Bull Braz Math Soc, New Series (2026) 57:12
https://doi.org/10.1007/s00574-026-00505-7

On the Geometry of Holomorphic Curves and Complex Surface

Amanda Dias Falqueto[1] · **Farid Tari**[1]

Received: 11 December 2025 / Accepted: 20 February 2026

Abstract
We investigate the geometry of holomorphic curves and complex surfaces from a singularity theory viewpoint. We show that with the choice of the holomorphic metric, the families of functions and mappings that measure the contact between curves or surfaces with model objects are holomorphic. This allows the application of singularity theory techniques to pick up the geometry that is invariant under translations and complex rotations.

Keywords Holomorphic curves · Complex surfaces · Contact · Curvature · Evolute · Singularities

Mathematics Subject Classification 57R45 · 53B99

1 Introduction

Singularity theory made extensive and deep contributions to the local differential geometry of submanifolds of the Euclidean and Minkowski spaces (see, for example, Bruce and Giblin 1984; Cipolla and Giblin 2000; Damon et al. 2016; Izumiya et al. 2015, 2003; Porteous 1994). The geometry of the submanifold is captured by its contact with degenerate objects, such as those with zero or constant Gaussian curvature. The contact is given by the singularity type of some mappings on the surface. These mappings come in natural families that generally are versal deformations, so they yield important properties of the submanifold.

Our aim in this paper is to show how this singularity theory approach can be used to study the local differential geometry of holomorphic curves in $\mathbb{C}^2$ and $\mathbb{C}^3$, and of

✉ Farid Tari
faridtari@icmc.usp.br

Amanda Dias Falqueto
amandafalqueto@gmail.com

[1] Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Avenida Trabalhador são-carlense, 400, Centro, São Carlos, SP CEP: 13566-590, Brazil

Published online: 02 March 2026

complex surfaces in $\mathbb{C}^3$. This, we believe, paves the way for the study of complex submanifolds in $\mathbb{C}^n$ from a singularity theory viewpoint.

Usually, the metric induced by the Hermitian inner product is used when studying geometric properties of submanifolds that are invariant under the action of the unitary group on $\mathbb{C}^n$ (see, for example, Nomizu and Smyth 1968). However, with this inner product, the families of functions and mappings of interest are not holomorphic. For instance, the Gauss map of a hypersurface is not holomorphic, and the family of height functions and orthogonal projections are also not holomorphic. Therefore, the results of singularity theory cannot be used to capture the geometry of submanifolds of $\mathbb{C}^n$ derived from the Hermitian inner product.

Our key observation is to use the holomorphic inner product

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i, \tag{1}$$

which turns $\mathbb{C}^n$ into a holomorphic Riemannian manifold with metric $ds^2 = \sum_{i=1}^{n} dz_i^2$. Its isometry group is given by $E(n, \mathbb{C}) = \mathbb{C}^n \rtimes O(n, \mathbb{C})$, which is the semi-direct product of the group of translations with the complex orthogonal group. More details on this holomorphic metric can be found in Pessers and Van der Veken (2016). The authors in Breiding et al. (2024) also make use of the holomorphic metric to study metric problems in $\mathbb{R}^n$ using methods from complex algebraic geometry.

It is worth observing that there are non-zero vectors $v \in \mathbb{C}^n$ with zero length, that is $\langle v, v\rangle = 0$. Such vectors are called isotropic vectors. Isotropic curves are curves with isotropic tangent vectors at all points. In the case $n = 2$, isotropic curves are lines, and through each point pass two such lines. In the case $n = 3$, there are isotropic curves that are not lines; see (Struik 1950, Section 1.12). Struik also provided a classification of isotropic surfaces, that is, those whose tangent planes are isotropic at all points (equivalently, whose normal vector is isotropic). These surfaces are isotropic planes, isotropic cylinders, isotropic cones, or tangent surfaces to an isotropic curve (Struik 1950, Section 5.6).

Our work is set within the framework of generic geometry, which studies generic properties of submanifolds. For generic regular holomorphic curves in $\mathbb{C}^n$, with $n = 2, 3$, which we deal with in Sects. 3 and 4, we expect the points with isotropic tangent vectors to be isolated points along the curve. We call such points isotropic points. On the other hand, for generic surfaces in $\mathbb{C}^3$ (see sect. 5), the points where the tangent planes are isotropic form a regular curve, which we call the isotropic locus of the surface.

The concepts of inflection points, vertices, and the evolute of a plane curve can be defined in terms of the contact of the curve with lines and circles. The holomorphic inner product (1) allows us to define the curvature of a curve and derive from it the concepts mentioned above (see Sect. 4). For space curves, we also introduce the notion of torsion, together with the Frenet-Serret frame and its corresponding differential equations (Sect. 4).

We deal in Sect. 5 with complex surfaces in $\mathbb{C}^3$. We define the Gauss map and show that its derivative is a symmetric operator. The concepts of the first and second fundamental forms can also be defined for such surfaces, and with that, all the concepts on the geometry of surfaces apply (see Pessers and Van der Veken 2016; Struik 1950). For both the curve and surface cases, we define the families of height functions, distance squared functions and orthogonal projections. All these families are holomorphic and singularity theory results can be used to derive geometric information about the curve and surface.

In Sect. 3, we show that the envelope of the normal lines to a plane curve, with respect to the Hermitian inner product, is empty, whereas the envelope of the normal lines, with respect to the holomorphic metric, is not (Theorem 3.9). This shows that the holomorphic metric is the natural choice for our study. The envelope of the normal lines is precisely the evolute of the curve. We show that the evolute extends smoothly to a regular curve at isotropic points and is tangent to the original curve at those points. On the other hand, one of the focal sheets of a surface in $\mathbb{C}^3$ extends smoothly to a regular surface along the isotropic locus of the surface and is tangent to it along that curve.

It is worth emphasising that our study is not merely concerned with the complexification of real-analytic curves and surfaces. First, not all holomorphic curves and complex surfaces arise in this way. Second, real curves and surfaces do not possess isotropic points; consequently, phenomena such as the behaviour of the evolute and the focal set at isotropic points have no analogue in the real setting.

In Joiţa et al. (2024), the authors studied the local component of the bifurcation set of the distance-squared function on algebraic curves $X \subset \mathbb{C}^2$, including at points at infinity, called the *ED discriminant* of X. They showed that this discriminant contains the evolute of X, as well as additional components that do not appear in the real case.

Regarding the generic singularities that arise in the families of functions and mappings considered here, these are determined by the dimensions of the corresponding parameter spaces. As a result, the types of singularities occurring in the complex setting are similar to those appearing in the real case.

Finally, when computing invariants of analytic curves and surfaces in Euclidean space, such as the number of inflection points and vertices accumulated at a curve singularity, or the number of umbilics accumulated at a degenerate umbilic point, one complexifies in order to obtain an upper bound for their real counterparts. The approach adopted in this paper assigns geometric meaning to the resulting complexified objects through the curvatures introduced here.

2 Preliminaries

We consider the holomorphic inner product in $\mathbb{C}^n$ defined in (1), which is a complex bilinear, symmetric and non-degenerate form.

The holomorphic inner product (1) has a property similar to that of the Lorentzian inner product: there are non-zero vectors that satisfy $\langle v, v\rangle = 0$. These vectors are called **isotropic** (Struik 1950).

Two non-zero vectors v and w are said to be **orthogonal** if $\langle v, w\rangle = 0$.

A hyperplane π in $\mathbb{C}^n$ through some point p_0 has equation $\langle z - p_0, v\rangle = 0$. The vector v is called its normal vector. The hyperplane is **isotropic** if its normal vector is isotropic. In that case, v is parallel to π.

The **length of a vector** $v \in \mathbb{C}^n$ is defined as $\langle v, v\rangle^{\frac{1}{2}}$. This is a bi-valued set as it depends on the choice of the branch of the square root, that is,

$$\langle v, v\rangle^{\frac{1}{2}} = \left\{ |\langle v, v\rangle|^{\frac{1}{2}} e^{i\frac{\arg_{[0,2\pi)}\langle v,v\rangle}{2}},\ |\langle v, v\rangle|^{\frac{1}{2}} e^{i\frac{\arg_{(-\pi,\pi]}\langle v,v\rangle}{2}} \right\},$$

where $\arg_I$ means the choice of the argument in the interval I.

The case $I = (-\pi, \pi]$ is called the principal branch of the square root function, and the case $I = [0, 2\pi)$ here is called the "other branch".

Let C be a holomorphic curve in $\mathbb{C}^n$ parametrised locally by $\gamma : D \to \mathbb{C}^n$, where γ is a holomorphic function and D is a simply connected open set in $\mathbb{C}$. Suppose that $\gamma'(t)$ is not an isotropic vector for all $t \in D$, and let

$$\mathcal{B}^- = \left\{ t \in D \,\middle|\, \mathrm{Im}(\langle \gamma'(t), \gamma'(t)\rangle) = 0,\ \mathrm{Re}(\langle \gamma'(t), \gamma'(t)\rangle) < 0 \right\}, \tag{2}$$

$$\mathcal{B}^+ = \left\{ t \in D \,\middle|\, \mathrm{Im}(\langle \gamma'(t), \gamma'(t)\rangle) = 0,\ \mathrm{Re}(\langle \gamma'(t), \gamma'(t)\rangle) > 0 \right\}. \tag{3}$$

As γ is holomorphic, $\mathcal{B}^{\pm}$ is a real semi-analytic subset of D, in particular, it has measure zero for generic curves.

To make the length of $\gamma'(t)$ in Sects. 3 and 4 a single-valued function, we use the principal branch of the square root when $t \in D \setminus \mathcal{B}^-$ and the other branch when $t \in D \setminus \mathcal{B}^+$. We show that the geometric features derived from the functions used here are independent of the choice of branch of the square root.

A complex sphere of centre p and radius $r \in \mathbb{C}$, $r \neq 0$, is defined as the set of points $z \in \mathbb{C}^n$ that satisfy

$$\langle z - p, z - p\rangle = r^2.$$

The complex unit sphere $\mathbb{C}S^{n-1}$ is the set

$$\mathbb{C}S^{n-1} = \{z \in \mathbb{C}^n : \langle z, z\rangle = 1\},$$

so every point in $\mathbb{C}S^{n-1}$ represents a unit vector.

When $n = 2$, $\mathbb{C}S^1$ is the complex unit circle and has equation $z_1^2 + z_2^2 = 1$, where $z = (z_1, z_2) \in \mathbb{C}^2$. Complex circles are conics that pass through the circular points $(1 : i : 0)$ and $(1 : -i : 0)$ at infinity in $\mathbb{C}P^2$. These represent the two lines $z_2 = \pm i z_1$ in the affine plane $\mathbb{C}^2 \subset \mathbb{C}P^2$. The tangent vectors to these lines are along the isotropic vectors in $\mathbb{C}^2$.

We use the usual singularity theory concepts and notation about the actions of the Mather groups on the set of germs of holomorphic mappings (see, for example, Wall 1981).

Two germs of holomorphic functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are $\mathcal{R}$-equivalent if $g = f \circ h^{-1}$, where $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of holomorphic diffeomorphism.

In this paper, we make use of singularities of type $A_k, k \geq 1$, and $D_k, k \geq 4$, whose normal forms are given by

$$A_k : z_1^k + \sum_{i=2}^{k} z_i^2, \quad D_k : z_1^2 z_2 + z_2^{k-1} + \sum_{i=3}^{k} z_i^2.$$

Two germs of families of holomorphic functions $F, G : (\mathbb{C}^n \times \mathbb{C}^m, (0,0)) \to (\mathbb{C}, 0)$ are said to be $\mathcal{R}^+$-equivalent if

$$G(z, v) = F(\phi(z, v), \psi(v)) + c(v),$$

where $(\phi, \psi) : (\mathbb{C}^n \times \mathbb{C}^m, (0,0)) \to (\mathbb{C}^n \times \mathbb{C}^m, (0,0))$ is a germ of a holomorphic diffeomorphism and $c : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is a germ of a holomorphic function.

3 Plane Curves

We define some basic concepts of regular holomorphic curves C, including curvature, vertices and evolute of such curves. We then consider the contact of C with lines and circles, as is done for real plane case in Bruce and Giblin (1984).

All the concepts treated here are local in nature, so we take a local parametrisation $\gamma : D \to \mathbb{C}^2$ of C at a given point, where γ is a holomorphic function and D is a simply connected open set in $\mathbb{C}$. We write $\gamma(t) = (z_1(t), z_2(t))$.

Suppose that $\gamma'(t)$ is not an isotropic vector. We choose a branch of the square root function and call the vector

$$T(t) = \frac{\gamma'(t)}{\langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}} = \frac{(z_1'(t), z_2'(t))}{(z_1'(t)^2 + z_2'(t)^2)^{\frac{1}{2}}}$$

the **unit tangent vector** to γ at t.

The unit vector

$$N(t) = \frac{(-z_2'(t), z_1'(t))}{(z_1'(t)^2 + z_2'(t)^2)^{\frac{1}{2}}}$$

is orthogonal to $T(t)$ and is called the **unit normal vector** to γ at t.

The map $N : D \to \mathbb{C}S^1$ is called the **Gauss map** of the curve γ.

We say that γ is a **unit speed parametrisation** if $\langle \gamma'(t), \gamma'(t)\rangle = 1$, that is, $\gamma'(t) \in \mathbb{C}S^1$, for all $t \in D$.

A point $\gamma(t)$ is an **isotropic point** if $\gamma'(t)$ is an isotropic vector.

Remark 3.1 (1) As the inner product (1) is holomorphic, the isotropic points are isolated in D or are the whole D. In the latter case, the curve is parallel to one of the isotropic lines $z_2 = \pm i z_1$.

(2) At an isotropic point, a tangent vector to γ is also orthogonal to the curve. Consequently, the orthonormal frame $\{T, N\}$ cannot be extended to such a point.

Theorem 3.2 *Let $\gamma : D \to \mathbb{C}^n$ be a local parametrisation of a regular holomorphic curve. Suppose that $t_0 \in D$ is not an isotropic point. Then there is a simply connected open subset D' of D containing t_0 such that $\gamma|_{D'}$ can be reparametrised by unit speed.*

Proof The proof follows the same steps as that for curves in the real plane; however some necessary adjustment need to be made. Let

$$l(t) = \int_{t_0}^{t} \langle \gamma'(u), \gamma'(u)\rangle^{\frac{1}{2}} du,$$

where the integral is taken along any path that joins t_0 and t in $D \setminus \mathcal{B}^-$ or $D \setminus \mathcal{B}^+$ depending on the choice of the branch of the square root function. The function $l : D \to \mathbb{C}$ is holomorphic in some neighbourhood of t_0 where $\langle \gamma'(t), \gamma'(t)\rangle \neq 0$.

Since $l'(t_0) = \langle \gamma'(t_0), \gamma'(t_0)\rangle^{\frac{1}{2}} \neq 0$, it follows by the inverse function theorem that there exists a simply connected open set $D' \subset D$ containing t_0 such that $l : D' \to l(D')$ is biholomorphic.

The local reparametrisation $\beta(s) = \gamma(l^{-1}(s))$ of the curve γ from $l(D') \to \mathbb{C}^n$ is unit speed since

$$\beta'(s) = \frac{\gamma'(l^{-1}(s))}{l'(l^{-1}(s))} = \frac{\gamma'(l^{-1}(s))}{\langle \gamma'(l^{-1}(s)), \gamma'(l^{-1}(s))\rangle^{\frac{1}{2}}}.$$

Remark 3.3 In the real case, $l'(u) \neq 0$ for all u in the interval of definition of γ, so l is strictly monotonous. Therefore, it has an inverse $l^{-1} : l(I) \to I$. In the complex case, the biholomorphicity of l is valid only locally at t_0.

We now shrink D if necessary and take $\gamma : D \to \mathbb{C}^2$ to be a unit speed local parametrisation. We denote by s the unit speed parameter and by t a general parameter.

Differentiation $\langle T(s), T(s)\rangle = 1$ gives $\langle T'(s), T(s)\rangle = 0$. This means that $T'(s)$ is parallel to $N(s)$, so there exists a scalar function $\kappa(s)$ such that

$$T'(s) = \kappa(s)N(s).$$

We call $\kappa(s)$ the **curvature** of γ at s.

Proposition 3.4 *Let $\gamma : D \to \mathbb{C}^2$ be a unit speed local parametrisation of a holomorphic plane curve, with $\gamma(t) = (z_1(t), z_2(t))$. Then,*

(1) $\kappa(s) = \langle T'(s), N(s)\rangle = (z_1' z_2'' - z_2' z_1'')(s)$.

(2) $N'(s) = -\kappa(s)T(s)$.

Proof Statement (1) is immediate.

For (2), differentiating $\langle T, T\rangle = z_1'^2 + z_2'^2 = 1$ gives $z_1' z_1'' + z_2' z_2'' = 0$. That, with $z_1' z_2'' - z_2' z_1'' = \kappa$ yields $z_1'' = -\kappa z_2'$ and $z_2'' = \kappa z_1'$.

Differentiating $N = (-z_2', z_1')$, we get $N' = (-z_2'', z_1'') = -\kappa(z_1', z_2') = -\kappa T$. □

Remark 3.5 1. For a general parametrisation γ, one can re-parametrise by unit speed, and show that the curvature is given by

$$\kappa = \frac{z_1' z_2'' - z_2' z_1''}{(z_1'^2 + z_2'^2)^{\frac{3}{2}}}, \tag{4}$$

so,

$$\kappa' = \frac{(z_1' z_2''' - z_2' z_1''')(z_1'^2 + z_2'^2) - 3(z_1' z_1'' + z_2' z_2'')(z_1' z_2'' - z_2' z_1'')}{(z_1'^2 + z_2'^2)^{\frac{5}{2}}}. \tag{5}$$

2. The curve γ is a part of a line if and only if $\kappa \equiv 0$. It is a part of a complex circle if and only if $\kappa \equiv const \neq 0$.

Definition 3.6 A point $t_0 \in D$ on γ is called an **inflection of order** l if $\kappa(t_0) = \kappa'(t_0) = \ldots = \kappa^{(l-1)}(t_0) = 0$ and $\kappa^{(l)}(t_0) \neq 0$. When $l = 1$, it is called an ordinary inflection.

The point t_0 is called a **vertex of order** l if $\kappa'(t_0) = \ldots = \kappa^{(l)}(t_0) = 0$ and $\kappa^{(l+1)}(t_0) \neq 0$. When $l = 1$, it is called an ordinary vertex.

Remark 3.7 1. It follows from the general expressions for the curvature in (4) and its derivative in (5) that the notions of ordinary inflection and vertex do not depend on the choice of the branch of the square root function. The same holds for inflections and vertices of any order, by induction on the expression of the derivative of the curvature function in (4).

2. Inflections of a curve are affine invariant and are defined as points where the curve has at least 3-point contact with its tangent line, equivalently, the intersection multiplicity of the curve with its tangent line is greater than 2. It is not difficult to show that this agrees with Definition 3.6 above (see Sect. 3.1).

3. Vertices of plane curves can be defined in a similar way using the contact of the curve with complex circles, see Bruce et al. (2025); Piene et al. (2025); Viro (1995, 1998) and Sect. 3.2.

We have the following obvious observation as a corollary of Proposition 3.4.

Corollary 3.8 *The Gauss map is a local bi-holomorphic diffeomorphism away from inflection points. It has an A_1-singularity at ordinary inflection points.*

We next make an observation concerning the H-normal lines of a curve in $\mathbb{C}^2$ with respect to the Hermitian inner product

$$\langle v, w \rangle_H = v_1 \overline{w}_1 + v_2 \overline{w}_2.$$

The equation of the H-normal line at t of a curve γ is given by

$$G_t(p) = \langle \gamma(t) - p, \gamma'(t) \rangle_H = 0, \ \ p \in \mathbb{C}^2.$$

The family of functions $G(t, p) = G_t(p)$ is not holomorphic, so we cannot define its envelope as a discriminant (Bruce 1986). We identify $\mathbb{C}^2$ with $\mathbb{R}^4$ and use the following definition of an envelope from Thom (1962); see also Dias Carneiro (1983). A germ of a q-parameter family of submanifolds of codimension $m-n+q$ in $\mathbb{R}^m$ is a diagram of smooth map-germs of the form

$$\mathbb{R}^q, 0 \stackrel{\pi}{\leftarrow} \mathbb{R}^n, 0 \stackrel{f}{\rightarrow} \mathbb{R}^m, 0$$

that satisfies the following conditions:

(a) π is a germ of a fibration;

(b) f restricted to each fibre $\pi^{-1}(q)$ is a germ of a one to one immersion.

If $S(f) = \{p \in \mathbb{R}^n : df_p \text{ is not surjective}\}$, then $E = f(S(f))$ is the envelope of the family.

Theorem 3.9 *Let $\gamma : D \to \mathbb{C}^2$ be a parametrisation of a regular holomorphic curve. Then, the envelope of the family of the normal lines to γ, with respect to the Hermitian inner product, is empty.*

Proof We take γ a unit speed local parametrisation (away from isotropic points) and consider the following diagram

$$\mathbb{C} \stackrel{\pi}{\leftarrow} \mathbb{C} \times \mathbb{C} \stackrel{f}{\rightarrow} \mathbb{C}^2,$$

where $\pi(s, v) = s$ and $f(s, v) = \gamma(s) + v\gamma'(s)^\perp$, where $\gamma'(s)^\perp = (-\overline{z_2'(s)}, \overline{z_1'(s)})$ is orthogonal to $\gamma'(s)$ with respect to the Hermitian inner product. Then, the image of the fibre $\pi^{-1}(s)$ is the H-normal line to γ at s.

We consider the family of the H-normal lines to γ as a family of planes in $\mathbb{R}^4$ and show that their envelope is empty.

We write $s = s_1 + is_2$, $v = v_1 + iv_2$, and $\gamma(s) = (z_1(s), z_2(s))$. Using the Wirtinger derivatives, we get

$$\begin{aligned} \frac{\partial f}{\partial s_1} &= (z_1', z_2') + v(-\overline{z_2''}, \overline{z_1''}), & \frac{\partial f}{\partial v_1} &= (-\overline{z_2'}, \overline{z_1'}), \\ \frac{\partial f}{\partial s_2} &= i(z_1', z_2') + iv(\overline{z_2''}, -\overline{z_1''}), & \frac{\partial f}{\partial v_2} &= i(-\overline{z_2'}, \overline{z_1'}). \end{aligned}$$

Identifying $\mathbb{C}^k$ with $\mathbb{R}^{2k}$, the determinant of the Jacobian matrix of $df_{(s,v)}$ is $1 + ||v\kappa(s)||^2 > 0$. Therefore, $S(f)$ is empty. Consequently, the H-normal lines to γ do not have an envelope. □

3.1 Contact with Lines

A line in $\mathbb{C}^2$, with an orthogonal vector v, has equation

$$\langle z, v \rangle = c,$$

for some constant $c \in \mathbb{C}$.

In view of Remarks 3.7 (1), we consider t in the entire domain of the parametrisation of the curve and define the **family of height functions** $H : D \times \mathbb{C}S^1 \to \mathbb{C}$ on γ by

$$H(t, v) = \langle \gamma(t), v \rangle.$$

The function H is holomorphic. For v fixed, the function $H_v(t) = H(t, v)$ is the height function along v. Its singularities measure the contact of the curve γ with the lines orthogonal to v.

We have the following result; see Bruce and Giblin (1984) for the real case analogue.

Theorem 3.10 *Let $\gamma : D \to \mathbb{C}^2$ be a local parametrisation of a regular holomorphic curve without isotropic points. Then,*

(1) *The function H_v is singular at t_0 if and only if $v \parallel N(t_0)$.*

(2) *For $v \parallel N(t_0)$, the singularity of H_v at t_0 is of type A_1 if and only if $\kappa(t_0) \neq 0$. It is of type A_2 if and only if $\kappa(t_0) = 0$ and $\kappa'(t_0) \neq 0$, that is, t_0 is an ordinary inflection point of the curve.*

(3) *For generic curves, H_v can only have an A_1 or an A_2 singularity and these are $\mathcal{R}^+$-versally unfolded by the family H.*

Remark 3.11 The singularities of the function $t \mapsto \langle \gamma(t), v \rangle$ depend only on the direction of $v \in \mathbb{C}^2$, so the family of height functions can be defined at an isotropic point t_0 as $H : D \times \mathbb{C}P^1 \to \mathbb{C}$, with $H(t, \mathbf{v}) = \langle \gamma(t), v \rangle$, where $v \in \mathbb{C}^2$ is any representative of $\mathbf{v}$. The function $t \mapsto \langle \gamma(t), v \rangle$ is singular at t_0 if and only if $v \parallel \gamma'(t_0)$. For generic curves, the singularity is an A_1 as we can assume that $\langle \gamma''(t_0), \gamma'(t_0) \rangle \neq 0$.

3.2 Contact with Circles

The contact of the curve γ at t_0 with the circle of centre c passing through $\gamma(t_0)$ is measured by the singularity of the (contact) function

$$d_c(t) = \langle \gamma(t) - c, \gamma(t) - c \rangle \tag{6}$$

at t_0.

The function d_c is holomorphic, and we call it the **distance squared function**.

Varying $c \in \mathbb{C}^2$, yields the **family of distance squared functions** $d : D \times \mathbb{C}^2 \to \mathbb{C}$, given by

$$d(t, c) = d_c(t), \tag{7}$$

which is also a holomorphic function.

Away from isotropic points, we parametrise γ by unit speed, using one of the branches of the square root function. Then, the function d_c is singular at s_0 (i.e., has an $A_{\geq 1}$-singularity) if and only if

$$\frac{1}{2} d_c'(s_0) = \langle T(s_0), \gamma(s_0) - c \rangle = 0,$$

that is, $c = \gamma(s_0) + \lambda N(s)$, for some $\lambda \in \mathbb{C}$. This means that the circle is tangent to the curve at s_0 if and only if its centre c belongs to the line through $\gamma(s_0)$ and parallel to $N(s_0)$. We call this line the **normal line** to γ.

Differentiating twice, gives

$$\frac{1}{2}d_c''(s_0) = \kappa(s_0)\langle N(s_0), \gamma(s_0) - c\rangle + 1.$$

Suppose that s_0 is not an inflection point. Then, the singularity of d_c at s_0 is of type $A_{\geq 2}$ if and only if $d_c'(s_0) = d_c''(s_0) = 0$, equivalently,

$$\begin{aligned} c = e(s_0) &= \gamma(s_0) + \frac{1}{\kappa(s_0)}N(s_0) \\ &= (z_1(s_0), z_2(s_0)) + \frac{(z_1'^2 + z_2'^2)(s_0)}{(z_1'z_2'' - z_2'z_1'')(s_0)}(-z_2'(s_0), z_1'(s_0)). \end{aligned} \tag{8}$$

Varying s_0 in D, we get a parametrised plane curve $s \mapsto e(s)$, called the **evolute** of γ. It is clear from (8) that the evolute is independent of the choice of branch of the square root function.

The circle of centre $e(s_0)$ tangent to the curve at $\gamma(s_0)$ is called the **osculating circle** of γ at s_0.

With $c = e(s_0)$, we have $\frac{1}{2}d_c'''(s_0) = 0$ if and only if $\kappa'(s_0) = 0$. Thus, the singularity of d_c at t_0 is of type $A_{\geq 3}$ if and only if $c = e(s_0)$ and t_0 is a vertex of γ.

We have the following result; see Bruce and Giblin (1984) for the real case analogue.

Theorem 3.12 *Let $\gamma : D \to \mathbb{C}^2$ be a local parametrisation of a regular holomorphic curve without inflections and isotropic points. Then, the distance squared function d_c has a singularity at t_0 of type*

$A_1 \iff c \neq e(t_0)$ is on the normal line of γ at t_0.

$A_2 \iff c = e(t_0)$ and t_0 is not a vertex of γ.

$A_3 \iff c = e(t_0)$ and t_0 is an ordinary vertex of γ.

For a generic γ, the function d_c has only the above singularities and these are $\mathcal{R}^+$-versally unfolded by the family d.

The local component of the bifurcation set of the family d is precisely the evolute of γ. Consequently, the evolute of a generic curve can only have a cusp singularity.

As $e'(t) = -\frac{\kappa'(t)}{\kappa(t)^2}N(t)$, we have the following consequence.

Corollary 3.13 *Away from vertices, the tangent line to the evolute is the normal line to the curve at the associated point. Consequently, the evolute is the envelope of the normal lines of γ.*

The following result describes the behaviour of the evolute at isotropic points.

Proposition 3.14 *Let $\gamma : D \to \mathbb{C}^2$ be a regular holomorphic curve. Suppose that $t_0 \in D$ is an isotropic point and that the curve has an ordinary contact with its tangent line at t_0. Then, the evolute extends to a holomorphic curve at t_0 that is tangent to γ at this point.*

Proof The proof follows from the parametrisation of the evolute in (8). The assumption on the curve γ having an ordinary contact with its tangent line at t_0 is equivalent to $(z_1'z_2'' - z_2'z_1'')(t_0) \neq 0$. □

3.3 Algebraic Curves

Let $f(z_1, z_2) = 0$ be a regular algebraic curve of degree d in $\mathbb{C}^2$. A general $f = 0$ has $3d(d-2)$ inflection points and $2d(3d-5)$ vertices; see (Breiding et al. 2024, Chapter 6) for proofs and references.

Isotropic points of the curve $f = 0$ are solutions of the system

$$\begin{cases} f(z_1, z_2) = 0, \\ (f_{z_1}^2 + f_{z_2}^2)(z_1, z_2) = 0. \end{cases} \tag{9}$$

We write

$$f(z_1, z_2) = f_d(z_1, z_2) + f_{d-1}(z_1, z_2) + \cdots + f_0(x, y),$$

where each $f_j(z_1, z_2)$ is homogeneous of degree j.

Theorem 3.15 *Let $f = 0$ be a regular algebraic curve of degree d in $\mathbb{C}^2$. Suppose the circular points at infinity $(1 : \pm i : 0)$ are not roots of f_d, and that f_d is reduced. Then the curve $f = 0$ has $2d(d-1)$ isotropic points.*

Proof To count the number of solutions of the systems (9), we projectivise and apply Bézout's theorem to obtain the number of intersections of the corresponding curves in $\mathbb{C}P^2$. We then consider possible solutions that lie on the line at infinity.

Denote by $F(z_1 : z_2 : z_3) = 0$ the homogenization of $f(z_1, z_2) = 0$ in $\mathbb{C}P^2$, given by

$$F(z_1 : z_2 : z_3) = f_d(z_1, z_2) + z_3 f_{d-1}(z_1, z_2) + \cdots + z_3^d f_0(x, y). \tag{10}$$

Projectivising the second equation in (9), we count the intersections of the curves

$$F = 0 \quad \text{and} \quad (F_{z_1} + iF_{z_2})(F_{z_1} - iF_{z_2}) = 0$$

in $\mathbb{C}P^2$.

The isotropic points on the line at infinity are given by

$$\begin{cases} f_d = 0, \\ (f_{d_{z_1}} + if_{d_{z_2}})(f_{d_{z_1}} - if_{d_{z_2}}) = 0. \end{cases}$$

Let $\alpha = (\alpha_1 : \alpha_2 : 0) \in \mathbb{C}P^2$ be a solution of the system of equations above. It follows from Euler's identity that $\alpha_1 f_{d_{z_1}}(\alpha) + \alpha_2 f_{d_{z_2}}(\alpha) = d.f_d(\alpha) = 0$. Substituting in the second equation of the system above yields $(\pm i\alpha_1 - \alpha_2) f_{d_{z_2}}(\alpha) = 0$. Therefore, either α is a circular point at infinity or f_d has a repeated factor. As both are excluded by hypothesis, it follows that none of the intersections of the curves lies on the line at infinity.

Now, $F_{z_1} \pm i F_{z_2} = 0$ are of degree $d-1$ and $F = 0$ is regular, so the curves $F = 0$ and $F_{z_1}^2 + F_{z_2}^2 = 0$ have no common component and the result follows by Bézout's theorem. □

Example 3.16 1. Consider the following conic in $\mathbb{C}^2$ given by

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1, \tag{11}$$

with $a \neq 0, b \neq 0, a^2 \neq b^2$. The normal vector at (z_1, z_2) is along $(\frac{z_1}{a^2}, \frac{z_2}{b^2})$ and is isotropic if and only if $\frac{z_1^2}{a^4} + \frac{z_2^2}{b^4} = 0$. Substituting in equation(11) gives $z_1^2 = \frac{a^4}{a^2-b^2}$. It follows that the conic has four isotropic points (compare Theorem 3.15).

When $a^2 = b^2$, the conic is a complex circle with equation

$$z_1^2 + z_2^2 = a^2.$$

It has no isotropic points. Projectivising, it passes through the circular points at infinity and is tangent to the isotropic lines through them. Hence, Theorem 3.15 does not apply.

2. Consider the following non-singular cubic in $\mathbb{C}^2$ given by

$$z_1^3 + z_2^3 - 3az_1z_2 + 1 = 0, \tag{12}$$

with $a^3 \neq 1$.

A point (z_1, z_2) on the cubic is isotropic if and only if

$$(z_1^2 - az_2)^2 + (z_2^2 - az_1)^2 = 0. \tag{13}$$

Using Maple, the resultant of the two polynomials on the left-hand sides of equations (12) and (13), with respect to, say, z_2, is given by

$$2z_1^{12} - 4(3a^3-1)z_1^9 - 6a(a^3-1)z_1^8 + 2(10a^6 - 7a^3 + 3)z_1^6 + 12a(2a^6 - 3a^3 + 1)z_1^5 \\ +9a^2(a^3-1)^2z_1^4 - 4(a^6 + 2a^3 - 1)z_1^3 - 6a(a^3-1)z_1^2 + a^6 + 1.$$

When the discriminant of the above resultant is non-zero, the cubic (12) has twelve distinct isotropic points (compare Theorem 3.15).

4 Space Curves

Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{C}^3$. The cross product of two vectors $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$ is the vector

$$z \times w = \begin{vmatrix} e_1 & e_2 & e_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (z_2w_3 - z_3w_2, z_3w_1 - z_1w_3, z_1w_2 - z_2w_1).$$

The **osculating plane** of a regular holomorphic space curve $\gamma : D \to \mathbb{C}^3$ at a point $t_0 \in D$ is the plane through $\gamma(t_0)$ parallel to $\gamma'(t_0)$ and $\gamma''(t_0)$. This osculating plane is isotropic if and only if $\gamma'(t_0) \times \gamma''(t_0)$ is isotropic, equivalently,

$$\langle \gamma'(t_0), \gamma'(t_0)\rangle\langle \gamma''(t_0), \gamma''(t_0)\rangle - \langle \gamma'(t_0), \gamma''(t_0)\rangle^2 = 0.$$

We have the following observation.

Lemma 4.1 *Suppose that γ is parametrised by unit speed at a non-isotropic point s_0, and let T be the unit tangent vector of γ. Then the osculating plane π at s_0 is isotropic if and only if $T'(s_0)$ is isotropic.*

Proof Assume first that π is isotropic. Then π has an isotropic normal vector $v \nparallel T(s_0)$, so it is generated by $T(s_0)$ and v, with $\langle T(s_0), v\rangle = 0$.

As $T'(s_0)$ is parallel to π, we can write $T'(s_0) = \alpha T(s_0) + \beta v$ for some $\alpha, \beta \in \mathbb{C}$. Differentiating the identity $\langle T, T\rangle = 1$ gives $\langle T', T\rangle = 0$, hence $\alpha = 0$. Thus, $T'(s_0) = \beta v$ which is isotropic. (We are assuming π to be an isotropic plane, so $\beta \neq 0$.)

Conversely, suppose that $T'(s_0)$ is an isotropic vector. For any $\alpha, \beta \in \mathbb{C}$, we have $\langle \alpha T(s_0) + \beta T'(s_0), T'(s_0)\rangle = 0$. Thus, π is an isotropic plane with normal $T'(s_0)$. □

Struik (1950, Section 1-12) describes space curves that are isotropic at every point. He also presents a proof of a theorem, attributed to E. Study, stating that if the osculating plane of a space curve is isotropic at every point, then the curve is either an isotropic curve or lies entirely in an isotropic plane.

In this section, we consider holomorphic space curves without isotropic points and whose osculating plane is nowhere isotropic. We parametrise the curve by unit speed (Theorem 3.2).

Under these assumptions, and by Lemma 4.1, there exist a scalar function κ and a unit vector N such that

$$T'(s) = \kappa(s)N(s).$$

Observe that $\kappa = \langle T', N\rangle$ is a holomorphic function, and $\kappa(s) \neq 0$ for all $s \in D$. From $\langle T'(s), T(s)\rangle = 0$, we get $\kappa(s)\langle N(s), T(s)\rangle = 0$. Therefore,

$$\langle N(s), T(s)\rangle = 0.$$

It follows that T and N are orthogonal. We call the vector $N(s)$ the **unit normal vector** to γ at s and call $\kappa(s)$ the **curvature** of γ at s.

We define the **binormal vector** $B(s)$ of γ at s as the vector $B(s) = T(s) \times N(s)$.

It follows from the properties of determinants that the vectors $T(s), N(s), B(s)$ form an orthonormal basis of $\mathbb{C}^3$, for all s. We call the frame $\{T, N, B\}$ the **Frenet-Serret frame** of γ.

We have $B' = T' \times N + T \times N' = T \times N'$, so $\langle B', T\rangle = \langle T \times N', T\rangle = 0$.

Writing $B' = \lambda_1 T + \lambda_2 N + \lambda_3 B$, it follows from $\langle B', T\rangle = 0$, and from the orthogonality of the vectors of the frame, that $\lambda_1 = 0$.

From $\langle B, B\rangle = 1$, we get $\langle B', B\rangle = 0$. It follows that $\langle \lambda_2 N + \lambda_3 B, B\rangle = \lambda_3 = 0$. Therefore, B' is parallel to N. We write

$$B'(s) = \tau(s)N(s),$$

and call $\tau(s)$ the **torsion** of γ at s. Clearly, $\tau = \langle B', N\rangle$ is a holomorphic function.

One can show that $T = N \times B$ and $N = B \times T$. Then,

$$N' = B' \times T + B \times T' = -\tau B - \kappa T.$$

Proposition 4.2 *The Frenet-Serret formulae for the moving frame T, N, B are:*

$$\begin{cases} T' = \kappa N, \\ N' = -\kappa T - \tau B, \\ B' = \tau N. \end{cases}$$

Of course one can define the Frenet-Serret frame, the curvature and torsion for regular holomorphic curves not parametrised by unit speed, assuming that they are without isotropic points and that the osculating plane is not isotropic at all points. Then,

$$\kappa = \frac{\langle \gamma' \times \gamma'', \gamma' \times \gamma''\rangle^{\frac{1}{2}}}{\langle \gamma', \gamma'\rangle^{\frac{3}{2}}} \quad \text{and} \quad \tau = -\frac{\langle \gamma' \times \gamma'', \gamma'''\rangle}{\langle \gamma' \times \gamma'', \gamma' \times \gamma''\rangle}.$$

Remark 4.3 It is worth observing that the torsion is a holomorphic function and is independent of the square root function and its branches.

4.1 Contact with Planes

For a space curve $\gamma : D \to \mathbb{C}^3$, without isotropic points and with the assumption that the osculating plane is not isotropic at all points, we define the family of height functions on γ as the holomorphic function $H : D \times \mathbb{C}S^2 \to \mathbb{C}$, where

$$H(t, v) = \langle \gamma(t), v\rangle.$$

We take, for simplicity, γ to be a unit speed local parametrisation. Then, as shown above, $\kappa(s) \neq 0$ and T, N, B form an orthonormal frame.

The height function along v is defined as $H_v(s) = H(s, v)$ and measures the contact of the curve γ with planes orthogonal to v.

We have $H_v' = \langle T, v\rangle$, and it vanishes at s_0 if and only if $\langle v, T(s_0)\rangle = 0$, that is, $v = \lambda N(s_0) + \mu B(s_0)$, for some $\lambda, \mu \in \mathbb{C}$, with $\lambda^2 + \mu^2 = 1$.

Now $H_v'' = \kappa\langle N, v\rangle$, so $H_v'(s_0) = H_v''(s_0) = 0$ if and only if $v = \pm B(s_0)$.

Differentiating again, we get $H_v''' = \kappa'\langle N, v\rangle - \kappa\langle \kappa T + \tau B, v\rangle$. Thus, $H_v'(s_0) = H_v''(s_0) = H_v'''(s_0) = 0$ if and only if $v = \pm B(s_0)$ and $\tau(s_0) = 0$. With these conditions, one can show that $H_v^{(4)}(s_0) \neq 0$ if and only if $\tau'(s_0) \neq 0$

We call the plane orthogonal to $T(s)$ the **normal plane** of γ at $\gamma(s)$. We have the following result (see Bruce and Giblin 1984, for real curves analogue).

Theorem 4.4 *Let $\gamma : D \to \mathbb{C}^3$ be a local parametrisation of a generic regular holomorphic curve without isotropic points and with the osculating plane not isotropic at all points. Then, the height function H_v can only have singularities of type A_1, A_2, A_3 and these occur at a given point t_0 when:*

$A_1 \iff$ *v belongs to the normal plane of γ at t_0 and $v \nparallel (\gamma' \times \gamma'')(t_0)$.*

$A_2 \iff v \parallel (\gamma' \times \gamma'')(t_0)$ *and* $\tau(t_0) \neq 0$.

$A_3 \iff v \parallel (\gamma' \times \gamma'')(t_0)$, $\tau(t_0) = 0$ *and* $\tau'(t_0) \neq 0$.

The above singularities of H_v are $\mathcal{R}^+$-versally unfolded by the family H.

Remark 4.5 Here too one can consider the contact of a space curve with isotropic planes by defining locally the family of height functions as $H : D \times \mathbb{C}P^2 \to \mathbb{C}$, with $H(t, \mathbf{v}) = \langle \gamma(t), v \rangle$ and $v \in \mathbb{C}^3$ any representative of $\mathbf{v}$.

4.2 Contact with Lines

We defined the **orthogonal projection** along $v \in \mathbb{C}S^2$ as the parallel projection along v to the plane through the origin and orthogonal to v.

The orthogonal projection $P_v(p) = p + \lambda v$ of a point $p \in \mathbb{C}^3$ along v satisfies $\langle p + \lambda v, v \rangle = 0$, so $\lambda = -\langle p, v \rangle$.

Observe that the orthogonal plane to v can be identified with the tangent plane to the unit circle $\mathbb{C}S^2$ at v. By varying v in $\mathbb{C}S^2$, we obtain the **family of orthogonal projections** $P : \mathbb{C}^3 \times \mathbb{C}S^2 \to T\mathbb{C}S^2$, given by

$$P(p, v) = (v, P_v(p)) = (v, p - \langle p, v \rangle v).$$

Clearly, P is a holomorphic map. The family of orthogonal projections of a space curve γ is the restriction of P to γ.

We take γ without isotropic points and with the osculating plane not isotropic at all points. We also assume for simplicity that it is a unit speed parametrisation.

The orthogonal projection $P_v(s) = \gamma(s) - \langle \gamma(s), v \rangle v$ can be considered locally at $s_0 \in D$ as a map-germ $(\mathbb{C}, s_0) \to (\mathbb{C}^2, P_v(\gamma(s_0)))$.

We have

$$P_v'(s) = T(s) - \langle T(s), v \rangle v.$$

This means that P_v is singular at s_0 if and only if $v = \pm T(s_0)$. Differentiating again gives

$$P_v''(s) = \kappa(s)(N(s) - \langle N(s), v \rangle v).$$

Therefore, at a singularity s_0 of P_v, we have $P_v''(s_0) = \kappa(s_0)N(s_0)$. It follows that the singularity of the defining equation of $P_v(\gamma)$ is of type A_k (since $\kappa(s_0) \neq 0$).

We have

$$P_v'''(s) = \kappa'(s)[N(s) - \langle N(s), v \rangle v] - \kappa(s)[\kappa(s)T(s) + \tau(s)B(s) - \langle \kappa(s)T(s) + \tau(s)B(s), v \rangle v],$$

so at a singularity s_0 of P_v we have

$$P_v'''(s_0) = \kappa'(s_0)N(s_0) - \kappa(s_0)\tau(s_0)B(s_0).$$

The vectors $P_v''(s_0)$ and $P_v'''(s_0)$ are linearly independent if and only if $\tau(s_0) \neq 0$. In that case, the singularity of P_v at s_0 is $\mathcal{A}$-equivalent to the cusp $s \mapsto (s^2, s^3)$ (this is refereed to as A_2-singularity since it is the zero set of a germ of a function with an A_2-singularity).

With $v = \pm T(s_0)$ and $\tau(s_0) = 0$, we have

$$\begin{aligned} P_v^{(4)}(s_0) &= (\kappa''(s_0) - \kappa^3(s_0))N(s_0) - \kappa(s_0)\tau'(s_0)B(s_0), \\ P_v^{(5)}(s_0) &= (\kappa'''(s_0) - 6\kappa'(s_0)\kappa^2(s_0))N(s_0) - (3\kappa'(s_0)\tau'(s_0) + \kappa(s_0)\tau''(s_0))B(s_0). \end{aligned}$$

The singularity of P_v is $\mathcal{A}$-equivalent to $s \mapsto (s^2, s^5)$ (i.e., it is of type A_4) if and only if $(\kappa'\tau' - 3\kappa\tau'')(s_0) \neq 0$.

We have the following results (see David 1983, for the real case analogue).

Theorem 4.6 *Let $\gamma : D \to \mathbb{C}^3$ be a local parametrisation of a generic regular holomorphic curve without isotropic points and with the osculating plane not isotropic at all points. Then, the orthogonal projection P_v along v can only have local singularities of type A_2 or A_4. We get, at t_0, a singularity of type*

$A_2 \iff v$ is tangent to γ at t_0 and $\tau(t_0) \neq 0$.

$A_4 \iff v$ is tangent to γ at t_0, $\tau(t_0) = 0$ and $(\kappa'\tau' - 3\kappa\tau'')(t_0) \neq 0$.

The above singularities are $\mathcal{A}_e$-versally unfolded by the family P.

Remark 4.7 The torsion is a holomorphic function and does not depend on the choice of the branch of the square root (Remark 4.3). The numerator of the function $\kappa'\tau' - 3\kappa\tau''$ is holomorphic, so the results in Theorem 4.6 do not depend on the choice of the branch of the square root and do not require the curve to be parametrized by unit speed.

5 Surfaces in $\mathbb{C}^3$

Let M be a surface in $\mathbb{C}^3$, that is, a complex two-dimensional submanifold of $\mathbb{C}^3$. We study the extrinsic geometry of M and work locally at a given point on M. We take a local parametrisation $\phi : U \to \mathbb{C}^3$ of M at that point. The map ϕ is a regular holomorphic map, and we write $M = \phi(U)$.

The **first fundamental form** of M at $p = \phi(q) \in M$, where $q = (z_1, z_2)$, is the quadratic form $I_p(v) = \langle v, v\rangle$, for all $v \in T_pM$. We denote

$$E = \langle \phi_{z_1}, \phi_{z_1}\rangle, \quad F = \langle \phi_{z_1}, \phi_{z_2}\rangle, \quad G = \langle \phi_{z_2}, \phi_{z_2}\rangle,$$

the coefficients of the first fundamental form I. Then, for any $v = a\phi_{z_1}(q) + b\phi_{z_2}(q) \in T_pM$, we have

$$I_p(v) = a^2E + 2abF + b^2G.$$

There may exist points on the surface where the quadratic form I_p is degenerate, that is, points $p = \phi(q)$ such that

$$\delta(q) = (EG - F^2)(q) = 0.$$

We call the set of such points the **isotropic locus (IL)** of M, and the points on this set are called **isotropic points**. We identify the IL with its preimage in U, so that

$$IL = \{(z_1, z_2) \in U : (EG - F^2)(z_1, z_2) = 0\}.$$

In fact, a point p is isotropic if and only if the tangent plane T_pM is isotropic. Surfaces whose points are all isotropic are the isotropic planes, isotropic cylinders, isotropic cones, or tangent surfaces to isotropic curves (see (Struik 1950, Sections 5-6)).

Example 5.1 1. Consider the example of the generic quadric in $\mathbb{C}^3$ given by

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} + \frac{z_3^2}{c^2} = 1, \tag{14}$$

with $a \neq 0, b \neq 0, c \neq 0$.

A normal vector to the quadric at a point (z_1, z_2, z_3) is parallel to $(\frac{z_1}{a^2}, \frac{z_2}{b^2}, \frac{z_3}{c^2})$. It is isotropic if and only if

$$\frac{z_1^2}{a^4} + \frac{z_2^2}{b^4} + \frac{z_3^2}{c^4} = 0. \tag{15}$$

If $a^2 = b^2 = c^2$, equation (14) represents a complex sphere, and this has no isotropic points.

Assume now that the quadric is not a complex sphere. Then equation (15) can be written as

$$z_3^2 = -c^4(\frac{z_1^2}{a^4} + \frac{z_2^2}{b^4}).$$

Substituting in equation in (14) yields

$$\frac{a^2 - c^2}{a^4} z_1^2 + \frac{b^2 - c^2}{b^4} z_2^2 = 1. \tag{16}$$

Therefore, the IL is the locus of intersection of the quadric (14) with the quadric (16).

If $c^2 \neq a^2$ and $c^2 \neq b^2$, the quadric (16) is a complex cylinder.

If, say, $c^2 = a^2$ and $c^2 \neq b^2$, the quadric (16) is the union of two planes with equation $z_2^2 = \frac{b^4}{b^2 - a^2}$, and the IL is the union of two complex circles with equation

$$z_1^2 + z_3^2 = \frac{a^4}{a^2 - b^2},$$

in each of these planes.

2. Consider the non-singular cubic surface in $\mathbb{C}^3$ given by

$$z_1^3 + z_2^3 + z_3^3 = 1.$$

Its isotropic locus is its intersection set with the surface of degree 4

$$z_1^4 + z_2^4 + z_3^4 = 0.$$

Calculating the resultant between the two polynomials (using Maple), with respect to, say, z_3, gives the following curve of degree twelve which represents the IL in the (z_1, z_2) chart:

$$2z_1^{12} + 4z_1^9 z_2^3 + 3z_1^8 z_2^4 + 6z_1^6 z_2^6 + 3z_1^4 z_2^8 + 4z_1^3 z_2^9 + 2z_2^{12} - 4(z_1 + z_2)^3 (z_1^2 - z_1 z_2 + z_2^2)^3 + 6(z_1 + z_2)^2 (z_1^2 - z_1 z_2 + z_2^2)^2 - 4z_1^3 - 4z_2^3 + 1 = 0.$$

As pointed out previously, our aim is to study the geometry of generic submanifolds. We therefore make the following standing assumption to avoid repeated statements. This assumption holds for generic holomorphic surfaces (indeed, any surface that does not satisfy it can be made to do so by a small deformation).

Assumption. We assume that the isotropic locus IL of M is either empty or consists of a regular curve on M.

At points $p = \phi(z_1, z_2)$ on $M \setminus IL$, the vector $\phi_{z_1} \times \phi_{z_2}$ is non-isotropic and orthogonal to $T_p M$. We shrink U if necessary, choose a branch of the square root function and define the **Gauss map** $N : M \setminus IL \to \mathbb{C}S^2$, by

$$N(z_1, z_2) = \frac{\phi_{z_1} \times \phi_{z_2}}{\langle \phi_{z_1} \times \phi_{z_2}, \phi_{z_1} \times \phi_{z_2} \rangle^{\frac{1}{2}}}(z_1, z_2).$$

Differentiating the identity $\langle N, N \rangle = 1$, gives $\langle N_{z_1}, N \rangle = \langle N_{z_2}, N \rangle = 0$. It follows that $-dN_p : T_p M \to T_{N(p)}\mathbb{C}S^2 \simeq T_p M$ is a linear operator on $T_p M$, called the **shape operator** of the surface.

From $\langle \phi_{z_1}, N \rangle = \langle \phi_{z_2}, N \rangle = 0$, we get

$$\langle \phi_{z_1 z_2}, N \rangle + \langle \phi_{z_1}, N_{z_2} \rangle = 0,$$
$$\langle \phi_{z_2 z_1}, N \rangle + \langle \phi_{z_2}, N_{z_1} \rangle = 0.$$

It follows that $\langle \phi_{z_1}, N_{z_2} \rangle = \langle \phi_{z_2}, N_{z_1} \rangle$, so $\langle dN_p(\phi_{z_1}), \phi_{z_2} \rangle = \langle dN_p(\phi_{z_2}), \phi_{z_1} \rangle$. Therefore, the bilinear form on $T_p M$, given by

$$II_p(u, v) = \langle -dN_p(u), v \rangle,$$

is symmetric. The quadratic form $II_p(v, v)$ is called the **second fundamental form** of M, and is denoted $II_p(v)$. Its coefficients are given by

$$\begin{aligned} l &= -\langle N_{z_1}, \phi_{z_1}\rangle = \langle N, \phi_{z_1 z_1}\rangle = \frac{\bar{l}}{\langle \phi_{z_1} \times \phi_{z_2}\rangle^{\frac{1}{2}}}, \\ m &= -\langle N_{z_1}, \phi_{z_2}\rangle = \langle N, \phi_{z_1 z_2}\rangle = \frac{\overline{m}}{\langle \phi_{z_1} \times \phi_{z_2}\rangle^{\frac{1}{2}}}, \\ n &= -\langle N_{z_2}, \phi_{z_2}\rangle = \langle N, \phi_{z_2 z_2}\rangle = \frac{\overline{n}}{\langle \phi_{z_1} \times \phi_{z_2}\rangle^{\frac{1}{2}}}, \end{aligned}$$

where

$$\begin{aligned} \bar{l} &= \langle \phi_{z_1} \times \phi_{z_2}, \phi_{z_1 z_1}\rangle, \\ \overline{m} &= \langle \phi_{z_1} \times \phi_{z_2}, \phi_{z_1 z_2}\rangle, \\ \overline{n} &= \langle \phi_{z_1} \times \phi_{z_2}, \phi_{z_2 z_2}\rangle \end{aligned}$$

are holomorphic functions that do not depend on the choice of the branch of the square root function.

For any $v = a\phi_{z_1}(q) + b\phi_{z_2}(q) \in T_pM$, we have

$$II_p(v) = a^2 l + 2abm + b^2 n.$$

The direction $v \in T_pM$ is **asymptotic** if $II_p(v) = 0$, equivalently,

$$a^2\bar{l} + 2ab\overline{m} + b^2\overline{n} = 0.$$

Following the same calculations as those for surfaces in the Euclidean space (see, for example, Carmo 1976), the matrix of the shape operator $-dN_p$, with respect to the basis ϕ_{z_1}, ϕ_{z_2} of T_pM, is

$$A_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

Since II_p is a symmetric bilinear form, its matrix A_p has two eigenvalues κ_1, κ_2 and two orthogonal eigenvectors e_1, e_2. We call κ_1, κ_2 the **principal curvatures** and e_1, e_2 the **principal directions**. A direction $a\phi_{z_1}(q) + b\phi_{z_2}(q) \in T_pM$ is principal if and only if

$$\begin{vmatrix} b^2 & -ab & a^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0, \tag{17}$$

equivalently,

$$\begin{vmatrix} b^2 & -ab & a^2 \\ E & F & G \\ \bar{l} & \overline{m} & \overline{n} \end{vmatrix} = (F\overline{n} - G\overline{m})b^2 + (E\overline{n} - G\bar{l})ab + (E\overline{m} - F\bar{l})a^2 = 0. \tag{18}$$

We denote by $K = \kappa_1\kappa_2 = \det(A_p)$ the **Gaussian curvature** of M. We have

$$K = \frac{ln - m^2}{EG - F^2}.$$

Following the same arguments as for surfaces in the Euclidean 3-space (Carmo 1976), one can show that the Gaussian curvature is an intrinsic property of $M \setminus IL$, and depends only on the first fundamental form.

The set of points on M where $K(p) = 0$ (equivalently, $\bar{l}\bar{n} - \bar{m}^2 = 0$) is called the **parabolic set**.

A point p is called **umbilic** if $\kappa_1(p) = \kappa_2(p)$. Umbilic points are also the points where the coefficients of the quadratic equation (18) vanish. The vanishing of two of these coefficients implies the vanishing of the third, so such points are generically isolated.

To each non-umbilic point $p = \phi(q)$ are associated two **focal points** $c_i = \phi(q) + \frac{1}{\kappa_i(q)}N(q)$, $i = 1$ or $i = 2$. As q varies, the focal points trace the two sheets of the **focal set** of M. The **ridge** is the preimage on the surface of the singular set of the focal set.

Remark 5.2 Clearly, the concepts of asymptotic and principal directions, as well as parabolic and umbilic points, are independent of the choice of branch of the square root function used to define the Gauss map. One can also show that the focal points are independent of this choice.

With the choice of the holomorphic metric (1), we can study the contact of the surface $M \setminus IL$ with planes (these have zero Gaussian curvature), complex spheres (which have constant Gaussian curvature) and lines. The resulting families of mappings that measure the contact between M with these objects are holomorphic (which is not the case when using the Hermitian inner product).

5.1 Contact with Planes

We define, as for plane curves, the family of height functions on the image of a local parametrization $\phi : U \setminus IL \to \mathbb{C}^3$ of a surface as the holomorphic function $H : U \setminus IL \times \mathbb{C}S^2 \to \mathbb{C}$, where

$$H(q, v) = \langle \phi(q), v \rangle.$$

The height function $H_v(q) = H(q, v)$ is singular at q if and only if $v \parallel N(q)$. The singularity is of type A_1 if and only if $ln - m^2 \neq 0$, that is, q is not a parabolic point. We have the following result; see (Izumiya et al. 2015, Chapter 6) for the real case analogue.

Theorem 5.3 *For a generic complex surface M in $\mathbb{C}^3$, the height function H_v on $M \setminus IL$ can have local singularities of type A_1, A_2, A_3 and these are $\mathcal{R}^+$-versally unfolded by the family of height functions. The singularities occur at a point $p \in M \setminus IL$ when:*

$A_1 \iff v \parallel N(p)$ and p is not a parabolic point.
$A_2 \iff v \parallel N(p)$, p is parabolic point and the unique asymptotic direction at p is transverse to the parabolic curve.
$A_3 \iff v \parallel N(p)$, p is parabolic point and the unique asymptotic direction at p is tangent to the parabolic curve at p.

We can choose a suitable coordinate system in $\mathbb{C}^3$ and parametrise the surface M locally at a given point $p_0 \in M$ in Monge form $\phi(z_1, z_2) = (z_1, z_2, f(z_1, z_2))$, for some holomorphic function f with zero 1-jet at the origin. The normal vector to M at the origin is $(0, 0, 1)$ and nearby vectors in $\mathbb{C}S^2$ can be parametrised by $(v_1, v_2, 1)$ with $(v_1, v_2) \in (\mathbb{C}^2, 0)$. We get the germ of the family of height functions $H : (\mathbb{C}^2 \times \mathbb{C}^2, (0, 0)) \to (\mathbb{C}, 0)$, given by

$$H(z_1, z_2, v_1, v_2) = z_1 v_1 + z_2 v_2 + f(z_1, z_2). \tag{19}$$

The algebraic conditions on the Taylor expansion of f at the origin for $H_{(0,0)}$ to have one of the singularities listed in Theorem 5.3, and for these singularities to be $\mathcal{R}^+$-versally unfolded by the family H in (19), are as given in (Izumiya et al. 2015, Chapter 6) (interpreting the coefficients in Izumiya et al. (2015) as complex numbers).

Remark 5.4 The family of height functions can be defined at points on the IL. As the normal to the surface at such points is isotropic, we define locally the family of height functions as $H : U \times \mathbb{C}P^2 \to \mathbb{C}$, where $H(q, \mathbf{v}) = \langle \phi(q), v \rangle$ and v is any representative in $\mathbb{C}^3$ of $\mathbf{v}$.

5.2 Contact with Spheres

The contact of the surface $M = \phi(U)$ at p with the complex sphere of centre c passing through p is measured by the singularities of the (contact) function

$$d_c(q) = \langle \phi(q) - c, \phi(q) - c \rangle. \tag{20}$$

The function d_c is holomorphic, and we call it the **distance squared function**.
The family of distance squared functions $d : U \times \mathbb{C}^3 \to \mathbb{C}$ is given by

$$d(q, c) = d_c(q) = \langle \phi(q) - c, \phi(q) - c \rangle. \tag{21}$$

The function d_c measures the contact of M with the complex spheres of centre c. We have the following result, analogous to that for surfaces in the Euclidean 3-space; see (Izumiya et al. 2015, Chapter 6) for the real case analogue.

Theorem 5.5 *For a generic complex surface M in $\mathbb{C}^3$, the distance squared function d_c on $M \setminus IL$ can have local singularities of type A_1, A_2, A_3, A_4, D_4 and these are $\mathcal{R}^+$-versally unfolded by the family of distance squared functions.*
The singularities occur at a point $p = \phi(q) \in M \setminus IL$ when:
A_1 : c is on the normal line of M at p.

A_2 : *c is a focal point.*
A_3 : *c is a focal point and is a generic point on the ridge curve.*
A_4 : *c is a special point on the ridge curve.*
D_4 : $\kappa_1(q) = \kappa_2(q)$, *that is, p is a umbilic, and* $c = 1/\kappa_1(p)$.

The structure of the focal set can be obtained from the fact that it is the local component of the bifurcation set of the family of distance-squared functions. Consequently, it is smooth at an A_2-singularity of d_c, diffeomorphic to a cuspidal edge at an A_3-singularity, and to a swallowtail at an A_4-singularity.

As in the case of plane curves, the focal set of a surface $M \setminus IL$ extends to the IL. The family of distance-squared functions is well defined at points on the IL, and its bifurcation set provides an extension of the focal set to the IL. We obtain the following result, analogous to Tari (2012) for surfaces in the Minkowski 3-space.

Theorem 5.6 *Let M be a generic complex surface in $\mathbb{C}^3$ and suppose that the isotropic direction at a point $p \in IL$ is transverse to the IL. Then locally at p, one sheet of its focal set extends to a smooth complex surface along the IL and is tangent to M at that locus. The IL of M is also the IL of that focal sheet. The other sheet of the focal set is either smooth or a cuspidal edge surface.*

Proof The proof is similar to that of Theorem 4.2 in Tari (2012), viewing the Minkowski inner product used there as the holomorphic inner product considered here (so lightlike points and directions in Tari (2012) correspond to isotropic points and directions in the present setting). We give an outline of the proof and refer to Tari (2012) for the details.

Under the genericity assumption, one can choose a local parametrisation of the surface such that ϕ_{z_1} is isotropic along the IL, that is, $E = F = 0$ along that curve. This considerably simplifies the calculations and makes them more transparent.

The function d_c has a singularity at $p = \phi(q) \in IL$ that is more degenerate than A_1 if and only if $c = p + \mu\phi_{z_1} \times \phi_{z_2}$ and

$$(F^2 - EG) - \mu(\overline{n}E - 2\overline{m}F + \overline{l}G) + \mu^2(\overline{m}^2 - \overline{l}\overline{n}) = 0. \tag{22}$$

If p lies on the closure of the parabolic curve, that is, if $(\overline{m}^2 - \overline{l}\overline{n})(q) = 0$, then one of the focal points is at infinity. We assume that this is not the case (the case when it is follows similarly).

As $(F^2 - EG)(q) = 0$, equation (22) has two solutions

$$\mu_1 = 0,\ \mu_2 = \frac{\overline{n}E - 2\overline{m}F + \overline{l}G}{\overline{m}^2 - \overline{l}\overline{n}}.$$

This means that the IL is a curve on one of the focal sheets, say $\mathcal{F}_1$. The condition that the isotropic direction at $p \in IL$ is transverse to the IL ensures that $\mu_2 \neq 0$, so the other focal point lies away from p. It also implies that d_c has an A_2-singularity, and hence $\mathcal{F}_1$ is a regular surface along the IL. We can then paramatrise $\mathcal{F}_1$ at p by

$$\psi(z_1, z_2) = \phi(z_1, z_2) + \mu(z_1, z_2)(\phi_{z_1} \times \phi_{z_2})(z_1, z_2),$$

with $\mu = 0$ along the IL. We have

$$\psi_{z_1} = \phi_{z_1} - \mu_{z_1}\phi_{z_1} \times \phi_{z_2} - \mu(\phi_{z_1} \times \phi_{z_2})_{z_1},$$
$$\psi_{z_2} = \phi_{z_2} - \mu_{z_2}\phi_{z_1} \times \phi_{z_2} - \mu(\phi_{z_1} \times \phi_{z_2})_{z_2}.$$

At $p = \phi(z_1, z_2) \in IL$, we have $\phi_{z_1} \times \phi_{z_2} \in T_pM$, and since $\mu = 0$ along the IL, it follows that ψ_{z_1} and ψ_{z_2} also belong to T_pM. Therefore, the focal sheet $\mathcal{F}_1$ is tangent to the surface M along the IL.

Because the tangent planes of M are isotropic along the IL, the induced holomorphic metric on the focal sheet is degenerate along that curve. Hence, the IL of M coincides with that of the focal sheet. □

If we take M locally in Monge form (as in Sect. 5.1), we get the germ of the family of distance squared functions $d : (\mathbb{C}^2 \times \mathbb{C}^3, (0, (0, 0, c_0))) \to (\mathbb{C}, 0)$, given by

$$d(z_1, z_2, (a, b, c)) = (z_1 - a)^2 + (z_1 - a)^2 + (f(z_1, z_2) - (c + c_0))^2. \tag{23}$$

The algebraic conditions on the Taylor expansion of f at the origin for $d_{(0,0,c_0)}$ to have one of the singularities listed in Theorem 5.5, and for these singularities to be $\mathcal{R}^+$-versally unfolded by the family d in (23), are as given in (Izumiya et al. 2015, Chapter 6) (interpreting the coefficients in Izumiya et al. (2015) as complex numbers).

5.3 Contact with Lines

The family of orthogonal projections in $\mathbb{C}^3$ is as given in Sect. 4.2 and is as follows:

$$\begin{aligned} P : \mathbb{C}^3 \times \mathbb{C}S^2 &\to T\mathbb{C}S^2 \\ (p, v) &\mapsto P(p, v) = (v, P_v(p)) = (v, p - \langle p, v\rangle v), \end{aligned}$$

Theorem 5.7 *For a generic complex surface M in $\mathbb{C}^3$, the orthogonal projection P_v on $M \setminus IL$ can have local singularities of $\mathcal{A}_e$-codimension ≤ 2 and these are $\mathcal{A}_e$-versally unfolded by the family P of orthogonal projections on $M \setminus IL$.*

Take M locally in Monge form (as in Sect. 5.1) and project along directions close to $v_0 = (0, 1, 0)$ to a fixed plane orthogonal to v_0. The directions are parametrised by $(v_1, 1, v_2)$ and the germ of the family of orthogonal projections can be taken as $P : (\mathbb{C}^2 \times \mathbb{C}^2, (0, 0)) \to (\mathbb{C}^2, 0)$, and given, after a change of coordinate in the source, by

$$P(z_1, z_2, v_1, v_2) = (z_1, f(z_1 + v_1 z_2, z_2) - v_2 z_2). \tag{24}$$

Here too, the algebraic conditions on the Taylor expansion of f at the origin for $P_{(v_1,v_2)}$ to have one of $\mathcal{A}_e$-codimension ≤ 2 singularities of map-germs from the plane to the plane, and for these singularities to be $\mathcal{A}_e$-versally unfolded by the family P in (23), are as given in (Izumiya et al. 2015, Chapter 6) (interpreting the coefficients in Izumiya et al. (2015) as complex numbers).

Acknowledgements ADF was supported by FAPESP, the São Paulo Research Foundation, process number 2023/11669-0. FT was supported by the FAPESP Thematic project grant 2019/07316-0. The authors thank the referee for valuable comments and Bill Bruce, Igor Mencattini, Juan Nuño Ballesteros, Toru Ohmoto and Raul Oset Sinha for valuable conversations and suggestions.

Author Contributions The authors contributed equally to this work.

Funding The Article Processing Charge (APC) for the publication of this research was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) (ROR identifier: 00x0ma614).

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no Conflict of interest.

References

Breiding, P., Kohn, K., Sturmfels, B.: Metric Algebraic Geometry. Oberwolfach Semin., 53. Birkhäuser/Springer, Cham, (2024)

Bruce, J.W., Giblin, P.J.: Curves and singularities, 2nd edn. Cambridge University Press, Cambridge, 1984. (1992)

Bruce, J.W.: Envelopes and characteristics. Math. Proc. Cambridge Philos. Soc. **100**, 475–492 (1986)

Bruce, J.W., Fernandes, M.A.C., Tari, F.: Geometric invariants of singular plane curves. Res. Math. Sci. **12**, 81 (2025)

Cipolla, R., Giblin, P.: Visual motion of curves and surfaces. Cambridge University Press, Cambridge (2000)

Damon, J., Giblin, P., Haslinger, G.: Local feature in natural images via singularity theory. Lecture Notes in Mathematics, vol. 2165. Springer, Berlin (2016)

David, J.M.S.: Projection-generic curves. J. London Math. Soc. **27**, 552–562 (1983)

Dias Carneiro, M.J.: Singularities of envelopes of families of submanifolds in $\mathbb{R}^n$. Ann. Sci. École Norm. Sup. **16**, 173–192 (1983)

do Carmo, M.P.: Differential geometry of curves and surfaces. Prentice-Hall Inc, Englewood Cliffs, NJ, (1976)

Izumiya, S., Romero Fuster, M.C., Ruas, M.A.S., Tari, F.: Differential geometry from a singularity theory viewpoint. World Scientific Publishing Company, (2015)

Izumiya, S., Pei, D.-H., Sano, T.: Singularities of hyperbolic Gauss maps. Proc. London Math. Soc. **86**, 485–512 (2003)

Joiţa, C., Siersma, D., Tibăr, M.: Euclidean distance discriminants and Morse attractors. (2024). arXiv:2412.16957

Nomizu, K., Smyth, B.: Differential geometry of complex hypersurfaces II. J. Math. Soc. Japan **20**, 498–521 (1968)

Pessers, V., Van der Veken, J.: On holomorphic Riemannian geometry and submanifolds of Wick-related spaces. J. Geom. Phys. **104**, 163–174 (2016)

Piene, R., Riener, C., Shapiro, B.: Return of the plane evolute. Ann. Inst. Fourier, Grenoble **75**, 1685–1751 (2025)

Porteous, I.R.: Geometric differentiation for the intelligence of curves and surfaces. Cambridge University Press, Cambridge (1994)

Struik, D.J.: Lectures on classical differential geometry. Addison-Wesley Press Inc, Cambridge, MA (1950)

Tari, F.: Caustics of surfaces in the Minkowski 3-space. Q. J. Math. **63**, 189–209 (2012)

Thom, R.: Sur la théorie des enveloppes. J. Math. Pures Appl. **41**, 177–192 (1962)

Viro, A.O.: Differential geometry "in the large" of plane algebraic curves, and integral formulas for invariants of singularities Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **231**, 255–268 (1995)

Viro, A.O.: Differential geometry "in the large" of plane algebraic curves, and integral formulas for invariants of singularities. J. Math. Sci. (New York) **91**, 3499–3507 (1998)

Wall, C.T.C.: Finite determinacy of smooth map-germs. Bull. Lond. Math. Soc. **13**, 481–539 (1981)