_4)$ and we discuss three possibilities: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

***1:** The possibility* $\mathbf{D} < 0$. Then $Z_4 > 0$ and systems (42) possess four real singularities and it is necessary to know the positions of the singularities $M_{3,4}$ with respect to M_1 and M_2. We calculate

$$\begin{aligned} (x_1-x_3)(x_1-x_4) &= \frac{\alpha_4}{4},\quad (x_1-x_3)+(x_1-x_4) = 1+2g,\\ (x_2-x_3)(x_2-x_4) &= -\frac{Z_4\beta_4}{4g^2(1+g)^2},\quad (x_2-x_3)+(x_2-x_4) = -\frac{(1+2g)Z_4}{g(1+g)}. \end{aligned}$$

Therefore considering the condition $Z_4 > 0$ we obtain

$$\begin{gathered} \text{sign}\,\big((x_1-x_3)(x_1-x_4)\big) = \text{sign}\,(\alpha_4),\quad \text{sign}\,\big((x_1-x_3)+(x_1-x_4)\big) = \text{sign}\,(1+2g);\\ \text{sign}\,\big((x_2-x_3)(x_2-x_4)\big) = -\text{sign}\,(\beta_4),\\ \text{sign}\,\big((x_2-x_3)+(x_2-x_4)\big) = -\text{sign}\,(g(1+g)(1+2g)). \end{gathered}$$

Clearly we need invariant polynomials governing the signs of α_4 and β_4. For systems (42) we calculate:

$$\xi_{14} = 1235\alpha_4\beta_4/2,\quad \xi_{30} = 1235\big[Z_4\beta_4 - g^2(1+g)^2\alpha_4\big]/2,\quad \zeta_2 = 4g(1+g).$$

And we have

$$\text{sign}\,(\xi_{14}) = \text{sign}\,(\alpha_4\beta_4),\quad \text{sign}\,(\zeta_2) = \text{sign}\,(g(1+g)).$$

Moreover in the case $\xi_{14} < 0$ (i.e. $\alpha_4\beta_4 < 0$) and $\mathbf{D} < 0$ (i.e. $Z_4 > 0$) we obtain

$$\text{sign}\,(\xi_{30}) = \text{sign}\,(Z_4\beta_4 - g^2(1+g)^2\alpha_4) = \text{sign}\,(\beta_4).$$

On the other hand considering the form of the invariant parabola $\Phi_2(x, y) = 0$ we have

$$\begin{aligned} y = &-\frac{1}{Z_4}\Big[(1+g)(1+4g+3g^2+4n)x - \frac{1}{2}(1+4g+3g^2+2n)(1+4g+3g^2+4n)\Big] \\ &-\frac{2g(1+g)}{Z_4}x^2. \end{aligned} \tag{48}$$

Therefore since $Z_4 > 0$ we deduce that the invariant parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ are tangent at the infinity to the same part (respectively to different parts) of the invariant line at infinity $Z = 0$ if $\zeta_2 < 0$ (respectively $\zeta_2 > 0$). So we consider these two cases separately.

1.1: *The case* $\zeta_2 < 0$. Then $g(g + 1) < 0$ and considering the above relations in this case we obtain the following configurations:

$\xi_{14} < 0,\ \beta_4 < 0,\ 2g + 1 < 0$ (i.e. $\beta_4 < 0,\ \alpha_4 > 0,\ -1 < g < -1/2$) $\Rightarrow$
$x_2 - x_3 > 0,\ x_2 - x_4 > 0,\ x_1 - x_3 > 0,\ x_1 - x_4 > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.57*;
$\xi_{14} < 0,\ \beta_4 < 0,\ 2g + 1 > 0$ (i.e. $\beta_4 < 0,\ \alpha_4 > 0,\ -1/2 < g < 0$) $\Rightarrow$
$x_2 - x_3 < 0,\ x_2 - x_4 < 0,\ x_1 - x_3 < 0,\ x_1 - x_4 < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.57*;
$\xi_{14} < 0,\ \beta_4 > 0$ (i.e. $\beta_4 > 0,\ \alpha_4 < 0,\ -1 < g < 0$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0,\ (x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.58*;
$\xi_{14} > 0,\ \beta_4 < 0,\ 2g + 1 < 0$ (i.e. $\beta_4 < 0,\ \alpha_4 < 0,\ -1 < g < -1/2$) $\Rightarrow$
$x_2 - x_3 > 0,\ x_2 - x_4 > 0,\ (x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0,\ \beta_4 < 0,\ 2g + 1 > 0$ (i.e. $\beta_4 < 0,\ \alpha_4 < 0,\ -1/2 < g < 0$) $\Rightarrow$
$x_2 - x_3 < 0,\ x_2 - x_4 < 0,\ (x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0,\ \beta_4 > 0,\ 2g + 1 < 0$ (i.e. $\beta_4 > 0,\ \alpha_4 > 0,\ -1 < g < -1/2$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0,\ x_1 - x_3 > 0,\ x_1 - x_4 > 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0,\ \beta_4 > 0,\ 2g + 1 > 0$ (i.e. $\beta_4 > 0,\ \alpha_4 > 0,\ -1/2 < g < 0$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0,\ x_1 - x_3 < 0,\ x_1 - x_4 < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.59.*

1.2: *The case* $\zeta_2 > 0$. Then $g(g + 1) > 0$ and we obtain the following configurations:

$\xi_{14} < 0,\ \beta_4 < 0,\ 2g + 1 < 0$ (i.e. $\beta_4 < 0,\ \alpha_4 > 0,\ g < -1$) $\Rightarrow$
$x_2 - x_3 < 0,\ x_2 - x_4 < 0,\ x_1 - x_3 > 0,\ x_1 - x_4 > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.60*;
$\xi_{14} < 0,\ \beta_4 < 0,\ 2g + 1 > 0$ (i.e. $\beta_4 < 0,\ \alpha_4 > 0,\ g > 0$) $\Rightarrow$
$x_2 - x_3 > 0,\ x_2 - x_4 > 0, x_1 - x_3 < 0,\ x_1 - x_4 < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.60*;
$\xi_{14} > 0,\ \beta_4 < 0,\ 2g + 1 < 0$ (i.e. $\beta_4 < 0,\ \alpha_4 < 0,\ g < -1$) $\Rightarrow$
$x_2 - x_3 < 0,\ x_2 - x_4 < 0,\ (x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.61*;
$\xi_{14} > 0,\ \beta_4 < 0,\ 2g + 1 > 0$ (i.e. $\beta_4 < 0,\ \alpha_4 < 0,\ g > 0$) $\Rightarrow$
$x_2 - x_3 > 0,\ x_2 - x_4 > 0,\ (x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.61*;
$\xi_{14} > 0,\ \beta_4 > 0,\ 2g + 1 < 0$ (i.e. $\beta_4 > 0,\ \alpha_4 > 0,\ g < -1$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0,\ x_1 - x_3 > 0,\ x_1 - x_4 > 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.61*;
$\xi_{14} > 0,\ \beta_4 > 0,\ 2g + 1 > 0$ (i.e. $\beta_4 > 0,\ \alpha_4 > 0,\ g < -1$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0,\ x_1 - x_3 < 0,\ x_1 - x_4 < 0$ $\Rightarrow\ \simeq$ *Config.* $\mathcal{P}$*.61.*

Applying the Mathematica function "FindInstance" (or "Reduce") we detect that the conditions $\mathbf{D} < 0$, $\zeta_2 > 0$, $\xi_{14} < 0$ and $\beta_4 > 0$ (i.e. $Z_4 > 0$, $g(g+1) > 0$, $\alpha_4 < 0$ and $\beta_4 > 0$) are incompatible.

We observe that in both cases (i.e. $\zeta_2 < 0$ and $\zeta_2 > 0$) the configurations do not depend on the sign $(1+2g)$. As a result we obtain the following lemma.

Lemma 4. *Assume that for systems* (42) *the condition* $\mathbf{D} < 0$ *holds. Then these systems possess the following configurations if and only if the corresponding conditions are satisfied:*

$\zeta_2 < 0,\ \xi_{14} < 0,\ \xi_{30} < 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.57;*
$\zeta_2 < 0,\ \xi_{14} < 0,\ \xi_{30} > 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.58;*
$\zeta_2 < 0,\ \xi_{14} > 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.59;*
$\zeta_2 > 0,\ \xi_{14} < 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.60;*
$\zeta_2 > 0,\ \xi_{14} > 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.61.*

2: The possibility $\mathbf{D} > 0$. Then $Z_4 < 0$ and systems (42) possess only two real singularities: M_1 (located on the parabola $\Phi_1(x,y) = 0$) and M_2 (located on the parabola $\Phi_2(x,y) = 0$). As it was mentioned earlier the direction of the second invariant parabola depends on the sign of $g(1+g)$ (see (48)).

So considering the condition $\mathbf{D} > 0$ (i.e. $Z_4 < 0$) we arrive at the configuration *Config.* $\mathcal{P}$*.62* if $\zeta_2 < 0$ (i.e. $g(g+1) < 0$) and *Config.* $\mathcal{P}$*.63* if $\zeta_2 > 0$ (i.e. $g(g+1) > 0$).

3: The possibility $\mathbf{D} = 0$. Then considering (46), (44) and the condition $\zeta_5\mathcal{R}_2 \neq 0$ (i.e. $Z_4(1+4g+3g^2+4n) \neq 0$) we conclude that the condition $\mathbf{D} = 0$ implies $\alpha_4\beta_4 = 0$. We have the next lemma.

Lemma 5. *For systems* (42) *the condition* $\beta_4 = 0$ *could be brought via an affine transformation to the condition* $\alpha_4 = 0$.

Proof: We apply to systems (42) the transformation

$$x_1 = \delta x - \frac{(1+g)(1+4g+3g^2+4n)}{2Z_4}, \quad y_1 = \delta y - \frac{(1+g)(1+3g)(1+4g+3g^2+4n)}{4Z_4}, \qquad t_1 = 1/\delta, \quad \delta = -\frac{2g(1+g)}{Z_4} \tag{49}$$

and setting the notation

$$n_1 = -\frac{(1+g)(1+11g+31g^2+21g^3+4n+20gn)}{4(1+6g+5g^2+4n)} \Rightarrow \qquad n = -\frac{(1+g)(1+11g+31g^2+21g^3+4n_1+20gn_1)}{4(1+6g+5g^2+4n_1)} \tag{50}$$

we arrive at the family of systems

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4}(1+3g)(1+4g+3g^2+2n_1) + n_1x_1 - \frac{1}{2}(1+g)y_1 + gx_1^2 + x_1y_1, \\ \dot{y}_1 &= \frac{1}{2}(1+3g)(1+4g+3g^2+2n_1)\,x_1 + 2n_1y_1 + (g-1)x_1y_1 + 2y_1^2. \end{aligned} \tag{51}$$

We observe that this family of systems coincide with (42) up to notation of the variables and parameters. Considering (50) for the above systems we calculate:

$$\beta_4(g,n_1) = (1+g)(1+3g)(1+6g+7g^2) + 4(1+2g)^2 n_1 =$$
$$\frac{g^2(1+g)^2(5+22g+21g^2+4n)}{1+6g+5g^2+4n} = -\frac{g^2(1+g)^2\alpha_4(g,n)}{Z_4};$$
$$\alpha_4(g,n_1) = (5+22g+21g^2+4n_1) =$$
$$\frac{4(1+10g+34g^2+46g^3+21g^4+4n+16gn+16g^2n)}{1+6g+5g^2+4n} = -\frac{4\beta_4(g,n)}{Z_4}.$$

We observe that due to $g(g+1)Z_4 \neq 0$ the condition $\beta_4(g,n_1)=0$ (respectively $\alpha_4(g,n_1)=0$) for systems (51) implies $\alpha_4(g,n)=0$ (respectively $\beta_4(g,n)=0$) for systems (42). This completes the proof of the lemma. ∎

Thus in what follows we assume that the condition $\alpha_4 = 5+22g+21g^2+4n = 0$ holds and this gives us $n = -(1+3g)(5+7g)/4$. Then we obtain the following family of systems

$$\begin{aligned}\dot{x} &= -\frac{1}{8}(1+3g)^2(3+5g) + \frac{1}{4}(1+3g)(5+7g)\,x - \frac{1}{2}(1+g)\,y + gx^2 + xy,\\ \dot{y} &= -\frac{1}{4}(1+3g)^2(3+5g)\,x + \frac{1}{2}(1+3g)(5+7g)\,y + (g-1)xy + 2y^2\end{aligned} \tag{52}$$

possessing the two invariant parabolas: $\Phi_1(x,y) = x^2 - y = 0$ and

$$\Phi_2 = -(1+3g)^2(2+3g)(3+5g) - 4(1+g)(1+3g)(2+3g)x + 4g(1+g)x^2 + 8(1+2g)^2 y = 0.$$

Considering (45) we detect that for $\alpha_4 = 0$ the singular point M_4 coalesce with M_1 producing a double finite singularity. So we obtain that systems (52) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ (M_1 is double) with the coordinates

$$x_1 = -\frac{1+3g}{2},\ y_1 = \frac{(1+3g)^2}{4};\quad x_2 = \frac{(1+3g)(4+13g+11g^2)}{2g(1+g)},$$
$$y_2 = -\frac{(1+3g)^2(3+5g)}{4(1+g)};\quad x_3 = \frac{3+5g}{2},\quad y_3 = -\frac{(3+5g)^2}{4}.$$

Considering the investigation of the singularities of systems (42) we did earlier we deduce that the singular points $M_1(\equiv M_4)$ and M_3 are the points of intersection of the invariant parabolas whereas M_2 is located on the parabola $\Phi_2(x,y)=0$. We calculate

$$(x_2-x_1)(x_2-x_3) = \frac{(1+2g)^2(1+3g)(2+3g)(2+7g+7g^2)}{g^2(1+g)^2},$$
$$(x_2-x_1)+(x_2-x_3) = \frac{4(1+2g)^3}{g(1+g)}$$

and since $\text{Discrim}\,[2+7g+7g^2, g] = -7 < 0$ we obtain

$$\text{sign}\big((x_2-x_1)(x_2-x_3)\big) = \text{sign}\big((1+3g)(2+3g)\big),$$
$$\text{sign}\big((x_2-x_1)+(x_2-x_3)\big) = \text{sign}\big(g(1+g)(1+2g)\big).$$

We point out that $\text{sign}\,(1+2g)$ is necessary only if $(1+3g)(2+3g) > 0$.

On the other hand for systems (52) we have

$$\zeta_2 = 4g(1+g), \quad \zeta_5 = 19(g-2)(3+g)(1+3g)^2(2+3g)^2,$$
$$\xi_3 = 217993032\, g(1+g)(1+3g)^3(2+3g)^3(2+7g+7g^2)^2$$

and due to $\zeta_2\zeta_5 \neq 0$ we obtain

$$\operatorname{sign}(\xi_3) = \operatorname{sign}\big(g(1+g)(1+3g)(2+3g)\big), \quad \operatorname{sign}(\zeta_2) = \operatorname{sign}(g(1+g)).$$

We claim that the condition $\zeta_2 > 0$ implies $\xi_3 > 0$. Indeed assume $\zeta_2 > 0$ and suppose the contrary, that $\xi_3 < 0$. This implies $(1+3g)(2+3g) < 0$ i.e. $-2/3 < g < -1/3$ and therefore we get $-1 < g < 0$, i.e. $g(g+1) < 0$. This means $\zeta_2 < 0$ and this contradiction proves our claim.

Thus considering the above relations for systems (52) we obtain the following configurations:

$\zeta_2 < 0,\ \xi_3 < 0,\ 2g+1 < 0$ (i.e. $-1 < g < -2/3$) $\Rightarrow x_2 - x_1 > 0,\ x_2 - x_3 > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.64*;
$\zeta_2 < 0,\ \xi_3 < 0,\ 2g+1 > 0$ (i.e. $-1/3 < g < 0$) $\Rightarrow x_2 - x_1 < 0,\ x_2 - x_3 < 0 \Rightarrow \simeq$ *Config.* $\mathcal{P}$*.64*;
$\zeta_2 < 0,\ \xi_3 > 0$ (i.e. $-2/3 < g < -1/3$) $\Rightarrow (x_2 - x_1)(x_2 - x_3) < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.65.*
$\zeta_2 > 0,\ 2g+1 < 0$ (i.e. $g < -1$) $\Rightarrow x_2 - x_1 < 0,\ x_2 - x_3 < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.66*;
$\zeta_2 > 0,\ 2g+1 > 0$ (i.e. $g > 0$) $\Rightarrow x_2 - x_1 > 0,\ x_2 - x_3 > 0 \Rightarrow \simeq$ *Config.* $\mathcal{P}$*.66.*

We observe that the detected configurations do not depend $\operatorname{sign}(2g+1)$ and we arrive at the following lemma.

Lemma 6. *Assume that for systems* (42) *the condition* $\mathbf{D} = 0$ *holds. Then these sytems possess the following configurations if and only if the corresponding conditions are satisfied:*

$\zeta_2 < 0,\ \xi_3 < 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.64;*
$\zeta_2 < 0,\ \xi_3 > 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.65;*
$\zeta_2 > 0 \Leftrightarrow$ *Config.* $\mathcal{P}$*.66.*

3.1.2.2 The case $B_1 = 0$. Considering (44) and the condition $\zeta_2\zeta_3\mathcal{R}_2 \neq 0$ (i.e. $g(g+1)(2g+1)(1+6g+5g^2+4n) \neq 0$) we conclude that the condition $B_1 = 0$ is equivalent to

$$\begin{aligned}&(1+3g)(2+3g)(1+4g+3g^2+2n)(1+6g+6g^2+4n)\\&\times(1+6g+9g^2+4n)(5+14g+9g^2+4n)/32 = 0.\end{aligned} \tag{53}$$

However due to some transformations we could reduce the number of the cases provided by the condition $B_1 = 0$. We have the next lemma.

Lemma 7. *The condition* (53) *could be transferred via affine transformations and time rescaling to the condition*

$$(1+3g)(1+4g+3g^2+2n) = 0. \tag{54}$$

Proof: To proof this lemma we follow two steps: ***(i)*** we apply a transformation which replaces the line $y = 0$ with $y = x$ and keeps the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$ and ***(ii)*** we apply a transformation which transfers the invariant parabola $\Phi_2(x,y) = 0$ (see (43)) to the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$.

Step (i) Applying to systems (42) the transformation

$$x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4$$

we obtain the systems

$$\dot{x}_1 = -\frac{1}{8}(2+3g)(1+6g+6g^2+4n) + \frac{1}{4}(1+2g+4n)x_1 + \frac{g}{2}y_1 - (1+g)x_1^2 + x_1y_1,$$
$$\dot{y}_1 = -\frac{1}{4}(2+3g)(1+6g+6g^2+4n)x_1 + \frac{1}{2}(1+2g+4n)y_1 - (g+2)x_1y_1 + 2y_1^2.$$

Then setting the new parameters

$$n_1 = \frac{1}{4}(1+2g+4n), \quad g_1 = -(1+g) \ \Rightarrow$$
$$n = \frac{1}{4}(1+2g_1+4n_1) \quad g = -(1+g_1), \tag{55}$$

we obtain the family of systems

$$\dot{x}_1 = \frac{1}{4}(1+3g_1)(1+4g_1+3g_1^2+2n_1) + n_1x_1 - \frac{1+g_1}{2}y_1 + g_1x_1^2 + x_1y_1,$$
$$\dot{y} = \frac{1}{2}(1+3g_1)(1+4g_1+3g_1^2+2n_1)x_1 + 2n_1y_1 + (g_1-1)x_1y_1 + 2y_1^2$$

which coincide with family (42) (up to notations). Then considering (55) calculations yield:

$$2+3g = -(1+3g_1), \quad 1+6g+6g^2+4n = 2(1+4g_1+3g_1^2+2n_1),$$
$$5+14g+9g^2+4n = 1+6g_1+9g_1^2+4n_1$$

and clearly we reduce the condition (53) to the condition

$$(1+3g)(1+4g+3g^2+2n)(1+6g+9g^2+4n) = 0.$$

Step (ii) As it was shown in the proof of Lemma 5 via the transformation (49) systems (42) can be brought to the same canonical form (51) but with a new parameter n_1 of the form (50). Then calculations yield

$$1+6g+9g^2+4n_1 = \frac{8g^2(1+4g+3g^2+2n)}{4(1+6g+5g^2+4n)}$$

and we conclude that due to $g \neq 0$ the condition $1+6g+9g^2+4n = 0$ could be transferred to $1+4g+3g^2+2n = 0$. As a result we arrive at the condition (54) and this completes the proof of Lemma 7. ∎

For systems (42) we calculate

$$\xi_{15} = 2(1+3g)(2+3g)$$

and we discuss two possibilities: $\xi_{15} \neq 0$ and $\xi_{15} = 0$.

1: *The possibility* $\xi_{15} \neq 0$. Then $1+3g \neq 0$ and considering (54) and Lemma 7 we deduce that the condition $B_1 = 0$ implies $1+4g+3g^2+2n = 0$.

This yields $n = -(1+g)(1+3g)/2$ and we obtain the following 1-parameter family of systems

$$\dot{x} = -\frac{1}{2}(1+g)(1+3g)x - \frac{1}{2}(g+1)y + gx^2 + xy,$$
$$\dot{y} = -y(1+4g+3g^2+x-gx-2y) \tag{56}$$

which besides the invariant parabolas

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=(1+g)(1+3g)x-2gx^2-(1+g)y=0$$

possesses the invariant line $y=0$. Considering Lemmas 1 and 2 we calculate

$$\theta=-8(g-1)(2+g),\quad B_2=243g(1+g)^4(1+2g)^2(2+3g)y^4/2. \tag{57}$$

So we examine the cases $B_2\neq 0$ and $B_2=0$.

2.1: *The case* $B_2\neq 0$. Then by Lemma 1 we could not have invariant lines in other direction than $y=0$. However by Lemma 2 we could have parallel invariant lines if $\theta=0$.

2.1.1: *The subcase* $\theta\neq 0$. We determine that systems (56) possess four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ with the coordinates

$$x_1=0,\ y_1=0;\quad x_2=\frac{(1+g)(1+3g)}{2g},\ y_2=0;\quad x_3=1+g,\ y_3=(1+g)^2,$$
$$x_4=-\frac{1+3g}{2},\ y_4=\frac{(1+3g)^2}{4}.$$

Considering the conditions provided by the statement $(\boldsymbol{\mathcal{A}_2})$ for systems (56) we have:

$$\zeta_1\zeta_2\zeta_3\zeta_5\mathcal{R}_2\neq 0\ \Leftrightarrow\ g(1+g)(g-2)(3+g)(1+2g)(1+3g)(8+27g+27g^2)\neq 0. \tag{58}$$

We observe that the invariant parabolas have two points of intersection: M_1 and M_3. Moreover we observe that the invariant line $y=0$ has the contact point M_1 with the parabola $\Phi_1(x,y)=0$ and two points of intersection M_1 and M_2 with the parabola $\Phi_2(x,y)=0$.

So three finite singularities are fixed as the intersections of invariant curves and their positions are determined by the values of the parameter g.

On the other hand the singular point M_4 is located on the invariant parabola $\Phi_1(x,y)=0$ and it is floating. So we need to determine its position with respect to the other two singularities located on the same invariant curve. So we calculate:

$$(x_4-x_1)(x_4-x_3)=\frac{1}{4}(1+3g)(3+5g),\quad (x_4-x_1)+(x_4-x_3)=-2(1+2g).$$

Therefore we obtain

$$\operatorname{sign}\big((x_4-x_1)(x_4-x_3)\big)=\operatorname{sign}\big((1+3g)(3+5g)\big),$$
$$\operatorname{sign}\big((x_4-x_1)+(x_4-x_3)\big)=-\operatorname{sign}(1+2g).$$

We observe that the direction of the second invariant parabola depends on $\operatorname{sign}(g(g+1))$.

On the other hand for systems (56) calculations yields:

$$\zeta_2=4g(g+1),\quad \xi_{16}=\frac{3705}{2}g^2(1+g)^2(1+3g)(3+5g),$$
$$\xi_{17}=\frac{3705}{4}(1+g)^2(1+2g)(1+3g)(3+5g)$$

and hence we have

$$\operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(g+1)\big),\ \operatorname{sign}(\xi_{16}) = \operatorname{sign}\big((1+3g)(3+5g)\big),$$
$$\operatorname{sign}(\xi_{17}) = \operatorname{sign}\big((1+2g)(1+3g)(3+5g)\big).$$

Thus we determine the following configurations:

$\zeta_2<0,\ \xi_{16}<0$ (i.e. $-3/5<g<-1/3$) $\Rightarrow (x_4-x_1)(x_4-x_3)<0 \Rightarrow$ *Config.* $\mathcal{P}.67$;
$\zeta_2<0,\ \xi_{16}>0,\ \xi_{17}<0$ (i.e. $-1<g<-3/5$) $\Rightarrow x_4>x_1,\ x_4>x_3 \Rightarrow$ *Config.* $\mathcal{P}.68$;
$\zeta_2<0,\ \xi_{16}>0,\ \xi_{17}>0$ (i.e. $-1/3<g<0$) $\Rightarrow x_4<x_1,\ x_4<x_3 \Rightarrow$ *Config.* $\mathcal{P}.69$;
$\zeta_2>0,\ \xi_{17}<0$ (i.e. $g<-1$) $\Rightarrow x_4>x_1,\ x_4>x_3 \Rightarrow$ *Config.* $\mathcal{P}.70$;
$\zeta_2>0,\ \xi_{17}>0$ (i.e. $g>0$) $\Rightarrow x_4<x_1,\ x_4<x_3 \Rightarrow$ *Config.* $\mathcal{P}.71$.

2.1.2: *The subcase* $\theta=0$. This condition implies $(g-1)(g+2)=0$ and since for systems (56) we have

$$\zeta_6=(g-1)(1+g)(2+5g+5g^2)/8$$

we consider two possibilities: $\zeta_6\neq0$ and $\zeta_6=0$.

2.1.2.1: *The possibility* $\zeta_6\neq0$. In this case the condition $\theta=0$ implies $g=-2$. Then we get the system

$$\dot{x}=-\frac{5x}{2}+\frac{y}{2}-2x^2+xy,\quad \dot{y}=-y(5+3x-2y)$$

which besides the invariant parabolas

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=5x+4x^2+y=0$$

possesses only one invariant line $y=0$. This means that the condition $g=-2$ does not imply the appearance of an additional parallel invariant line. So since we have $g=-2<-1$ we arrive at the configuration *Config.* $\mathcal{P}.70$ (detected above).

2.1.2.2: *The possibility* $\zeta_6=0$. Then $\theta=0$ implies $g=1$ and we arrive at the system

$$\dot{x}=-4x-y+x^2+xy,\quad \dot{y}=2y(y-4),$$

possessing the invariant lines $y=0$ and $y=4$ as well as the invariant parabolas

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=-4x+x^2+y=0.$$

In this case we obtain the configuration *Config.* $\mathcal{P}.72$.

2.2: *The case* $B_2=0$. Considering (57) and (58) the condition $B_2=0$ implies $g=-2/3$ and we arrive at the system

$$\dot{x}=\frac{x}{6}-\frac{y}{6}-\frac{2x^2}{3}+xy,\quad \dot{y}=\frac{1}{3}y(1-5x+6y). \tag{59}$$

possessing the invariant lines $y=0$ and $y=x-1/4$ as well as the invariant parabolas

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=x-4x^2+y=0.$$

We determine that the invariant line $y=x-1/4$ is tangent to the invariant parabola $\Phi_1(x,y)=0$ at the point $M_4(1/2,1/4)$ as well as to the parabola $\Phi_2(x,y)=0$ at the point $M_2(1/4,0)$. So it is not too difficult to find out that in this case we get configuration *Config.* $\mathcal{P}.73$.

Thus in the case $B_1=0$ and $\xi_{15}\neq0$ we proved the following lemma.

Lemma 8. *Assume that for systems* (42) *the conditions* $B_1 = 0$ *and* $\xi_{15} \neq 0$ *hold. Then these systems possess the following configurations if the corresponding conditions are satisfied:*

$B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} < 0 \Rightarrow$ *Config.* $\mathcal{P}.67$*;*
$B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} > 0,\ \xi_{17} < 0 \Rightarrow$ *Config.* $\mathcal{P}.68$*;*
$B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} > 0,\ \xi_{17} > 0 \Rightarrow$ *Config.* $\mathcal{P}.69$*;*
$B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 > 0,\ \xi_{17} < 0\ \xi_{17} > 0 \Rightarrow$ *Config.* $\mathcal{P}.70$*;*
$B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 > 0,\ \xi_{17} > 0\ \xi_{17} > 0 \Rightarrow$ *Config.* $\mathcal{P}.71$*;*
$B_2 \neq 0,\ \theta = 0,\ \zeta_6 \neq 0 \Rightarrow$ *Config.* $\mathcal{P}.70$*;*
$B_2 \neq 0,\ \theta = 0,\ \zeta_6 = 0 \Rightarrow$ *Config.* $\mathcal{P}.72$*;*
$B_2 = 0 \Rightarrow$ *Config.* $\mathcal{P}.73$*.*

2: *The possibility* $\xi_{15} = 0$. Then considering the proof of Lemma 7 we may assume $1+3g = 0$. Then $g = -1/3$ and we arrive at the 1-parameter family of systems

$$\dot{x} = nx - \frac{y}{3} - \frac{x^2}{3} + xy, \quad \dot{y} = \frac{2}{3}y(3n - 2x + 3y), \tag{60}$$

which besides the invariant parabolas

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 9n^2 - 6nx + x^2 + (9n-1)y = 0,$$

possesses the invariant line $y = 0$. Considering Lemmas 1 and 2 we calculate

$$B_2 = -8(1+3n)(9n-1)(12n-1)y^4/9, \quad \mathbf{D} = 4096n^6(9n-1)/243,$$
$$\theta = 160/9 \neq 0, \quad \zeta_5 = -4256n^2/9, \quad \mathcal{R}_2 = 28(9n-1)/81.$$

So we discuss the cases $B_2 \neq 0$ and $B_2 = 0$.

2.1: *The case* $B_2 \neq 0$. Then by Lemmas 1 and 2 we could not have another invariant line.

On the other hand considering (45) systems (60) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$) with the coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_2 = 3n,\ y_2 = 0; \quad x_{3,4} = \frac{1}{3}\big(1 \pm \sqrt{1-9n}\,\big),\ y_{3,4} = \frac{1}{9}\big(2 - 9n \pm 2\sqrt{1-9n}\,\big).$$

We observe that in this case we have $n(1-9n) \neq 0$ due to $\zeta_5\mathcal{R}_2 \neq 0$ and we conclude that all finite singularities are distinct. Moreover we determine that the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x,y) = 0$ at the singular point $M_1(0,0)$ as well as to the parabola $\Phi_2(x,y) = 0$ at $M_2(3n, 0)$.

Since $\mathbf{D} \neq 0$ and $\text{sign}\,(1-9n) = -\text{sign}\,(\mathbf{D})$ we examine two subcases: $\mathbf{D} < 0$ and $\mathbf{D} > 0$.

2.1.1: *The subcase* $\mathbf{D} < 0$. Then $1 - 9n > 0$ (i.e. $n < 1/9$) and we arrive at the unique configuration *Config.* $\mathcal{P}.74$ independently of the position of the singularity $M_2(3n, 0)$ with respect of $M_1(0,0)$.

2.1.2: *The subcase* $\mathbf{D} > 0$. This implies $1 - 9n < 0$, i.e. the singularities M_3 and M_4 are complex and in this case we arrive at the configuration *Config.* $\mathcal{P}.75$.

2.2: *The case* $B_2 = 0$. Since $9n - 1 \neq 0$ due to $\mathcal{R}_2 \neq 0$ the condition $B_2 = 0$ yields

$$(1 + 3n)(12n - 1) = 0.$$

1.2.1: *The subcase* $1 + 3n = 0$. Then $n = -1/3$ and this leads to the system

$$\dot{x} = -\frac{1}{3}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{2}{3}y(-1 - 2x + 3y).$$

Then applying the transformation $x_1 = (1 - x)/4$, $y_1 = (y - x)/4$, $t_1 = 4t$ we get system (59) possessing the configuration *Config.* $\mathcal{P}.73$.

1.2.2: *The subcase* $12n - 1 = 0$. This implies $n = 1/12$ and we arrive at the system

$$\dot{x} = \frac{1}{12}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{1}{6}y(1 - 8x + 12y),$$

which via the affine transformation $x_1 = -x + 1/2$, $y_1 = y - x + 1/4$ can be brought to the system (59) possessing the configuration *Config.* $\mathcal{P}.73$.

3.1.3 The statement ($\mathcal{A}_3$)

In this case the conditions $\zeta_4 = \zeta_5 = 0$ and considering (44) and the condition $\zeta_1 \neq 0$ (i.e. $(g - 2)(g + 3)$) we get the condition

$$1 + 4g + 3g^2 + 4n = 0 \;\Rightarrow\; n = -\frac{1}{4}(1 + g)(1 + 3g).$$

This leads to the family of systems

$$\begin{aligned} \dot{x} =& \frac{1}{8}(1 + g)(1 + 3g)^2 - \frac{1}{4}(1 + g)(1 + 3g)x - \frac{1}{2}(1 + g)y + gx^2 + xy, \\ \dot{y} =& \frac{1}{4}(1 + g)(1 + 3g)^2\, x - \frac{1}{2}(1 + g)(1 + 3g)y + (g - 1)xy + 2y^2 \end{aligned} \tag{61}$$

possessing the parabola $\Phi(x, y) = x^2 - y = 0$ which is of multiplicity 2.

Following the statement **($\mathcal{A}_3$)** for the above systems we calculate

$$\begin{aligned} \zeta_1 =& 2(g - 2)(3 + g), \quad \zeta_2 = 4g(1 + g), \quad \zeta_3 = 8(1 + 2g)^2, \quad \zeta_4 = \zeta_5 = 0, \\ \mathcal{R}_2 =& -g(1 + g)(g - 2)(3 + g)(8 + 27g + 27g^2)/8, \\ B_1 =& g^4(1 + g)^4(1 + 2g)(1 + 3g)^3(2 + 3g)^3/8. \end{aligned} \tag{62}$$

Therefore since the quadratic polynomial $8 + 27g + 27g^2$ has negative discriminant, for systems (61) we have the condition:

$$\zeta_1\zeta_2\zeta_3\mathcal{R}_2 \neq 0 \;\Rightarrow\; g(1 + g)(g - 2)(3 + g)(1 + 2g) \neq 0. \tag{63}$$

According to Lemma 1 for the existence of an invariant line of systems (61) the condition $B_1 = 0$ is necessary. So we discuss two cases: $B_1 \neq 0$ and $B_1 = 0$.

3.1.3.1 The case $B_1 \neq 0$. Then we could not have any invariant line. We determine that systems (61) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with the coordinates

$$x_1 = -\frac{1+3g}{2},\ y_1 = \frac{(1+3g)^2}{4};\quad x_{2,3} = \frac{1}{2}\big(1 + g \pm \sqrt{-2g(1+g)}\,\big),$$
$$y_{2,3} = \frac{g+1}{4}\big(1 - g \pm 2\sqrt{-2g(1+g)}\,\big). \tag{64}$$

We point out that M_1 is a multiple singularity of systems (61). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\dot{x} = -\frac{1}{2}g(3g+1)x - (2g+1)y + gx^2 + xy,$$
$$\dot{y} = \frac{1}{2}g(3g+1)^2x + (2g+1)(3g+1)y + (g-1)xy + 2y^2,$$

where $M_0(0, 0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$ and

$$\mu_2 = \frac{1}{2}g(g+1)(3g+1)(3g+2)\big[g(1+3g)x^2 + 4gxy + 2y^2\big] \neq 0,$$

due to the condition $B_1 \neq 0$. By [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity exactly 2.

We calculate

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0$$

and clearly all three singularities are located on the invariant parabola. On the other hand the singular points M_2 and M_3 could be either complex or real or coinciding, depending on the value of the product $g(g+1) \neq 0$ (due to $\zeta_2 \neq 0$). Since $\zeta_2 = 4g(g+1)$ we consider two subcases: $\zeta_2 < 0$ and $\zeta_2 > 0$.

1: *The subcase* $\zeta_2 < 0$. This implies $g(g+1) < 0$, i.e. $-1 < g < 0$. In this case all three singularities located on the invariant parabola are real and we need to determine the position of the double singularity M_1 with respect to the simple singularities M_2 and M_3. So considering (64) we calculate:

$$(x_2 - x_1)(x_3 - x_1) = (1+3g)(2+3g)/2 \ \Rightarrow\ \text{sign}\big((x_2 - x_1)(x_3 - x_1)\big) = \text{sign}\big((1+3g)(2+3g)\big).$$

On the other hand for systems (61) we calculate

$$\xi_{15} = 2(1+3g)(2+3g) \neq 0$$

due to $B_1 \neq 0$. Therefore in the case $\xi_{15} < 0$ the double point M_1 is located between the singularities M_2 and M_3 and we arrive at the configuration *Config.* $\mathcal{P}$*.76*.

In the case $\xi_{15} > 0$ the double point M_1 is located outside the curvilinear interval (M_2, M_3) and we get the configuration *Config.* $\mathcal{P}$*.77*.

2: *The subcase* $\zeta_2 > 0$ Then $g(g+1) > 0$ and clearly the singularities M_2 and M_3 are complex. In this case evidently we can get the unique configuration *Config.* $\mathcal{P}$*.78*.

3.1.3.2 The case $B_1 = 0$. Considering (62) and the condition (63) we conclude that the condition $B_1 = 0$ is equivalent to $(1+3g)(2+3g) = 0$.

If $g = -1/3$ then we arrive at the system

$$\dot{x} = -(y + x^2 - 3xy)/3, \quad \dot{y} = -2y(2x - 3y)/3 \tag{65}$$

possessing the invariant line $y = 0$ which is tangent to the double invariant parabola at the singular point $M_1(0,0)$. Moreover in this case the singular point M_3 coalesced with M_1 producing a triple singularity. As a result we get the configuration *Config.* $\mathcal{P}.79$.

Assume now $g = -2/3$. Then we get the system

$$\dot{x} = (1 + 2x - 4y - 16x^2 + 24xy)/24, \quad \dot{y} = (x + 2y - 20xy + 24y^2)/12$$

which via the affine transformation $x_1 = -x + 1/2$, $y_1 = y - x + 1/4$ can be brought to the system (65) possessing the configuration *Config.* $\mathcal{P}.79$.

Thus we have proved the following lemma.

Lemma 9. *Assume that for a quadratic system the conditions* $(\boldsymbol{\mathcal{A}_3})$ *are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$B_1 \neq 0,\ \zeta_2 < 0,\ \xi_{15} < 0 \ \Rightarrow$ *Config.* $\mathcal{P}.76$;
$B_1 \neq 0,\ \zeta_2 < 0,\ \xi_{15} > 0 \ \Rightarrow$ *Config.* $\mathcal{P}.77$;
$B_1 \neq 0,\ \zeta_2 > 0,\ \xi_{15} > 0 \ \Rightarrow$ *Config.* $\mathcal{P}.78$;
$B_1 = 0 \ \Rightarrow$ *Config.* $\mathcal{P}.79$.

3.1.4 The statement $(\boldsymbol{\mathcal{A}_4})$

In this case the condition $\zeta_4 = \mathcal{R}_2 = 0$ holds and considering (44) and the condition $\zeta_1 \neq 0$ (i.e. $(g-2)(g+3) \neq 0$) we get $(8 + 27g + 27g^2)(1 + 6g + 5g^2 + 4n) = 0$. However the discriminant of the quadratic polynomial $8 + 27g + 27g^2$ equals $-135 < 0$. So we obtain the condition

$$1 + 6g + 5g^2 + 4n = 0 \ \Rightarrow \ n = -\frac{1}{4}(1+g)(1+5g).$$

This leads to the family of systems

$$\begin{aligned} \dot{x} =& \frac{1}{8}(1 + g - 2x)(1 + 4g + 3g^2 - 4gx - 4y), \\ \dot{y} =& \frac{1}{4}(1+g)^2(1+3g)\,x - \frac{1}{2}(1+g)(1+5g)y + (g-1)xy + 2y^2, \end{aligned} \tag{66}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$ and the invariant line $x = (g+1)/2$.

Following the statement $(\boldsymbol{\mathcal{A}_4})$ for the above systems we calculate

$$\begin{aligned} \zeta_1 =& 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g), \quad \zeta_3 = 8(1+2g)^2, \\ \zeta_4 =& 0 = \mathcal{R}_2, \quad \zeta_5 = 19(g-2)g^2(1+g)^2(3+g), \\ B_1 =& 0, \quad B_2 = -648g^5(1+g)^5(1+3g)(2+3g)x^4. \end{aligned} \tag{67}$$

Therefore for systems (61) we have the condition:

$$\zeta_1\zeta_2\zeta_3\zeta_5 \neq 0 \;\Rightarrow\; g(1+g)(g-2)(3+g)(1+2g) \neq 0. \tag{68}$$

We discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. In this case by Lemma 1 systems (66) could not possess invariant lines in other directions than the invariant line $x = (g+1)/2$.

On the other hand by Lemma 2 these systems could possess an invariant line parallel to the existent one if $\theta = (g-1)(g+2) = 0$. However a straightforward calculation shows us that neither the condition $g = 1$ nor $g = -2$ does not imply the appearance of an additional parallel invariant line.

Next we determine that systems (66) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with the coordinates

$$x_1 = \frac{1+g}{2},\; y_1 = \frac{(1+g)^2}{4};\quad x_2 = \frac{1+g}{2},\; y_2 = \frac{1}{4}(1+g)(1+3g);$$
$$x_3 = -\frac{1+3g}{2},\; y_3 = \frac{(1+3g)^2}{4}.$$

It is not difficult to detect that the invariant line $x = (g+1)/2$ intersect the invariant parabola at the singular point M_1.

We point out that M_1 is a multiple singularity of systems (66). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\dot{x} = gx^2 + xy, \qquad \dot{y} = g(1+g)^2x - g(1+g)y + (-1+g)xy + 2y^2,$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$ and

$$\mu_2 = g^2(1+g)^2(1+2g)x(gx+y) \neq 0,$$

due to the condition (68). By [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity exactly 2.

On the other hand the singularity M_2 is located on the invariant line whereas M_3 is located on the invariant parabola and both these singularities are floating. So we need to determine the position of these points with respect to the double singularity M_1. So we calculate:

$$y_2 - y_1 = g(1+g)/2 \;\Rightarrow\; \operatorname{sign}(y_2 - y_1) = \operatorname{sign}\big(g(g+1)\big),$$
$$x_3 - x_1 = -(1+2g) \;\Rightarrow\; \operatorname{sign}(x_3 - x_1) = -\operatorname{sign}(1+2g),$$

and we observe that for systems (66) we have $\zeta_2 = 4g(g+1)$ and hence $\operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(g+1)\big)$.

Thus we arrive at the following configurations:

$\zeta_2 < 0,\ 2g+1 < 0$ (i.e. $-1 < g < -1/2$) $\Rightarrow$ $y_2 < y_1,\ x_3 > x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.80*;
$\zeta_2 < 0,\ 2g+1 > 0$ (i.e. $-1/2 < g < 0$) $\Rightarrow$ $y_2 < y_1,\ x_3 < x_1$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.80*;
$\zeta_2 > 0,\ 2g+1 < 0$ (i.e. $g < -1$) $\Rightarrow$ $y_2 > y_1,\ x_3 > x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$*.81*;
$\zeta_2 > 0,\ 2g+1 > 0$ (i.e. $g > 0$) $\Rightarrow$ $y_2 > y_1,\ x_3 < x_1$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.81.*

2: The possibility $B_2 = 0$. Considering (67) and the condition (68) we conclude that the condition $B_2 = 0$ is equivalent to $(1 + 3g)(2 + 3g) = 0$.

If $g = -1/3$ then we arrive at the system

$$\dot{x} = -(3x - 1)(x - 3y)/9, \quad \dot{y} = 2y(1 - 6x + 9y)/9 \tag{69}$$

possessing additionally the invariant line $y = 0$ which is tangent to the invariant parabola at the singular point $M_3(0, 0)$ and intersects the invariant line $x = 1/3$ at the singular point $M_2(1/3, 0)$. Therefore we obtain the configuration *Config.* $\mathcal{P}$*.82*.

Assume now $g = -2/3$. This leads to the system

$$\dot{x} = -(6x - 1)(8x - 1 - 12y)/72, \quad \dot{y} = (-x + 14y - 60xy + 72y^2)/36$$

which via the affine transformation $x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4$ we could be brought to system (69) possessing the configuration *Config.* $\mathcal{P}$*.82*.

Thus we have proved the following lemma.

Lemma 10. *Assume that for a quadratic system the conditions* $(\boldsymbol{\mathcal{A}_4})$ *are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$B_2 \neq 0, \ \zeta_2 < 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}$*.80;*
$B_2 \neq 0, \ \zeta_2 > 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}$*.81;*
$B_2 = 0, \ \zeta_2 > 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}$*.82.*

3.1.5 The statement $(\boldsymbol{\mathcal{A}_5})$

According to Proposition 2 the condition $\zeta_3 = 0$ must be fulfilled. Considering (8) we get $g = -1/2$ and we arrive at the 2-parameter family of systems

$$\dot{x} = m + nx - y/4 - x^2/2 + xy, \quad \dot{y} = 2mx + 2ny - 3xy/2 + 2y^2 \tag{70}$$

possessing the parabola $\Phi(x, y) = x^2 - y = 0$. For these systems we calculate

$$\begin{aligned} \zeta_1 &= -25/2, \quad \zeta_2 = -1, \quad \zeta_3 = 0, \quad \zeta_4 = 25(32m + 8n - 1)/512, \\ \mathcal{R}_1 &= 375(32m + 8n - 1)/256, \\ B_1 &= m(16m - 4n - 1)(4m + 3n)(16m + 8n - 1)(32m + 8n - 1)/512. \end{aligned} \tag{71}$$

3.1.5.1 The case $B_1 \neq 0$. We observe that the family of systems (70) is a subfamily of (7) defined by the condition $g = -1/2$. Therefore it is clear that systems (70) possess four finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ $(i = 1, 2, 3, 4)$ where considering (10) we have

$$\tilde{x}_i = x_i\big|_{\{g=-1/2\}}, \quad \tilde{y}_i = y_i\big|_{\{g=-1/2\}}, \quad i = 1, 2, 3, 4.$$

As it was proved for the family (7) in the case $\mathbf{D} \neq 0$ these systems could possess only two distinct configurations: *Config.* $\mathcal{P}$*.1* if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}$*.2* if $\mathbf{D} > 0$. The same two configurations could be obtained in the particular case $g = -1/2$, because this value of the parameter g is not a bifurcation value for the mentioned two configurations.

Assuming $\mathbf{D} = 0$ we get $\mathcal{F}_1\mathcal{F}_2 = 0$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given in (9). So taking $g = -1/2$ we follow the examination of the two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: The possibility $\xi_1 \neq 0$. Then $\mathcal{F}_1 \neq 0$ and hence the condition $\mathbf{D} = 0$ implies $\mathcal{F}_2 = 0$. In this case we arrive at the systems (11) which for $g = -1/2$ become

$$\begin{aligned} \dot{x} =& \frac{(3+2v)^2(4v-3)}{1728} - \frac{4v^2-9}{48}x - \frac{1}{4}y - \frac{1}{2}x^2 + xy, \\ \dot{y} =& \frac{(3+2v)^2(4v-3)}{864}x - \frac{4v^2-9}{24}y - \frac{3}{2}xy + 2y^2. \end{aligned} \tag{72}$$

For the above systems we calculate

$$\xi_1 = 25 \cdot 2^{-12}3^{-8}v^3(2v-3)^2(3+2v)^2(9+2v^2), \quad \xi_2 = 2^{-11}3^{-8}v^2(2v-3)^2(3+2v)^2(9+2v^2)^2,$$

and clearly the condition $\xi_1 \neq 0$ implies $\xi_2 \neq 0$. Therefore following the examination of the 2-parameter family of systems (11) we conclude that the 1-parameter family of systems (72) could possess the unique configuration *Config.* $\mathcal{P}$*.3*.

2: The possibility $\xi_1 = 0$. Then $\mathcal{F}_1 = 0$ and we arrive at the systems (14) which for $g = -1/2$ become

$$\begin{aligned} \dot{x} =& \frac{(1+2u)^2}{64} + \frac{1-4u^2}{16}x - \frac{1}{4}y - \frac{1}{2}x^2 + xy, \\ \dot{y} =& \frac{(1+2u)^2}{32}x + \frac{1-4u^2}{8}y - \frac{3}{2}xy + 2y^2, \end{aligned}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$. For these systems we calculate:

$$\xi_2 = (4u^2-1)^2(2+u^2)/2048, \quad \xi_3 = \frac{27249129}{2048}(4u^2-1)^3(2+u^2)$$
$$B_1 = 2^{-18}u(u^2-1)(4u^2-1)^3$$

and due to the condition $B_1 \neq 0$ we have $\xi_2 > 0$ and $\xi_3 \neq 0$.

So according to the investigation done for systems (14) in the case $g = -1/2$ we get the configuration *Config.* $\mathcal{P}$*.6* for $\xi_3 < 0$ and *Config.* $\mathcal{P}$*.7* for $\xi_3 > 0$.

3.1.5.2 The case $B_1 = 0$.

Considering (71) and the condition $\zeta_4 \neq 0$ we deduce that $B_1 = 0$ is equivalent to

$$m(16m - 4n - 1)(4m + 3n)(16m + 8n - 1) = 0.$$

However considering Lemma 3 we have the following corollary.

Corollary 1. [Lemma 3] *The condition* $(16m - 4n - 1)(16m + 8n - 1) = 0$ *for systems* (70) *could be transferred to the condition* $m(4m + 3n) = 0$ *via an affine transformation.*

It is important to point out that in the proof of Lemma 3 the transformed systems the parameter $g_1 = -(1 + g)$ (see (18)) in the case $g = -1/2$ we get $g_1 = -1/2$ and therefore the homogeneous quadratic part of systems (70) is conserved.

Taking into account the above corollary we conclude that it is sufficient to consider the condition

$$m(4m+3n)=0. \tag{73}$$

For systems (70) we have

$$\zeta_4 = 25(32m+8n-1)/512, \quad \xi_4 = 328125m(16m+8n-1)(32m+8n-1)/16 \tag{74}$$

and we consider two subcases: $\xi_4 \neq 0$ and $\xi_4 = 0$.

3.1.5.2.1 The subcase $\xi_4 \neq 0$. Then $m \neq 0$ and from (73) we obtain $4m+3n=0$. This implies $m=-3n/4$ and we arrive at the family of systems

$$\dot{x} = -\,3n/4 + nx - y/4 - x^2/2 + xy, \quad \dot{y} = -(n+y)(3x-4y)/2 \tag{75}$$

which is a subfamily of (25) defined by the condition $g=-1/2$. Considering (27) for $g=-1/2$ we obtain the following four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ of systems (25) with the coordinates

$$x_1=\sqrt{-n},\ y_1=-n;\quad x_2=-\sqrt{-n},\ y_2=-n;\quad x_3=\frac{3}{4},\ y_3=\frac{9}{16};$$
$$x_4=-4n,\ y_4=-3n.$$

As it was mentioned for systems (25) the finite singular points M_1, M_2 and M_3 are located on the invariant parabola $\Phi(x,y)=x^2-y=0$. Moreover M_1 and M_2 are the points of intersection of the invariant line $y=-n$ with the parabola.

On the other hand we have $\Phi(x_4,y_4)=n(3+16n)$. Therefore we conclude that M_4 could be located on the invariant parabola if and only if either $n=0$ or $n=-3/16$.

For systems (75) we calculate

$$B_2 = -81(1+4n)(1+16n)^2y^4/128, \quad \xi_4 = -984375n(1+4n)(1+16n)/64$$

and due to $\xi_4 \neq 0$ we have $B_2 \neq 0$. We observe that due to $\xi_4 \neq 0$ (i.e. $n \neq 0$) the singularities M_1 and M_2 could be either complex or real. So we calculate

$$\mathbf{D} = 3n^3(3+16n)^2(9+16n)^2/256$$

and hence in the case $\mathbf{D} \neq 0$ we have $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(n)$. Moreover for $\mathbf{D} \neq 0$ we have $n(3+16n) \neq 0$ and hence the singular point M_4 could not be located on the invariant parabola. So we discuss three possibilities: $\mathbf{D}<0$, $\mathbf{D}>0$ and $\mathbf{D}=0$.

***1:** The possibility* $\mathbf{D}<0$. This implies $n<0$ and then the finite singularities M_1 and M_2 are real. In order to determine the position of M_3 with respect to M_1 and M_2 we calculate

$$(x_3-x_1)(x_3-x_2) = (9+16n)/16 \ \Rightarrow\ \operatorname{sign}\big((x_3-x_1)(x_3-x_2)\big) = \operatorname{sign}(9+16n),$$
$$(x_3-x_1)+(x_3-x_2) = 3/2 > 0.$$

Therefore we conclude that in the case $(x_3-x_1)(x_3-x_2)>0$ we have $x_3-x_1>0$ and $x_3-x_2>0$.

On the other hand for systems (75) calculations yield

$$\xi_9 = \frac{698726655}{2048}(1+16n)^2(9+16n)$$

and since $\xi_9 \neq 0$ (due to $\mathbf{D}\zeta_4 \neq 0$) we obtain $\operatorname{sign}(\xi_9) = \operatorname{sign}(9+16n)$.

So in the case $\mathbf{D} < 0$ we arrive at the configuration *Config.* $\mathcal{P}$*.17* if $\xi_9 < 0$ and *Config.* $\mathcal{P}$*.18* if $\xi_9 > 0$.

2: *The possibility* $\mathbf{D} > 0$. Then $n > 0$ and the singularities M_1 and M_2 are complex. Since due to $\mathbf{D} > 0$ (i.e. $n(3+16n) \neq 0$) the singularity M_4 does not lie on the invariant parabola we get the configuration *Config.* $\mathcal{P}$*.20*.

3: *The possibility* $\mathbf{D} = 0$. Since $\xi_4 \neq 0$ we have $n \neq 0$ and therefore the condition $\mathbf{D} = 0$ implies $(3+16n)(9+16n) = 0$. We observe that $\xi_9 = 0$ if and only if $9+16n = 0$ because $1+16n \neq 0$ due to $\xi_4 \neq 0$.

Thus we arrive at the configuration *Config.* $\mathcal{P}$*.23* if $\xi_9 \neq 0$ and *Config.* $\mathcal{P}$*.26* if $\xi_9 = 0$.

3.1.5.2.2 The subcase $\xi_4 = 0$. Then considering (74) and the condition $\zeta_4 \neq 0$ we obtain $m(16m+8n-1) = 0$. Taking into consideration Corollary 1 it is sufficient to examine the case $m = 0$. Then we obtain the 1-parameter family of systems

$$\dot{x} = nx - y/4 - x^2/2 + xy, \quad \dot{y} = y(4n - 3x + 4y)/2 \tag{76}$$

which is a subfamily of (31) defined by the condition $g = -1/2$. Considering (34) for $g = -1/2$ we obtain that the above systems possess the following four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = 0,\ y_1 = 0;\quad x_2 = 2n,\ y_2 = 0;\quad x_{3,4} = \frac{1}{8}(3 \pm \sqrt{9-64n}),\ y_{3,4} = \frac{1}{32}(9 - 32n \pm 3\sqrt{9-64n}).$$

As it was mentioned for systems (31) the singular points M_3 and M_4 are located on the invariant parabola. Moreover the singularity M_2 lies on the invariant line $y = 0$ and its position with respect to M_1 depends on $\operatorname{sign}(n)$. It is clear that M_2 coalesces with M_1 if and only if $n = 0$. The singularities M_3 and M_4 are complex (respectively, real) if $9 - 64n < 0$ (respectively, $9 - 64n > 0$) and they coincide (producing a multiple point) if $9 - 64n = 0$.

On the other hand for systems (76) we calculate:

$$\begin{gathered}\zeta_4 = 25(8n-1)/512, \quad \mathbf{D} = 3n^6(64n-9)/4, \\ B_2 = -81(1+4n)(8n-1)^2y^4/128, \quad \xi_{11} = 524906375\, n(8n-1)^2/81\end{gathered} \tag{77}$$

and evidently due to $\zeta_4 \neq 0$ we have $\operatorname{sign}(\xi_{11}) = \operatorname{sign}(n)$ and if $\mathbf{D} \neq 0$ we obtain $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(64n-9)$.

So considering Lemma 1 we discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. Then by Lemma 1 systems (76) could not have invariant lines in other directions different from $y = 0$. We examine three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} < 0$. This implies $9 - 64n > 0$ and hence the singularities M_3 and M_4 are real. In order to determine the position of M_3 and M_4 with respect to M_1 we calculate

$$(x_3 - x_1)(x_4 - x_1) = n \;\Rightarrow\; \text{sign}\big((x_3 - x_1)(x_4 - x_1)\big) = \text{sign}\,(n),$$
$$(x_3 - x_1) + (x_4 - x_1) = 3/2 > 0.$$

Therefore we conclude that in the case $(x_3 - x_1)(x_4 - x_1) > 0$ we have $x_3 - x_1 > 0$ and $x_4 - x_1 > 0$.

So it is not too difficult to determine that in this case systems (76) possess the configuration *Config.* $\mathcal{P}$*.31* if $\xi_{11} < 0$ (i.e. $n < 0$) and *Config.* $\mathcal{P}$*.35* if $\xi_{11} > 0$ (i.e. $n > 0$).

1.2: *The case* $\mathbf{D} > 0$. Then $9 - 64n < 0$ and systems (76) possess two real and two complex finite singularities. Since the condition $9 - 64n < 0$ implies $n > 0$ we arrive at the configuration *Config.* $\mathcal{P}$*.37*.

1.3: *The case* $\mathbf{D} = 0$. Then we obtain $n(9 - 64n) = 0$ and since $\xi_{11} = 0$ is equivalent to $n = 0$, in the case $\xi_{11} \neq 0$ we have $9 - 64n = 0$ and the singular points M_3 and M_4 coalesce producing a double singular point and we arrive at the configuration *Config.* $\mathcal{P}$*.40*.

Assume now $\xi_{11} = 0$, i.e. $n = 0$. Then we have a coalescence of three finite singularities: $M_4 = M_2 = M_1$. As a result we get the configuration *Config.* $\mathcal{P}$*.41*.

2: *The possibility* $B_2 = 0$. Considering (77) and the condition $\zeta_4 \neq 0$ we obtain $n = -1/4$. Then we arrive at the system

$$\dot{x} = -x/4 - y/4 - x^2/2 + xy, \quad \dot{y} = y(-1 - 3x + 4y)/2$$

which belongs to the family of systems (39). Therefore since $g = -1/2$ we have $-1 < g < 0$ and considering the examination of the family (39) we obtain the configuration *Config.* $\mathcal{P}$*.52*.

Thus we have proved the following lemma.

Lemma 11. *Assume that for a quadratic system the conditions* $(\boldsymbol{\mathcal{A}_5})$ *are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$B_1 \neq 0,\ \mathbf{D} < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.1;*
$B_1 \neq 0,\ \mathbf{D} > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.2;*
$B_1 \neq 0,\ \mathbf{D} = 0,\ \xi_1 \neq 0 \Rightarrow$ *Config.* $\mathcal{P}$*.3;*
$B_1 \neq 0,\ \mathbf{D} = 0,\ \xi_1 = 0,\ \xi_3 < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.6;*
$B_1 \neq 0,\ \mathbf{D} = 0,\ \xi_1 = 0,\ \xi_3 > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.7;*
$B_1 = 0,\ \xi_4 \neq 0,\ \mathbf{D} < 0,\ \xi_9 < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.17;*
$B_1 = 0,\ \xi_4 \neq 0,\ \mathbf{D} < 0,\ \xi_9 > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.18;*
$B_1 = 0,\ \xi_4 \neq 0,\ \mathbf{D} > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.20;*
$B_1 = 0,\ \xi_4 \neq 0,\ \mathbf{D} = 0,\ \xi_9 \neq 0 \Rightarrow$ *Config.* $\mathcal{P}$*.23;*
$B_1 = 0,\ \xi_4 \neq 0,\ \mathbf{D} = 0,\ \xi_9 = 0 \Rightarrow$ *Config.* $\mathcal{P}$*.26;*
$B_1 = 0,\ \xi_4 = 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \xi_{11} < 0 \Rightarrow$ *Config.* $\mathcal{P}$*.31;*
$B_1 = 0,\ \xi_4 = 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \xi_{11} > 0 \Rightarrow$ *Config.* $\mathcal{P}$*.35;*

$B_1 = 0,\ \xi_4 = 0,\ B_2 \neq 0,\ \mathbf{D} > 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}.37$;
$B_1 = 0,\ \xi_4 = 0,\ B_2 \neq 0,\ \mathbf{D} = 0,\ \xi_{11} \neq 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}.40$;
$B_1 = 0,\ \xi_4 = 0,\ B_2 \neq 0,\ \mathbf{D} = 0,\ \xi_{11} = 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}.41$;
$B_1 = 0,\ \xi_4 = 0,\ B_2 = 0 \quad \Rightarrow \quad$ *Config.* $\mathcal{P}.52$.

3.1.6 The statements $(\mathcal{A}_6)$, $(\mathcal{A}_7)$ and $(\mathcal{A}_8)$

According to Proposition 2 all these three statements have the common condition $\zeta_3 = \zeta_4 = 0$. Considering (71) we have to force the condition $32m + 8n - 1 = 0$ and we get $m = (1-8n)/32$. This leads to one-parameter family

$$\dot{x} =(4x-1)(8n-1-4x+8y)/32, \quad \dot{y} = (x-8nx+32ny-24xy+32y^2)/16 \tag{78}$$

which possess the following two invariant parabolas: $\Phi_1(x, y) = x^2 - y = 0$ and

$$\Phi_2 = (8n-1)(16n-1) - 4(16n-1)x + 16x^2 + 8(16n-3)y = 0.$$

Moreover these systems have the invariant line $x = 1/4$.

Following the statements $(\mathcal{A}_6)$ – $(\mathcal{A}_8)$ for the above systems we calculate

$$\zeta_1 = -\,25/2, \quad \zeta_2 = -1, \quad \zeta_3 = \zeta_4 = 0,\ \zeta_5 = -\frac{475}{256}(16n-1)^2, \ \ \mathcal{R}_2 = \frac{125}{1024}(16n-3). \tag{79}$$

Since we have the unique parameter n, according to Proposition 2 we arrive at the statements:

$$(\mathcal{A}_6) \ \text{if } \zeta_5\mathcal{R}_2 \neq 0; \qquad (\mathcal{A}_7) \ \text{if } \zeta_5 = 0; \qquad (\mathcal{A}_8) \ \text{if } \mathcal{R}_2 = 0.$$

We examine each one of these three possibilities.

3.1.6.1 The possibility $\zeta_5\mathcal{R}_2 \neq 0$.

Then $(16n-1)(16n-3) \neq 0$ and by Proposition 2 (statement $(\mathcal{A}_6)$) the invariant parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ are distinct. For these systems we calculate

$$B_1 = 0, \quad B_2 = -\frac{81}{2048}(8n-1)^2(1+16n)^2x^4, \quad \theta = 18 \neq 0,$$

and since $\theta \neq 0$ by Lemma 2 these systems could possess an invariant line parallel to the existent one.

On the other hand according to Lemma 1 for the existence of an invariant line in other direction different than $x = 0$ the condition $B_2 = 0$ is necessary. So we discuss two cases: $B_2 \neq 0$ and $B_2 = 0$.

1: The case $B_2 \neq 0$. We determine that systems (78) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{gathered} x_1 = \frac{1}{4},\ y_1 = \frac{1}{16}; \quad x_2 = \frac{1}{4}, \quad y_2 = \frac{1-8n}{8}; \\ x_{3,4} = \frac{1}{4}\big(1 \pm \sqrt{3-16n}\big), \quad y_{3,4} = \frac{1}{8}\big(2 - 8n \pm \sqrt{3-16n}\big). \end{gathered} \tag{80}$$

We observe that the invariant parabolas have two points of intersection: M_3 and M_4. Moreover we observe that the invariant line $x = 1/4$ intersects the parabola $\Phi_1(x, y) = 0$ at the point M_1 and the parabola $\Phi_2(x, y) = 0$ at the point M_2.

So all four finite singularities are fixed as the intersections of invariant curves and their positions are determined by the values of the parameter n. We observe that the finite singularities M_3 and M_4 are real if $3-16n>0$ and complex if $3-16n<0$.

On the other hand for systems (78) we have

$$\mathbf{D}=3(16n-3)^3(16n-1)^2/1048576\neq 0$$

due to $\zeta_5\mathcal{R}_2\neq 0$. Therefore $\operatorname{sign}(\mathbf{D})=\operatorname{sign}(16n-3)$.

So in the case $B_2\neq 0$ we arrive at the configuration *Config.*$\mathcal{P}$*.83* if $\mathbf{D}<0$ and *Config.*$\mathcal{P}$*.84* if $\mathbf{D}>0$.

2: The case $B_2=0$. Considering (79) this implies $(8n-1)(1+16n)=0$.

2.1: The subcase $8n-1=0$. Then $n=1/8$ and we get the system

$$\dot{x}=-(4x-1)(x-2y)/8,\quad \dot{y}=y(1-6x+8y)/4 \tag{81}$$

possessing the following five invariant curves (two parabolas and three invariant lines):

$$\Phi_1(x,y)=x^2-y,\ \ \Phi_2(x,y)=-x+4x^2-2y,\ \ x=1/4,\ \ y=0,\ \ y=x-1/4.$$

So it is easy to detect the unique configuration *Config.*$\mathcal{P}$*.85*.

2.2: The subcase $1+16n=0$. In this case $n=-1/16$ and we arrive at the system

$$\dot{x}=-(4x-1)(3+8x-16y)/64,\quad \dot{y}=-(16y-1)(3x-4y)/32$$

which can be brought to the system (81) via the following affine transformation:

$$x_1=x/2+1/8,\ \ y_1=y/2-1/32,\ \ t_1=2t.$$

So in this case we get the same configuration *Config.*$\mathcal{P}$*.87*.

3.1.6.2 The possibility $\zeta_5=0$. Considering (79) we get $n=1/16$ and this leads to the system

$$\dot{x}=-(4x-1)(1+8x-16y)/64,\quad \dot{y}=(x+4y-48xy+64y^2)/32.$$

possessing the invariant line $x=1/4$ and the double invariant parabola $\Phi_1(x,y)=x^2-y=0$ because the conditions provided by the statement $\boldsymbol{(\mathcal{A}_7)}$ of Proposition 2 are fulfilled. In this case the singular points M_1 and M_2 coalesced and it is not difficult to detect that in this case we get the configuration *Config.*$\mathcal{P}$*.86*.

3.1.6.3 The possibility $\mathcal{R}_2=0$. Considering (79) this implies $n=3/16$ and we obtain the system

$$\dot{x}=-(4x-1)(-1+8x-16y)/64,\quad \dot{y}=(-x+12y-48xy+64y^2)/32$$

possessing the unique invariant parabola $\Phi_1(x,y)=x^2-y=0$ (according to the statement $\boldsymbol{(\mathcal{A}_8)}$ of Proposition 2) and the invariant line $x=1/4$. Considering (80) we determine that in this case the singularities M_3 and M_4 coalesced with M_1 producing a triple singularity. So we arrive at the configuration *Config.*$\mathcal{P}$*.87*.

Thus we have proved the following lemma.

Lemma 12. *Assume that for a quadratic system the conditions* $\eta > 0$, $\chi_1 = 0$, $\zeta_1\zeta_2 \neq 0$ *and* $\zeta_3 = \zeta_4 = 0$ *hold. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$\mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0,\ B_2 \neq 0,\ \mathbf{D} < 0 \Rightarrow$ *Config.* $\mathcal{P}.83$;
$\mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0,\ B_2 \neq 0,\ \mathbf{D} > 0 \Rightarrow$ *Config.* $\mathcal{P}.84$;
$\mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0,\ B_2 = 0 \Rightarrow$ *Config.* $\mathcal{P}.85$;
$\mathcal{R}_2 \neq 0,\ \zeta_5 = 0 \Rightarrow$ *Config.* $\mathcal{P}.86$;
$\mathcal{R}_2 = 0 \Rightarrow$ *Config.* $\mathcal{P}.87$.

3.1.7 The statement $(\mathcal{A}_9)$

According to Proposition 2 the condition $\zeta_2 = 0$ must be fulfilled. Considering (8) we get $g(g+1) = 0$.

Following the proof of Lemma 3 we conclude that the condition $g+1 = 0$ could be brought to the condition $g = 0$ due to an affine transformation (see formulas (18)).

So it is sufficient to examine only the case $g = 0$. In this case we arrive at the 2-parameter family of systems

$$\dot{x} = m + nx - y/2 + xy, \quad \dot{y} = 2mx + 2ny - xy + 2y^2 \tag{82}$$

possessing the parabola $\Phi(x, y) = x^2 - y = 0$. Considering the statement $(\mathcal{A}_9)$ for these systems we calculate

$$\begin{aligned} \zeta_1 =& -12, \quad \zeta_2 = 0, \quad \zeta_6 = (2m+n)/2, \quad \mathcal{R}_1 = 0, \quad \mathcal{R}_2 = 6(2m+n), \\ B_1 =& 2m(4m-1-2n)(2m+n)^3. \end{aligned} \tag{83}$$

Remark 5. *Following [1, Lemma 5.2, statement (i)] for systems* (82) *we calculate*

$$\mu_0 = 0, \quad \mu_1 = -2(2m+n)y \neq 0$$

due to the condition $\zeta_6 \neq 0$*. Therefore according to [1, Lemma 5.2, statement (i)] we conclude that one of the singular points of systems* (82) *has gone to infinity and coalesced with the infinite singularity* $N[1:0:0]$ *producing a double infinite singularity of the type* $(1, 1)$*.*

3.1.7.1 The case $B_1 \neq 0$.

We observe that the family of systems (82) is a subfamily of (7) defined by the condition $g = 0$. Considering the finite singularities of (7) given in (10) we remark that in the case $g = 0$ the singular point $M_1(x_1, y_1)$ with the coordinates

$$x_1 = -\frac{2m+n+gn}{g(1+g)}, \ y_1 = \frac{2m}{1+g}$$

has gone to infinity. According to Remark 5 this singularity coalesced with the infinite singularity $N[1:0:0]$ producing a double infinite singularity of the type $(1, 1)$.

Therefore we deduce that systems (82) possess three finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ $(i = 2, 3, 4)$ where considering (10) we have

$$\tilde{x}_i = x_i\big|_{\{g=0\}}, \quad \tilde{y}_i = y_i\big|_{\{g=0\}}, \quad i = 2, 3, 4.$$

So taking into consideration [1, Proposition 5.1] for systems (82) we calculate

$$\mu_0 = 0, \quad \mathbf{D} = 48(2m+n)^4(-2m + 108m^2 + 36mn - n^2 + 16n^3), \quad \mathbf{R} = 12(2m+n)^2y^2.$$

We observe that $\mathbf{R} \neq 0$ due to $\zeta_6 \neq 0$ and by Proposition 5.1 from [1] we have three finite distinct real singularities if $\mathbf{D} < 0$ and one real and two complex if $\mathbf{D} > 0$. Considering the double point at infinity of the type $(1,1)$ we arrive at the configuration *Config.* $\mathcal{P}.88$ if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}.89$ if $\mathbf{D} > 0$.

Assume now that for systems (82) the condition $\mathbf{D} = 48\mathcal{F}_1^2\mathcal{F}_2 = 0$ where

$$\mathcal{F}_1 = -(2m+n)^2, \quad \mathcal{F}_2 = -2m + 108m^2 + 36mn - n^2 + 16n^3.$$

Since $\mathcal{F}_1 \neq 0$ due to $\zeta_6 \neq 0$ we conclude that the condition $\mathbf{D} = 0$ is equivalent to $\mathcal{F}_2 = 0$.

Setting a new parameter v similarly as in the generic case (see page 27) we arrive at the 1-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{(1+v)^2(2v-1)}{216} - \frac{v^2-1}{12}x - \frac{1}{2}y + xy, \\ \dot{y} =& \frac{(1+v)^2(2v-1)}{108}x - \frac{v^2-1}{6}y - xy + 2y^2, \end{aligned} \tag{84}$$

which is a subfamily of (11) defined by the condition $g = 0$.

We recall that systems (11) possess three finite singularities given in (13) and M_1 is of multiplicity at least two. We observe that for $g = 0$ the singular point M_3 has gone to infinity and according to Remark 5 this singularity coalesces with the infinite singularity $N[1:0:0]$ producing a double infinite singularity of the type $(1,1)$.

Thus we deduce that systems (84) possess at most two different finite singularities $M_1(\tilde{x}_1, \tilde{y}_1)$ (multiple) and $M_2(\tilde{x}_2, \tilde{y}_2)$ where considering (13) we have

$$\tilde{x}_1 = \frac{1+v}{6}, \ \tilde{y}_1 = \frac{(1+v)^2}{36}; \quad \tilde{x}_2 = \frac{1-2v}{6}, \ \tilde{y}_2 = \frac{(1-2v)^2}{36}.$$

We observe that the singular point M_2 coalesces with the double point M_1 if and only if $v = 0$.

Thus considering the condition $B_1 \neq 0$ (i.e. systems (84) do not have any invariant line) we arrive at the configuration *Config.* $\mathcal{P}.90$ if $v \neq 0$ and *Config.* $\mathcal{P}.91$ if $v = 0$.

On the other hand for systems (84) we calculate

$$\xi_2 = \frac{1}{209952}v^2(v-2)^6(1+v)^2, \quad \zeta_6 = \frac{1}{108}(v-2)^2(1+v)$$

and due to $\zeta_6 \neq 0$ we conclude that the condition $v \neq 0$ is equivalent to $\xi_2 \neq 0$. Therefore we get the configuration *Config.* $\mathcal{P}.90$ if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}.91$ if $\xi_2 = 0$.

3.1.7.2 The case $B_1 = 0$. Considering (83) and the condition $\zeta_6 \neq 0$ (i.e. $2m + n \neq 0$) we deduce that the condition $B_1 = 0$ is equivalent to $m(4m - 1 - 2n) = 0$.

On the other hand for systems (82) we calculate:

$$\xi_1 = 9(4m-1-2n)(2m+n)^2/4, \quad \zeta_6 = (2m+n)/2$$

and due to $\zeta_6 \neq 0$ we conclude that the condition $4m - 1 - 2n = 0$ is equivalent to $\xi_1 = 0$. So we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: *The subcase* $\xi_1 \neq 0$. Then the condition $B_1 = 0$ yields $m = 0$ and this leads to the 1-parameter family of systems

$$\dot{x} = \frac{1}{2}(2nx - y + 2xy), \quad \dot{y} = y(2n - x + 2y) \tag{85}$$

possessing the parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $y = 0$. Calculations yield

$$B_2 = -324n^2(1 + 2n)y^4$$

and considering Lemma 1 we discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1.1: The possibility $B_2 \neq 0$. We determine that systems (85) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with the coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_{2,3} = \frac{1}{4}\big(1 \pm \sqrt{1 - 16n}\,\big), \quad y_{2,3} = \frac{1}{8}\big(1 - 8n \pm \sqrt{1 - 16n}\,\big). \tag{86}$$

According to Remark 5 the forth finite singularity coalesced with an infinite one and we have a singular point of type $(1, 1)$.

We observe that M_1 is the point of tangency of $y = 0$ with the invariant parabola and that M_2 and M_3 are either real or complex or coinciding, depending on the value of $1 - 16n$.

So in order to determine the positions of the real singularities M_2 and M_3 with respect to M_1 we calculate:

$$(x_2 - x_1)(x_3 - x_1) = n, \quad (x_2 - x_1) + (x_3 - x_1) = 1/2 > 0.$$

On the other hand for systems (85) calculations yield:

$$\mathbf{D} = -48n^6(1 - 16n), \quad \mathcal{R}_2 = 6n$$

and therefore due to $\mathcal{R}_2 \neq 0$ we have $\operatorname{sign}(\mathcal{R}_2) = \operatorname{sign}(n)$ and $\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(1 - 16n)$.

Thus in the case $B_2 \neq 0$ we arrive at the following four configurations:

$\mathbf{D} < 0\ \mathcal{R}_2 < 0 \ \Rightarrow\ $ *Config.* $\mathcal{P}$*.92*;
$\mathbf{D} < 0\ \mathcal{R}_2 > 0 \ \Rightarrow\ $ *Config.* $\mathcal{P}$*.93*;
$\mathbf{D} > 0 \ \Rightarrow\ $ *Config.* $\mathcal{P}$*.94*;
$\mathbf{D} = 0 \ \Rightarrow\ $ *Config.* $\mathcal{P}$*.95.*

1.2: The possibility $B_2 = 0$. This implies $(1 + 2n)n = 0$ and since $n \neq 0$ (due to $\zeta_6 = n/2 \neq 0$) we get $1 + 2n = 0$. Then $n = -1/2$ and we obtain the system

$$\dot{x} = \frac{1}{2}(-x - y + 2xy), \quad \dot{y} = y(-1 - x + 2y), \tag{87}$$

possessing two invariant lines: $y = 0$ and $y = x$. Considering (86) we get three real finite singularities and this leads to the configuration *Config.* $\mathcal{P}$*.96.*

2: *The subcase* $\xi_1 = 0$. This implies $m = (1 + 2n)/4$ and we arrive at the 1-parameter family of systems

$$\dot{x} = \frac{1}{4}(1 + 2n + 4nx - 2y + 4xy), \quad \dot{y} = \frac{1}{2}(x + 2nx + 4ny - 2xy + 4y^2), \tag{88}$$

possessing the parabola $\Phi(x,y)=x^2-y=0$ and the invariant line $y=x-(2n+1)/2$. Calculations yield

$$B_2=-81(1+2n)(1+4n)^2(x-y)^4$$

and considering Lemma 1 we examine two possibilities: $B_2\neq 0$ and $B_2=0$.

2.1: The possibility $B_2\neq 0$. We determine that the above systems possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates

$$x_1=-\frac{1}{2},\ y_1=\frac{1}{4};\quad x_{2,3}=\frac{1}{2}\big(1\pm\sqrt{-(1+4n)}\,\big),\quad y_{2,3}=\frac{1}{2}\big(-2n\pm\sqrt{-(1+4n)}\,\big).$$

We observe that the singularities M_2 and M_3 are the points of intersection of the invariant line $y=x-(2n+1)/2$ with the invariant parabola and they are real (respectively complex; coinciding) if $1+4n<0$ (respectively $1+4n>0$; $1+4n=0$). And again in the case of real singularities we calculate

$$(x_2-x_1)(x_3-x_1)=(5+4n),\quad (x_2-x_1)+(x_3-x_1)=2>0.$$

On the other hand for systems (88) calculations yield:

$$\mathbf{D}=3(1+4n)^5(5+4n)^2/4,\quad \xi_3=8164197(1+4n)^3(5+4n)/2$$

and therefore for $\mathbf{D}\neq 0$ we have $\operatorname{sign}(\mathbf{D})=\operatorname{sign}(1+4n)$ and $\operatorname{sign}(\xi_3)=\operatorname{sign}\big((1+4n)(5+4n)\big)$.

Thus in the case $B_2\neq 0$ and $\mathbf{D}\neq 0$ we arrive at the following configurations:

$\mathbf{D}<0,\ \xi_3<0\ \Rightarrow$ *Config.* $\mathcal{P}$*.97*;
$\mathbf{D}<0,\ \xi_3>0\ \Rightarrow$ *Config.* $\mathcal{P}$*.98*;
$\mathbf{D}>0\ \Rightarrow$ *Config.* $\mathcal{P}$*.99*.

Assume now $\mathbf{D}=0$. This implies $(1+4n)(5+4n)=0$. Since $\zeta_6=(1+4n)/4\neq 0$ we get $5+4n=0$ and this means that one of the singularities M_2 or M_3 coalesces with M_1. So we arrive at the configuration *Config.* $\mathcal{P}$*.100*.

2.2: The possibility $B_2=0$. This implies $(1+2n)(1+4n)=0$ and since $1+4n\neq 0$ (due to $\zeta_6\neq 0$) we get $1+2n=0$. Then $n=-1/2$ and we arrive at system (87) the configuration *Config.* $\mathcal{P}$*.96*.

Thus we have proved the following lemma.

Lemma 13. *Assume that for a quadratic system the conditions* $(\boldsymbol{\mathcal{A}_9})$ *are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$B_1\neq 0,\ \mathbf{D}<0\ \Rightarrow$ *Config.* $\mathcal{P}$*.88;*
$B_1\neq 0,\ \mathbf{D}>0\ \Rightarrow$ *Config.* $\mathcal{P}$*.89;*
$B_1\neq 0,\ \mathbf{D}=0,\ \xi_2\neq 0\ \Rightarrow$ *Config.* $\mathcal{P}$*.90;*
$B_1\neq 0,\ \mathbf{D}=0,\ \xi_2=0\ \Rightarrow$ *Config.* $\mathcal{P}$*.91;*
$B_1=0,\ \xi_1\neq 0,\ B_2\neq 0,\ \mathbf{D}<0,\ \mathcal{R}_2<0\ \Rightarrow$ *Config.* $\mathcal{P}$*.92;*
$B_1=0,\ \xi_1\neq 0,\ B_2\neq 0,\ \mathbf{D}<0,\ \mathcal{R}_2>0\ \Rightarrow$ *Config.* $\mathcal{P}$*.93;*
$B_1=0,\ \xi_1\neq 0,\ B_2\neq 0,\ \mathbf{D}>0\ \Rightarrow$ *Config.* $\mathcal{P}$*.94;*

$$\begin{array}{ll}
B_1=0,\ \xi_1\neq 0,\ B_2\neq 0,\ \mathbf{D}=0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.95;\\
B_1=0,\ \xi_1\neq 0,\ B_2=0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.96;\\
B_1=0,\ \xi_1=0,\ B_2\neq 0,\ \mathbf{D}<0,\ \xi_3<0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.97;\\
B_1=0,\ \xi_1=0,\ B_2\neq 0,\ \mathbf{D}<0,\ \xi_3>0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.98;\\
B_1=0,\ \xi_1=0,\ B_2\neq 0,\ \mathbf{D}>0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.99;\\
B_1=0,\ \xi_1=0,\ B_2\neq 0,\ \mathbf{D}=0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.100;\\
B_1=0,\ \xi_1=0,\ B_2=0 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.96.
\end{array}$$

Since all the statements provided by Proposition 2 are considered hence this proposition is proved. ■

3.2 Systems in $\mathbf{QSP}_{(\eta>0)}$ with the condition $\zeta_1=0$

In what follows we examine each one of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ provided by Proposition 3.

According to this proposition a system satisfying the conditions provided by one of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ could be brought to the form:

$$\dot{x}=m+nx-\frac{3y}{2}+2x^2+xy,\quad \dot{y}=2mx+2ny+xy+2y^2, \tag{89}$$

and this system possesses the invariant parabola $\Phi_1(x,y)=x^2-y=0$.

3.2.1 The statement $(\mathcal{B}_1)$

According to this statement for systems (89) we calculate $\chi_3=0$ and

$$\chi_4=61875\,\mathcal{U}_1\mathcal{U}_3,\quad \zeta_7=-\frac{52875}{2}\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3,\quad \mathcal{R}_3=\ \ 3850561006875\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \tag{90}$$

where

$$\mathcal{U}_1=1+4m+2n,\quad \mathcal{U}_2=4m-147-14n,$$
$$\mathcal{U}_3=18m+1372m^2-84mn+27n^2+144n^3.$$

On the other hand following Lemma 1 we calculate

$$B_1=m(2m-n)(2m+3n+9)(4m-6n-9)(1+4m+2n) \tag{91}$$

and considering Lemma 1 we discuss two cases: $B_1\neq 0$ and $B_1=0$.

3.2.1.1 The case $B_1\neq 0$. We observe that the family of systems (89) is a subfamily of (7) defined by the condition $g=2$. Therefore it is clear that systems (89) possess four finite singularities $M_i(\tilde{x}_i,\,\tilde{y}_i)$ $(i=1,2,3,4)$ where considering (10) we have

$$\tilde{x}_i=x_i\big|_{\{g=2\}},\quad \tilde{y}_i=y_i\big|_{\{g=2\}},\quad i=1,2,3,4.$$

On the other hand for systems (89) we have

$$\mathbf{D}=48\widetilde{\mathcal{F}}_1^2\widetilde{\mathcal{F}}_2,\quad \widetilde{\mathcal{F}}_1=\mathcal{F}_1\big|_{\{g=2\}},\quad \widetilde{\mathcal{F}}_1=\mathcal{F}_1\big|_{\{g=2\}},$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given in (9).

As it was proved for the family (7) in the case $\mathbf{D} \neq 0$ these systems could possess only two distinct configurations: *Config.* $\mathcal{P}.1$ if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}.2$ if $\mathbf{D} > 0$. The same two configurations could be obtained in the particular case $g = 2$, because this values of the parameter g is not a bifurcation value for the mentioned two configurations.

Assume now $\mathbf{D} = 0$. This implies $\widetilde{\mathcal{F}}_1\widetilde{\mathcal{F}}_2 = 0$ and we have to distinguish what factor vanishes. We point out that the invariant polynomial ξ_1 which governed the condition $\mathcal{F}_1 = 0$ for systems (7) in generic case (i.e. $g \neq 2$) vanishes for $g = 2$. So we have to use another invariant polynomial and for systems (89) we calculate:

$$\xi_{18} = 17969284698750\, \mathcal{U}_2\mathcal{U}_3\, \widetilde{\mathcal{F}}_1.$$

Therefore due to the condition $\zeta_7 \neq 0$ we obtain that the condition $\widetilde{\mathcal{F}}_1 = 0$ is equivalent to $\xi_{18} = 0$. So we examine two subcases: $\xi_{18} \neq 0$ and $\xi_{18} = 0$.

1: The possibility $\xi_{18} \neq 0$. Then $\widetilde{\mathcal{F}}_1 \neq 0$ and hence the condition $\mathbf{D} = 0$ implies $\widetilde{\mathcal{F}}_2 = 0$. Following the investigation of the family of systems (7) in the particular case $g = 2$ we arrive at the systems (11) which for $g = 2$ become

$$\begin{aligned}\dot{x} =&\frac{1}{216}(2v+1)(v-1)^2 - \frac{1}{12}(v^2-1)x - \frac{3y}{2} + 2x^2 + xy,\\ \dot{y} =&\frac{1}{108}(2v+1)(v-1)^2x - \frac{1}{6}(v^2-1)y + xy + 2y^2.\end{aligned} \tag{92}$$

For the above systems we calculate

$$\begin{gathered}\xi_2 = 2^{-5}3^{-8}(v-4)^2(v-1)^2v^2(v^2-20v-8)^2,\\ \xi_{18} = 998293594375(v-10)(v-4)^4(v-1)^4(20+v)^2(4+5v)^2(v^2-20v-8)/3188646\end{gathered}$$

and we observe that due to $\xi_{18} \neq 0$ the condition $\xi_2 = 0$ is equivalent to $v = 0$.

Therefore following the examination of the 2-parameter family of systems we conclude that the 1-parameter family of systems (92) in the case $B_1\xi_{18} \neq 0$ possesses the configuration *Config.* $\mathcal{P}.3$ if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}.4$ if $\xi_2 = 0$.

2: The possibility $\xi_{18} = 0$. Then we have $\widetilde{\mathcal{F}}_1 = 0$ and this implies $\mathbf{D} = 0$. Following the investigation of the family of systems (7) in the particular case $g = 2$ we arrive at the systems (14) which for $g = 2$ become

$$\begin{aligned}\dot{x} =&\frac{3}{8}(u-2)^2 - \frac{1}{4}(u^2-4)x - \frac{3y}{2} + 2x^2 + xy,\\ \dot{y} =&\frac{3}{4}(u-2)^2x - -\frac{1}{2}(u^2-4)y + xy + 2y^2.\end{aligned} \tag{93}$$

For the above systems we calculate

$$\begin{gathered}\chi_4 = -556875(u-3)^5(u-2)^2(13+u)/4, \quad \xi_2 = 9(u-3)^2(u-2)^2\widetilde{Z}_1/2,\\ \xi_3 = 490484322\tilde{\alpha}_1^3\widetilde{Z}_1, \quad \widetilde{Z}_1 = Z_1\big|_{\{g=2\}}, \quad \tilde{\alpha}_1 = \alpha_1\big|_{\{g=2\}},\end{gathered}$$

where Z_1 and α_1 are the polynomials defined for systems (14) (see (16) and (17)).

We observe that due to the condition $\chi_4 \neq 0$ we have

$$\operatorname{sign}(\xi_2) = \operatorname{sign}(\widetilde{Z}_1), \quad \operatorname{sign}(\xi_3) = \operatorname{sign}(\widetilde{Z}_1\tilde{\alpha}_1),$$

and following the examination of the 2-parameter family of systems (14) we conclude that the 1-parameter family of systems (93) in the case $B_1 \neq 0$ possesses the following configurations if and only if the corresponding conditions are satisfied:

$\xi_2 < 0 \Rightarrow$ *Config.* $\mathcal{P}.5$;
$\xi_2 > 0,\ \xi_3 < 0 \Rightarrow$ *Config.* $\mathcal{P}.6$;
$\xi_2 > 0,\ \xi_3 > 0 \Rightarrow$ *Config.* $\mathcal{P}.7$;
$\xi_2 = 0 \Rightarrow$ *Config.* $\mathcal{P}.8$.

3.2.1.2 The case $B_1 = 0$. Considering (91) and the condition $\chi_4 \neq 0$ (i.e. $4m + 2n + 1 \neq 0$) we observe that the condition $B_1 = 0$ is equivalent to

$$m(2m - n)(2m + 3n + 9)(4m - 6n - 9) = 0.$$

For systems (89) calculations yield:

$$\begin{aligned} \xi_{19} &= -12870000\, m\,(2m - n)\,(2m + 3n + 9),\\ \xi_{20} &= -540\, m\,(2m - n)\,\mathcal{U}_1\mathcal{U}_3, \quad \xi_{21} = -110106\, m\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \end{aligned} \tag{94}$$

and we consider two subcases: $\xi_{19} \neq 0$ and $\xi_{19} = 0$.

3.2.1.2.1 The subcase $\xi_{19} \neq 0$. Then $m(2m-n)(2m+3n+9) \neq 0$ and therefore the condition $B_1 = 0$ yields $4m - 6n - 9 = 0$. This implies $m = 3(3 + 2n)/4$ and we arrive at the 1-parameter family of systems

$$\begin{aligned} \dot{x} &= (9 + 6n + 4nx + 8x^2 - 6y + 4xy)/4,\\ \dot{y} &= (9x + 6nx + 4ny + 2xy + 4y^2)/2, \end{aligned} \tag{95}$$

which possess the invariant line $y = x - (3+2n)/2$ and four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 4)$ with the coordinates

$$\begin{aligned} x_1 &= -\frac{3}{2},\ y_1 = \frac{9}{4};\quad x_2 = -\frac{3 + 4n}{4},\ y_2 = \frac{3 + 2n}{4};\\ x_{3,4} &= \frac{1}{2}\big(1 \pm \sqrt{-(5 + 4n)}\,\big),\ y_{3,4} = \frac{1}{2}\big(-2 - 2n \pm \sqrt{-(5 + 4n)}\,\big). \end{aligned}$$

We determine that the singularities M_1, M_3 and M_4 are located on the invariant parabola. Moreover M_3 and M_4 are the points of intersections of the invariant line $y = x - (3+2n)/2$ with the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. We calculate

$$\Phi_1(x_2, y_2) = \frac{1}{16}(4n - 3)(5 + 4n)$$

and we conclude that M_2 could be located on the parabola if and only if $4n - 3 = 0$, because for systems (95) we have

$$\chi_4\zeta_7\mathcal{R}_3 \neq 0 \ \Leftrightarrow\ (5 + 4n)(9 + 4n)(69 + 4n) \neq 0. \tag{96}$$

In order to determine the position of the singularity M_1 with respect to M_3 and M_4 (when they are real) we calculate

$$\begin{gathered} (x_1 - x_3)(x_1 - x_4) = (21 + 4n)/4 \ \Rightarrow\ \operatorname{sign}\big((x_1 - x_2)(x_1 - x_3)\big) = \operatorname{sign}(21 + 4n),\\ (x_1 - x_3) + (x_1 - x_4) = -4 < 0. \end{gathered}$$

Thus we observe that for the parameter n we have the following possible bifurcation values: $n \in \{-21/4, -5/4, 3/4\}$. Moreover we point out that due to the condition (96) the condition $5+4n \neq 0$ must hold (i.e. $n \neq -5/4$) and hence the singularities M_3 and M_4 could not coincide.

On the other hand according to Lemma 1 for the existence of an invariant line in other direction different than $y = x$ the condition $B_2 = 0$ is necessary. For systems (95) we calculate

$$B_2 = -729(3+2n)(9+4n)^2(x-y)^4, \quad \mathbf{D} = 243(4n-3)^2(5+4n)^3(21+4n)^2/4,$$
$$\xi_9 = 16299895407840(9+4n)^2(21+4n),$$

and in the case $\mathbf{D} \neq 0$ we have

$$\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(5+4n), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}(21+4n).$$

Considering Lemma 1 we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. We observe that for $\mathbf{D} \neq 0$ all four finite singular points of systems (95) are distinct. So we discuss two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} \neq 0$. Considering the bifurcation values of the parameter n mentioned above we determine for systems (95) the following configurations (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_9 < 0$ (i.e. $n < -21/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.17*;
$\mathbf{D} < 0$, $\xi_9 > 0$ (i.e. $-21/4 < n < -5/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.19*;
$\mathbf{D} > 0$ (i.e. $n > -5/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.20.*

1.2: *The case* $\mathbf{D} = 0$. Then due to the condition (96) we get $(4n-3)(21+4n) = 0$ and we observe that the condition $21 + 4n = 0$ is governed by the invariant polynomial ξ_9. Therefore we arrive at the configuration *Config.* $\mathcal{P}$*.21* if $\xi_9 \neq 0$ and *Config.* $\mathcal{P}$*.27* if $\xi_9 = 0$.

2: *The possibility* $B_2 = 0$. Considering the condition (96) we get $n = -3/2$ and this leads to the system

$$\dot{x} = \frac{1}{2}(-3x + 4x^2 - 3y + 2xy), \quad \dot{y} = y(-3 + x + 2y),$$

possessing two invariant lines $y = x$ and $y = 0$ besides the invariant parabola. So it is not too difficult to determine that this system possesses the configuration *Config.* $\mathcal{P}$*.54*.

3.2.1.2.2 The subcase $\xi_{19} = 0$. Then $m(2m-n)(2m+3n+9) = 0$ and we examine two possibilities: $\xi_{20} \neq 0$ and $\xi_{20} = 0$.

1: *The possibility* $\xi_{20} \neq 0$. In this case considering (94) we obtain $m(2m-n) \neq 0$ and therefore we get $2m + 3n + 9 = 0$. This implies $m = -3(3+n)/2$ and we arrive at the 1-parameter family of systems

$$\dot{x} = \frac{1}{2}(2x-3)(3+n+2x+y), \quad \dot{y} = -3(3+n)x + 2ny + xy + 2y^2, \tag{97}$$

which is a subfamily of (20) defined by the condition $g = 2$. The family (20) was investigated earlier and considering (21) for $g = 2$ (i.e. for systems (97)) we have

$$Z_2 = -4(2+n), \quad \alpha_2 = 21+4n, \quad \beta_2 = 33+4n,$$
$$B_2 = -5832(3+n)(9+4n)^2(17+4n)x^4, \quad \mathbf{D} = -972 Z_2 \alpha_2^2 \beta_2^2,$$
$$B_3 = -9(9+4n)x^2(12x^2+4nx^2-24xy-8nxy-5y^2)/2,$$
$$\xi_7 = 1057164750000 Z_2 \alpha_2^2 \beta_2, \quad \xi_8 = 55640250000 Z_2 \alpha_2 \beta_2^2,$$
$$\xi_{18} = -7277560302993750(6+n)(9+4n)(57+4n)\alpha_2\beta_2.$$

We observe that for the parameter n we have the following possible bifurcation values: $n \in \{-33/4, -21/4, -2\}$. Considering (90) for systems (97) we obtain

$$\chi_4 \zeta_7 \mathcal{R}_3 \neq 0 \;\Leftrightarrow\; (6+n)(9+4n)(17+4n)(57+4n)\beta_2 \neq 0 \qquad (98)$$

and therefore we have $\beta_2 B_3 \neq 0$. Moreover due to the above condition we obtain that $\xi_{18} = 0$ if and only if $\alpha_2 = 0$.

Thus in the case $B_2 \neq 0$, following the investigation of the family (20) for $g = 2$ we get the following configurations (depending on the parameter n):

- $\mathbf{D} < 0$, $\xi_7 < 0$ (i.e. $n < -33/4$) $\Rightarrow$ *Config.* $\mathcal{P}$*.9*;
- $\mathbf{D} < 0$, $\xi_7 > 0$, $\xi_8 < 0$ (i.e. $-33/4 < n < -21/4$) $\Rightarrow$ *Config.* $\mathcal{P}$*.11*;
- $\mathbf{D} < 0$, $\xi_7 > 0$, $\xi_8 > 0$ (i.e. $-21/4 < n < -2$) $\Rightarrow$ *Config.* $\mathcal{P}$*.12*;
- $\mathbf{D} > 0$ (i.e. $n > -2$) $\Rightarrow$ *Config.* $\mathcal{P}$*.13*;
- $\mathbf{D} = 0$, $\xi_{18} \neq 0$ (i.e. $n = -2$) $\Rightarrow$ *Config.* $\mathcal{P}$*.14*;
- $\mathbf{D} = 0$, $\xi_{18} = 0$ (i.e. $n = -21/4$) $\Rightarrow$ *Config.* $\mathcal{P}$*.16*.

Assuming $B_2 = 0$ and considering the condition (98) we get $n = -3$ and we arrive at the system

$$\dot{x} = \frac{1}{2}(2x-3)(2x+y), \quad \dot{y} = y(-6+x+2y),$$

possessing two invariant lines $x = 3/2$ and $y = 0$ besides the invariant parabola. So it is not too difficult to determine that this system possesses the configuration *Config.* $\mathcal{P}$*.49*.

2: *The possibility* $\xi_{20} = 0$. Then from (94) we obtain $m(2m-n) = 0$ and we discuss two cases: $\xi_{21} \neq 0$ and $\xi_{21} = 0$.

2.1: *The case* $\xi_{21} \neq 0$. Then $m \neq 0$ and we obtain $m = n/2$. This leads to the following 1-parameter family of systems

$$\dot{x} = \frac{n}{2} + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = (n+y)(x+2y), \qquad (99)$$

which is a subfamily of (25) defined by the condition $g = 2$. The family (25) was investigated earlier and considering (27) for $g = 2$ (i.e. for systems (99)) we conclude that these systems possess four singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$x_1 = \sqrt{-n},\ y_1 = -n; \quad x_2 = -\sqrt{-n},\ y_2 = -n; \quad x_3 = -\frac{1}{2},\ y_3 = \frac{1}{4}; \quad x_4 = -\frac{2n}{3},\ y_4 = \frac{n}{3}.$$

For the above systems we calculate

$$\theta = -32 \neq 0, \quad B_2 = -162(1+4n)(9+4n)^2 y^4, \quad \chi_4 = 61875n(1+4n)(9+4n)(1+36n)$$

and therefore $B_2 \neq 0$ due to $\chi_4 \neq 0$. Following the examination of the configurations of systems (25) for $g = 2$ we obtain:

$$\alpha_3 = 4n - 3, \quad \beta_3 = 1 + 4n, \ \mathbf{D} = 768n^3\alpha_3^2\beta_3^2, \quad \xi_9 = 3219732426240(9 + 4n)^2\beta_3$$

and due to $\chi_4 \neq 0$ in the case $\mathbf{D} \neq 0$ we have

$$\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(n), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}(\beta_3).$$

Moreover due to $\chi_4 \neq 0$ (i.e. $n\beta_3 \neq 0$) the condition $\alpha_3 = 0$ is equivalent to $\mathbf{D} = 0$.

Thus we arrive at the following configurations: (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_9 < 0$ (i.e. $n < -1/4$) $\Rightarrow$ *Config.* $\mathcal{P}$.*17*;
$\mathbf{D} < 0$, $\xi_9 > 0$ (i.e. $-1/4 < n < 0$) $\Rightarrow$ *Config.* $\mathcal{P}$.*19*.
$\mathbf{D} > 0$ (i.e. $3/4 \neq n > 0$) $\Rightarrow$ *Config.* $\mathcal{P}$.*20*;
$\mathbf{D} = 0$ (i.e. $n = 3/4$) $\Rightarrow$ *Config.* $\mathcal{P}$.*21*.

2.2: *The case* $\xi_{21} = 0$. Then $m = 0$ and we obtain the 1-parameter family of systems

$$\dot{x} = nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(2n + x + 2y), \tag{100}$$

which is a subfamily of (31) defined by the condition $g = 2$. The family (31) was investigated earlier and considering (34) for $g = 2$ (i.e. for systems (100)) we have

$$Z_3 = 1 - 16n, \quad B_2 = -1458(3 + n)(1 + 2n)(3 + 2n)y^4, \quad \theta = -32 \neq 0,$$
$$B_3 = -9n(7 + 4n)x^2y^2/2 - 9(3 + n)xy^3 + 9(3 + n)y^4/2 \neq 0,$$
$$\mathbf{D} = -3888n^6Z_3, \quad \xi_{22} = 1050n, \quad \chi_4 = 556875n^2(1 + 2n)(3 + 16n).$$

So due to the condition $\chi_4 \neq 0$ in the case $\mathbf{D} \neq 0$ we have

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Z_3), \quad \operatorname{sign}(\xi_{22}) = \operatorname{sign}(n).$$

Thus in the case $B_2 \neq 0$, following the investigation of the family (31) for $g = 2$ we get the following configurations (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_{22} < 0$ (i.e. $n < 0$) $\Rightarrow$ *Config.* $\mathcal{P}$.*34*;
$\mathbf{D} < 0$, $\xi_{22} > 0$ (i.e. $0 < n < 1/16$) $\Rightarrow$ *Config.* $\mathcal{P}$.*33*;
$\mathbf{D} > 0$ (i.e. $n > 1/16$) $\Rightarrow$ *Config.* $\mathcal{P}$.*36*;
$\mathbf{D} = 0$ (i.e. $n = 1/16$) $\Rightarrow$ *Config.* $\mathcal{P}$.*39*.

Assume now $B_2 = 0$. Since the condition $\chi_7 \neq 0$ implies $1 + 2n \neq 0$ we get $(3 + 2n)(3 + n) = 0$. Since for systems (100) we have $\xi_{23} = -225(3 + n)/4$ we arrive at the configuration *Config.* $\mathcal{P}$.*54* in the case $\xi_{23} \neq 0$ (then $n = -3/2$) and *Config.* $\mathcal{P}$.*49* in the case $\xi_{23} \neq 0$ (i.e. $n = -3$).

3.2.2 The statements ($\mathcal{B}_2$), ($\mathcal{B}_3$) and ($\mathcal{B}_4$)

According to Proposition 3 all these three statements have the common condition $\zeta_7 = 0$. Considering (90) and the condition $\chi_4 \neq 0$ we have to force the condition $\mathcal{U}_2 = 4m - 147 - 14n = 0$ and we get $m = 7(21 + 2n)/4$. This leads to 1-parameter family

$$\begin{aligned} \dot{x} =&(147 + 14n + 4nx + 8x^2 - 6y + 4xy)/4, \\ \dot{y} =&(147x + 14nx + 4ny + 2xy + 4y^2)/2, \end{aligned} \tag{101}$$

for which we calculate

$$\begin{aligned}\chi_4 &= 61875(37+4n)(69+4n)(357+4n)(301+36n),\\ \zeta_8 &= 5(21+4n)^2/4, \quad \mathcal{R}_4 = 19500(33+4n).\end{aligned} \tag{102}$$

So following Proposition 3 we have to distinguish three possibilities: $\zeta_8\mathcal{R}_4 \neq 0$ (statement $(\boldsymbol{\mathcal{B}_2})$), $\zeta_8 = 0$ (statement $(\boldsymbol{\mathcal{B}_3})$) and $\mathcal{R}_4 = 0$ (statement $(\boldsymbol{\mathcal{B}_4})$). We examine each one of these possibilities.

3.2.2.1 The possibility $\zeta_8\mathcal{R}_4 \neq 0$. Then $33+4n \neq 0$ and then systems (101) possess the following two invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = -(21+2n)(21+4n) + 6(21+4n)x + 24x^2 - 2(33+4n)y = 0.$$

We observe that systems (101) is a subfamily of (42) defined by the condition $g = 2$. The family (42) was investigated earlier and considering (45) and (47) for $g = 2$ (i.e. for systems (101)) we have

$$\begin{gathered} Z_4 = -(33+4n), \quad \alpha_4 = 133+4n, \quad \beta_4 = 861+100n, \quad \theta = -32 \neq 0,\\ \mathbf{D} = -3(21+4n)^2 Z_4\alpha_4^2\beta_4^2/4, \quad \zeta_2 = 24 > 0, \quad \xi_{14} = 35100000\,\alpha_4\beta_4,\\ B_1 = 105(21+2n)(33+4n)(37+4n)(49+4n)(69+4n)/2. \end{gathered}$$

We observe that due to $\zeta_8\mathcal{R}_4 \neq 0$ we have $Z_4(21+4n) \neq 0$ and in the case $\mathbf{D} \neq 0$ we obtain

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Z_4), \quad \operatorname{sign}(\xi_{14}) = \operatorname{sign}(\alpha_4\beta_4).$$

Moreover the direction of the invariant parabola $\Phi_2(x,y) = 0$ depends on $\operatorname{sign}(33+4n)$.

According to Lemma 1 for the existence of an invariant line of systems (102) the condition $B_1 = 0$ is necessary. So we consider two cases: $B_1 \neq 0$ and $B_1 = 0$.

3.2.2.1.1 The case $B_1 \neq 0$. Then we could not have any invariant line. In this case for the parameter n we detect only three possible bifurcation values: $n \in \{-133/4, -861/100, -33/4\}$. Moreover we point out that due to the condition $\mathcal{R}_4 \neq 0$ we have $Z_4 \neq 0$ (i.e. $n \neq -33/4$).

So considering these possible bifurcation values of the parameter n in the case $B_1 \neq 0$ for systems (101) we determine the following configurations (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_{14} < 0$ (i.e. $-133/4 < n < -861/100$) $\Rightarrow$ *Config.* $\mathcal{P}.60$;
$\mathbf{D} < 0$, $\xi_{14} > 0$, $\beta_4 < 0$ (i.e. $n < -133/4$) $\Rightarrow$ *Config.* $\mathcal{P}.61$;
$\mathbf{D} < 0$, $\xi_{14} > 0$, $\beta_4 > 0$ (i.e. $-861/100 < n < -33/4$) $\Rightarrow$ *Config.* $\mathcal{P}.61$;
$\mathbf{D} > 0$ (i.e. $n > -33/4$) $\Rightarrow$ *Config.* $\mathcal{P}.63$;
$\mathbf{D} = 0$, $\beta_4 \neq 0$ (i.e. $n = -133/4$) $\Rightarrow$ *Config.* $\mathcal{P}.66$;
$\mathbf{D} = 0$, $\beta_4 = 0$ (i.e. $n = -861/100$) $\Rightarrow$ *Config.* $\mathcal{P}.66$.

We observe that we could join the above conditions as follows:

$\mathbf{D} < 0$, $\xi_{14} < 0$ $\Rightarrow$ *Config.* $\mathcal{P}.60$;
$\mathbf{D} < 0$, $\xi_{14} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}.61$;
$\mathbf{D} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}.63$;
$\mathbf{D} = 0$ $\Rightarrow$ *Config.* $\mathcal{P}.66$.

3.2.2.2 The case $B_1 = 0$. Considering (102) and the condition $\chi_4\zeta_8\mathcal{R}_4 \neq 0$ we observe that the condition $B_1 = 0$ is equivalent to $49+4n = 0$. This implies $n = -49/4$ and the corresponding system (101) possesses the invariant line $y = 49/4$ and we obtain the configuration which is equivalent to *Config.* $\mathcal{P}$*.71.*

3.2.2.3 The possibility $\zeta_8 = 0$. Considering (102) this condition gives us $n = -21/4$ and we arrive at the system

$$\begin{aligned}\dot{x} =&(147 - 42x - 12y + 16x^2 + 8xy)/8,\\ \dot{y} =&(147x - 42y + 4xy + 8y^2)/4.\end{aligned} \tag{103}$$

On the other hand for $n = -21/4$ we obtain

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 24(x^2 - y) = 0,$$

i.e. the above system has a double invariant parabola. Moreover in this case we have one real singular point $M_1(x_1, y_1)$ and two complex $M_{2,3}(x_{2,3}, y_{2,3})$, where

$$x_1 = -\frac{7}{2}, \quad y_1 = \frac{49}{4}; \quad x_{2,3} = \frac{3}{2} \pm i\sqrt{3}, \quad y_{2,3} = -\frac{3}{4} \pm 3i\sqrt{3}.$$

We point out that M_1 is a double singularity of systems (103) being located on the double invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. Therefore we arrive at the configuration *Config.* $\mathcal{P}$*.78.*

3.2.2.4 The possibility $\mathcal{R}_4 = 0$. Considering (102) this condition implies $n = -33/4$ and we obtain the system

$$\dot{x} =(2x - 3)(-21 + 8x + 4y)/8, \quad \dot{y} = (63x - 66y + 4xy + 8y^2)/4.$$

We observe that for $n = -33/4$ the second invariant parabola becomes the reducible conic: $\Phi_2(x, y) = 6(2x - 3)^2 = 0$.

So the above system possesses the invariant line $x = 3/2$ and the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$ and it is not too difficult to determine that we have a configuration which is equivalent to *Config.* $\mathcal{P}$*.81.*

3.2.3 The statement ($\mathcal{B}_5$)

According to Proposition 3 we must have the condition $\chi_4 = 0$ and $\zeta_5\zeta_9 \neq 0$. So for systems (89) we calculate:

$$\chi_4 = 61875(1 + 4m + 2n)\mathcal{V}, \quad \xi_{24} = 14062(4m - 14n - 147)\mathcal{V},$$
$$\zeta_5 = 25(196m - 46n - 3)(4m - 147 - 14n)/16,$$
$$\zeta_9 = -2970000(4m - 14n - 147)(10m + 196m^2 - 88mn + 15n^2),$$

where

$$\mathcal{V} = 18m + 1372m^2 - 84mn + 27n^2 + 144n^3.$$

We consider two possibilities: $\xi_{24} \neq 0$ and $\xi_{24} = 0$.

3.2.3.1 The possibility $\xi_{24} \neq 0$. Then $\mathcal{V} \neq 0$ and the condition $1 + 4m + 2n = 0$ implies $m = -(1 + 2n)/4$. This leads to the 1-parameter family of systems

$$\dot{x} = -\frac{1}{4}(2n+1) + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = -\frac{1}{2}(2n+1)x + 2ny + xy + 2y^2, \tag{104}$$

which possess the invariant line $y = x - 1/4$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -1 - 2n + 2(1 + 4n)x - 2(-1 + 4n)y - 8y^2 = 0.$$

For these systems we have

$$\begin{aligned} \zeta_5 &= 25(37 + 4n)(13 + 36n), \quad \xi_{24} = -140625(5 + 4n)^2(37 + 4n)(13 + 36n)/16, \\ \zeta_9 &= 11 \cdot 30^4(1 + 4n)(37 + 4n)(13 + 36n), \quad B_1 = 0, \quad \theta = -32 \neq 0, \\ B_2 &= -81(1 + 2n)(1 + 4n)(17 + 4n)(x - y)^4, \quad \mathbf{D} = 3(1 + 4n)(5 + 4n)^6/4. \end{aligned} \tag{105}$$

Therefore following Lemma 1 we discuss two subcases: $B_2 \neq 0$ and $B_2 = 0$.

1: *The subcase $B_2 \neq 0$.* Then by Lemmas 1 and 2 (since $\theta \neq 0$) we conclude that systems (104) could possess only one invariant line (which is $y = x - 1/4$).

We determine that systems (104) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = \frac{1}{2},\ y_1 = \frac{1}{4};\ \ x_2 = \frac{1 - 4n}{12},\ \ y_2 = -\frac{1 + 2n}{6};$$
$$x_{3,4} = -\frac{1}{2}\big(1 \pm \sqrt{-1 - 4n}\,\big),\ \ y_{3,4} = \frac{1}{2}\big(- 2n \pm \sqrt{-1 - 4n}\,\big).$$

We observe that the invariant parabolas have three points of intersection: M_1, M_3 and M_4 and the singularities M_3 and M_4 could be real or complex depending on the value of $1 + 4n \neq 0$ (due to $\zeta_9 \neq 0$). Moreover the direction of the invariant parabola $\Phi_2(x, y) = 0$ also depends on the value of $1 + 4n$.

It is easy to determine that the invariant line $y = x - 1/4$ is tangent to the both invariant parabola at the point M_1. The singularity M_2 is located on the invariant line and we calculate

$$\Phi_1(x_2, y_2) = \frac{1}{144}(5 + 4n)^2 \neq 0, \quad \Phi_2(x_2, y_2) = -\frac{1}{18}(5 + 4n)^2 \neq 0$$

due to $\xi_{23} \neq 0$. Considering (105) that $\mathbf{D} \neq 0$ due to $\zeta_9\xi_{24} \neq 0$ and $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(5 + 4n)$.

On the other hand we need to know the position of the singularity M_2 with respect to M_1 and we calculate:

$$x_2 - x_1 = \frac{1 - 4n}{12} - \frac{1}{2} = -\frac{5 + 4n}{12} \ \Rightarrow\ \operatorname{sign}(x_2 - x_1) = -\operatorname{sign}(5 + 4n).$$

So all finite singularities (except M_2) are fixed as the intersection points of invariant curves and their positions are determined by the values of the parameter n. More exactly in the case $B_2 \neq 0$ for the parameter n we have the following bifurcation values: $n \in \{-5/4, -1/4\}$.

So considering these possible bifurcation values of the parameter n in the case $B_2 \neq 0$ for systems (101) we determine the following configurations:

$\mathbf{D} < 0$ and $n < -5/4$ $\Rightarrow$ *Config.* $\mathcal{P}$*.101*;
$\mathbf{D} < 0$ and $n > -5/4$ $\Rightarrow$ $\simeq$ *Config.* $\mathcal{P}$*.101*;
$\mathbf{D} > 0$ (i.e. $n > -1/4$) $\Rightarrow$ *Config.* $\mathcal{P}$*.102*.

Thus we obtain the configuration *Config.$\mathcal{P}$.101* if $\mathbf{D} < 0$ and *Config.$\mathcal{P}$.102* if $\mathbf{D} > 0$.

2: The subcase $B_2 = 0$. Considering the condition $\zeta_9 \neq 0$ we get $(1+2n)(17+4n) = 0$.

If $1+2n = 0$ we get $n = -1/2$ and we arrive at the system

$$\dot{x} = (-x + 4x^2 - 3y + 2xy)/2, \quad \dot{y} = y(-1 + x + 2y),$$

possessing the following four invariant curves (two parabolas and two invariant lines):

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = x - 3y + 4y^2 = 0, \quad y = x - 1/4, \quad y = 0.$$

As a result we arrive at the configuration *Config.$\mathcal{P}$.103*.

Assuming $n = -17/4$ we get the system

$$\dot{x} = (-3 + 2x)(-5 + 8x + 4y)/8, \quad \dot{y} = (15x - 34y + 4xy + 8y^2)/4$$

possessing the invariant lines $y = x - 1/4$ and $x = 3/2$ and the invariant parabolas

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 15 - 64x + 72y - 16y^2.$$

We observe that the line $x = 3/2$ is tangent to $\Phi_2(x,y) = 0$ at the singular point $M_4(3/2, 9/4)$. As a result we get the configuration equivalent to *Config.$\mathcal{P}$.103*.

3.2.3.2 The possibility $\xi_{24} = 0$.

This implies $\mathcal{V} = 0$ and we calculate

$$\text{Discrim}\,[\mathcal{V}, m] = 36(1 - 28n)(3 + 28n)^2 \equiv \gamma(n)$$

and since the parameters m and n are real the condition $\gamma(n) \geq 0$ is necessary.

We claim that the condition $\zeta_5 \neq 0$ implies that $3 + 28n \neq 0$. Indeed setting $n = -3/28$ we calculate

$$\mathcal{V} = \frac{(2744m + 27)^2}{5488} = 0 \quad \Rightarrow \quad m = -\frac{27}{2744},$$

and this implies $\zeta_5 = 0$. So our claim is proved.

Therefore and the condition $1-28n \geq 0$ is necessary for the existence of real roots of the polynomial $\mathcal{V}$. Then we can set a new parameter u as follows: $1 - 28n = u^2 \geq 0$. Then $n = (1 - u^2)/28$ and we obtain

$$\mathcal{V} = \frac{1}{5488}(15 + 2744m + 24u + 3u^2 - 6u^3)(15 + 2744m - 24u + 3u^2 + 6u^3) = 0.$$

Due to the change $u \to -u$ we may assume that the second factor vanishes and we get

$$m = -3(u-1)^2(5 + 2u)/2744.$$

This leads to the 1-parameter family of systems

$$\begin{aligned} \dot{x} &= -\frac{3(u-1)^2(5+2u)}{2744} - \frac{u^2-1}{28}x - \frac{3y}{2} + 2x^2 + xy, \\ \dot{y} &= -\frac{3(u-1)^2(5+2u)}{1372}x - \frac{u^2-1}{14}y + xy + 2y^2, \end{aligned} \tag{106}$$

which possess two invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 3(u-1)^4 + 112(u-1)^3x + 1176(u-1)^2y - 38416y^2 = 0.$$

For these systems we have

$$\begin{aligned}
\zeta_5 &= 25(u-50)(u-22)(u^2-4)(3u-10)(3u+46)/38416,\\
\zeta_9 &= 2227500(u-50)(u-22)(u^2-4)(u-1)^3(3u-10)(3u+46)/823543,\\
\zeta_{10} &= -15(354160 + 48u - 1336u^2 + 64u^3 + 27u^4 - 19u^5 + 3u^6)/38416,\\
\mathbf{D} &= -3^5 7^{-18}(u-1)^6(6+u)^8(34+u)^2(2+5u)^2
\end{aligned} \tag{107}$$

On the other hand for systems (106) we have

$$B_1 = -3^3\, 2^{-5}\, 7^{-15}(u-22)(u-8)^3(u-1)^3(6+u)^3(13+u)(20+u)(5+2u)(3u-10)(4+3u) \tag{108}$$

and following Lemma 1 we discuss two subcases: $B_1 \neq 0$ and $B_1 = 0$.

1: *The case* $B_1 \neq 0$. Then by Lemma 1 systems (106) could not possess any invariant line.

We determine that systems (106) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned}
x_1 &= -\frac{5+2u}{14},\ y_1 = \frac{(5+2u)^2}{196};\quad x_2 = \frac{(u-1)(22+26u+u^2)}{1372},\quad y_2 = -\frac{(u-1)^2(5+2u)}{1372};\\
x_3 &= \frac{1-u}{14},\quad y_3 = \frac{(1-u)^2}{196};\quad x_4 = \frac{3(u-1)}{14},\quad y_4 = \frac{9(u-1)^2}{196}.
\end{aligned} \tag{109}$$

We observe that the invariant parabolas have two points of intersection: M_3 and M_4. Moreover the finite singularity M_1 (respectively, M_2) is located on the invariant parabola $\Phi_1(x,y) = 0$ (respectively $\Phi_2(x,y) = 0$) and the direction of the invariant parabola $\Phi_2(x,y) = 0$ depends on the value of $u - 1 \neq 0$ (due to $\zeta_9 \neq 0$).

Since the condition $\mathbf{D} = 0$ implies the existence of a multiple singularity we examine two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The subcase* $\mathbf{D} \neq 0$. Then all the singularities of systems (106) are different and in order to determine the positions of the singularities M_1 and M_2 with respect to the common points of intersection M_3 and M_4 of the parabolas we calculate:

$$\begin{aligned}
x_3 - x_1 &= \frac{6+u}{14},\quad x_3 - x_2 = -\frac{(u-1)(6+u)(20+u)}{1372},\\
x_4 - x_1 &= \frac{2+5u}{14},\quad x_4 - x_2 = -\frac{(u-8)(u-1)(34+u)}{1372}.
\end{aligned}$$

Moreover the singular point M_2 which lies on the parabola $\Phi_2(x,y) = 0$ could be located above or below its axis $y = y_v$ where y_v is the ordinate of the vertex of this parabola. For the parabola $\Phi_2(x,y) = 0$ we obtain $y_v = 3(u-1)^2/196$ and then we calculate

$$y_2 - y_v = -(u-1)^2(13+u)/686 \ \Rightarrow\ \mathrm{sign}\,(y_2 - y_v) = -\mathrm{sign}\,(13+u).$$

On the other hand we have

$$\Phi_1(x_2, y_2) = \frac{(u-1)^2(6+u)^3(34+u)}{1882384}, \quad \Phi_2(x_1, y_1) = -(6+u)^3(2+5u)$$

and we observe that for the parameter u we distinguish the following possible bifurcation values: $u \in \{-34, -20, -13, -6, -2/5, 1, 8\}$. We point out that due to the condition $\zeta_5\zeta_9 B_1 \mathbf{D} \neq 0$ we have

$$(u-8)(u-1)(u+20)(u+13)(u+6)(34+u)(2+5u) \neq 0.$$

So considering the possible bifurcation values of the parameter u in the case $B_1 \neq 0$ and $\mathbf{D} \neq 0$ for systems (106) we determine the following configurations:

$$\begin{array}{lll}
u < -34 & \Rightarrow x_3 > x_4,\ x_1 > x_3,\ x_2 < x_4,\ y_2 > y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.104;\\
-34 < u < -20 & \Rightarrow x_3 > x_4,\ x_1 > x_3,\ x_4 < x_2 < x_3,\ y_2 > y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.105;\\
-20 < u < -13 & \Rightarrow x_3 > x_4,\ x_1 > x_3,\ x_4 < x_3 < x_2,\ y_2 > y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.105;\\
-13 < u < -6 & \Rightarrow x_3 > x_4,\ x_1 > x_3,\ x_4 < x_3 < x_2,\ y_2 < y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.105;\\
-6 < u < -2/5 & \Rightarrow x_3 > x_4,\ x_4 < x_1 < x_3,\ x_4 < x_2 < x_3,\ y_2 < y_v & \Rightarrow \simeq \textit{Config.}\,\mathcal{P}.105;\\
-2/5 < u < 1 & \Rightarrow x_3 > x_4,\ x_1 < x_3,\ x_4 < x_2 < x_3,\ y_2 < y_v & \Rightarrow \simeq \textit{Config.}\,\mathcal{P}.104;\\
1 < u < 8 & \Rightarrow x_3 > x_4,\ x_1 < x_3,\ x_4 < x_2 < x_3,\ y_2 < y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.106;\\
u > 8 & \Rightarrow x_3 > x_4,\ x_1 < x_3,\ x_4 < x_2 < x_3,\ y_2 < y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.106.
\end{array}$$

So considering the above obtained configurations and the corresponding conditions we deduce the following common conditions:

$$\begin{array}{lll}
\textit{Config.}\,\mathcal{P}.105 & \Leftrightarrow & (u+34)(5u+2) < 0;\\
\textit{Config.}\,\mathcal{P}.104 & \Leftrightarrow & (u+34)(5u+2) > 0 \text{ and } u-1 < 0;\\
\textit{Config.}\,\mathcal{P}.106 & \Leftrightarrow & (u+34)(5u+2) > 0 \text{ and } u-1 > 0.
\end{array}$$

On the other hand for systems (106) we have

$$\xi_{25} = 2^{-3}7^{-9}5913(u-1)^3(6+u)^4(34+u)(2+5u),$$
$$\zeta_5\zeta_9 = 2^{-2}7^{-11}3^4 5^6 11(u-1)^3(-50+u)^2(-22+u)^2(-2+u)^2(2+u)^2(-10+3u)^2(46+3u)^2,$$

and due to $\mathbf{D}\zeta_5\zeta_9 \neq 0$ we have

$$\xi_{25} \neq 0, \quad \operatorname{sign}(\xi_{25}) = \operatorname{sign}\big((u-1)(34+u)(2+5u)\big), \quad \operatorname{sign}(\zeta_5\zeta_9) = \operatorname{sign}(u-1).$$

This leads to the following invariant conditions:

$$\begin{array}{lll}
\xi_{25} < 0 & \Leftrightarrow & \textit{Config.}\,\mathcal{P}.104;\\
\xi_{25} > 0,\ \zeta_5\zeta_9 < 0 & \Leftrightarrow & \textit{Config.}\,\mathcal{P}.105;\\
\xi_{25} > 0,\ \zeta_5\zeta_9 > 0 & \Leftrightarrow & \textit{Config.}\,\mathcal{P}.106.
\end{array}$$

1.2: *The subcase* $\mathbf{D} = 0$. Considering the condition $B_1 \neq 0$ this implies $(34+u)(2+5u) = 0$. Taking into consideration the position of the invariant parabolas and the coordinates (109) of systems (106) we obtain:

$$\begin{array}{lll}
u = -34 & \Rightarrow x_3 > x_4,\ x_1 > x_3,\ x_2 = x_4,\ y_2 > y_v & \Rightarrow \textit{Config.}\,\mathcal{P}.107;\\
u = -2/5 & \Rightarrow x_3 > x_4,\ x_1 = x_4,\ x_4 < x_2 < x_3,\ y_2 < y_v & \Rightarrow \simeq \textit{Config.}\,\mathcal{P}.107.
\end{array}$$

So we deduce that in the case $B_1 \neq 0$ and $\mathbf{D} = 0$ we get the unique configuration *Config.* $\mathcal{P}.107$.

2: *The case* $B_1 = 0$. Considering (108) and (107) we conclude that due to $\zeta_9 \neq 0$ the condition $B_1 = 0$ is equivalent to

$$(u-8)(6+u)(13+u)(20+u)(5+2u)(4+3u) = 0.$$

However we could decrease the number of the factors.

First of all we give the following remark.

Remark 6. *We remark that in the case* $u - 1 \neq 0$ *(i.e. when the second parabola exists) applying the transformation*

$$x_1 = \frac{343}{(u-1)^3}\,y - \frac{21}{4(u-1)}, \quad y_1 = \frac{343}{(u-1)^3}\,x + \frac{147}{4(u-1)^2}, \quad t_1 = \frac{(u-1)^3}{343}\,t$$

we arrive at the family of systems of the same form (106)*:*

$$\begin{aligned}
\dot{x}_1 &= -\frac{3(u_1-1)^2(5+2u_1)}{2744} - \frac{u_1^2-1}{28}\,x_1 - \frac{3y_1}{2} + 2x_1^2 + x_1y_1,\\
\dot{y}_1 &= -\frac{3(u_1-1)^2(5+2u_1)}{1372}\,x_1 - \frac{u_1^2-1}{4}\,y + x_1y_1 + 2y_1^2
\end{aligned}$$

with the parameter $u_1 = \dfrac{48+u}{u-1}$ *(then* $u = \dfrac{48+u_1}{u_1-1}$*).*

Considering Remark 6 and the relation $u = \dfrac{48+u_1}{u_1-1}$ we calculate

$$(u+13) = \frac{7(5+2u_1)}{u_1-1}; \qquad (u+20) = \frac{7(3u_1+4)}{u_1-1}.$$

So to determine the configurations given by the condition $B_1 = 0$ it is sufficient to consider the conditions provided by the equality

$$(u-8)(6+u)(5+2u)(4+3u) = 0.$$

2.1: *The possibility* $u = -4/3$. This leads to the system

$$\dot{x} = (-1 - 2x + 144x^2 - 108y + 72xy)/72, \quad \dot{y} = (x+2y)(-1+36y)/36$$

possessing the invariant line $y = 1/36$ and two invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 1 - 16x + 72y - 432y^2.$$

We determine that te configuration of the above system correspond to *Config.* $\mathcal{P}$*.108.*

2.2: *The possibility* $u = 8$. This leads to the system

$$\dot{x} = (-3+2x)(3+8x+4y)/8, \quad \dot{y} = (x+2y)(-9+4y)/4$$

possessing three invariant lines $y = 9/4$, $y = x + 3/4$ and $x = 3/2$ and two invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 3 + 16x + 24y - 16y^2.$$

We observe that all five invariant curves intersect at the singular point $M_4(3/2, 9/4)$. So we get the configuration *Config. $\mathcal{P}$.109*.

2.3: *The possibility* $u = -5/2$. In this case arrive at the system

$$\dot{x} = (-3x + 32x^2 - 24y + 16xy)/16, \quad \dot{y} = y(-3 + 8x + 16y)/4$$

possessing the invariant line $y = 0$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 32x + 96y - 256y^2.$$

We observe that the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x, y) = 0$ at the point $M_1(0, 0)$ and intersect the second parabola at $M_2(3/32, 0)$. It is not too difficult to determine that in this case we have the configuration *Config. $\mathcal{P}$.110*.

2.4: *The possibility* $u = -6$. In this case we get the system

$$\dot{x} = (3 - 10x + 16x^2 - 12y + 8xy)/8, \quad \dot{y} = (3x - 10y + 4xy + 8y^2)/4$$

possessing the invariant line $y = x - 1/4$ and the invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 16x + 24y - 16y^2 = 0.$$

Considering the coordinates (109) of systems (106) we observe that for $u = -6$ the singular points M_2, M_3 and M_1 coalesced producing a triple singular point. Moreover this triple singularities is a point of tangency of the invariant line $y = x - 1/4$ with both parabolas. As a result we get the configuration *Config. $\mathcal{P}$.111*.

On the other hand for systems (106) we have

$$\begin{aligned}
\zeta_9 &= \frac{2227500}{823543}(u - 50)(u - 22)(u^2 - 4)(u - 1)^3(3u - 10)(3u + 46),\\
\zeta_6 &= -\frac{3}{1372}(6 + u)(292 - 52u + 5u^2), \quad \text{Discrim}\,[292 - 52u + 5u^2] = -3136 < 0,\\
\xi_{26} &= \frac{3}{941192}(u - 50)(u^2 - 4)(u - 1)(3u + 46)(6 + u)(13 + u)(5 + 2u),\\
\xi_{27} &= \frac{1}{2744}(u - 8)(380 + 52u + 9u^2), \quad \text{Discrim}\,[380 + 52u + 9u^2] = -10976 < 0.
\end{aligned}$$

We observe that due do $\zeta_9 \neq 0$ the condition $\xi_{26} \neq 0$ is equivalent to $(6 + u)(13 + u)(5 + 2u) \neq 0$. Moreover considering Remark 6 we conclude that for $\xi_{26} = 0$ we may assume $(6 + u)(5 + 2u) = 0$ because the condition $13 + u = 0$ could be brought to $5 + 2u = 0$ via an affine transformation and time rescalling.

Thus in the case $B_1 = 0$ systems (106) possess the following configurations if and only if the corresponding conditions are satisfied:

$\xi_{26} \neq 0$, $\xi_{27} \neq 0$ (then $u = -4/3$) $\Leftrightarrow$ *Config. $\mathcal{P}$.108*;
$\xi_{26} \neq 0$, $\xi_{27} = 0$ (then $u = 8$) $\Leftrightarrow$ *Config. $\mathcal{P}$.109*;
$\xi_{26} = 0$, $\zeta_6 \neq 0$ (then $u = -5/2$) $\Leftrightarrow$ *Config. $\mathcal{P}$.110*;
$\xi_{26} = 0$, $\zeta_6 = 0$ (then $u = -6$) $\Leftrightarrow$ *Config. $\mathcal{P}$.111*.

3.2.4 The statement ($\mathcal{B}_6$)

In this case the condition $\chi_4 = \zeta_9 = 0$ must be fulfilled. Due to the condition $\zeta_5 \neq 0$ we obtain that $\zeta_9 = 0$ is equivalent to $\mathcal{W} = 0$. Straightforward calculations gives us that the systems of equations $\chi_4 = 0$ and $\mathcal{W} = 0$ could have only the following solutions $\mathcal{S}_i = (m_i, n_i)$ $(i = 1, 2, 3, 4)$:

$$\mathcal{S}_1 = (0,0), \quad \mathcal{S}_2 = \Big(-\frac{1}{8}, -\frac{1}{4}\Big), \quad \mathcal{S}_3 = \Big(-\frac{5}{72}, -\frac{13}{36}\Big), \quad \mathcal{S}_4 = \Big(-\frac{27}{2944}, -\frac{3}{8}\Big).$$

However we have

$$\chi_4(\mathcal{S}_i) = \zeta_9(\mathcal{S}_i) = 0, \quad i = 1, 2, 3, 4,$$
$$\zeta_5(\mathcal{S}_1) \neq 0, \quad \zeta_5(\mathcal{S}_2) \neq 0, \quad \zeta_5(\mathcal{S}_3) = \zeta_5(\mathcal{S}_4) = 0,$$

and hence only the solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the conditions of statement ($\mathcal{B}_6$). Therefore we examine only these two solutions.

We observe that each one of them gives us a concrete system (without parameters) and it remains to construct the corresponding system having a single fixed configuration of the invariant parabolas and lines.

For systems (89) we calculate:

$$\begin{aligned}\xi_9 =& 3^6 17252510\big[65536m^4 - 32m^3(6131 + 3252n) - 16m^2(-32110 - 7953n + 484n^2)\\ &+ 6m(10221 - 53292n + 5540n^2 + 4336n^3) - 9(-2304 - 7857n - 12140n^2 + 836n^3 + 240n^4)\big].\end{aligned}$$

We observe that under the conditions of statement ($\mathcal{B}_6$) the condition $\xi_9 = 0$ leads to the solution $\mathcal{S}_2$, because $\xi_9(\mathcal{S}_2) = 0$ and $\xi_9(\mathcal{S}_1) \neq 0$. Then we examine two possibilities: $\xi_9 \neq 0$ and $\xi_9 = 0$.

3.2.4.1 The possibility $\xi_9 \neq 0$. Then the conditions provided by statement ($\mathcal{B}_6$) lead to the solution $\mathcal{S}_1$ and therefore we have $m = n = 0$. In this case we arrive at the system

$$\dot{x} = -\frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(x + 2y), \tag{110}$$

possessing the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

We determine that the above system possesses the following two singular points: $M_1(0, 0)$ and $M_2(-1/2, 1/4)$. We observe that the point M_1 is the point of tangency of the invariant line with the parabola. Moreover this point is a triple singularity of system (110), because we have

$$\mu_4 = \mu_3 = \mu_2 = 0, \quad \mu_1 = -3(x + 2y) \neq 0,$$

and by [1, Lemma 5.2, statement (ii)] the point M_1 is of multiplicity exactly 3. As a result we get the configuration *Config. $\mathcal{P}$.42.*

3.2.4.2 The possibility $\xi_9 = 0$. In this case we get the solution $\mathcal{S}_2$, i.e. $m = -1/8$ and $n = -1/4$. So we arrive at the system

$$\dot{x} = -\frac{1}{8} - \frac{x}{4} - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = \frac{1}{4}(4y - 1)(x + 2y),$$

possessing the invariant lines $y = 1/4$ and $y = x - 1/4$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$. As a result we get the configuration *Config. $\mathcal{P}$.112.*

3.2.5 The statement $(\mathcal{B}_7)$

In this case the condition $\chi_4 = \zeta_5 = 0$ and $\zeta_6 \neq 0$ must be fulfilled. Straightforward calculations gives us that the systems of equations $\chi_4 = 0$ and $\zeta_5 = 0$ could have only the following solutions $\widetilde{\mathcal{S}}_i = (m_i, n_i)$ $(i = 1, \ldots, 6)$:

$$\widetilde{\mathcal{S}}_1 = \Big(-\frac{27}{2744}, -\frac{3}{28}\Big),\ \widetilde{\mathcal{S}}_2 = \Big(-\frac{2205}{8}, -\frac{357}{4}\Big),\ \widetilde{\mathcal{S}}_3 = \Big(\frac{539}{72}, -\frac{301}{36}\Big),$$
$$\widetilde{\mathcal{S}}_4 = \Big(\frac{35}{8}, -\frac{37}{4}\Big),\ \widetilde{\mathcal{S}}_5 = \Big(-\frac{5}{72}, -\frac{13}{36}\Big),\ \widetilde{\mathcal{S}}_6 = \Big(-\frac{189}{8}, -\frac{69}{4}\Big),$$

We split these solutions into two sets:

$$\mathcal{G}_1 = \{\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2, \widetilde{\mathcal{S}}_3\},\quad \mathcal{G}_2 = \{\widetilde{\mathcal{S}}_4, \widetilde{\mathcal{S}}_5, \widetilde{\mathcal{S}}_6\}.$$

Lemma 14. *Assume that the conditions of statement* $(\mathcal{B}_7)$ *are satisfied and then the system of equations* $\chi_4 = \zeta_5 = 0$ *generates six solutions* $\widetilde{\mathcal{S}}_i = (m_i, n_i)$ $(i = 1, \ldots, 6)$ *given above. In this case the invariant polynomial* ξ_6 *distinguishes the set* $\mathcal{G}_1$ *from the set* $\mathcal{G}_2$*.*

Proof: To prove this lemma it is sufficient to evaluate ξ_6 for the elements of each one of the sets. For systems (89) we calculate

$$\xi_6 = 2^9 3^4 2877985 m(1 + 4m + 2n)(-147 + 50m + 61n + 8n^2),$$

and we obtain

$$\xi_6(\widetilde{\mathcal{S}}_i) \neq 0,\quad i = 1, 2, 3,\qquad \xi_6(\widetilde{\mathcal{S}}_j) = 0,\quad j = 4, 5, 6,$$

and we complete the proof of the lemma. ∎

According to the above lemma we discuss two cases: $\xi_6 \neq 0$ and $\xi_6 = 0$.

3.2.5.1 The case $\xi_6 \neq 0$. Then we have to examine the elements of the first set $\mathcal{G}_1$.

1: *The subcase* $\widetilde{\mathcal{S}}_1$. Then we have $m = -27/2744$ and $n = -3/28$ and we get the system

$$\dot{x} = 2x^2 + xy - \frac{3x}{28} - \frac{3y}{2} - \frac{27}{2744},\quad \dot{y} = xy - \frac{27x}{1372} + 2y^2 - \frac{3y}{14}, \tag{111}$$

possessing three invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and

$$\Phi_2(x, y) = 3 + 112x + 1176y - 38416y^2 = 0,\quad \Phi_3(x, y) = -243 + 3024x - 10584y + 38416y^2 = 0.$$

We observe that the singular point $M_1(3/14, 9/196)$ is the point of intersection of all three invariant parabolas. So we get the configuration *Config.* $\mathcal{P}$*.113*.

Next we prove that the systems generated by $\widetilde{\mathcal{S}}_2$ and $\widetilde{\mathcal{S}}_3$ could be brought to system (111) via an affine transformation and a time rescaling.

Consider first the solution $\widetilde{\mathcal{S}}_2$, i.e. $m = -2205/8$ and $n = -357/4$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{357x}{4} - \frac{3y}{2} - \frac{2205}{8},\quad \dot{y} = xy - \frac{2205x}{4} + 2y^2 - \frac{357y}{2},$$

which via the transformation

$$x_1 = -3/28 + y/343, \quad y_1 = 3/196 + x/343, \quad t_1 = 343t,$$

could be brought to the system (111).

Analogously, taking the solution $\widetilde{\mathcal{S}}_3$, i.e. $m = 539/72$ and $n = -301/36$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{301x}{36} - \frac{3y}{2} + \frac{539}{72}, \quad \dot{y} = xy + \frac{539x}{36} + 2y^2 - \frac{301y}{18},$$

which via the transformation

$$x_1 = 9/28 - 27y/343, \quad y_1 = 27/196 - 27x/343, \quad t_1 = -343t/27,$$

could be brought to the system (111).

3.2.5.2 The case $\xi_6 = 0$.

Then we have to examine the elements of the first set $\mathcal{G}_2$.

1: *The subcase* $\widetilde{\mathcal{S}}_4$. Then we have $m = 35/8$ and $n = -37/4$ and we get the system

$$\dot{x} = 2x^2 + xy - \frac{37x}{4} - \frac{3y}{2} + \frac{35}{8}, \quad \dot{y} = xy + \frac{35x}{4} + 2y^2 - \frac{37y}{2}, \tag{112}$$

possessing the invariant line $y = x - 1/4$ and three invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and

$$\Phi_2(x, y) = 5 - 12x + 3x^2 + y = 0, \quad \Phi_3(x, y) = -35 + 144x - 152y + 16y^2 = 0.$$

We observe that the singular point $M_1(1/2, 1/4)$ is the point of intersection of all four invariant curves. So we get the configuration *Config.*$\mathcal{P}$*.114*.

Next we prove that the systems generated by $\widetilde{\mathcal{S}}_5$ and $\widetilde{\mathcal{S}}_6$ could be brought to system (112) via an affine transformation and a time rescaling.

Consider first the solution $\widetilde{\mathcal{S}}_5$, i.e. $m = -5/72$ and $n = -13/36$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{13x}{36} - \frac{3y}{2} - \frac{5}{72}, \quad \dot{y} = -((5x)/36) - (13y)/18 + xy + 2y^2,$$

which via the transformation

$$x_1 = 11/4 - 9y, \quad y_1 = 19/4 - 9x, \quad t_1 = -t/9,$$

could be brought to the system (112).

Analogously, taking the solution $\widetilde{\mathcal{S}}_6$, i.e. $m = -189/8$ and $n = -69/4$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{69x}{4} - \frac{3y}{2} - \frac{189}{8}, \quad \dot{y} = xy - \frac{189x}{4} + 2y^2 - \frac{69y}{2},$$

which via the transformation

$$x_1 = 2 - x/3, \quad y_1 = 7 - y/3, \quad t_1 = -3t,$$

could be brought to the system (112).

3.3 Configurations of systems in $\mathbf{QSP}_{(\eta<0)}$

In what follows we examine the configurations of the systems in $\mathbf{QSP}_{(\eta<0)}$ in each one of the cases provided by Proposition 4. According to this proposition we consider the canonical form (6), i.e. the systems

$$\dot{x} = m + (2n-1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \tag{113}$$

with $C_2 = x(x^2+y^2)$, possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

In what follows we examine the configurations of the systems in $\mathbf{QSP}_{(\eta<0)}$ in each one of the cases provided by Proposition 4.

3.3.1 The statement ($\mathcal{E}_1$)

For systems (113) we calculate

$$\begin{aligned} \zeta_4 =& (25+g^2)(3g+9g^3-4m-6gn)/16, \\ \mathcal{R}_1 =& 15(1+g^2)(25+g^2)(3g+9g^3-4m-6gn)/2. \end{aligned} \tag{114}$$

3.3.1.1 The case $B_1 \neq 0$. Then according to Lemma 1 systems (113) could not possess any invariant line.

Let us examine the finite singularities of these systems. Following [1, Proposition 5.1] we calculate the invariant polynomial $\mathbf{D} = 12\mathcal{F}_1'^2\mathcal{F}_2'$, where

$$\begin{aligned} \mathcal{F}_1' =& -2gm - 2g^3m + 4m^2 - n - g^2n - 4gmn + g^2n^2, \\ \mathcal{F}_2' =& 8 - g^2 - 72gm + 8g^3m + 432m^2 - 48n + 4g^2n + 144gmn + 96n^2 - 4g^2n^2 - 64n^3, \end{aligned} \tag{115}$$

and we discuss two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

3.3.1.1.1 The subcase $\mathbf{D} \neq 0$. We determine that systems (113) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{aligned}
x_1 =& -\frac{gn-2m}{g^2+1}, \; y_1 = \frac{2gm+n}{g^2+1}; \quad x_2 = \frac{1}{6\mathcal{Y}^{1/3}}\big[\mathcal{Y}^{2/3} + \mathcal{Y}^{1/3}g - 3g^2 + 2\mathcal{Z}\big], \\
y_2 =& \frac{1}{6\left(\sqrt[3]{\mathcal{W}}\,(2\mathcal{Z}-3g^2) + 4g\mathcal{W}^{2/3} + \mathcal{W}\right)}\big[3\sqrt{3\mathcal{F}_2'}\mathcal{Z} + 3\mathcal{W}^{2/3}\left(g^3 + 10gn - 5g + 12m\right) \\
& + \mathcal{W}^{1/3}\mathcal{Z}\left(2\mathcal{Z} - 3g^2\right) + 3\left(g\left(g^4 + 22g^2n - 11g^2 + 84n^2 - 84n + 21\right) + 36m\mathcal{Z}\right) + g\mathcal{W}^{4/3}\big]; \\
x_3 =& \frac{1}{12\mathcal{Y}^{1/3}}\big[\left(-1+i\sqrt{3}\right)\mathcal{Y}^{2/3} + 2g\mathcal{Y}^{1/3} - \left(1+i\sqrt{3}\right)\left(2\mathcal{Z}-3g^2\right)\big], \\
y_3 =& \frac{1}{-48g\mathcal{Y}^{2/3} + 6\left(1+i\sqrt{3}\right)\mathcal{Y}^{1/3}\left(2\mathcal{Z}-3g^2\right) + 6\left(1-i\sqrt{3}\right)\mathcal{Y}}\big[-6\mathcal{Y}^{2/3}\left(g^3 + 10gn - 5g + 12m\right) \\
& + \left(1+i\sqrt{3}\right)\mathcal{Y}^{4/3}g + \left(1+i\sqrt{3}\right)\mathcal{Y}^{1/3}\mathcal{Z}\left(2\mathcal{Z}-3g^2\right) \\
& + \left(1-i\sqrt{3}\right)\mathcal{Y}\mathcal{Z} + \left(1-i\sqrt{3}\right)g\left(2\mathcal{Z}-3g^2\right)^2\big];
\end{aligned}$$

$$x_4 = \frac{1}{12\mathcal{Y}^{1/3}}\Big[\Big(-1-i\sqrt{3}\Big)\mathcal{Y}^{2/3}+2g\mathcal{Y}^{1/3}-\Big(1-i\sqrt{3}\Big)\big(2\mathcal{Z}-3g^2\big)\Big],$$

$$\begin{aligned}y_4 =& \frac{1}{-48g\mathcal{Y}^{2/3}+6\,(1-i\sqrt{3})\,\mathcal{Y}^{1/3}\,(2\mathcal{Z}-3g^2)+6\,(1+i\sqrt{3})\,\mathcal{Y}}\Big[-6\mathcal{Y}^{2/3}\,(g^3+10gn-5g+12m)\\ &+\Big(1-i\sqrt{3}\Big)\mathcal{Y}^{4/3}g+\Big(1-i\sqrt{3}\Big)\mathcal{Y}^{1/3}\mathcal{Z}\,(2\mathcal{Z}-3g^2)\\ &+\Big(1+i\sqrt{3}\Big)\mathcal{Y}\mathcal{Z}+\Big(1+i\sqrt{3}\Big)\,g\,\big(2\mathcal{Z}-3g^2\big)^2\Big],\end{aligned}$$

where

$$\mathcal{X} = 24-3g^2-216gm+24g^3m+1296m^2-144n+12g^2n+432gmn+288n^2-12g^2n^2-192n^3,$$

$$\mathcal{Y} = g^3+18gn-9g+108m+3\sqrt{\mathcal{X}},\quad \mathcal{W} = -9g+g^3+108m+18gn+3\sqrt{3\mathcal{F}_2'},$$

$$\mathcal{Z} = -3+2g^2+6n.$$

Calculations yield:

$$\Phi(x_2,y_2)=\Phi(x_3,y_3)=\Phi(x_4,y_4)=0,\quad \Phi(x_1,y_1)=\frac{\mathcal{F}_1'}{(1+g^2)^2}$$

and therefore we deduce that three singularities M_2, M_3 and M_4 of systems (113) are located at the invariant parabola. Moreover M_1 is located outside the parabola and could belong to it if and only if the condition $\mathcal{F}_1'=0$ holds, where $\mathcal{F}_1'$ is given in (115). However we have $\mathbf{D}=12\mathcal{F'}_1^2\mathcal{F}_2'\neq 0$ and hence on the parabola we always have three distinct singularities.

On the other hand according to [1, Proposition 5.1] if $\mathbf{D}>0$ systems (113) possess two real and two complex finite singularities. For $\mathbf{D}<0$ we could have either 4 real or 4 complex finite singularities. However since M_1 is a real singular point for these systems we conclude that in the case $\mathbf{D}<0$ we have 4 real finite distinct singularities.

Thus since the real singularity M_1 is outside the invariant parabola and all three finite singularities on the parabola (real or complex) are distinct and furthermore we could not have any invariant line we arrive at the configuration *Config.*$\mathcal{P}$*.115* if $\mathbf{D}<0$ and *Config.*$\mathcal{P}$*.116* if $\mathbf{D}>0$.

3.3.1.1.2 The subcase $\mathbf{D}=0$. This implies $\mathcal{F}_1'\mathcal{F}_2'=0$ and we calculate:

$$\xi_1=-6\zeta_4\mathcal{F}_1'.$$

Therefore we deduce that due to $\zeta_4\neq 0$ the condition $\mathcal{F}_1'=0$ is equivalent to $\xi_1=0$. So we examine two possibilities: $\xi_1\neq 0$ and $\xi_1=0$.

1: *The possibility* $\xi_1\neq 0$. In this case the condition $\mathbf{D}=0$ implies $\mathcal{F}_2'=0$. Since this polynomial is quadratic with respect to the parameter m we calculate

$$\text{Discrim}\,[\mathcal{F}_2',m]=64(g^2+12n-6)^3.$$

Therefore since the parameters m, n and g of systems (113) must be real we conclude that the condition $g^2+12n-6\geq 0$ has to be fulfilled. So setting a new parameter v: $g^2+12n-6=v^2\geq 0$ we get $n=(6-g^2+v^2)/12$ and then we calculate

$$\mathcal{F}_2'=\frac{1}{108}\big[216m-(g+v)^2(g-2v)\big]\big[216m-(g-v)^2(g+2v)\big]=0$$

and due to the change $v \to -v$ we could force the first factor to vanish. Then we obtain

$$m = (g-2v)(g+v)^2/216$$

and considering the expression for the parameters m and n we arrive at the 2-parameter family of systems

$$\begin{aligned}\dot{x} =& \frac{(g-2v)(g+v)^2}{216} - \frac{g^2-v^2}{12}x - \frac{g}{2}y + gx^2 - xy,\\ \dot{y} =& \frac{(g-2v)(g+v)^2}{108}x + \frac{6-g^2+v^2}{6}y + gxy - 2y^2,\end{aligned} \tag{116}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$. We observe that for the above systems we have the following conditions on the parameters g and v:

$$\begin{aligned}\xi_1 \neq 0 \ \Leftrightarrow\ & (8g-v)^2(4g+v)\left(2g^2-8gv-v^2+18\right)\left(g^2+2gv+v^2+9\right) \neq 0;\\ B_1 \neq 0 \ \Leftrightarrow\ & (2g-v)(4g+v)(36+4g^2-4gv+v^2)(g^2+2gv+v^2+9)\\ & \times (g^2-4gv+4v^2+9) \neq 0.\end{aligned} \tag{117}$$

We determine that systems (116) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates

$$\begin{aligned}x_1 &= \frac{-27g+5g^3-6gv^2-v^3}{54(1+g^2)},\ y_1 = \frac{54-9g^2+g^4+9v^2-3g^2v^2-2gv^3}{108(1+g^2)};\\ x_2 &= \frac{g-2v}{6},\ y_2 = \frac{(g-2v)^2}{36};\ \ x_3 = \frac{g+v}{6},\ y_3 = \frac{(g+v)^2}{36}.\end{aligned}$$

We calculate

$$\Phi(x_2,y_2) = \Phi(x_3,y_3) = 0,\quad \Phi(x_1,y_1) = -\frac{\left(2g^2-8gv-v^2+18\right)\left(g^2+2gv+v^2+9\right)^2}{2916\left(g^2+1\right)^2}$$

and we conclude that the singular points M_2 and M_3 are located on the invariant parabola, whereas M_1 is outside the invariant parabola due to the conditions (117).

We claim that M_3 is a multiple singularity of systems (116). Indeed, applying the corresponding translation, we could place M_3 at the origin of coordinates and we arrive at the systems

$$\begin{aligned}\dot{x} =& \frac{1}{18}(g+v)(4g+v)x - \frac{1}{6}(4g+v)y + gx^2 - xy,\\ \dot{y} =& \frac{1}{54}(g+v)\left(2g^2+gv-v^2-18\right)x + \frac{1}{18}\left(v^2-2g^2-gv+18\right)y + gxy - x^2 - 2y^2,\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_3.

Considering [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$ and

$$\mu_2 = \frac{1}{324}v\left[(g+v)^2+9\right]\left[(v-2g)x+6y\right]\left[(3-g^2-gv)x+(4g+v)y\right],$$

and by [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity at least 2. We observe that due to the condition $B_1 \neq 0$ we have $\mu_2 = 0$ if and only if $v = 0$. In this case we calculate

$$\mu_2 = 0,\quad \mu_1 = -\frac{1}{27}\left[(5g^4+27)x - 32g^3y\right] \neq 0.$$

According to [1, Lemma 5.2, statement (ii)] we have a double point if $v \neq 0$ and a triple one if $v = 0$.

On the other hand for systems (116) we calculate

$$\xi_2 = \frac{1}{209952} v^2 (18 + 2g^2 - 8gv - v^2)^2 (9 + g^2 + 2gv + v^2)^2,$$

and due to (117) we obtain that the condition $v = 0$ is equivalent to $\xi_2 = 0$.

Thus for systems (116) we obtain the configuration *Config.* $\mathcal{P}$*.117* if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}$*.118* if $\xi_2 = 0$.

2: The possibility $\xi_1 = 0$. We obtain $\mathcal{F}_1' = 0$ and since this polynomial is quadratic with respect to the parameter m, we calculate

$$\text{Discrim}\,[\mathcal{F}_1', m] = 4(1 + g^2)^2 (g^2 + 4n).$$

It is clear that for the existence of real solutions of the equation $\mathcal{F}_1' = 0$, the condition $g^2 + 4n \geq 0$ must hold. So we set a new parameter u as follows: $g^2 + 4n = u^2 \geq 0$ and we get $n = (u^2 - g^2)/4$. Then calculation yields

$$\mathcal{F}_1' = -\frac{1}{16}\big[8m - (g - u)(2 + g^2 - gu)\big]\big[8m - (g + u)(2 + g^2 + gu)\big] = 0$$

and due to the change $u \to -u$ we could force the first factor to vanish. In this case we obtain

$$m = (g - u)(2 + g^2 - gu)/8$$

and considering the expression for the parameters m and n we arrive at the 2-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{(g - u)(2 + g^2 - gu)}{8} + \frac{u^2 - g^2 - 2}{4} x - \frac{g}{2} y + gx^2 - xy, \\ \dot{y} =& \frac{(g - u)(2 + g^2 - gu)}{4} x - \frac{g^2 - u^2}{2} y + gxy - 2y^2, \end{aligned} \tag{118}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$. We determine that systems (118) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with the coordinates

$$\begin{aligned} x_1 = \frac{g - u}{2},\ y_1 = \frac{(g - u)^2}{4};\ \ x_{2,3} = \frac{1}{4}\big(u \pm \sqrt{Y_1}\,\big), \\ y_{2,3} = \frac{1}{8}\big[u^2 + 2gu - 2g^2 - 4 \pm u\sqrt{Y_1}\,\big],\ \ Y_1 = u^2 + 4gu - 4g^2 - 8. \end{aligned}$$

We calculate

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0$$

and therefore all three singularities are located on the invariant parabola. Moreover, we point out that M_1 is a singularity of systems (118) having multiplicity at least 2. Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\begin{aligned} \dot{x} =& \frac{1}{2}(g^2 - gu - 1)x + \frac{1}{2}(u - 2g)y + gx^2 - xy, \\ \dot{y} =& \frac{1}{2}(g - u)(g^2 - gu - 1)x + \frac{1}{2}(g - u)(2g - u)y - x^2 + gxy - 2y^2, \end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$ and

$$\mu_2 = \frac{1}{2}(g^2+1)\big[(g-u)^2+1\big]\big[(g^2-gu+1)x^2+(u-2g)xy+2y^2\big].$$

We observe that $\mu_2 \neq 0$ and by [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity exactly 2.

On the other hand, the singularities M_2 and M_3 could be either real or complex depending on the value of Y_1. In order to determine the position of the double singularity M_1 with respect to M_2 and M_3 in the case they are real (i.e., $Y_1 > 0$), we calculate:

$$(x_2-x_1)(x_3-x_1) = \big((g-u)^2+1\big)/2 > 0.$$

Therefore we deduce that in the case $Y_1 > 0$ both singularities M_2 and M_3 are located on the same side of the double point M_1. It is clear that for $Y_1 = 0$ the points M_2 and M_3 coalesce and we obtain two double points located on the invariant parabola.

For systems (118) calculations yield

$$\xi_2 = \frac{1}{8}(1+g^2)^2\big[1+(g-u)^2\big]^2 Y_1,$$

and hence we obtain that $\operatorname{sign}(\xi_2) = \operatorname{sign}(Y_1)$ if $\xi_2 \neq 0$.

Thus we obtain that systems (118) possess the configuration *Config.* $\mathcal{P}$*.119* if $\xi_2 < 0$; *Config.* $\mathcal{P}$*.120* if $\xi_2 > 0$ and *Config.* $\mathcal{P}$*.121* if $\xi_2 = 0$.

3.3.1.2 The case $B_1 = 0$. For systems (113) we calculate

$$B_1 = -\frac{1}{64}\big[g+g^3+4m-2gn\big]\big[(g-8m)^2+(1-4n)^2\big]\Psi(g,m,n),$$

where

$$\Psi(g,m,n) = 16m^2+8gm(3+2n)+(4+g^2)(1+g^2+4n+4n^2).$$

On the other hand we calculate

$$\xi_4 = \frac{13125}{2}(25+g^2)(3g+9g^3-4m-6gn)\big[(g-8m)^2+(1-4n)^2\big],$$
$$\zeta_4 = (25+g^2)(3g+9g^3-4m-6gn)/16$$

and due to $\zeta_4 \neq 0$ we obtain that $\xi_4 = 0$ is equivalent to $(g-8m)^2+(1-4n)^2 = 0$. So we examine two subcases: $\xi_4 \neq 0$ and $\xi_4 = 0$.

3.3.1.2.1 The subcase $\xi_4 \neq 0$. Then the condition $B_1 = 0$ implies either $g+g^3+4m-2gn = 0$ or $\Psi(g,m,n) = 0$. We have the next lemma.

Lemma 15. *For systems* (113)*, if $\xi_4 \neq 0$ then the condition $B_1 = B_2 = 0$ is equivalent to $\Psi(g,m,n) = 0$.*

Proof: Assume first that the condition $\Psi(g,m,n) = 0$ holds and we calculate $\text{Discrim}\,[\Psi, m] = -64(g^2-4n-2)^2 \le 0$. So in order to have a real solution with respect to m it is necessary that the condition $g^2-4n-2=0$ holds. This yields $n=(g^2-2)/4$ and we obtain

$$\Psi(g,m) = \frac{1}{4}(4g+g^3+8m)^2, \quad B_2 = (4g+g^3+8m)^2\phi(g,m,x,y),$$

where $\phi(g,m,x,y)$ is a polynomial of degree four in x and y. Therefore clearly the condition $\Psi(g,m)=0$ implies $B_2=0$.

Assume now that for systems (113) the conditions $B_1 = B_2 = 0$ and $\xi_4 \ne 0$ are fulfilled and suppose the contrary that the condition $\Psi(g,m,n)\ne 0$ holds. Then the condition $B_1=0$ yields $g+g^3+4m-2gn=0$. This gives us $m=-g(g^2-2n+1)/4$ and we calculate

$$\begin{gathered}\Psi(g,n) = (1+g^2)(g^2-4n-2)^2,\\ B_2 = -\frac{81}{2}\big(g^2+1\big)^2\big(g^2-4n-2\big)^2\big[4g^4+g^2(8-16n)+(1-4n)^2\big]x^4,\\ \text{Discrim}\,[4g^4+g^2(8-16n)+(1-4n)^2, n] = -256g^2<0,\end{gathered}$$

due to $\zeta_4\ne 0$. Therefore the condition $B_2=0$ implies $\Psi(g,n)=0$ and the contradiction we obtained completes the proof of Lemma 15. ∎

So in what follows we discuss two possibilities: $B_2\ne 0$ and $B_2=0$.

1: *The possibility* $B_2\ne 0$. Then by Lemma 15 we have $\Psi(g,m,n)\ne 0$ and the condition $B_1=0$ implies $g+g^3+4m-2gn=0$. This gives us $m=-g(g^2-2n+1)/4$ and we arrive at the 2-parameter family of systems:

$$\begin{aligned}\dot{x} &= -\frac{1}{4}g(g^2-2n+1)+\frac{1}{2}(2n-1)x-\frac{g}{2}\,y+gx^2-xy,\\ \dot{y} &= -\frac{1}{2}g(g^2-2n+1)\,x+2ny-x^2+gxy-2y^2,\end{aligned} \tag{119}$$

possessing the invariant line $L_1(x,y)=2x+g=0$ besides the invariant parabola $\Phi(x,y)=x^2-y=0$.

For the above systems we calculate

$$\zeta_4 = g(g^2+25)(5g^2-4n+2)/8, \quad \theta=-8(g^2+9).$$

Since $\theta\ne 0$, according to Lemma 2 systems (119) could not have an invariant line parallel with $2x+g=0$. On the other hand, according to Lemma 1 we could not have invariant line in another direction because $B_2\ne 0$.

Next we determine that systems (119) possess four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ with the coordinates

$$\begin{gathered}x_1=-\frac{g}{2},\ y_1=\frac{g^2}{4};\quad x_2=-\frac{g}{2},\ y_2=\frac{2n-g^2}{2};\\ x_{3,4}=\frac{1}{2}\big(g\pm\sqrt{Y_2}\big),\quad y_{3,4}=\frac{1}{2}\big(2n-1\pm g\sqrt{Y_2}\,\big),\quad Y_2=4n-g^2-2.\end{gathered}$$

We calculate

$$\Phi(x_1,y_1)=\Phi(x_3,y_3)=\Phi(x_4,y_4)=L_1(x_1,y_1)=L_1(x_2,y_2)=0,\quad \Phi(x_2,y_2)=(3g^2-4n)/4,$$

and we deduce that the singularity M_1 is the point of intersection of the invariant line with the invariant parabola. Moreover the point M_2 is located on the invariant line and it could belong to the invariant parabola if and only if $3g^2 - 4n = 0$. Finally, we observe that M_3 and M_4 could be either real or complex or coinciding (depending on the value of Y_2) and they lie on the invariant parabola.

In order to detect the reciprocal positions of the finite singularities we calculate

$$(x_3 - x_1)(x_4 - x_1) = (5g^2 - 4n + 2)/4 = \gamma_1/4, \quad y_2 - y_1 = (4n - 3g^2)/4 = \delta_1/4.$$

Therefore we deduce that in the case $Y_2 > 0$ the singularities M_2 and M_3 are located on the same side (respectively opposite sides) with respect to the singular point M_1 if $\gamma_1 > 0$ (respectively $\gamma_1 < 0$). We observe that $\gamma_1 \neq 0$ due to the condition $\zeta_4 \neq 0$.

On the other hand we note that $y_2 > y_1$ if $\delta_1 > 0$ and $y_2 < y_1$ if $\delta_1 < 0$. Moreover $y_2 = y_1$ if $\delta_1 = 0$ and in this case the point of intersection of the invariant line with the parabola is a double singular point of systems (119).

We determine that the invariant polynomial $\mathbf{D}$ which is responsible for the existence of multiple finite singularities for systems (119) has the form

$$\mathbf{D} = -\frac{3}{4}(g^2+1)^4 Y_2 \gamma_1^2 \delta_1^2, \quad \zeta_4 = g(25+g^2)\gamma_1/8,$$
$$\xi_7 = 4698510000 g^2 (1+g^2)^2 Y_2 \gamma_1 \delta_1^2, \quad \xi_8 = -247290000 g^2 (1+g^2)^2 Y_2 \delta_1 \gamma_1^2.$$

So due to $\zeta_4 \neq 0$ in the case $\mathbf{D} \neq 0$ we obtain

$$\text{sign}\,(\mathbf{D}) = -\text{sign}\,(Y_2), \quad \text{sign}\,(\xi_7) = \text{sign}\,(Y_2\gamma_1), \quad \text{sign}\,(\xi_8) = -\text{sign}\,(Y_2\delta_1),$$

and we examine two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} \neq 0$. Then $Y_2 \neq 0$ and systems (119) have four distinct finite singularities.

Remark 7. *We observe that* $\gamma_1 + \delta_1 = 2(g^2+1) > 0$. *Therefore the conditions* $\gamma_1 < 0$ *and* $\delta_1 < 0$ *are incompatible.*

Considering this remark in the case $\mathbf{D} \neq 0$ we determine that systems (119) have the following configurations:

$\mathbf{D} < 0,\ \xi_7 < 0$	$\Rightarrow$	$(x_3 - x_1)(x_4 - x_1) < 0,\ y_2 > y_1$	$\Rightarrow$	*Config.* $\mathcal{P}.122$;
$\mathbf{D} < 0,\ \xi_7 > 0,\ \xi_8 < 0$	$\Rightarrow$	$(x_3 - x_1)(x_4 - x_1) > 0,\ y_2 > y_1$	$\Rightarrow$	*Config.* $\mathcal{P}.123$;
$\mathbf{D} < 0,\ \xi_7 > 0,\ \xi_8 > 0$	$\Rightarrow$	$(x_3 - x_1)(x_4 - x_1) > 0,\ y_2 < y_1$	$\Rightarrow$	*Config.* $\mathcal{P}.124$;
$\mathbf{D} > 0,\ \xi_8 < 0$	$\Rightarrow$	$y_2 < y_1$	$\Rightarrow$	*Config.* $\mathcal{P}.125$;
$\mathbf{D} > 0,\ \xi_8 > 0$	$\Rightarrow$	$y_2 > y_1$	$\Rightarrow$	*Config.* $\mathcal{P}.126$.

1.2: *The case* $\mathbf{D} = 0$. Since $\gamma_1 \neq 0$ due to $\zeta_4 \neq 0$ this condition implies $Y_2\delta_1 = 0$ and we calculate

$$\xi_1 = -3g(1+g^2)^2(25+g^2)\gamma_1\delta_1/16$$

and since $\gamma_1 \neq 0$ we conclude that the condition $\delta_1 = 0$ is equivalent to $\xi_1 = 0$. So we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.2.1: *The subcase* $\xi_1 \neq 0$. Then $Y_2 = 0$ (i.e. $n = (g^2+2)/4$) and we obtain that M_3 coalesces with M_4 producing a double singular point on the parabola. Moreover the position of M_2 is determined by the value of δ_1. For systems (119) with $n = (g^2+2)/4$ we calculate:

$$\xi_1\zeta_8 = -3g^4(1+g^2)^3(25+g^2)\delta_1/2, \quad \zeta_4 = g^3(25+g^2)/2.$$

So due to $\zeta_4 \neq 0$ we obtain that $\operatorname{sign}(\xi_1\zeta_8) = -\operatorname{sign}(\delta_1)$ and therefore in the case $\mathbf{D} = 0$ and $\xi_1 \neq 0$ we arrive at the configuration *Config.* $\mathcal{P}$*.127* if $\xi_1\zeta_8 < 0$ and *Config.* $\mathcal{P}$*.128* if $\xi_1\zeta_8 > 0$.

1.2.2: *The subcase* $\xi_1 = 0$. This condition implies $n = 3g^2/4$ and then the singular point M_2 coalesces with M_1 and we observe that in this case $\gamma_1 = 2(g^2+1) > 0$.

On the other hand for $n = 3g^2/4$ we have

$$Y_2 = 2(g^2-1), \quad \xi_2 = (g^2-1)(1+g^2)^4$$

and clearly the condition $\xi_2 = 0$ is equivalent to $Y_2 = 0$. This implies the coalescence of M_3 with M_4 obtaining two double singularities located on the invariant parabola.

Thus in the case $\mathbf{D} = \xi_1 = 0$ we get the configuration *Config.* $\mathcal{P}$*.129* if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}$*.130* if $\xi_2 = 0$.

2: *The possibility* $B_2 = 0$. So we have $B_1 = B_2 = 0$ and by Lemma 15 the condition $\Psi(g,m,n) = 0$ holds. Considering the proof of Lemma 15 we arrive at the following conditions:

$$g^2 - 4n - 2 = 0 \;\Rightarrow\; n = (g^2-2)/4; \quad 4g + g^3 + 8m = 0 \;\Rightarrow\; m = -g(4+g^2)/8.$$

This leads to the 1-parameter family of systems

$$\begin{aligned} \dot{x} &= -\frac{1}{8}g(g^2+4) + \frac{1}{4}(g^2-4)x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= -\frac{1}{4}g(g^2+4)\,x + \frac{1}{2}(g^2-2)y - x^2 + gxy - 2y^2, \end{aligned} \tag{120}$$

possessing three invariant lines

$$L_1(x,y) = 2x + g = 0, \quad L_{2,3}(x,y) = 4(y \pm ix) - g(g \mp 2i) = 0,$$

besides the invariant parabola. We determine that systems (120) possess four finite singularities $M_i(x_i,y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{gathered} x_1 = -\frac{g}{2},\; y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2},\; y_2 = -\frac{g^2+2}{4}; \\ x_{3,4} = \frac{1}{2}(g \pm 2i), \quad y_{3,4} = \frac{1}{4}(g \pm 2i)^2, \end{gathered}$$

We calculate

$$\begin{gathered} \Phi(x_1,y_1) = \Phi(x_3,y_3) = \Phi(x_4,y_4) = 0, \quad \Phi(x_2,y_2) = (g^2+1)/2, \\ L_1(x_1,y_1) = L_1(x_2,y_2) = L_2(x_1,y_1) = L_2(x_3,y_3) = L_3(x_4,y_4) = 0. \end{gathered}$$

and we deduce that the singularity M_1 is the point of intersection of all three invariant lines with the invariant parabola. Moreover the point M_2 is located on the invariant line and it could not belong to the invariant parabola because $g^2+1 \neq 0$. In addition, since $y_2 - y_1 = -(g^2+1)/2 < 0$ we deduce that the point M_2 on the vertical invariant line $L_1 = 0$ is located below M_1.

Thus we arrive at the unique configuration *Config.* $\mathcal{P}$*.131* which the systems (120) could possess.

3.3.1.2.2 The subcase $\xi_4 = 0$. This implies $(g-8m)^2 + (1-4n)^2 = 0$ and we get $m = g/8$ and $n = 1/4$. Then we obtain the following 1-parameter family of systems

$$\dot{x} = \frac{g}{8} - \frac{x}{4} - \frac{gy}{2} + gx^2 - xy, \quad \dot{y} = \frac{gx}{4} + \frac{y}{2} - x^2 + gxy - 2y^2, \tag{121}$$

which besides the invariant parabola $\Phi(x,y) = x^2 - y = 0$ possess two complex invariant lines $L_{1,2}(x,y) = 4(y \pm ix) - 1 = 0$.

For these systems we calculate

$$\zeta_4 = g(25+g^2)(1+9g^2)/16, \quad B_3 = -3g(1+g^2)(x^2+y^2)^2/4,$$

and since $\zeta_4 \neq 0$ ($g \neq 0$) we obtain $B_3 \neq 0$. So according to Lemma 1, these systems could not have invariant line in the third (real) direction.

We determine that systems (121) possess four finite singularities $M_i(x_i, y_i)$ ($i = 1,2,3,4$) with the coordinates

$$x_1 = 0, \ y_1 = \frac{1}{4}; \quad x_2 = \frac{g}{2}, \ y_2 = \frac{g^2}{4}; \quad x_{3,4} = \pm\frac{i}{2}, \ y_{3,4} = -\frac{1}{4}.$$

We calculate

$$\Phi(x_2,y_2) = \Phi(x_3,y_3) = \Phi(x_4,y_4) = 0, \quad \Phi(x_1,y_1) = -1/4,$$

and we observe that the real singular point M_1 is a point of intersection of the complex invariant lines and it is located outside of the invariant parabola. The second real singular point M_2 is located on the parabola and its position is governed by the real parameter $g \neq 0$. As a result we arrive at the unique configuration *Config.* $\mathcal{P}$*.132*.

3.3.2 The statement ($\mathcal{E}_2$)

According to Proposition 4 in this case the conditions $\zeta_4 = 0$ and $\mathcal{R}_7\zeta_5 \neq 0$ must be fulfilled. Considering (114), the condition $\zeta_4 = 0$ implies

$$3g + 9g^3 - 4m - 6gn = 0 \ \Rightarrow \ m = 3g(3g^2 - 2n + 1)/4,$$

and we arrive at the following family of systems

$$\begin{aligned} \dot{x} &= \frac{3}{4}g\left(3g^2 - 2n + 1\right) + \frac{1}{2}(2n-1)x - \frac{gy}{2} + gx^2 - xy, \\ \dot{y} &= \frac{3}{2}g\left(3g^2 - 2n + 1\right)x + 2ny - x^2 + gxy - 2y^2, \end{aligned} \tag{122}$$

possessing two invariant parabolas $\Phi_1(x,y) = x^2 - y = 0$ and

$$\Phi_2(x,y) = (3g^2 - 4n)(1 + 3g^2 - 2n) + 2g(3g^2 - 4n)x - 4(1+g^2)x^2 + 2(2 + 5g^2 - 4n)y = 0.$$

For systems (122) we calculate

$$\begin{aligned} \zeta_5 &= 19(g^2+25)(3g^2-4n)^2/4, \quad \mathcal{R}_7 = 16120(3g^2+1)(5g^2-4n+2), \\ \theta &= -8(g^2+9), \quad B_1 = -g(g^2+1)(9g^2+1)(5g^2-4n+2)\Psi_2\Psi_3/32, \end{aligned} \tag{123}$$

where

$$\Psi_2(g,n) = 81g^4 + g^2(28 - 72n) + 4(1+2n)^2, \quad \Psi_3(g,n) = 36g^4 + g^2(16 - 48n) + (1-4n)^2.$$

According to Lemma 1 systems (122) could have at least one invariant line only if $B_1 = 0$. So we discuss two possibilities: $B_1 \neq 0$ and $B_1 = 0$.

3.3.2.1 The possibility $B_1 \neq 0$. We determine that systems (121) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned} & x_1 = \frac{3g}{2},\ y_1 = \frac{9g^2}{4};\ \ x_2 = \frac{g(3+9g^2-8n)}{2(1+g^2)},\ y_2 = \frac{9g^4+g^2(3-6n)+2n}{2(1+g^2)}; \\ & x_{3,4} = \frac{1}{2}\big(-g \pm \sqrt{Y_3}\big),\ \ y_{3,4} = \frac{1}{2}\big(2n-1-2g^2 \mp g\sqrt{Y_3}\big),\ \ Y_3 = 4n-5g^2-2. \end{aligned} \tag{124}$$

We calculate

$$\Phi_1(x_1,y_1) = \Phi_1(x_3,y_3) = \Phi_1(x_4,y_4) = 0, \qquad \Phi_2(x_2,y_2) = \Phi_2(x_3,y_3) = \Phi_2(x_4,y_4) = 0,$$

and we deduce that the singularities M_3 and M_4 are the points of intersection of both invariant parabolas. Moreover the point M_1 (respectively M_2) is located on the invariant parabola $\Phi_1 = 0$ (respectively $\Phi_2 = 0$).

In order to detect the positions of the singularities M_1 and M_2 with respect to M_3 and M_4 (in the case $Y_3 > 0$) we calculate:

$$\begin{aligned} (x_3-x_1)(x_4-x_1) &= (2+21g^2-4n)/4 = \gamma_2/4, \\ (x_3-x_2)(x_4-x_2) &= -\frac{Y_3}{4\,(g^2+1)^2}\big[21g^4+2g^2(5-8n)+1\big] = -\frac{Y_3}{4\,(g^2+1)^2}\delta_2. \end{aligned}$$

We observe that

$$\operatorname{sign}\big((x_3-x_1)(x_4-x_1)\big) = \operatorname{sign}(\gamma_2), \qquad \operatorname{sign}\big((x_3-x_2)(x_4-x_2)\big) = -\operatorname{sign}(Y_3\delta_2).$$

So we deduce that the point M_1 (respectively M_2) is located on the invariant parabola $\Phi_1 = 0$ (respectively $\Phi_2 = 0$) between M_3 and M_4 if and only if $\gamma_2 < 0$ (respectively $Y_3\delta_2 < 0$).

On the other hand for systems (121) we calculate

$$\mathbf{D} = -3(3g^2-4n)^2 Y_3 \gamma_2^2 \delta_2^2/4,\ \ \xi_{14} = 1235\gamma_2\delta_2/2,\ \ \xi_{30} = 1235\big(Y_3\delta_2 - (1+g^2)^2\gamma_2\big)/2. \tag{125}$$

We observe that in the case $\mathbf{D} \neq 0$ we have

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Y_3), \ \ \operatorname{sign}(\xi_{14}) = \operatorname{sign}(\gamma_2\delta_2).$$

Moreover in the case $\xi_{14} < 0$ (i.e. $\gamma_2\delta_2 < 0$) and $\mathbf{D} < 0$ (i.e. $Y_3 > 0$) we obtain

$$\operatorname{sign}(\xi_{30}) = \operatorname{sign}(Y_3\delta_2 - (1+g^2)^2\gamma_2) = \operatorname{sign}(\delta_2).$$

So we discuss two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

3.3.2.1.1 The case $\mathbf{D} \neq 0$. We observe that in the case $\mathbf{D} > 0$ the singular points M_3 and M_4 are complex and clearly in this case it is not necessary to distinguish the signs of the polynomials γ_2 and δ_2.

Thus taking into account the information we mentioned above we detect that in the case $\mathbf{D} \neq 0$ we arrive at the following configurations:

$$
\begin{array}{llll}
\mathbf{D}<0,\ \gamma_2<0,\ \delta_2>0 & \Rightarrow\ (x_3-x_1)(x_4-x_1)<0,\ (x_3-x_2)(x_4-x_2)<0 & \Rightarrow & \textit{Config.}\,\mathcal{P}.\textit{134};\\
\mathbf{D}<0,\ \gamma_2>0,\ \delta_2<0 & \Rightarrow\ (x_3-x_1)(x_4-x_1)>0,\ (x_3-x_2)(x_4-x_2)>0 & \Rightarrow & \textit{Config.}\,\mathcal{P}.\textit{133};\\
\mathbf{D}<0,\ \gamma_2<0,\ \delta_2<0 & \Rightarrow\ (x_3-x_1)(x_4-x_1)<0,\ (x_3-x_2)(x_4-x_2)>0 & \Rightarrow & \textit{Config.}\,\mathcal{P}.\textit{135};\\
\mathbf{D}<0,\ \gamma_2>0,\ \delta_2>0 & \Rightarrow\ (x_3-x_1)(x_4-x_1)>0,\ (x_3-x_2)(x_4-x_2)<0 & \Rightarrow & \simeq\textit{Config.}\,\mathcal{P}.\textit{135};\\
\mathbf{D}>0 & \Rightarrow & \Rightarrow & \textit{Config.}\,\mathcal{P}.\textit{136}.
\end{array}
$$

We observe that in the case $\gamma_2\delta_2 > 0$ we obtain two equivalent configurations given by *Config.* $\mathcal{P}$*.135* and we deduce that in the case $\mathbf{D} \neq 0$ systems (122) possess the following configuration if and only if the corresponding invariant conditions are satisfied:

$$
\begin{array}{ll}
\mathbf{D}<0,\ \xi_{14}<0,\ \xi_{30}<0 & \Rightarrow \textit{Config.}\,\mathcal{P}.\textit{133};\\
\mathbf{D}<0,\ \xi_{14}<0,\ \xi_{30}>0 & \Rightarrow \textit{Config.}\,\mathcal{P}.\textit{134};\\
\mathbf{D}<0,\ \xi_{14}>0 & \Rightarrow \textit{Config.}\,\mathcal{P}.\textit{135};\\
\mathbf{D}>0 & \Rightarrow \textit{Config.}\,\mathcal{P}.\textit{136}.
\end{array}
$$

3.3.2.1.2 The case $\mathbf{D} = 0$. Considering the value of the invariant polynomials given above and the conditions $\zeta_5 \neq 0$ (i.e. $3g^2 - 4n \neq 0$) and $\mathcal{R}_7 \neq 0$ (i.e. $Y_2 \neq 0$) we obtain that $\mathbf{D} = 0$ implies $\gamma_2\delta_2 = 0$. Taking into account (125) we deduce that this condition is equivalent to $\xi_{14} = 0$.

We claim that in the case $\xi_{14} = 0$ systems (122) possess the configuration *Config.* $\mathcal{P}$*.137* if $\xi_{30} < 0$; *Config.* $\mathcal{P}$*.138* if $\xi_{30} > 0$ and *Config.* $\mathcal{P}$*.139* if $\xi_{30} = 0$.

Indeed assume $\xi_{14} = 0$, i.e. $\gamma_2\delta_2 = 0$. In order to prove our claim we examine each one of these two possibilities.

1: *The subcase* $\delta_2 = 0$. This condition implies

$$
n = (1 + 10g^2 + 21g^4)/(16g^2), \qquad \gamma_2 = \frac{(7g^2-1)\,(9g^2+1)}{4g^2}
$$

and we determine that the singular point M_2 coalesces with M_4. If in addition $\gamma_2 = 0$ then M_1 coalesces with M_3 and we obtain two double singularities on the parabola.

So we detect that systems (121) possess the configuration which is equivalent to *Config.* $\mathcal{P}$*.137* if $\gamma_2 > 0$; to *Config.* $\mathcal{P}$*.138* if $\gamma_2 < 0$ and to *Config.* $\mathcal{P}$*.139* if $\gamma_2 = 0$.

Considering (125) we observe that for $\delta_2 = 0$ we have $\operatorname{sign}(\xi_{30}) = -\operatorname{sign}(\gamma_2)$ and our claim is proved in this case.

2: *The subcase* $\gamma_2 = 0$. This implies $n = (2 + 21g^2)/4$ and we observe that in this case the singular point M_1 coalesces with M_3 and we calculate

$$
\delta_2 = (1 - 7g^2)(1 + 9g^2).
$$

It is not too difficult to detect that in the case $\gamma_2 = 0$ systems (121) possess the configuration *Config.* $\mathcal{P}$*.137* if $\delta_2 < 0$; to *Config.* $\mathcal{P}$*.138* if $\delta_2 > 0$ and to *Config.* $\mathcal{P}$*.139* if $\delta_2 = 0$.

It remains to observe that considering (125) for $\gamma_2 = 0$ we obtain $Y_3 = 16g^2 > 0$ and therefore we have $\operatorname{sign}(\xi_{30}) = \operatorname{sign}(\delta_2)$ and this completes the proof of our claim.

3.3.2.2 The possibility $B_1 = 0$.

Considering (123) this condition implies $g(5g^2-4n+2)\Psi_2\Psi_3 = 0$. We claim that due to the condition $\mathcal{R}_7 \neq 0$ the condition $B_1 = 0$ is equivalent to $g = 0$. Indeed, assuming $g \neq 0$ we obtain

$$\text{Discrim}\,[\Psi_2, n] = -4096g^2 < 0, \quad \text{Discrim}\,[\Psi_3, n] = -256g^2 < 0,$$

and hence the equation $\Psi_2 = 0$ as well as $\Psi_3 = 0$ could not have real solution with respect to the parameter n. This completes the proof of our claim.

Thus we have $g = 0$ and we arrive at the 1-parameter family of systems

$$\dot{x} = \frac{1}{2}x(2n - 2y - 1), \quad \dot{y} = -x^2 + 2ny - 2y^2, \tag{126}$$

possessing the invariant line $x = 0$ and the invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = x^2 + (2n - 1)y - n(2n - 1) = 0.$$

For these systems we calculate

$$\zeta_5 = 1900n^2, \;\; \mathcal{R}_7 = -32240(2n - 1), \;\; B_2 = -162(2n + 1)^2(4n - 1)^2x^4.$$

We discuss two cases: $B_2 \neq 0$ and $B_2 = 0$.

3.3.2.2.1 The case $B_2 \neq 0$.

Then by Lemma 1 we could not have invariant lines in another direction. Considering (124) we obtain for $g = 0$ that systems (126) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = 0,\; y_1 = 0;\;\; x_2 = 0,\; y_2 = n;$$
$$x_{3,4} = \frac{1}{2}\big(\pm\sqrt{Y_3}\big),\;\; y_{3,4} = \frac{1}{2}\big(2n - 1\big),\;\; Y_3 = 2(2n - 1).$$

We observe that the invariant line $x = 0$ intersects the invariant parabola $\Phi_1 = 0$ at the point M_1 and the invariant parabola $\Phi_2 = 0$ at the point M_2.

For systems (126) we have

$$\mathbf{D} = -12n^2Y_3^3 \;\Rightarrow\; \text{sign}\,(\mathbf{D}) = -\text{sign}\,(Y_3).$$

Since all the singular points are fixed as points of intersection of invariant curves and $\mathbf{D} \neq 0$ due to the condition $\zeta_5\mathcal{R}_7 \neq 0$ we arrive at the configuration *Config.* $\mathcal{P}$*.140* if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}$*.141* if $\mathbf{D} > 0$.

3.3.2.2.2 The case $B_2 = 0$.

This implies $(2n + 1)(4n - 1) = 0$.

Assume $4n - 1 = 0$. This implies $n = 1/4$ and we arrive at the system

$$\dot{x} = -\frac{1}{4}x(4y + 1), \quad \dot{y} = \frac{1}{2}\left(-2x^2 - 4y^2 + y\right), \tag{127}$$

possessing three invariant lines $L_1(x, y) = x = 0$ and $L_{2,3}(x, y) = y \pm ix - 1/4 = 0$ and the invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and $\Phi_2(x, y) = -4x^2 + 2y - 1/2 = 0$.

We observe that the point $M_2(0,1/4)$ is the point of intersection of the above complex lines. As a result we obtain that the above system possesses the configuration *Config.* $\mathcal{P}$*.142*.

Now if $2n+1=0$ (i.e. $n=-1/2$) we arrive at the system

$$\dot{x}=-\,x(1+y),\qquad \dot{y}=-x^2-y-2y^2,$$

which can be brought to system (127) via the affine transformation and time rescaling

$$x_1=x/2,\ y_1=y/2+1/4,\ t_1=2t,$$

possessing the configuration *Config.* $\mathcal{P}$*.142*.

3.3.3 The statement ($\mathcal{E}_3$)

According to Proposition 4 in this case the conditions $\zeta_4=\zeta_5=0$ and $\mathcal{R}_7\neq 0$ hold. Considering (123), the condition $\zeta_5=0$ implies

$$3g^2-4n=0\ \Rightarrow\ n=3g^2/4,$$

and we get the following 1-parameter family of systems

$$\begin{aligned}\dot{x}=&\frac{3}{8}g\left(3g^2+2\right)+\frac{1}{4}\left(3g^2-2\right)x-\frac{gy}{2}+gx^2-xy,\\ \dot{y}=&\frac{3}{4}\left(3g^2+2\right)gx+\frac{3g^2y}{2}-x^2+gxy-2y^2,\end{aligned}\tag{128}$$

possessing according to Proposition 4 the double invariant parabola $\Phi(x,y)=x^2-y=0$.

For the above systems we compute

$$\begin{gathered}\zeta_4=\zeta_5=0,\ \ \mathcal{R}_7=32240(1+g^2)(1+3g^2)\neq 0,\ \ \theta=-8(g^2+9),\\ B_1=-g(1+g^2)^4(1+9g^2)^3/4.\end{gathered}$$

According to Lemma 1 systems (128) could have at least one invariant line only if $B_1=0$. So we discuss two possibilities: $B_1\neq 0$ and $B_1=0$.

1: *The possibility* $B_1\neq 0$. We determine that systems (128) possess the following three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates:

$$\begin{gathered}x_1=\frac{3g}{2},\ y_1=\frac{9g^2}{4};\ \ x_{2,3}=-\frac{1}{2}\big[g\pm i\sqrt{2(1+g^2)}\big],\\ y_{2,3}=\frac{1}{4}\big[-2-g^2\pm 2gi\sqrt{2(1+g^2)}\big].\end{gathered}\tag{129}$$

It is clear that the real singular point M_1 is double as it is located on the double invariant parabola and this could be proved directly.

It could be checked that the complex singular points M_2 and M_3 are also situated on the invariant parabola. However this is not relevant according to the Definition 1 of a configuration.

Then we deduce that in the case $B_1\neq 0$ systems (128) have the unique configuration *Config.* $\mathcal{P}$*.143*.

2: *The possibility* $B_1=0$. Then $g=0$ and system (128) with $g=0$ possesses the additional invariant line $x=0$. So considering the singularities (129) for $g=0$ we arrive at the unique configuration *Config.* $\mathcal{P}$*.144*.

3.3.4 The statement ($\mathcal{E}_4$)

According to Proposition 4 in this case the conditions $\zeta_4 = \mathcal{R}_7 = 0$ and $\zeta_5 \neq 0$ hold. Considering (123), the condition $\mathcal{R}_7 = 0$ implies

$$5g^2 - 4n + 2 = 0 \;\Rightarrow\; n = (2 + 5g^2)/4,$$

and we arrive at the following 1-parameter family of systems

$$\begin{aligned}\dot{x} =& \frac{1}{8}(g + 2x)\left(3g^2 + 4gx - 4y\right),\\ \dot{y} =& \frac{3g^3x}{4} + \frac{1}{2}\left(5g^2 + 2\right)y - x^2 + gxy - 2y^2,\end{aligned} \tag{130}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the invariant line $2x + g = 0$. For these systems we have $B_2 = -648(1 + g^2)^5(1 + 9g^2)x^4 \neq 0$ and by Lemma 1 we could not have other invariant lines.

On the other hand we determine that systems (130) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with the coordinates:

$$x_1 = -\frac{g}{2},\; y_1 = \frac{g^2}{4};\;\; x_2 = -\frac{g}{2},\;\; y_2 = \frac{1}{4}\left(3g^2 + 2\right);\;\; x_3 = \frac{3g}{2},\;\; y_3 = \frac{9g^2}{2}.$$

We claim that the singular point M_1 is a multiple singularity of systems (130). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\begin{aligned}\dot{x} =& x(gx - y),\\ \dot{y} =& g(1 + g^2)x - x^2 + (1 + g^2)y + gxy - 2y^2,\end{aligned}$$

where $M_0(0, 0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials:

$$\begin{aligned}\mu_4 = \mu_3 = 0,\;\; \mu_2 = 2g(1 + g^2)^2x(gx - y),\\ \mu_1 = -(1 + g^2)(x + 5g^2x - 4gy),\end{aligned}$$

and by [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity at least 2. We observe that $\mu_2 = 0$ if and only if $g = 0$. But in this case $\mu_1 \neq 0$. Therefore according to [1, Lemma 5.2, statement (ii)] we have a double point if $g \neq 0$ and a triple one if $g = 0$. We observe that this condition is governed by the invariant polynomial $\zeta_3 = 32g^2$.

We determine that the multiple singular point M_1 is the point of intersection of the line $2x+g = 0$ with the invariant parabola and the point M_3 is also located on the invariant parabola. Moreover the singular point M_3 coalesces with M_1 for $g = 0$ producing a triple finite singularity of systems (130).

On the other hand the singularity M_2 is located on the invariant line above the point M_1 because $y_2 - y_1 = (g^2 + 1)/2 > 0$. Therefore we arrive at the configuration *Config.*$\mathcal{P}$*.145* if $\zeta_3 \neq 0$ and *Config.*$\mathcal{P}$*.146* if $\zeta_3 = 0$.

Acknowledgments

N. Vulpe thanks the Instituto de Ciências Matemáticas e de Computação (USP) and Universidade Federal de São Carlos (UFSCar), São Carlos, Brazil, for the hospitallity and for the financial support during his visit to both institutes. R. D. S. Oliveira was supported by CNPq grant “Projeto Universal” 407454/2023-3, CNPq grant “Produtividade em Pesquisa” 310857/2023-6 and by Projeto Temático FAPESP number 2019/21181-0. A. C. Rezende was partially supported by Projeto Temático FAPESP number 2019/21181-0. D. Schlomiuk and N. Vulpe were supported by the NSERC Grant RN000355. N. Vulpe was supported by IRP of USM for 2024-2027 No.011303 and partially supported by the grant number 21.70105.31 ŞD.

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