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SURVEY

Prime gaps and the Firoozbakht Conjecture

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Abstract
This expository paper presents the statement of the Firoozbakht Conjecture, some of its relations with prime gaps and shows a consequence of Zhang's theorem concerning the Firoozbakht Conjecture.

Keywords Prime gaps · Firoozbakht Conjecture · Zhang's theorem

Mathematics Subject Classification 11A41 · 11N05

1 Introduction

What makes a math problem a good one? For the authors of this note, the answer to this question involves three major ingredients:

- It's easy to understand;
- It's very difficult to solve;
- It has several strong implications.

An example of a problem satisfying the three above conditions is the Firoozbakht Conjecture (see [20]). In 1982 (see [18]), the Iranian mathematician Farideh Firoozbakht, from the University of Isfahan, conjectured the following:
Firoozbakht Conjecture *Let* $\{p_n\}_{n\in\mathbb{N}}$ *the sequence of prime numbers. Then the sequence* $\{\sqrt[n]{p_n}\}_{n\in\mathbb{N}}$ *is strictly decreasing.*

Let's see the first cases:

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n	p_n	$\sqrt[n]{p_n}$
1	2	2.0000
2	3	1.7321
3	5	1.7100
4	7	1.6266
5	11	1.6154
6	13	1.5334
7	17	1.4989
8	19	1.4449
9	23	1.4168
10	29	1.4004

The Firoozbakht Conjecture has been verified for all primes below 4×10^{18} (see [11]), but there is no consensus about the truth of this conjecture. The authors of this paper, in particular, believe it is true. The importance of the Firoozbakht Conjecture is that it is the boldest (and reasonable) known statement about prime gaps. The n-th prime gap g_n is defined by the relation

$$g_n = p_{n+1} - p_n, \quad \forall\, n \in \mathbb{N}.$$

Using this language, the Firoozbakht Conjecture can be written in following way:

Firoozbakht Conjecture, prime gaps version $g_n < p_n(\sqrt[n]{p_n} - 1), \quad \forall\, n \in \mathbb{N}.$

Is this paper, we will present a very strong consequence of the Firoozbakht Conjecture concerning prime gaps and explain how this can solve a lot of other problems about prime numbers. We also show that Zhang's theorem implies Firoozbakht Conjecture is true for infinitely many values of n.

2 Relationship with prime gaps

The next lemma will be used further.

Lemma 2.1 $\dfrac{g_n}{g_n + p_n} < \ln(p_{n+1}) - \ln(p_n), \quad \forall\, n \in \mathbb{N}.$

Proof We know by calculus that $1 + \ln(x) < x, \ \forall\, x \in (0, 1)$. Taking $x = \dfrac{p_n}{p_{n+1}}$, we obtain the result. □

The major consequence of the Firoozbakht Conjecture is the following inequality:

Theorem 2.1 *If the Firoozbakht Conjecture is true, then*

$$g_n < \ln^2(p_n) - \ln(p_n) - 1, \quad \forall\, n \geq 10.$$

In particular,

$$g_n < \ln^2(p_n) - \ln(p_n), \quad \forall\, n \geq 5,$$

and

$$\limsup_{n\to\infty} \frac{g_n}{\ln^2(p_n)} \leq 1.$$

Proof (Following [12]) Suppose that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}}, \quad \forall\, n \in \mathbb{N}.$$

Taking the natural logarithm, we obtain

$$\frac{\ln(p_n)}{n} > \frac{\ln(p_{n+1})}{n+1}, \quad \forall\, n \in \mathbb{N}.$$

Isolating n, we obtain

$$n < \frac{\ln(p_n)}{\ln(p_{n+1}) - \ln(p_n)}, \quad \forall\, n \in \mathbb{N}.$$

Using the previous lemma, we have

$$n < \ln(p_n) + \frac{\ln(p_n) \cdot p_n}{g_n}, \quad \forall\, n \in \mathbb{N}.$$

On the other hand, a calculation shows that

$$\frac{x + \ln^2(x)}{\ln(x) - 1 - \frac{1}{\ln(x)}} < \frac{x}{\ln(x) - 1 - \frac{1}{\ln(x)} - \frac{1}{\ln^2(x)}}, \quad \forall\, x \geq 285{,}967.$$

If $\pi(x)$ denotes the prime-counting function, then [2, Corollary 3.6] states that

$$\frac{x}{\ln(x) - 1 - \frac{1}{\ln(x)} - \frac{1}{\ln^2(x)}} < \pi(x), \quad \forall\, x \geq 1{,}772{,}201.$$

Then

$$\frac{x + \ln^2(x)}{\ln(x) - 1 - \frac{1}{\ln(x)}} < \pi(x), \quad \forall\, x \geq 1{,}772{,}201.$$

Taking $x = p_n$ and remembering that $1{,}772{,}201 = p_{133{,}115}$, we get

$$\frac{p_n + \ln^2(p_n)}{\ln(p_n) - 1 - \frac{1}{\ln(p_n)}} < \pi(p_n) = n < \ln(p_n) + \frac{\ln(p_n) \cdot p_n}{g_n}, \quad \forall\, n \geq 133{,}115.$$

Isolating g_n, we get

$$g_n < \frac{p_n}{p_n + \ln(p_n) + 1} \cdot [\ln^2(p_n) - \ln(p_n) - 1], \quad \forall\, n \geq 133{,}115.$$

But

$$\frac{p_n}{p_n + \ln(p_n) + 1} < 1, \quad \forall\, n \in \mathbb{N}.$$

Then

$$g_n < \ln^2(p_n) - \ln(p_n) - 1, \quad \forall\, n \geq 133{,}115.$$

With the help of a computer for checking the remaining cases ($10 \leq n \leq 133{,}114$), we finish the proof. □

The combination of the previous theorem with the following result provides another tight relation between prime gaps and Firoozbakht Conjecture.

Theorem 2.2 *If* $g_n < \ln^2(p_n) - \ln(p_n) - 1.17, \quad \forall\, n \geq 10$, *then the Firoozbakht Conjecture is true.*

For a proof of this result we refer the reader to [12]. Now we will see how Theorem 2.1 can be used to solve long standing classical problems concerning prime numbers.

3 Other consequences

The best upper bound on prime gaps currently known is due to Baker, Harman and Pintz:

Theorem 3.1 (Baker, Harman, Pintz) $g_n \leq p_n^{0.525}, \quad \forall\, n \gg 0$.

It is easy to see that Firoozbakht Conjecture improves the Baker–Harman–Pintz's bound significantly. For a proof of the Baker–Harman–Pintz's Theorem we refer the reader to [3].

For the next consequence we will use the following lemma.

Lemma 3.1 *If Firoozbakht Conjecture is true, then* $g_n < \sqrt{n}, \quad \forall\, n \geq 3645$.

Proof A simple consequence of Prime Number Theorem is that

$$p_n < n^2, \quad \forall\, n \geq 2.$$

So, if Firoozbakht Conjecture is true, then

$$g_n < \ln^2(p_n) - \ln(p_n) - 1 < \ln^2(n^2) - \ln(n^2) - 1, \ \forall\, n \geq 10.$$

By calculus, we know that

$$\ln^2(n^2) - \ln(n^2) - 1 \le \sqrt{n}, \quad \forall\, n \ge 411{,}781.$$

So

$$g_n < \sqrt{n}, \quad \forall\, n \ge 411{,}781.$$

Verifying the remaining cases ($3645 \le n \le 411{,}780$) with the help of a computer, we finish the proof. □

Consequence 1 (Sierpinski) *Let n be an integer greater than 1. Write the numbers $1, 2, \ldots, n^2$ in a matrix in the following way*

$$\begin{array}{cccc} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ 2n+1 & 2n+2 & \ldots & 3n \\ \ldots & \ldots & \ldots & \ldots \\ (n-1)n+1 & (n-1)n+2 & \ldots & n^2 \end{array}$$

Then each row contains, at least, a prime number. Moreover, for each positive integer k exists another integer $n_0(k)$ such that if $n \ge n_0(k)$, then each row contains, at least, k prime numbers.

Proof Let n be an integer, $n \ge 34{,}123$. As $34{,}123 = p_{3645}$ is in the first row, suppose by absurd in the k-th row, $2 \le k \le n$, there is no prime number. So $(k-1)n + 1, (k-1)n + 2, \ldots, kn$ are all composite numbers. Let p_m be the largest prime less then $(k-1)n + 1$. Then $g_m \ge n$. But $m < p_m \le n^2$, which implies $\sqrt{m} < n$. Also, as $p_m \ge p_{3645}$, then $m \ge 3645$. By the previous lemma, $g_m < \sqrt{m} < n$, absurd. By checking the remaining cases ($2 \le n \le 34{,}122$) with the help of a computer, we finish the proof of the first statement.

The second statement follows the same argument, if the Firoozbakht Conjecture is true, since given $\varepsilon > 0$ we have $g_n < n^{\varepsilon}$ for every n sufficiently large. We leave the details to the reader. □

This conjecture appears for the first time in [17]. Other consequence that is easily obtained from the Firoozbakht Conjecture is Andrica Conjecture (see [1]):

Consequence 2 [Andrica] $g_n < 2\sqrt{p_n} + 1, \quad \forall\, n \in \mathbb{N}$.

Proof By calculus, we know that

$$\ln^2(x) - \ln(x) - 1 < 2\sqrt{x} + 1, \ \forall\, x \ge 1.$$

So, if the Firoozbakht Conjecture is true, then

$$g_n < \ln^2(p_n) - \ln(p_n) - 1 < 2\sqrt{p_n} + 1, \quad \forall\, n \ge 10.$$

Verifying the remaining cases ($1 \le n \le 9$), we finish the proof. □

Another consequence is Oppermann Conjecture, who posed it in 1882 (see [15]):

Consequence 3 (Oppermann) $\pi(n^2 - n) < \pi(n^2) < \pi(n^2 + n), \quad \forall\, n \in \mathbb{N} - \{1\}$.

Proof Oppermann Conjecture is true for $n \in \{2, \ldots, 73\}$. Now, suppose that there exists $n \in \mathbb{N}$, $n \geq 74$, such that $\pi(n^2 - n) = \pi(n^2)$, and let $m = \pi(n^2 - n)$. Then

$$p_m < n^2 - n < n^2 < p_{m+1}.$$

Therefore

$$g_m = p_{m+1} - p_m > n,$$

which implies

$$n < g_m < \ln^2(p_m) - \ln(p_m) - 1 < \ln^2(p_m) < \ln^2(n^2 - n).$$

But, by calculus,

$$x > \ln^2(x^2 - x), \quad \forall\, x \geq 74.$$

This is an absurd. So $\pi(n^2 - n) < \pi(n^2)$. A similar argument for the other inequality finishes the proof. □

A direct consequence of assuming Oppermann Conjecture is (see [21])

Consequence 4 (Legendre, strong form) *There are at least two prime numbers between two consecutive squares.*

This, in turn (as $p_{n+1} \geq p_n + 2, \quad \forall\, n \in \mathbb{N} - \{1\}$), implies (see [19])

Consequence 5 (Brocard) *There are at least four prime numbers between* p_n^2 *and* $p_{n+1}^2, \quad \forall\, n \in \mathbb{N} - \{1\}$.

To finish this section, we invite the reader to prove another consequence of the Firoozbakht Conjecture:

Consequence 6 (Legendre Conjecture for cubes, strong form) *There are at least four prime numbers between two consecutive cubes.*

Proof Left to the reader. □

4 Related conjectures

Recently, three other conjectures were made related to the Firoozbakht Conjecture. For the first two we refer the reader to [14]. The third conjecture was communicated directly to the first author by a private e-mail.

Conjecture 4.1 (Nicholson 2013) $\left(\frac{p_{n+1}}{p_n}\right)^n < n\ln(n), \ \forall\, n \geq 5.$

Conjecture 4.2 (Forgues 2014) $\left(\frac{\ln(p_{n+1})}{\ln(p_n)}\right)^n < e, \ \forall\, n \in \mathbb{N}.$

Conjecture 4.3 (Farhadian 2016) $p_n^{\left[\left(\frac{p_{n+1}}{p_n}\right)^n\right]} \leq n^{p_n}, \ \forall\, n \geq 5.$

To see how these conjectures are related to Firoozbakht Conjecture, we will remember the following inequalities:

Theorem 4.1 *$p_n > n\ln(n), \quad \forall\, n \in \mathbb{N}$; and*

$$\ln(n) + \ln(\ln(n)) - 1 < \frac{p_n}{n} < \ln(n) + \ln(\ln(n)), \quad \forall\, n \geq 6.$$

For a proof of the first inequality (Rosser's Theorem), we refer the reader to [16]. For the second, to [6].

Theorem 4.2 *Farhadian $\Rightarrow$ Nicholson $\Rightarrow$ Firoozbakht $\Rightarrow$ Forgues.*

Proof Suppose Farhadian Conjecture is true. Then, taking logarithms,

$$\left(\frac{p_{n+1}}{p_n}\right)^n \leq \frac{p_n \ln(n)}{\ln(p_n)}, \quad \forall\, n \geq 5.$$

By the last theorem,

$$\frac{p_n}{n} < \ln(n) + \ln(\ln(n)) = \ln(n\ln(n)) < \ln(p_n), \quad \forall\, n \geq 6,$$

which implies Nicholson Conjecture. Now, assuming Nicholson Conjecture, it follows from Rosser's Theorem that

$$\left(\frac{p_{n+1}}{p_n}\right)^n < p_n, \ \forall\, n \in \mathbb{N},$$

which is equivalent to

$$p_{n+1}^n < p_n^{n+1}, \quad \forall\, n \in \mathbb{N},$$

which is equivalent to Firoozbakht Conjecture. Now, taking logarithms in the last inequality, we obtain

$$\frac{\ln(p_{n+1})}{\ln(p_n)} < 1 + \frac{1}{n}.$$

Raising to the n-power, we get

$$\left(\frac{\ln(p_{n+1})}{\ln(p_n)}\right)^n < \left(1+\frac{1}{n}\right)^n < e, \quad \forall\, n \in \mathbb{N},$$

which is the Forgues Conjecture. □

5 A consequence of the Zhang's theorem

Zhang's theorem is the following celebrated statement.

Theorem 5.1 (Zhang) $\liminf g_n < \infty$.

For a proof of the theorem we refer the reader to [22]. One of the consequences of the Zhang's theorem is that the Firoozbakht Conjecture is true for infinitely many values of n.

Theorem 5.2 *There are infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}}$.*

Proof (Following [8]) Let

$$z = \liminf(p_{n+1} - p_n)$$

and

$$Z = \{n \in \mathbb{N};\; p_{n+1} - p_n = z\}.$$

Then Z is infinite and

$$\frac{p_n^{n+1}}{p_{n+1}^n} = \frac{p_n^{n+1}}{(p_n+z)^n} = \left(\frac{p_n}{p_n+z}\right)^n \cdot p_n = \left[\left(\frac{1}{1+\frac{z}{p_n}}\right)^{\frac{p_n}{z}}\right]^{\frac{zn}{p_n}} \cdot p_n, \quad \forall\, n \in Z.$$

As

$$\lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)^x = \frac{1}{e}$$

and

$$\lim_{n\to\infty}\frac{zn}{p_n} = 0$$

(by Rosser's Theorem), then

$$\left[\left(\frac{1}{1+\frac{z}{p_n}}\right)^{\frac{p_n}{z}}\right]^{\frac{zn}{p_n}} \to 1 \text{ if } n \to \infty,\; n \in Z.$$

So

$$\lim_{\substack{n\to\infty \\ n\in Z}} \frac{p_n^{n+1}}{p_{n+1}^n} = \infty.$$

In particular, $p_n^{n+1} > p_{n+1}^n$, for all sufficiently large $n \in Z$. This is equivalent to the statement of the theorem. □

6 Final remarks

It is worth commenting on the current state of the relationship of the Firoozba-kht Conjecture with two other rather famous problems in number theory: the Riemann Hypothesis and the Cramér Conjecture.

Cramér worked on Riemann Hypothesis in the 1920s and showed that it implies $g_n \ll \sqrt{p_n}\ln(p_n)$; for a proof of this result we refer the reader to [4] or [5]. On the other hand, Firoozbakht Conjecture implies that the function g_n has an order of growth $\ln^2(p_n)$, which is much smaller than $\sqrt{p_n}\ln(p_n)$. However, the authors do not know if the Riemann Hypothesis implies that the function g_n grows strictly slower than $\sqrt{p_n}\ln(p_n)$.

The statement of the Cramér Conjecture is $g_n \ll \ln^2(p_n)$. It is clear that Firoozbakht Conjecture implies Cramér Conjecture, according to Theorem 2.1. Cramér presented his conjecture based on a probabilistic model of prime numbers. In this model, Cramér showed that the relation

$$\limsup_{n\to\infty} \frac{g_n}{\ln^2(p_n)} = 1$$

is true with probability 1 (see [5]). However, as pointed out by Granville in [9], Maier's Theorem shows that the Cramér's model does not adequately describe the distribution of primes numbers on short intervals, and a refinement of the Cramér's model taking into account divisibility by small primes suggests that

$$\limsup_{n\to\infty} \frac{g_n}{\ln^2(p_n)} \geq 2e^{-\gamma} \approx 1.1229\ldots,$$

where

$$\gamma = \lim_{n\to\infty}\left(-\ln(n) + \sum_{k=1}^{n}\frac{1}{k}\right) \approx 0.5772\ldots$$

is the Euler–Mascheroni constant. More specifically, if $\pi(x)$ is the prime-counting function, then the Cramér's model predicts that

$$\lim_{n\to\infty} \frac{\pi\left(x + (\ln(x))^{\lambda}\right) - \pi(x)}{(\ln(x))^{\lambda-1}} = 1, \quad \forall\, \lambda \geq 2,$$

while Maier's Theorem states that if $\lambda > 1$, then

$$\liminf_{n\to\infty} \frac{\pi\left(x + (\ln(x))^{\lambda}\right) - \pi(x)}{(\ln(x))^{\lambda-1}} < 1$$

and

$$\limsup_{n\to\infty} \frac{\pi\left(x + (\ln(x))^{\lambda}\right) - \pi(x)}{(\ln(x))^{\lambda-1}} > 1.$$

For a proof of Maier's Theorem we refer the reader to [13]. All this shows how important is a deep study of the Firoozbakht Conjecture, or, more generally, of the function g_n.

Its worth to notice the recent papers [7,10], which contain other relations of the Firoozbakht Conjecture, and some variants, with prime gaps and indicate the current interest of the subject.

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