

**THE ULTRABORNOLOGICALNESS OF THE SPACE OF
DENJOY-PERRON-KURZWEIL INTEGRABLE FUNCTIONS AND OF OTHER
NATURAL NON-COMPLETE SUBSPACES OF $C([a, b], X)$ AND OF $\prod E_i$. II**

by

Antonio Gilioli

IME-USP

Professor Antonio Gilioli died tragically in a car accident on May 26, 1990 on his way back from the 31th Seminário Brasileiro de Análise that took place at São Carlos. Some days before the Seminary he gave us the manuscript extending, to dimension $n \geq 2$, the results he presented at the meeting. We publish here this manuscript as part II; the definitions, notations, etc. of part I [31° SBA (1990), 323-367] are kept.

Chaim S. Höning

§6. Generalization (with Applications) of §3

Lemma 6.1. Let E be a lts, P and Q continuous projections of E , such that $PQ = QP$. Then,

- a) PQ is a continuous projection of E such that $\text{Ker}(PQ) = (\text{Ker } P) + (\text{Ker } Q)$,
 $\text{Im}(PQ) = (\text{Im } P) \cap (\text{Im } Q)$;
- b) $R = I - (I - P)(I - Q)$ is a continuous projection of E such that $\text{Im } R = (\text{Im } P) + (\text{Im } Q)$; moreover, $\text{Im } R$ is topologically isomorphic to the product of the subspaces $\text{Im } P$ and $(\text{Im } Q) \cap (\text{Ker } P)$ of E ;
- c) if P_1, \dots, P_n are continuous projection of E such that $P_i P_j = P_j P_i$ for $i, j = 1, \dots, n$, then $S = I - \prod_{i=1}^n (I - P_i)$ is a continuous projection of E such that $\text{Im } S = \sum_{i=1}^n \text{Im } P_i$, and one may find linear subspaces F_i of $\text{Im } P_i$ such that $\text{Im } S$ is topologically isomorphic to the product $\prod_{i=1}^n F_i$. If, in this case,

D_m is a sequence of subsets of $\text{Im } S$ containing 0 such that $D_m + D_m \subset D_{m+1}$ for all m , and if none of the D_m is a 0-nghb in $\text{Im } S$, then there is some $i \leq n$ such that $D_m \cap \text{Im } P_i$ is not a 0-nghb in $\text{Im } P_i$ for all m (because $D_m \cap F_i$ is not a 0-nghb in F_i for all m).

Part b) of the above lemma gives a clue how to generalize the context of §3.

Let E be a lts, K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K , and for each $R \in \mathcal{R}$, $P_R : E \rightarrow E$ a continuous projection, such that:

$$(6.2) \left\{ \begin{array}{l} \text{i) } \phi \in \mathcal{R}, K \in \mathcal{R}, P_\phi = 0, P_K = I \\ \text{ii) if } R, S \in \mathcal{R}, \text{ then } R \cap S \in \mathcal{R} \text{ and } P_{R \cap S} = P_R P_S = P_S P_R \\ \text{iii) if } R \in \mathcal{R}, R_1, \dots, R_n \in \mathcal{R} \text{ and } R = \bigcup_{i=1}^n R_i, \text{ then } P_R = I - \prod_{i=1}^n (I - P_{R_i}) \\ \text{iv) if } R \in \mathcal{R} \text{ and } \text{diam } R > 0, \text{ there exists } n \in \mathbb{N} \text{ and } R_1, \dots, R_n \in \mathcal{R} \text{ such} \\ \quad \text{that } R = \bigcup_{i=1}^n R_i \text{ and } \text{diam } R_i \leq \frac{1}{2} \text{diam } R \text{ for } i = 1, \dots, n. \\ \text{v) if } q \in K, \text{ then either } q \text{ is an isolated point of } K, \text{ or there exists} \\ \quad m \in \mathbb{N} \text{ and } S_1, \dots, S_m \in \mathcal{R} \text{ such that } K = \bigcup_{i=1}^m S_i \text{ and } q \text{ is} \\ \quad \text{a frontier point (considered in } Z \text{) of each } S_i, i = 1, \dots, m \end{array} \right.$$

Remark that, by iv), if q is an isolated point of K , then $\{q\} \in \mathcal{R}$.

Suppose that G is a linear subspace of E , such that

$$(6.3) \quad P_R(G) \subset G, \quad \text{for all } R \in \mathcal{R}.$$

Then, we will denote $G^R = P_R(G)$. Of course, if R and R_i are as in (6.2) iii), then $G^R = \sum_{i=1}^n G^{R_i}$, and G^R is topologically isomorphic to a product of the type $\prod_{i=1}^n H_i$, with $H_i \subset G^{R_i}$ for each i (by Lemma 6.1 c)).

Let \mathcal{A} be the class of finite unions of elements of \mathcal{R} . If $A \in \mathcal{A}$, with $A = \bigcup_{i=1}^n R_i$, we define:

$$(6.4) \quad P_A = I - \prod_{i=1}^n (I - P_{R_i}).$$

It is easy to verify, applying (6.2) ii) and iii), that the definition (6.4) does not depend on the particular representation of A as a finite union of elements of \mathcal{R} . It is evident that, if $A = \bigcup_{i=1}^m A_i$, with $A_i \in \mathcal{A}$, then $A \in \mathcal{A}$ and $P_A = I - \prod_{i=1}^m (I - P_{A_i})$. It is not difficult to verify that, if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$ and $P_{A \cap B} = P_A P_B = P_B P_A$. If (6.3) holds, it follows from Lemma 6.1 c) that $P_A(G) \subset G$ for all $A \in \mathcal{A}$.

Let us denote by $\overset{\circ}{M}$ the interior of M (considered in Z).

Lemma 6.5. Let E be a lts, K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K , $P_R : E \rightarrow E$, $R \in \mathcal{R}$, continuous projections of E , satisfying (6.2). Let \mathcal{A} be the class of finite unions of elements of \mathcal{R} and assume definitions (6.4). Suppose that Ω is an open (in Z) subset of K , R_n and S_n two increasing sequences of elements of \mathcal{A} such that $\Omega = \cup R_n = \cup \overset{\circ}{R}_n = \cup S_n = \cup \overset{\circ}{S}_n$, and that $\{P_A : A \in \mathcal{A}\}$ is equicontinuous. Then, a) for each $x \in E$, there exists $\lim_{n \rightarrow \infty} P_{R_n}(x)$ if and only if there exists $\lim_{n \rightarrow \infty} P_{S_n}(x)$, and in this case the limits are equal, and will be denoted by $P_\Omega(x)$; b) if there exists $P_\Omega(x)$ for all $x \in E$, then P_Ω is a continuous projection of E , which commutes with P_A , for all $A \in \mathcal{A}$, and which has the following properties:

- i) if $A \in \mathcal{A}$ and $A \subset \Omega$, then $P_A P_\Omega = P_A$;
- ii) if $A \in \mathcal{A}$ and $\Omega \subset A$, then $P_A P_\Omega = P_\Omega$;
- iii) if θ is another open (in Z) subset of K for which P_θ is defined on all E , and $\theta \subset \Omega$, then $P_\theta P_\Omega = P_\Omega P_\theta = P_\theta$.

Suppose now that we also have:

$$(6.6) \left\{ \begin{array}{l} \text{i) } \{P_A : A \in \mathcal{A}\} \text{ is equicontinuous} \\ \text{ii) for each } q \in K \text{ and } \varepsilon > 0, \text{ there exists } R \in \mathcal{R} \\ \text{such that } q \in \overset{\circ}{R} \text{ and } \text{diam } R < \varepsilon. \end{array} \right.$$

Then, every open (in Z) subset Ω of K may be written as the union of an increasing sequence R_n of elements of \mathcal{A} , such that $\Omega = \cup R_n = \cup \overset{\circ}{R}_n$. Hence, by Lemma 6.5 a),

we may define $P_\Omega = \lim_{n \rightarrow \infty} P_{R_n}$ on a linear subspace of E .

As a matter of fact, in several concrete examples we will not need Lemma 6.5 a) in order to define P_R° for $R \in \mathcal{R}$, because there will exist a canonical increasing sequence or family of subsets of $\overset{\circ}{R}$ whose interiors cover $\overset{\circ}{R}$, hence we may define P_R° as the limit of such sequence or family, without needing to know whether the limit is independent of the sequence or family chosen.

Usually, (6.6) i) is very restrictive, and it will be enough to have:

$$(6.6') \left\{ \begin{array}{l} \text{i) } \{P_R : R \in \mathcal{R}\} \text{ is equicontinuous} \\ \text{ii) if } R \in \mathcal{R}, \text{ there exists an increasing sequence } R_n \in \mathcal{R} \\ \text{such that } \cup R_n = \cup \overset{\circ}{R}_n = \overset{\circ}{R} \end{array} \right.$$

Suppose now that

$$(6.7) \quad \text{for each } R \in \mathcal{R}, P_R^\circ \text{ is defined on all } E.$$

By Lemma 6.5 b), P_R° commutes with P_A for all $A \in \mathcal{A}$, $P_A P_R^\circ = P_A$ if $A \subset \overset{\circ}{R}$ and $P_A P_R^\circ = P_R^\circ$ if $\overset{\circ}{R} \subset A$.

Suppose that G is a linear subspace of E such that, besides (6.3), also satisfies

$$(6.3') \quad P_R^\circ(G) \subset G \text{ for all } R \in \mathcal{R}.$$

We denote then $G_R = P_R^\circ(G)$ and $G_{\partial R} = (P_R - P_R^\circ)(G) = P_{\partial R}(G)$, and $P_{\partial R} = P_R - P_R^\circ$.

In the case $E = \Pi\{E_\alpha : \alpha \in K\}$, where E_α are lts, and $R \in \mathcal{R}$, we shall consider as the canonical P_R the projections $P_R : E \rightarrow E$ defined by $P_R(x) = y$, with $y_\alpha = x_\alpha$ if $\alpha \in R$ and $y_\alpha = 0$ if $\alpha \notin R$. It is easy to see that (6.2) iii) holds, and (6.2) i) and ii) also hold, provided ϕ , K and $R \cap S$ belong to \mathcal{R} , hence condition (6.2) reduces to the requirement that ϕ and K belong to \mathcal{R} , that \mathcal{R} is stable by finite intersections, and that iv) and v) of (6.2) hold. Moreover, (6.6) i) always holds, and if we assume (6.6) ii) then P_Ω is defined on all E for every open (in Z) subset of K , so in particular (6.7) is verified. In that case, $P_\Omega(x) = y$, with $y_\alpha = x_\alpha$ if $\alpha \in \Omega$ and $y_\alpha = 0$ otherwise.

The following Propositions 6.8 and 6.9 are the analogues (with similar proofs) of Proposition 3.7 and 3.8 to this new context.

Proposition 6.8. Let E be a lts, G a linear subspace of E , K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K and, for each $R \in \mathcal{R}$, let $P_R : E \rightarrow E$ be a continuous projection of E . Let D_m be a sequence of balanced subsets of G such that $D_m + D_m \subset D_{m+1}$ for every m , but such that none of the D_m is a 0-nghb in G .

a) If (6.2) and (6.3) hold, then there is $q \in K$ such that either

i) q is an isolated point of K (hence $\{q\} \in \mathcal{R}$) and $D_m \cap G^{(q)}$ is not a 0-nghb in $G^{(q)}$ for any m

or

ii) there is a strictly decreasing sequence $K_n \in \mathcal{R}$, with $\lim_{n \rightarrow \infty} (\text{diam } K_n) = 0$, such that $\{q\} = \bigcap K_n$, q belongs to the frontier of each K_n (in Z) and $D_m \cap G^{K_n}$ is not a 0-nghb in G^{K_n} , for any m and n .

b) If (6.2), (6.3), (6.6'), (6.7) and (6.3') hold (assuming definition (6.4)), if for each isolated point $q \in K$ there is some $m \in \mathbb{N}$ such that $D_m \cap G^{(q)}$ is a 0-nghb in $G^{(q)}$ and if, for each $R \in \mathcal{R}$ there is some $m \in \mathbb{N}$ such that $D_m \cap G_{\partial R}$ is a 0-nghb in $G_{\partial R}$, then, there is $p \in K$ and a decreasing sequence $K_n \in \mathcal{R}$, with $\lim_{n \rightarrow \infty} (\text{diam } K_n) = 0$, such that $p \in \partial K_n$ for all n and $D_m \cap G_{K_n}^{\circ}$ is not a 0-nghb in $G_{K_n}^{\circ}$ for any m and n .

Proposition 6.9. Let E be a lts, G a linear subspace of E , K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K , $P_R : E \rightarrow E$ continuous projections for all $R \in \mathcal{R}$, satisfying (6.2), (6.3), (6.6'), (6.7) and (6.3') (assuming definition (6.4)), and let F be a linear subspace of E such that $G \subset F$

and such that

$$(6.10) \quad \forall x \in E, \forall R \in \mathcal{R}, \text{ if } P_A(x) \in G \text{ for all } A \in \mathcal{A} \\ \text{such that } A \subset \overset{\circ}{R}, \text{ then } P_{\overset{\circ}{R}}(x) \in F.$$

If x_n is a sequence with property ℓ^∞ in E , then $B(\ell^\infty, x_n) \subset F$ in each of the following cases:

- a) there exists $p \in K$ and a decreasing sequence $K_n \in \mathcal{R}$ such that $\lim_{n \rightarrow \infty} (\text{diam } K_n) = 0$, $\{p\} = \bigcap K_n$, $p \in \partial K_1$ (hence $p \in \partial K_n$ for all n) and $x_n \in G_{K_n}^\circ$ for all n ;
- b) there exists $R \in \mathcal{R}$ and an increasing sequence $A_n \in \mathcal{A}$ such that $\bigcup A_n = \overset{\circ}{R}$ and $x_n \in (P_{\overset{\circ}{R}} - P_{A_n})(G)$ for all n .

It is useful to remark that, if $R \in \mathcal{R}$ is such that there is an increasing sequence $S_n \in \mathcal{R}$ such that $\overset{\circ}{R} = \bigcup S_n = \bigcup \overset{\circ}{S}_n$ and $x \in E$, then $P_A(x) \in G$ for all $A \in \mathcal{A}$ such that $A \subset \overset{\circ}{R}$ if and only if $P_{S_n}(x) \in G$ for all n and if and only if $P_S(x) \in G$ for all $S \in \mathcal{R}$ such that $S \subset \overset{\circ}{R}$.

Theorem 6.11. a) Suppose that either: 1) E is a metrizable complete lts or 2) $E = \Pi\{E_\alpha : \alpha \in K\}$. Let K be a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K , $P_R : E \rightarrow E$ continuous projections for all $R \in \mathcal{R}$, satisfying (6.2), (6.6') and (6.7) (assuming definition (6.4)); in the second case, we assume that P_R are the canonical P_R . Suppose also that G and F are linear subspaces of E , with $G \subset F$, such that G satisfies (6.3), (6.3'), (6.10) and also:

$$(6.12) \quad G \text{ is dense in } F.$$

$$(6.13) \quad G^{(\theta)} \text{ and } G_{\partial R} \text{ are } u\text{-BL (resp. BL, resp. barrelled) for every isolated point } q \in K \text{ and for every } R \in \mathcal{R}.$$

Then, F is u -BL (resp. BL, resp. barrelled).

b) If, furthermore, $G = F$ and
 (6.13') $G^{(g)}$ and $G_{\partial R}$ are u -ultrab. BL (resp. ultrab. BL, resp. ultrab.)
 for every isolated point $q \in K$ and for every $R \in \mathcal{R}$,
 then, F is u -ultrab. BL (resp. ultrab. BL, resp. ultrab.).

Proof. Similar to that of Theorem 3.11, utilizing Proof 6.8 and 6.9 a) instead of Prop. 3.7 and 3.8. ■

The Theorems 3.11' and 3.11" (as well as Corol. 3.23') are secondary results and we leave to the reader the formulation of the corresponding statements in the present context. We turn our attention to the analogue of Theorem 3.21, which needs the analogues of Definition 3.16 and Proposition 3.17.

Definition 6.14. Let E be lts, G a linear subspace of E , K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subsets of K and $P_R : E \rightarrow E$ continuous projections for each $R \in \mathcal{R}$, satisfying (6.2), (6.3), (6.6'), (6.7) and (6.3') (assuming definition (6.4)). If $x \in E$ and $R \in \mathcal{R}$, we say that x is of type G in R if $P_A(x) \in G$ for all $A \in \mathcal{A}$ such that $A \subset \overset{\circ}{R}$. We denote by $F(G)$ the \mathcal{R} -closure of G in E , which is defined to be the set of all $x \in E$ for which there is a finite number R_1, \dots, R_n of elements of \mathcal{R} which cover K , such that $P_{\partial R_i}(x) \in G$ for $i = 1, 2, \dots, n$, and such that, for each R_i , $i = 1, \dots, n$, x is of type G in R_i . One may also define the α - \mathcal{R} -closure $F^\alpha(G)$ of G in E for each ordinal number α , and define the total \mathcal{R} -closure of G in E .

Proposition 6.15. Assume the conditions of Definition 6.12 hold, and that $F(G)$ is the \mathcal{R} -closure of G in E . Then:

- i) $F(G)$ is a linear subspace of E , such that $G \subset F(G)$

- ii) G is dense in $F(G)$
- iii) $F(G)$ satisfies (6.10)
- iv) $F(G)$ is the smallest linear subspace of E containing G , for which (6.10) holds.

Theorem 6.16. Let E be a lcs, K a compact metric subspace of a topological Hausdorff space Z , \mathcal{R} a class of closed subspaces of K , $P_R : E \rightarrow E$ continuous projections for each $R \in \mathcal{R}$, G a linear subspace of E satisfying (6.2), (6.3), (6.6'), (6.7) and (6.3') (assuming Definition (6.4)), $F(G)$ the \mathcal{R} -closure of G in E and suppose that

$$(6.13'') \quad G^{(q)} \text{ and } G_{\theta R} \text{ are ultrabornological for every isolated point } q \in K \text{ and for every } R \in \mathcal{R}.$$

Then, $F(G)$ (or, more generally, $F^\alpha(G)$ for each ordinal number α) is ultrabornological in each of the following cases:

- 1) E is metrizable complete (and lcs);
- 2) $E = \prod \{E_\alpha : \alpha \in K\}$, where the E_α are lcs and the P_R are the canonical P_R .

Proof. Similar to that of Theorem 3.21, utilizing respectively Proposition 6.9 b), Theorem 6.11 and Proposition 6.15 instead of Proposition 3.8, Theorem 3.11 and Proposition 3.17.

In order to show that G is dense in $F(G)^\alpha$, take $x \in F(G)$, and let $R_1, \dots, R_m \in \mathcal{R}$ be the sets given by Definition 6.14. It is enough to show that $P_{R_i}^\bullet(x)$ belongs to the closure of G in $F(G)^\alpha$ for $i = 1, \dots, m$.

The claim corresponding to that of Theorem 3.21 is: there is $w \in G$, an increasing sequence $A_n \in \mathcal{A}$, such that $\cup A_n = \cup \overset{\circ}{A}_n = \overset{\circ}{R}_i$ and a sequence $x_n \in (P_{R_i}^\bullet - P_{A_n})(G)$ such that:

- i) $P_{R_i}^\bullet(x) = w + \sum_{n=1}^\infty x_n$;
- ii) the sequence $z_n = 2^n x_n$ has property ℓ^∞ in E
- iii) $B(\ell^\infty, z_n)$ is a Banach disk of E'' .

Applications to subspaces of $C([a, b] \times [c, d], X)$

Let $Z = \mathbb{R}^2$, $K = [a, b] \times [c, d]$ and \mathcal{R} the set of all rectangles $[\alpha, \beta] \times [\gamma, \delta]$, with $a \leq \alpha < \beta \leq b$, $c \leq \gamma < \delta \leq d$. Let X be a Banach space and $E = \ell^\infty(K, X)$. If $\lambda \in]a, b[$, we define $S_\lambda : E \rightarrow E$ by $(S_\lambda f)(x, y) = f(x, y)$ if $x \in [a, \lambda]$ and $(S_\lambda f)(x, y) = f(\lambda, y)$ if $x \in]\lambda, b[$, and we define $S_a : E \rightarrow E$ as the null mapping. If $\mu \in]c, d[$, we define $S^\mu : E \rightarrow E$ by $(S^\mu f)(x, y) = f(x, y)$ if $y \in [c, \mu]$ and $(S^\mu f)(x, y) = f(x, \mu)$ if $y \in]\mu, d[$, and we define $S^c : E \rightarrow E$ as the null mapping. It is easy to see that each pair of elements of $\{S_\lambda : \lambda \in [a, b]\} \cup \{S^\mu : \mu \in [c, d]\}$ commutes. If $R = [\alpha, \beta] \times [\gamma, \delta]$, we define $P_R : E \rightarrow E$ by $P_R = (S_\beta - S_\alpha)(S^\delta - S^\gamma)$.

One can easily verify (6.2); the space Z assures that (6.2) v) holds for \mathcal{R} , even at the vertices of K . Although each S_λ and each S^μ has norm ≤ 1 , we can only say that $\|P_R\| \leq 4$, so that (6.6') holds, but (6.6) i) is probably false. Remark that (6.7) is not satisfied by E ; if one considers $E_1 = G([a, b] \times [c, d], X)$, the space of regulated functions in two variables, then (6.7) is satisfied on E_1 . Consider also $E_2 = C([a, b] \times [c, d], X)$ and $E_3 = C_0([a, b] \times [c, d], X)$, the last one being the space of continuous functions $f : K \rightarrow X$ such that $f(x, y) = 0$ if $x = a$ or $y = c$. We have $(E_3)_{\partial R} = \{0\}$ for all $R \in \mathcal{R}$, hence (6.13), (6.13') and (6.13'') are automatically satisfied by a subspace G of E_3 , and $P_R = P_R$. Then, Theorem 6.11 reduces to:

Theorem 6.17. Let $E = C_0([a, b] \times [c, d], X)$, G and F linear subspaces of E such that $G \subset F$, and G is dense in F . Let \mathcal{R} be the set of all rectangles $[\alpha, \beta] \times [\gamma, \delta]$ contained in K and suppose that

$$(6.3) \quad P_R(G) \subset G \text{ for all } R \in \mathcal{R}$$

$$(6.10) \quad \forall x \in E, \forall R \in \mathcal{R}, \text{ if } P_S(x) \in G \text{ for all } S \in \mathcal{R} \text{ such that } S \subset \overset{\circ}{R}, \\ \text{then } P_R(x) \in F.$$

Then, F is u -BL. If, moreover, $G = F$, then F is u -ultrab. BL. Assuming only (6.3), then the \mathcal{R} -closure $F(G)$ of G in E is ultrabornological, as also are the α - \mathcal{R} -closure of G in E and the total closure of G in E .

If $f : K \rightarrow X$ is integrable, we put $\tilde{f}(s, t) = \int_R f$, where $R = [a, s] \times [c, t]$. Applying Theorem 6.17, one gets as in §4 the following

Example 6.18. The space

$$\tilde{F}_g = \{ \tilde{f} : f : K \rightarrow X \text{ is Kurzweil-integrable} \}$$

is u -ultrab. BL.

We leave to the reader the construction of other ultrabornological spaces.