UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

Polyhedral Structure on Flag Manifolds

Antonio Conde

Nº 127

NOTAS

Série Matemática

São Carlos – SP
Set./2001

Resumos

Para qualquer número natural n fixado, mergulhamos todas as variedades flag, que vem de uma partição de n, um espaço euclidiano, como órbitas de uma seção ortogonal. Cada uma destas variedades flag mergulhada F é associada a um poliedro P e a estrutura poliedral de P é transferida para F.

Departamento de Matemática
ICMC-USP
Caixa Postal 668
13560-970 - São Carlos-SP

Polyhedral Structure on Flag Manifolds

Antonio Conde

Departamento de Matemática ICMC-USP, C.P. 668

São Carlos-SP Brasil

Abstract: For any fixed natural number n, we embed all flag manifolds coming from a partition of n, into an Euclidean Space as orbits of an ortogonal action. Each of these embedded flags, F is associated to a polyhedron (polytope) P and the polyhedral structure of P is transferred to F.

Key Words: Orthogonal action, embedding, Flag manifolds permutahedra.

Let M_n be the real vector space of $n \times n$ real matrices, M_n^s the subspace of symmetric matrices and M_n^a the subspace of autisymmetric matrices. We consider in M_n the inner product

$$< A, B >= tr(AB^t)$$

M_n decomposes in the orthogonal direct sum

$$M_n = M_n^s \oplus M_n^a$$

The action of the ortogonal group 0(n) on M_n through conjugation

$$0(n) \times M_n \to M_n$$

$$(X, A) \to XAX^t$$

is an isometry of M_n that preserves the above decomposition. M_n^s and M_n^a are orthogonal to each other and such action preserves both of them. 0(n) is contained in M_n and M_n^a is the tangent space to $0(n)$ at the identihy matrix $\mathbb{1}$ and so M_n^s is normal to $0(n)$ at $\mathbb{1}$.

We restrict our attention to the action of $0(n)$ on M_n^s.

$$0(n) \times M_n^s \to M_n^s$$

$$(X, A) \to XAX^t$$

Let $D_n \subset M^s$ be the vector subspace of the diagonal matrices.

Since symmetric matrices are orthogonally diagonalizable, each orbit.

$$F(A) = \{XAX^t / X \in 0(n)\}$$

crosses D_n, say in $D \varepsilon D_n$. So $F(A) = F(D)$ and we think of orbits determined by diagonal matrices D. The values in the diagonal of D are the eigenvalues of any matrix is the orbit $F(D)$. Since these eigenvalues do not come in an order, we fix the increasing order.

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \qquad \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n$$

We let n_i be the multiplicity of λ_i. So

$$n_1 + n_2 + \ldots + n_s = n$$

and we heve a partition of n. Such a partition defines uniquely a flag manifold.

$$\frac{0(n)}{0(n_1) \times 0(n_2) * \ldots * 0(n_s)}$$

which is usually denoted by $F(n_1, n_2, \ldots, n_s)$, and if n_i appears p_i times in the partition we also use

$F(n_1^{P_1}, n_2^{P_2}, \ldots n_t^{P_t})$ for $F(n_1, n_2, \ldots, n_s)$

The orbit $F(D)$ of the diagonal matrix D with diagonal $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ and multiplicite $n_1, n_2, \ldots, n_s$ $(n_1 + n_2 + \ldots + n_s = n)$ crosses D_n in more points (diagonal matrices) obtained from D by permutation of the $\lambda_i' s$. For instance, if the $\lambda_i' s$ were distinct (2×2)

then the number of points in $D_n \cap F(D)$ would be $n!$. For the general case, we are treating it is

$$\frac{n!}{n_1!n_2!...n_s!}$$

If σ is a permutation of $J_n = \{1, 2, ..., n\}$ and $D_\sigma$is the diagonal matrix obtained by permuting the diagonal of D, using σ, D_σ is in the orbit of D and so there exists a matrix X in $0(n)$ such that $XDX^t = D_\sigma$. We can take X^t to be the permutation matrix obtained from the identity matrix $\mathbb{1}$ by permuting its colums according to σ.

When we apply such X to any matrix A in M_n^s as $X\Delta X^t$ it is an isometry of M_n^s which preserves each orbit and preserves D_n.

We want now to identify the orbit $F(D)$ through our D with the flag manifold $F(n_1, n_2, ..., n_s)$. To do this we need to identify the isotrofy group of D. So we have to specifg those matrices $X \in 0(n)$ such that

$$XDX^t = D$$

or $XD = DX$ since $X^t = X^{-1}$. So the isotropy group of D is compose of those matrices X of $0(n)$ which commutes with D. The multiplicity of λ_i being n_i we conclude that X has to belong to the subgroup.

$$0(n_1) \times 0(n_2) \times .... \times 0(n_s) \subset 0(n)$$

This product is understood as the set of matrices in $0(n)$ with ? of $0(n_i)$ in the diagonal

$$\begin{pmatrix} \boxed{O(n_1)} & & & \\ & \boxed{O(n_2)} & & \\ & & \ddots & \\ & & & \boxed{O(n_s)} \end{pmatrix}$$

Any element in this group stabilizes D. So this is the isotropy group of D. We conclude that the orbit $F(D)$ is then homeomorphic to the quocient $0(n)/0(n_1) \times ... \times 0(n_s)$ that is the flag manifold $F(n_1, n_2, ..., n_s)$. As for differentiable the map

$$\begin{array}{ccc} O(n) & \longrightarrow & F(D) \\ X & \longmapsto & X\,DX^t \end{array}$$

is more than C^∞.

For each partition of n, $n_1, n_2 + ... + n_s = n$ we can produce a diagonal matrix D, with diagonal $\lambda_1, \leq \lambda_2 \leq ... \leq \lambda_n$ and λ_i with multiplicity n_i and so $F(n_1, n_2, ..., n_s)$ gets embedded in M_n^s as the orbit $F(D)$.

Varying D in D_n we get embeddings of all flag manifolds coming from partitions of n in M_n^s. These embeddings are in fact a foliation of M_n^s with singularities once the dimensions of the leaves-vary.

$$dim\ F(n_1, n_2, ..., n_s) = \sigma_2(n_1, n_2, ..., n_s)$$

where σ_2 is the second elementary symmetric polynomial in s variables.

$$dim\ M_n^s = \frac{1}{2}\ n(n+1)$$

For the partition of n by $1's$, $1 + 1 + ... + 1 = n$ we have

$$dim\ F(1, 1, ..., 1) = dim\ (F(1^n)) = \frac{1}{2}\ n(n-1)$$

$F(1^n)$ is the highest dimmensional orbit in M_n^s.

The orthogonal projection of M_n^s onto D_n

$$\delta : M_n^s \longrightarrow D_n$$

is $\delta(A) = diagonal\ of\ A$.

If we fix $D \in D_n$ and take the orbit $F(D)$, we have the set $F(D) \cap D_n$ which is finite with cardinal

$$\frac{n!}{n_1!n_2!...n_s!}$$

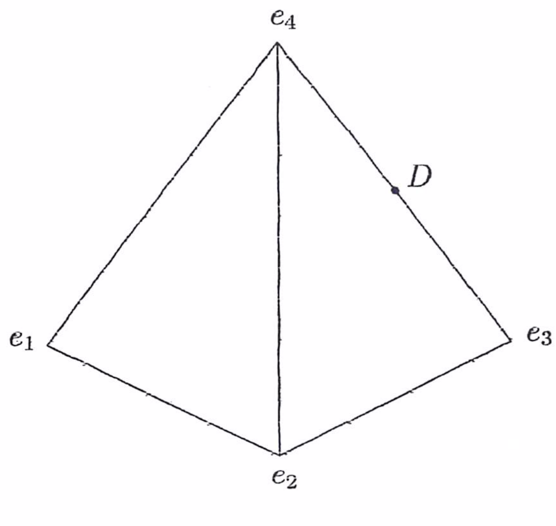
Where n_i is the multiplicity of the eigenvalue λ_i of D.

For example for $n = 4 = 2 + 2$ we can take $D = diag(0, 0, 1/2, 1/2)$. So 0 has multiplicity 2 and 1/2 also. Such a D produces an embedding of $F(2,2) = F(2^2)$, which is the Grassmann manifold of 2-planes in $I\!R^4$ (through the origin) in M_4^s as the orbit $F(D)$. The cardinal of $F(D) \cap D_n$ is

$$\frac{4!}{2!2!} = 6$$

The 4 points of $I\!R^4$ given by the canonical bases e_1, e_2, e_3, e_4 has, as its convex hull, the regular tetrahedron of the 3-dimensional hiperplane $x_1 + x_2 + x_3 + x_4 - 4 = 0$. This regular tetrahedron is the set of points (diagonal matrices) with non negative coordinates with trace 1.

In a picture we have

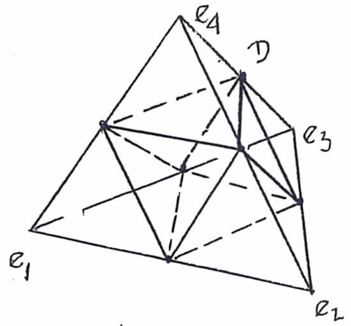


The point D corresponding to our matrix

$$diag(0, 0, 1/2, 1/2)$$

is the midle poind of the edge connecting e_3 to e_4.

To get the other points of $F(D) \cap D_4$, all we have to do is to permute the coordinates $(0, 0, 1/2, 1/2)$ using the symmetric group of $J_4 = \{1, 2, 3, 4\}$. This will give us the midle points of all the edges our tetrahedron, which has for convex hull the regular octrahedron



Coming back to our general diagonal matrix $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and λ_i with multiplicity n_i; its known [1] [3] [4] that the ortogonal projection of $F(D)$ onto D_n.

$$\delta : F(D) \longrightarrow D_n$$

$$\delta(A) = diagonal of A$$

has as immage the convex hull of

$$F(D) \cap D_n$$

In our above example, $F(D)$ is the Grassmann manifold $G_{2,2}$ embedded in M_4^s, whose orthogonal projection in D_4 is the regular tetrahedron.

We can assume our general diagonal matrix D having nonnegative eigenvalues, whose trace is 1. We can impose this restriction and still get as its orbit any type of flag manifold, for all we need is the partition of n, which determined by the multiplicites of the eigenvalues only. This way the point P whose coordinates are in the diagonal of D, is in the basic simplex which is the convex hull of the canonical base of D_n.

Our attention now is on the fact that the orthogonal projection of the orbit $F(D)$ into D_n

$$\delta : F(D) \longrightarrow D_n$$

$\delta(A) = diagonal\ of\ A$, has for image the convex hull of $F(D) \cap D_n$, which is a polyhedron (or polytope) contained in the basic simplex having as vértices the canonical base of D_n. Think of the particular example we gave with $D = diag(0,0,1/2,1/2)$ $F(D)$, being the Grassmann manifold $G_{2,2}$ and the polyhedron the octahedron contained in the tetrahedron.

The polyhedron (or polytope) bove, has its vértices obtained by permuting the coordinates in the diagonal of D. Such a polyhedron is called a permutahedron.

As we mentioned before, if σ is a permutation of $J_n = \{1, 2, ..., n\}$ and we apply σ to D obtaining D_σ, that has its diagonal as that of D but permuted by σ.

D_σ is a conjugate of D by the permutation matrix determined by σ, say X.

$$D_\sigma = XDX^t$$

Such a conjugation applyed to the orbit $F(D)$ preserves it. We want to verify that such action of X an $F(D)$ commutes with the projection δ.

For this we take a general $n \times n$ matrix A and see what happens with the diagonal entries of A when we conjugate A by X, XAX^t.

Let us specify X^t as the permutation matrix given by a transposition τ that interchanges l and k. So the operation on A given by AX^t interchanges the colums ℓ and k of A. The next operation on AX^t given by XAX^t interchanges the l and k rows of AX^t and as such the diagonal of XAX^t is the diagonal of A where a_{kk} and $a_{\ell\ell}$ are interchanged and we have that

$$\delta(A_\tau) = \tau\delta(A)$$

If it is true for transpositions τ, it is true for any permutation σ.

If σ is any permutation of the n-symmetric group and we set σ to act on D_n by permuting the diagonal entries, this gives as a linear isomorphism of D_n and at the same time σ preserves the set of vértices $F(D) \cap D_n$ which is determined by D through permutations. So the action of σ on the convex hull of $F(D) \cap D_n$ preserves it.

In conclusion we have an action of the n-symmetric group S_n as isometries of the permutahedron $P(D)$ which is the convex hull of $F(D) \cap D_n$.

Now we have all the elements to transfer the polyhedral structure of $P(D)$ to the flag manifold $F(D)$.

For each vertex of $P(D)$ we have

$\delta(V) = V$, and V is the only point of $F(D)$ with this property.

In fact if $Y \in F(D)$ and $\delta(Y) = V$ we have that $Y = XVX^t$ for some $X \in 0(n)$. Conjugation is isometry in M_n^s, Y has the same diagonal as V, having the same norm as V, Y cannot have any non-zero entrie out side the diagonal, so $Y = V$.

So for each vertex V of $P(D)$ we call $\{V\} = \delta^{-1}(V)$ a vertex of $F(D)$. Now we do the same $\delta^{-1}()$ for eddges, faces etc... getting a decomposition of $F(D)$ into subsets corresponding to the combinatorial structure of $P(D)$. Now the action of the symmetric group. S_n on $P(D)$ is canonically transfered to $F(D)$ where it interchanges the combinatorial elemetns isometricaly, as we wanted.

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