SPECTRAL PARTITION PROBLEMS WITH VOLUME AND INCLUSION CONSTRAINTS

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Abstract. In this paper we discuss a class of spectral partition problems with a measure constraint, for partitions of a given bounded connected open set. We establish the existence of an optimal open partition, showing that the corresponding eigenfunctions are locally Lipschitz continuous, and obtain some qualitative properties for the partition. The proof uses an equivalent weak formulation that involves a minimization problem of a penalized functional where the variables are functions rather than domains, suitable deformations, blowup techniques and a monotonicity formula.

Keywords: Optimal open partition, Shape optimization, Multiphase problems, Regularity of solutions, First eigenvalue.

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1. Introduction

In this article we study an optimal partition problem with volume and inclusion constraints, for a cost functional depending on the first Dirichlet eigenvalue of the Laplacian. Let Ω be a bounded connected open set of $\mathbb{R}^N$, for $N \geq 2$. For an integer $k \geq 2$ and $0 < a < |\Omega|$, we consider the multiphase shape optimization problem

$$\inf \left\{ \sum_{i=1}^{k} \lambda_1(\omega_i) \;\middle|\; \begin{array}{c} \omega_i \subset \Omega \text{ are nonempty open sets for all } i = 1, \dots, k, \\ \omega_i \cap \omega_j = \emptyset \text{ for all } i \neq j \text{ and } \sum_{i=1}^{k} |\omega_i| = a \end{array} \right\}, \tag{1.1}$$

where $\lambda_1(\cdot)$ denotes the first Dirichlet eigenvalue and $|\cdot|$ stands for the Lebesgue measure. The main goal of this paper is to prove the existence of an optimal open partition to (1.1), showing also that the corresponding eigenfunctions are locally Lipschitz continuous in Ω (see Theorem 1.2 below). The proof uses a weak formulation that involves a minimization problem of a penalized functional where the variables are functions rather than domains.

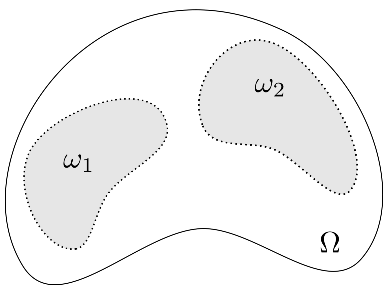

Fig. 1. An admissible partition (ω_1, ω_2) in $\mathcal{P}_a(\Omega)$ for a certain $0 < a < |\Omega|$ and $k = 2$.

Minimizing a functional with measure constraints appears in electromagnetic casting processes [21, 25]. The study of these functionals is motivated by their applications to industry and has attracted many mathematicians, physicists, and engineers. In particular, minimization problems with volume constraint involving the Dirichlet eigenvalues for the Laplace operator have been extensively studied by many authors; we refer to [10, 11, 13, 14, 15] and the references therein, as well as the books [9, 30, 31, 32, 47]. Another example of an optimization problem with measure constraint, this time appearing in the context of one-phase free boundary problems, is given by the paper by N. Aguilera-H. Alt and L. Caffarelli [1]. We point out that optimal *partition* problems with measure constraints can also be motivated considering situations where there is a limitation in several different resources; the case of cost functions depending on Dirichlet eigenvalues is a natural generalization of the classical one phase problem, see (1.2) below.

Problem (1.1) with one phase corresponds to

$$\inf\{\lambda_1(\omega),\ \omega \subset \Omega,\ \omega \text{ open},\ |\omega| = a\}. \tag{1.2}$$

Let us split the discussion of such a problem between the case when Ω is a bounded domain and when $\Omega = \mathbb{R}^N$.

For problem (1.2) with Ω a bounded domain, due to the lack of a suitable topology, the classical variational techniques are not appropriate to prove the existence and regularity of Dirichlet eigenvalues problems with measure constraints. One important notion that aids in dealing with this issue is the γ-convergence, introduced by G. Dal Maso and U. Mosco in [22, 23]. This type of convergence allowed G. Buttazzo and G. Dal Maso in [15, Example 2.6] to produce the first and classical existence result (actually, the authors prove it in a very general situation which includes the minimization of λ_k). The authors prove that there exists a minimizer to (1.2) in the class of quasi-open sets. Such a class of sets is, in fact, the largest family for which the Dirichlet eigenvalues of the Laplacian problem is still well-posed and inherits a strong maximum principle. For more details on quasi-open sets, see [9, Chapter 4].

To obtain *open* optimal sets is more challenging than having quasi-open ones (actually, there are even situations where an open solution does not exist; see, for instance, [29, Theorem 3.11]). Then, the fundamental question is to understand whether and when a solution has additional regularity properties. In [6], T. Briançon, M. Hayouni and M. Pierre prove the existence of an open solution ω to (1.2) when Ω is bounded, establishing at the same time locally Lipschitz regularity for the corresponding eigenfunctions. An important part of the strategy in [6] is to take a solution $\bar{u}$ to the problem

$$\inf\left\{\int_\Omega |\nabla u|^2 :\ u \in H_0^1(\Omega),\ \int_\Omega u^2 = 1,\ |\{u \neq 0\}| \leq a\right\}$$

and show that it is also a minimizer of the penalized functional

$$J(u) = \int_\Omega |\nabla u|^2 + \lambda_a \left(1 - \int_\Omega u^2\right)^+ + m(|\{u \neq 0\}| - a)^+, \quad \text{where } \lambda_a := \int_\Omega |\nabla \bar{u}|^2, \tag{1.3}$$

for m sufficiently large. In the end, the authors show the equivalence between these two problems, and the equivalence between (1.2) and

$$\min\{\lambda_1(\omega) + m(|\omega| - a)^+,\ \omega \subset \Omega \text{ open}\} \tag{1.4}$$

for large $m > 0$. Then, T. Briançon and J. Lamboley, in [7], prove that any open solution ω^* has a locally finite perimeter and that, up to a negligible set, $\partial\omega^* \cap \Omega$ is analytic. Finer results about the singular set are shown in [37] (which deals with a more general vectorial case).

Remark 1.1. *We observe that* [6] *also deals with Dirichlet energies related to problems of type* $-\Delta u = f$, $u \in H_0^1(\omega)$. *The regularity of the free boundary for such optimization problems is addressed in* [5] *by T. Briançon. When the operator is of divergence type, M. Hayouni (see* [29]*) shows the existence and Lipschitz continuity under the assumption that the state function is positive by increasing the admissible set and regularizing the volume constraint, respectively. We observe that, for a problem like* (1.2) *with eigenvalues of a divergence-type operator, E. Teixeira and S. Snelson obtain, in* [42]*, Hölder regularity of the eigenfunctions; they prove this when the diffusion coefficient is close, in a suitable sense, to the identity. The authors in* [29, 42] *use different approaches. Finally, E. Russ, B. Trey and B. Velichkov in* [41] *perform a complete study of problem* (1.2) *for eigenvalues of elliptic operators with drift.*

The study of the one phase problem (1.2) with $\Omega = \mathbb{R}^N$ corresponds to the problem appearing in the 19th century in the monograph [40]. By the Faber-Krahn inequality [26, 34], it is classical to show that the ball of volume a is the minimizer of (1.2). For the minimization problem of the second eigenvalue, it is known that the solution is a union of two disjoint balls with equal measure, by the Hong-Krahn-Szegö inequality [33, 35]. Using different strategies, D. Bucur in [8] and, more or less simultaneously, D. Mazzoleni and A. Pratelli in [36] obtain a general existence result in the whole space $\mathbb{R}^N$ for the minimization of the k-th eigenvalue with a prescribed measure in the class of quasi-open sets. For more details on the spectral problem see [31, Chapters 2 & 3].

Concerning now the multiphase case (1.1), our work is, up to our knowledge, the first to treat the case when Ω is a bounded set. There are, however, related problems, which we now describe. When $\Omega = \mathbb{R}^N$ it is easy to check, again by Faber-Krahn inequality, that the solution is a union of k disjoint balls (see for instance the proof of Theorem 1.3 below). If, instead, one is minimizing $L_\ell(\omega_1, \ldots, \omega_k) = \sum_{i=1}^k \lambda_\ell(\omega_i)$, for $\ell \geq 3$ then, up to our knowledge, nothing is known (for $\ell = 2$, the solution is a union of $2k$ disjoint balls).

Observe that, by a scaling argument (reasoning, for instance, as in [47, Proposition 6.3]), such problems are equivalent to

$$\min \left\{ \sum_{i=1}^{k} \lambda_\ell(\omega_i) + m|\omega_i| \;:\; \omega_i \subset \mathbb{R}^N,\ \omega_i \text{ open},\ \omega_i \cap \omega_j = \emptyset \text{ for } i \neq j \right\}, \tag{1.5}$$

for some $m > 0$. A similar problem but with partitions contained in a bounded domain Ω is studied in [12] by D. Bucur and B. Velichkov. More precisely, the authors treat the following minimization problem

$$\min \left\{ \sum_{i=1}^{k} \lambda_\ell(\omega_i) + m|\omega_i| \;:\; \omega_i \subset \Omega,\ \omega_i \text{ quasi-open},\ \omega_i \cap \omega_j = \emptyset \text{ for } i \neq j \right\}, \tag{1.6}$$

where $m > 0$, Ω is a given bounded open set. Notice that, for $m > 0$ sufficiently large, the solution will be a partition of Ω with an empty region and the sets ω_i will not cover the whole Ω. Hence, the geometry of the partitions in [12] is similar to the geometry of the partitions in $\mathcal{P}_a(\Omega)$. The authors prove qualitative properties for an optimal partition, such as inner density estimates, finite perimeter and absence of triple points,among others. For $\ell = 1, 2$, they also show the existence of open minimizers. A complete study of the regularity of the free boundary of optimal sets is obtained for $\ell = 1$ in [24, Corollary 1.3]. For results in the special case $N = 2$ and numerical simulations, see [4].

We emphasize that, when Ω is bounded, problems (1.1) and (1.6) for $\ell = 1$ are not equivalent (scaling arguments no longer work), and it seems that some deformation arguments used in (1.6) do not provide directly useful information for (1.1), due to the $a > 0$ in our measure constraint.

Observe that, when $a = |\Omega|$, problem (1.2) becomes a spectral partition problem without volume constraint, namely

$$\inf \left\{ \sum_{i=1}^{k} \lambda_1(\omega_i) \;\middle|\; \begin{array}{c} \omega_i \subset \Omega \text{ are nonempty open sets for all } i, \\ \omega_i \cap \omega_j = \emptyset \text{ for all } i \neq j \text{ and } \overline{\Omega} = \cup_{i=1}^k \overline{\omega_i} \end{array} \right\}. \tag{1.7}$$

Combining the results from [17, 19] (see also [45, Section 8]), it is known that optimal partitions exists, and the free boundary $\cup_i \partial\omega_i$ is, up to a singular set of lower dimension, regular. Finer results for the singular set are proved in the recent paper [2], namely that the $(N-2)$–Hausdorff dimension of the singular set is finite, together with a stratification result. The case of higher eigenvalues has been addressed in [39] (see also references therein).

To conclude this literature review, we refer to the papers [43, 44], where the authors consider the same cost functional as (1.1), with a distance constraint between elements of each partition (namely $\text{dist}(\omega_i, \omega_j) \geq r$ for every $i \neq j$) instead of a measure constraint.

1.1. **Statement of the main results and structure of the paper.** As already mentioned, the main goal of this paper is to prove the existence of (open) minimizers to (1.1), together with the local Lipschitz continuity of the corresponding eigenfunctions. For that, it is convenient to relax the measure constraint, dealing with

$$c_a = \inf_{(\omega_1,\dots,\omega_k) \in \mathcal{P}_a(\Omega)} \sum_{i=1}^{k} \lambda_1(\omega_i), \tag{1.8}$$

where

$$\mathcal{P}_a(\Omega) := \left\{ (\omega_1, \dots, \omega_k) \;\middle|\; \begin{array}{c} \omega_i \subset \Omega \text{ are nonempty open sets for all } i, \\ \omega_i \cap \omega_j = \emptyset \text{ for all } i \neq j \text{ and } \sum_{i=1}^k |\omega_i| \leq a \end{array} \right\}.$$

In this paper we do not use the notion of γ-convergence of quasi-open sets; instead, one important feature to study the existence and regularity of the solutions in this scenario is the equivalence between optimal partition problems and minimization problems involving a state functional . We introduce a weak formulation that involves a cost functional, where the variables are functions rather than domains, namely

$$\tilde{c}_a = \inf_{(u_1,\dots,u_k) \in H_a} J(u_1, \dots, u_k), \tag{1.9}$$

where

$$J(u_1,\dots,u_k) := \sum_{i=1}^k \int_\Omega |\nabla u_i|^2$$

and

$$H_a := \Big\{(u_1,\dots,u_k)\ \Big|\ u_i\in H_0^1(\Omega) \text{ and } \int_\Omega u_i^2 = 1 \text{ for every } i, u_i\cdot u_j\equiv 0 \text{ for } i\neq j,\ \sum_{i=1}^k |\Omega_{u_i}|\le a\Big\},$$

with $\Omega_{u_i} := \{x\in\Omega \mid u_i(x)\neq 0\}$ for all $i\in\{1,\dots,k\}$.

Our main result is the following.

Theorem 1.2. *The problem (1.8) admits a solution. Moreover:*

i) Given any optimal partition $(\omega_1,\dots,\omega_k)\in\mathcal{P}_a(\Omega)$, then each ω_i is connected and $\sum_{i=1}^k|\omega_i| = a$. Therefore, the problems (1.1) and (1.8) have the same solutions. In addition, if u_i is a first eigenfunction associated with ω_i, then u_i is locally Lipschitz continuous in Ω.

ii) Problems (1.8) and (1.9) are equivalent in the following sense:
  - *a) $c_a = \tilde{c}_a$;*
  - *b) if $(u_1,\dots,u_k)\in H_a$ is an optimal solution of (1.9) and $\Omega_{u_i} := \{u_i\neq 0\}$, then $(\Omega_{u_1},\dots,\Omega_{u_k})\in\mathcal{P}_a(\Omega)$ solves (1.8);*
  - *c) if $(\omega_1,\dots,\omega_k)\in\mathcal{P}_a(\Omega)$ is an optimal partition for (1.8) and u_i is a first eigenfunction, L^2-normalized, associated to the set ω_i, then $(u_1,\dots,u_k)\in H_a$ is a minimizer for (1.9).*

We prove the existence of an optimal partition by exploiting the equivalence between the problems (1.8) and (1.9), which plays a crucial role in overcoming technical difficulties to treat (1.8) directly. For instance, by applying the direct method of calculus of variations to (1.9), we can easily prove the existence of minimizers. Even though the sets Ω_{u_i} are quasi-open, we do not use this fact directly in this paper and the concept of γ-convergence of quasi-open sets; instead, we notice that the continuity of minimizers is a fundamental property in proving the equivalence between the problems. This is the content of Proposition 2.1 and Proposition 2.2.

In order to prove the continuity of minimizers, we adapt the techniques presented in [6]. However, several difficulties appear due to the fact we are dealing with partitions instead of only one set. Firstly, the generalization of (1.3) that works in our scenario is:

$$J_\mu(u_1,\dots,u_k) := \sum_{i=1}^k \frac{\displaystyle\int_\Omega |\nabla u_i|^2}{\displaystyle\int_\Omega u_i^2} + \mu\left[\sum_{i=1}^k |\Omega_{u_i}| - a\right]^+, \qquad \text{for } (u_1\dots,u_k)\in\overline{H},$$

where

$$\overline{H} := \Big\{(u_1,\dots,u_k)\in H_0^1(\Omega;\mathbb{R}^k)\ \Big|\ u_i\neq 0\ \forall i,\ u_i\cdot u_j\equiv 0\ \ \forall i\neq j\Big\}.$$

In fact, any tentative of producing a functional more similar to the one in (1.3) would result in products of the type $\Pi_{i\neq j}\int_\Omega u_i^2\int_\Omega|\nabla u_j|^2$ and $\left(1-\Pi_{i=1}^k\int_\Omega u_i^2\right)^+$, which do not seem easy to deal with. In particular, the latter product prevents us from concluding $\|u_i\|_2 = 1$, $i=1,\dots,k$, for the minimizes of J_μ. To extract information from this new penalized energy, we rely on some deformation arguments from [18, 19, 20] (see Appendix A). These were introduced for the study of the spectral partition *without* volume constraints, like (1.7) above. In the context of problem (1.7), these deformations provide that any solution should satisfy a set of inequalities (namely, they should belong to the class $\mathcal{S}_{\lambda_1,\dots,\lambda_k}$, see (3.8) below). Due to the presence of an empty region (related to the fact that $a<|\Omega|$), in our context, we obtain more involved inequalities, see Proposition 3.2 below, which is the key result in our paper.

Throughout Section 4, where we prove Lipschitz continuity, we use the continuity of the minimizers proved in Section 3, namely in the proof of Proposition 4.4. For the proof of Lipschitz continuity, we were not able to apply directly the ideas in [6] to our framework. In our case, we proceed as in [18, 19, 20] by using powerful tools such as blow-up methods, the Caffarelli-Jerison-Kenig monotonicity formula, suitable inequalities obtained via deformations (Proposition 3.2), and some properties of the class $\mathcal{S}_{\lambda_1,\dots,\lambda_k}$ mentioned before.

In the next result, we characterize the minimizers of (1.1), in the case we have enough space inside Ω.

Theorem 1.3. *There exists $\bar{a} = \bar{a}(\Omega, N, k)$ such that, for $a<\bar{a}$, then any solution of (1.1) is a partition made of k disjoint open balls, all with the same radius.*

Finally, for the case with $k=2$, we prove the existence of an optimal partition that inherits some symmetry from the box Ω.

Theorem 1.4. *Consider (1.1) with $k=2$, suppose Ω is a ball or an annulus centered at the origin and fix a unit vector e. Then there exists an optimal partition (ω_1,ω_2) for (1.1), with corresponding nonnegative eigenfunctions u_1,u_2, such that:*

a) ω_1 and ω_2 are axially symmetric with respect to e;
b) u_1 and u_2 are foliated Schwarz symmetric with respect to e and $-e$, respectively.

This paper is structured as follows: Section 2 is devoted to the equivalence between the minimization problems (1.8) and (1.9) under the assumption that minimizers are continuous (which will be proved later). Also, we prove the existence of the minimizers to (1.8) and (1.9) and define a penalized functional. In Section 3, we introduce some properties for the class $\mathcal{S}_{\lambda_1,\dots,\lambda_k}$ and show that the minimizers of (1.9) are bounded and continuous functions. Section 4 is dedicated to establishing the Lipschitz continuity for the corresponding eigenfunctions. In Section 5, we introduce the proofs of the main results. In Appendix A, we present some properties for some classes of deformations and gather some auxiliary results that are used in the manuscript.

To conclude, we point out that our strategy based on variations is flexible and can be applied in other contexts; a work regarding the study of (1.1) with eigenvalues associated to divergence type operators is currently in preparation.

2. Preliminaries and existence of minimizers for the weak formulation

In this part, we show the existence of minimizers to (1.9) and introduce a key step for the proof of Theorem 1.2, namely the equivalence of problem (1.9) with a penalized version J_μ defined below. We start by showing the equivalence (assuming that the minimizers are continuous functions) between problems (1.8) and (1.9). It is worth highlighting that throughout this paper, for an open set $\omega\subset\Omega$ and a function $u:\omega\to\mathbb{R}$, we also denote by u its extension to Ω as being zero outside ω.

Proposition 2.1. *It holds that $\tilde c_a\le c_a$. Moreover, if there exists $(\bar u_1,\dots,\bar u_k)\in H_a$ a minimizer to (1.9) and $\bar u_1,\dots,\bar u_k$ are continuous in Ω, then $(\Omega_{\bar u_1},\dots,\Omega_{\bar u_k})\in\mathcal{P}_a(\Omega)$ is a minimizer of the functional in (1.8), each $\bar u_i$ is a first Dirichlet eigenfunction in $\Omega_{\bar u_i}$ and $\tilde c_a=c_a$.*

Proof. We start by choosing a partition $(\omega_1,\dots,\omega_k)\in\mathcal{P}_a(\Omega)$. For each $i\in\{1,\dots,k\}$, consider u_i the first (positive) Dirichlet eigenfunction corresponding to $\omega_i\subseteq\Omega$, normalized in $L^2(\omega_i)$. Then $(u_1,\dots,u_k)\in H_a$ and

$$\tilde c_a\le\sum_{i=1}^k\int_\Omega|\nabla u_i|^2=\sum_{i=1}^k\int_{\omega_i}|\nabla u_i|^2=\sum_{i=1}^k\lambda_1(\omega_i). \tag{2.1}$$

Applying the infimum in (2.1) over the set $\mathcal{P}_a(\Omega)$, we get $\tilde c_a\le c_a$.

Now, assume there exists $(\bar u_1,\dots,\bar u_k)\in H_a$ which solves the minimization problem (1.9), and that $\bar u_1,\dots,\bar u_k$ are continuous in Ω. Observe that, for each i, $\Omega_{\bar u_i}=\{\bar u_i\neq0\}$ is an open set and that $\int_\Omega|\nabla\bar u_i|^2=\int_{\Omega_{\bar u_i}}|\nabla\bar u_i|^2\ge\lambda_1(\Omega_{\bar u_i})$ and $(\Omega_{\bar u_1},\dots,\Omega_{\bar u_k})\in\mathcal{P}_a(\Omega)$. Then

$$\tilde c_a=J(\bar u_1,\dots,\bar u_k)=\sum_{i=1}^k\int_\Omega|\nabla\bar u_i|^2\ge\sum_{i=1}^k\lambda_1(\Omega_{\bar u_i})\ge c_a.$$

Then $\tilde c_a=c_a$ and c_a is achieved. □

Therefore, in order to prove the main result of this paper, namely Theorem 1.2, it is sufficient to prove:

(C1) There exists a minimizer $(\bar u_1,\dots,\bar u_k)\in H_a$ to (1.9).

(C2) If $(\bar u_1,\dots,\bar u_k)\in H_a$ is a minimizer to (1.9), then each $\bar u_i$ is locally Lipschitz continuous in Ω.

The first condition is proved next in Proposition 2.2, while the second is shown in Sections 3 and 4.

Proposition 2.2. *The infimum $\tilde c_a$ is achieved, that is, there exists $(\bar u_1,\dots,\bar u_k)\in H_a$ such that*

$$\tilde c_a=J(\bar u_1,\dots,\bar u_k)\le J(u_1,\dots,u_k)\quad\text{for all }(u_1,\dots,u_k)\in H_a.$$

Proof. It is clear that $0 \le \tilde{c}_a < \infty$. Let $(u_{1,n}, \dots, u_{k,n})_{n\in\mathbb{N}} \subset H_a$ be a minimizing sequence for J and we may suppose that

$$\tilde{c}_a + 1 \ge J(u_{1,n}, \dots, u_{k,n}) = \sum_{i=1}^{k} \int_\Omega |\nabla u_{i,n}|^2 \quad \text{for all } n. \tag{2.2}$$

Therefore, for all $i = 1, \dots, k$, the sequences $(u_{i,n})_{n\in\mathbb{N}}$ are bounded in $H_0^1(\Omega)$ and there exists $\bar{u}_i \in H_0^1(\Omega)$ such that, up to subsequences, as $n \to \infty$,

$$u_{i,n} \to \bar{u}_i \qquad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega), \text{ a.e. in } \Omega.$$

Then

$$\int_\Omega \bar{u}_i^2 = \lim_{n\to\infty} \int_\Omega u_{i,n}^2 = 1 \text{ for every } i, \quad |\bar{u}_i(x)|^2|\bar{u}_j(x)|^2 = \lim_{n\to\infty} |u_{i,n}(x)|^2|u_{j,n}(x)|^2 = 0 \text{ a.e. } x \text{ in } \Omega,\ i \ne j$$

and, by applying Fatou's Lemma, we obtain

$$|\Omega_{\bar{u}_i}| = \int_\Omega \chi_{\Omega_{\bar{u}_i}}(x) \le \int_\Omega \liminf_{n\to\infty} \chi_{\Omega_{u_{i,n}}}(x) \le \liminf_{n\to\infty} \int_\Omega \chi_{\Omega_{u_{i,n}}}(x) = \liminf_{n\to\infty} |\Omega_{u_{i,n}}|.$$

Thus,

$$\sum_{i=1}^{k} |\Omega_{\bar{u}_i}| \le \liminf_{n\to\infty} \sum_{i=1}^{k} |\Omega_{u_{i,n}}| \le a.$$

Therefore $(\bar{u}_1, \dots, \bar{u}_k)$ belongs to H_a and so

$$\tilde{c}_a \le J(\bar{u}_1, \dots, \bar{u}_k) \le \liminf_{n\to\infty} J(u_{1,n}, \dots, u_{k,n}) = \tilde{c}_a,$$

which finishes the proof. □

Given $\mu > 0$, define the penalized functional $J_\mu : \overline{H} \to \mathbb{R}$ by

$$J_\mu(u_1, \dots, u_k) := \sum_{i=1}^{k} \frac{\displaystyle\int_\Omega |\nabla u_i|^2}{\displaystyle\int_\Omega u_i^2} + \mu \left[\sum_{i=1}^{k} |\Omega_{u_i}| - a \right]^+, \qquad \text{for } (u_1 \dots, u_k) \in \overline{H},$$

where

$$\overline{H} := \Big\{ (u_1, \dots, u_k) \in H_0^1(\Omega; \mathbb{R}^k) \ \Big| \quad u_i \ne 0\ \forall i,\ u_i \cdot u_j \equiv 0 \quad \forall i \ne j \Big\}.$$

Proposition 2.3. *Let $(\bar{u}_1, \dots, \bar{u}_k) \in H_a$ be a minimizer of (1.9) and take $\mu > \left(\frac{2^{(k-1)/2}\, \tilde{c}_a}{N|B_1|^{1/N}} a^{\frac{2-N}{2N}} \right)^2$. Then*

$$\sum_{i=1}^{k} \int_\Omega |\nabla \bar{u}_i|^2 \le J_\mu(u_1, \dots, u_k) \tag{2.3}$$

for all $(u_1, \dots, u_k) \in \overline{H}$. In particular, J and J_μ have a common minimizer.

Proof. Let $u_1, \dots, u_k \in H_0^1(\Omega) \setminus \{0\}$ such that $u_i \ne 0$ for all i, $u_i \cdot u_j = 0$ for all $i \ne j$ and $\sum_{i=1}^{k} |\Omega_{u_i}| \le a$. Since $\left(\frac{u_1}{\|u_1\|_2}, \dots, \frac{u_k}{\|u_k\|_2} \right) \in H_a$ and $(\bar{u}_1, \dots, \bar{u}_k)$ is a minimizer of (1.9),

$$\sum_{i=1}^{k} \int_\Omega |\nabla \bar{u}_i|^2 \le \sum_{i=1}^{k} \frac{\int_\Omega |\nabla u_i|^2}{\int_\Omega u_i^2}. \tag{2.4}$$

By the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ and reasoning exactly as in the proof of Proposition 2.2, we can find a minimizer $(u_{1,\mu}, \dots, u_{k,\mu}) \in \overline{H}$ of J_μ. With no loss of generality, we may assume $u_{i,\mu} \ge 0$ and $\|u_{i,\mu}\|_2 = 1$ for all i.

Let $\mu > \left(\frac{2^{(k-1)/2}\, \tilde{c}_a}{N|B_1|^{1/N}} a^{\frac{2-N}{2N}} \right)^2$. Suppose, by contradiction, that $\sum_{i=1}^{k} |\Omega_{u_{i,\mu}}| > a$ and consider the auxiliary functions $u_i^t := (u_{i,\mu} - t)^+$, for $t > 0$. Observe that, since $(u_{1,\mu}, \dots, u_{k,\mu}) \in \overline{H}$, we also have $(u_1^t, \dots, u_k^t) \in \overline{H}$ for $t > 0$. Since $(u_{1,\mu}, \dots, u_{k,\mu})$ is a minimizer of J_μ, we get

$$J_\mu(u_{1,\mu}, \dots, u_{k,\mu}) \le J_\mu(u_1^t, \dots, u_k^t).$$

Using the fact that $\sum_{i=1}^k |\Omega_{u_{i,\mu}}| > a$, for $t > 0$ sufficiently small, we obtain

$$\sum_{i=1}^k \int_\Omega |\nabla u_{i,\mu}|^2 + \mu \left[\sum_{i=1}^k |\Omega_{u_{i,\mu}}| - a \right] \leq \sum_{i=1}^k \frac{\int_\Omega |\nabla (u_{i,\mu} - t)^+|^2}{\int_\Omega |(u_{i,\mu} - t)^+|^2} + \mu \left[\sum_{i=1}^k |\Omega_{u_i^t}| - a \right].$$

By using Lemma A.1, we have

$$\sum_{i=1}^k \int_\Omega |\nabla u_{i,\mu}|^2 + \mu \sum_{i=1}^k |\{0 < u_{i,\mu} \leq t\}| \leq \sum_{i=1}^k \int_{\{u_{i,\mu} > t\}} |\nabla u_{i,\mu}|^2 + 2t \int_\Omega u_{i,\mu} \int_\Omega |\nabla (u_{i,\mu} - t)^+|^2 + o(t).$$

From the fact that $\|u_{i,\mu}\|_2 = 1$ for all $i = 1, \ldots, k$, combined with the Hölder inequality, we infer that

$$\sum_{i=1}^k \int_{\{0 < u_{i,\mu} \leq t\}} |\nabla u_{i,\mu}|^2 + \mu \sum_{i=1}^k |\{0 < u_{i,\mu} \leq t\}| \leq 2t \sum_{i=1}^k |\Omega_{u_{i,\mu}}|^{1/2} \int_\Omega |\nabla (u_{i,\mu} - t)^+|^2 + o(t).$$

Now, we approximate $u_{i,\mu}$ by a mollifier and apply the coarea formula with Sard's Lemma to obtain the inequality (through the limit) to $u_{i,\mu}$

$$\sum_{i=1}^k \int_0^t \int_{\{u_{i,\mu} = s\}} \left(|\nabla u_{i,\mu}| + \frac{\mu}{|\nabla u_{i,\mu}|} \right) d\mathcal{H}^{N-1} ds \leq 2t \sum_{i=1}^k |\Omega_{u_{i,\mu}}|^{1/2} \int_\Omega |\nabla (u_{i,\mu} - t)^+|^2 + o(t).$$

Minimizing the function $x \to x + \mu x^{-1}$ over the set $\{x > 0\}$ leads to

$$2\sqrt{\mu} \sum_{i=1}^k \int_0^t \int_{\{u_{i,\mu} = s\}} d\mathcal{H}^{N-1} ds \leq 2t \sum_{i=1}^k |\Omega_{u_{i,\mu}}|^{1/2} \int_\Omega |\nabla (u_{i,\mu} - t)^+|^2 + o(t).$$

At this point, we can use the isoperimetric inequality in the term $\int_{\{u_{i,\mu} = s\}} d\mathcal{H}^{N-1} = \text{per}(\{u_{i,\mu} > s\})$, divide the equation by t, let $t \to 0$ to conclude that

$$N|B_1|^{1/N} \sqrt{\mu} \sum_{i=1}^k |\Omega_{u_{i,\mu}}|^{\frac{N-1}{N}} \leq \sum_{i=1}^k |\Omega_{u_{i,\mu}}|^{1/2} \int_\Omega |\nabla u_{i,\mu}|^2.$$

Now, notice that

$$\sum_{i=1}^k \int_\Omega |\nabla u_{i,\mu}|^2 \leq J_\mu(u_{1,\mu}, \ldots, u_{k,\mu}) \leq J_\mu(\bar{u}_1, \ldots, \bar{u}_k) = J(\bar{u}_1, \ldots, \bar{u}_k) = \tilde{c}_a,$$

and $\tilde{c}_a$ does not depend on μ. Hence, by using Jensen's inequality we obtain

$$N|B_1|^{1/N} \sqrt{\mu} \left(\sum_{i=1}^k |\Omega_{u_{i,\mu}}| \right)^{\frac{N-1}{N}} \leq 2^{(k-1)/2}\, \tilde{c}_a \left(\sum_{i=1}^k |\Omega_{u_{i,\mu}}| \right)^{1/2}.$$

Therefore

$$\sqrt{\mu} \leq \frac{2^{(k-1)/2}\, \tilde{c}_a}{N|B_1|^{1/N}} \left(\sum_{i=1}^k |\Omega_{u_{i,\mu}}| \right)^{\frac{2-N}{2N}} \leq \frac{2^{(k-1)/2}\, \tilde{c}_a}{N|B_1|^{1/N}} a^{\frac{2-N}{2N}},$$

which contradicts the assumption on the size of μ. Hence, $\sum_{i=1}^k |\Omega_{u_{i,\mu}}| \leq a$.

Finally, we conclude that

$$J_\mu(u_{1,\mu}, \ldots, u_{k,\mu}) \leq J_\mu(\bar{u}_1, \ldots, \bar{u}_k) = \sum_{i=1}^k \int_\Omega |\nabla \bar{u}_i|^2 \leq J_\mu(u_{1,\mu}, \ldots, u_{k,\mu}),$$

where the last inequality follows from (2.4). ◻

3. Continuity of minimizers

Let $(\bar u_1,\dots,\bar u_k)\in H_a$ be a minimizer to (1.9). Up to replacing $\bar u_i$ by $|\bar u_i|$, we may suppose that $\bar u_i\geq 0$ in Ω for all $i=1,\dots,k$. This is done throughout this paper. Then, inspired by [18, 19, 20], we set

$$\hat u_i=\bar u_i-\sum_{j=1,j\neq i}^{k}\bar u_j,\quad \lambda_{\bar u_i}=\int_\Omega|\nabla\bar u_i|^2\quad i=1,\dots,k.$$

We start by showing that minimizers of (1.9) are bounded functions.

Proposition 3.1. *Let $(\bar u_1,\dots,\bar u_k)\in H_a$ be a (nonnegative) minimizer to (1.9). Then, for each $i=1,\dots,k$, $\bar u_i$ satisfies*

$$-\Delta\bar u_i\leq\lambda_{\bar u_i}\bar u_i\quad in\quad\Omega,\tag{3.1}$$

in the sense of distributions. In particular, for each $i=1,\dots,k$:

- *$\bar u_i$ is a bounded function;*
- *$\bar u_i$ is defined at every $x\in\Omega$, in the sense that each x is a Lebesgue point.*

Proof. Let $\varphi\in C_c^\infty(\Omega)$ with $\varphi\geq 0$ and set $\check u_t$ as in (A.2) for small t. Then, from Lemma A.2, $\check u_t\in H_a$. Using the fact that $(\bar u_1,\dots,\bar u_k)\in H_a$ is a minimizer to (1.9) and (A.1), we infer that

$$\begin{aligned}\int_\Omega|\nabla\bar u_1|^2&\leq\frac{\int_\Omega|\nabla(\bar u_1-t\varphi)^+|^2}{\|(\bar u_1-t\varphi)^+\|_2^2}\leq\frac{\int_\Omega|\nabla(\bar u_1-t\varphi)|^2}{\|(\bar u_1-t\varphi)^+\|_2^2}=\int_\Omega|\nabla(\bar u_1-t\varphi)|^2\left(1+2t\int_\Omega\bar u_1\varphi+O(t^2)\right)\\&=\int_\Omega|\nabla\bar u_1|^2-2t\int_\Omega\nabla\bar u_1\cdot\nabla\varphi+2t\int_\Omega|\nabla\bar u_1|^2\int_\Omega\bar u_1\varphi+O(t^2).\end{aligned}$$

Dividing the inequality above by t and letting $t\to 0$, we obtain

$$\int_\Omega\nabla\bar u_1\cdot\nabla\varphi\leq\lambda_{\bar u_1}\int_\Omega\bar u_1\varphi,\tag{3.2}$$

which implies (3.1). By classical elliptic estimates, see for instance the proof of [6, Lemma 4.2], we infer that $\bar u_1\in L^\infty(\Omega)$. Similarly, we can also ensure that $\bar u_i\in L^\infty(\Omega)$ for all $i=2,\dots,k$. Finally, the fact that every point is a Lebesgue point for each component $\bar u_i$ is a direct consequence of Proposition A.5. □

The following result is crucial for everything that follows.

Proposition 3.2. *Let $(\bar u_1,\dots,\bar u_k)$ be a minimizer of (1.9). Then, for each $i=1,\dots,k$, and for any nonnegative function $\varphi\in H_0^1(\Omega)$ such that $supp(\varphi)\subseteq B_r(x_0)\Subset\Omega$,*

$$\left\langle-\Delta\hat u_i-\lambda_{\bar u_i}\bar u_i+\sum_{j\neq i}\lambda_{\bar u_j}\bar u_j,\varphi\right\rangle\geq -C\left(r^{N-1}+\|\varphi\|_1+r\|\varphi\|_2^2+r\|\nabla\varphi\|_2^2+r\|\varphi\|_1\|\nabla\varphi\|_2^2+r\|\varphi\|_2^2\|\nabla\varphi\|_2^2\right),\tag{3.3}$$

where $C>0$ depends only on $\tilde c_a, N, \|\bar u_1\|_\infty,\dots,\|\bar u_k\|_\infty$ and a.

Proof. Consider

$$\tilde u_t=(\tilde u_{1,t},\dots,\tilde u_{k,t})=\left(\frac{(\hat u_1+t\varphi)^+}{\left\|(\hat u_1+t\varphi)^+\right\|_2},\frac{(\hat u_2-t\varphi)^+}{\left\|(\hat u_2-t\varphi)^+\right\|_2},\dots,\frac{(\hat u_k-t\varphi)^+}{\left\|(\hat u_k-t\varphi)^+\right\|_2}\right),$$

with $t\in(0,1)$ sufficiently small. By using (2.3) and Lemma A.3, we obtain

$$\sum_{i=1}^{k}\int_\Omega|\nabla\bar u_i|^2\leq\frac{\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2}{\|(\hat u_1+t\varphi)^+\|_2^2}+\sum_{i=2}^{k}\frac{\int_\Omega|\nabla(\hat u_i-t\varphi)^+|^2}{\|(\hat u_i-t\varphi)^+\|_2^2}+\mu|\Omega_\varphi|.\tag{3.4}$$

Since $\varphi\geq 0$, $0<t<1$, employing Lemma A.1, we get

$$\frac{1}{\|(\hat u_1+t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2\leq\int_\Omega|\nabla(\hat u_1+t\varphi)^+|^2\left(1-2t\int_\Omega\bar u_1\varphi+ct^2\|\varphi\|_2^2\right),$$

and

$$\frac{1}{\|(\hat u_i-t\varphi)^+\|_2^2}\int_\Omega|\nabla(\hat u_i-t\varphi)^+|^2=\int_\Omega|\nabla(\hat u_i-t\varphi)^+|^2\left(1+2t\int_\Omega\bar u_i\varphi+ct^2\|\varphi\|_2^2\right),$$

for all $i = 2, \dots, k$. In addition, since $0 < t \le 1$, we have that

$$\begin{aligned}
&\int_\Omega |\nabla(\hat{u}_1 + t\varphi)^+|^2 \le \int_\Omega |\nabla(\hat{u}_1 + t\varphi)|^2 \\
&\quad = \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + 2t \int_\Omega \nabla \bar{u}_1 \cdot \nabla \varphi - 2t \sum_{j=2}^k \int \nabla \bar{u}_j \cdot \nabla \varphi + t^2 \int_\Omega |\nabla \varphi|^2 \\
&\quad \le \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + t \int_\Omega |\nabla \bar{u}_1|^2 + t \int_\Omega |\nabla \varphi|^2 + t \sum_{j=2}^k \int_\Omega |\nabla \bar{u}_j|^2 + t \sum_{j=2}^k \int_\Omega |\nabla \varphi|^2 + t^2 \int_\Omega |\nabla \varphi|^2 \\
&\quad \le \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + t \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + (k+1)t \int_\Omega |\nabla \varphi|^2,
\end{aligned}$$

and, arguing similarly,

$$\int_\Omega |\nabla(\hat{u}_i - t\varphi)^+|^2 \le \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + t \sum_{j=1}^k \int_\Omega |\nabla \bar{u}_j|^2 + (k+1)t \int_\Omega |\nabla \varphi|^2,$$

for each $i = 2, \dots, k$. Hence, since $\sum_{i=1}^k \int_\Omega |\nabla \bar{u}_i|^2$ is bounded, $\lambda_{\bar{u}_i} = \int_\Omega |\nabla \bar{u}_i|^2$, and $\bar{u}_1$ is bounded in L^∞ from Proposition 3.1, we obtain

$$\begin{aligned}
\frac{1}{\|(\hat{u}_1 + t\varphi)^+\|_2^2} \int_\Omega |\nabla(\hat{u}_1 + t\varphi)^+|^2 \le \int_\Omega |\nabla(\hat{u}_1 + t\varphi)^+|^2 + O(t^2)\|\varphi\|_2^2 \|\nabla \varphi\|_2^2 + ct\|\varphi\|_1 \\
- 2t\lambda_{\bar{u}_1} \int_\Omega \bar{u}_1 \varphi + ct^2 \|\varphi\|_1 \|\nabla \varphi\|_2^2 + ct^2 \|\varphi\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\|(\hat{u}_i - t\varphi)^+\|_2^2} \int_\Omega |\nabla(\hat{u}_i - t\varphi)^+|^2 \le \int_\Omega |\nabla(\hat{u}_i - t\varphi)^+|^2 + O(t^2)\|\varphi\|_2^2 \|\nabla \varphi\|_2^2 + ct\|\varphi\|_1 \\
+ 2t\lambda_{\bar{u}_i} \int_\Omega \bar{u}_i \varphi + ct^2 \|\varphi\|_1 \|\nabla \varphi\|_2^2 + ct^2 \|\varphi\|_2^2.
\end{aligned}$$

It follows from the fact $|\nabla(\hat{u}_1 + t\varphi)^+|^2 + \sum_{i=2}^k |\nabla(\hat{u}_i - t\varphi)^+|^2 = |\nabla(\hat{u}_1 + t\varphi)|^2$, combined with (3.4), Lemma A.3 and the estimates above that

$$\begin{aligned}
0 \le 2t \int_\Omega \nabla \hat{u}_1 \cdot \nabla \varphi + t^2 \int_\Omega |\nabla \varphi|^2 - 2t\lambda_{\bar{u}_1} \int_\Omega \bar{u}_1 \varphi + 2t \sum_{i=2}^k \lambda_{\bar{u}_i} \int_\Omega \bar{u}_i \varphi + ct^2 \|\varphi\|_1 \|\nabla \varphi\|_2^2 \\
+ ct\|\varphi\|_1 + ct^2 \|\varphi\|_2^2 \|\nabla \varphi\|_2^2 + ct^2 \|\varphi\|_2^2 + \mu |\Omega_\varphi|.
\end{aligned}$$

Hence, by dividing the inequality above by $2t$, we obtain

$$\begin{aligned}
-\int_\Omega \nabla \hat{u}_1 \cdot \nabla \varphi + \lambda_{\bar{u}_1} \int_\Omega \bar{u}_1 \varphi - \sum_{i=2}^k \lambda_{\bar{u}_i} \int_\Omega \bar{u}_i \varphi \le ct\|\nabla \varphi\|_2^2 + ct\|\varphi\|_1 \|\nabla \varphi\|_2^2 + c\|\varphi\|_1 \\
+ ct\|\varphi\|_2^2 \|\nabla \varphi\|_2^2 + ct\|\varphi\|_2^2 + \mu \frac{|\Omega_\varphi|}{t}.
\end{aligned}$$

By choosing $t = r$ and using the fact that $|\Omega_\varphi| \le |B_r| \le cr^N$, we conclude that

$$\left\langle \Delta \hat{u}_1 + \lambda_{\bar{u}_1} \bar{u}_1 - \sum_{i=2}^k \lambda_{\bar{u}_i} \bar{u}_i, \varphi \right\rangle \le C \left(r^{N-1} + \|\varphi\|_1 + r\|\varphi\|_2^2 + r\|\nabla \varphi\|_2^2 + r\|\varphi\|_1 \|\nabla \varphi\|_2^2 + r\|\varphi\|_2^2 \|\nabla \varphi\|_2^2 \right).$$

Now, we do similar computations with the functions of the form

$$\frac{(\hat{u}_i + t\varphi)^+}{\left\|(\hat{u}_i + t\varphi)^+\right\|_2}, \frac{(\hat{u}_1 - t\varphi)^+}{\left\|(\hat{u}_1 - t\varphi)^+\right\|_2}, \dots, \frac{(\hat{u}_{i-1} - t\varphi)^+}{\left\|(\hat{u}_{i-1} - t\varphi)^+\right\|_2}, \frac{(\hat{u}_{i+1} - t\varphi)^+}{\left\|(\hat{u}_{i+1} - t\varphi)^+\right\|_2}, \dots, \frac{(\hat{u}_k - t\varphi)^+}{\left\|(\hat{u}_k - t\varphi)^+\right\|_2},$$

to obtain

$$\left\langle \Delta \hat{u}_i + \lambda_{\bar{u}_i} \bar{u}_i - \sum_{j \ne i}^k \lambda_{\bar{u}_j} \bar{u}_j, \varphi \right\rangle \le C \left(r^{N-1} + r\|\varphi\|_1 + r\|\varphi\|_2^2 + r\|\nabla \varphi\|_2^2 + r\|\varphi\|_1 \|\nabla \varphi\|_2^2 + r\|\varphi\|_2^2 \|\nabla \varphi\|_2^2 \right),$$

for every $i = 1, \dots, k$. ∎

In what follows, we present a rescaled version of the previous proposition that will be useful later on.

Proposition 3.3. *Let $(\bar u_1,\dots,\bar u_k)$ be a minimizer of (1.9). Consider $R>0$, $x_0\in\Omega$ and a sequence $(r_n)_{n\in\mathbb{N}}$ such that $B_{r_nR}(x_0)\Subset\Omega$. For each $i=1,\dots,k$, define*

$$u_{i,n}(x)=\bar u_i(x_0+r_nx).$$

Then, given a nonnegative $\varphi\in H_0^1(\Omega)$ with $supp(\varphi)\subseteq B_R$, we have

$$\left\langle -\Delta\hat u_{i,n}-r_n^2\lambda_{\bar u_i}\bar u_{i,n}+r_n^2\sum_{j\neq i}\lambda_{\bar u_j}\bar u_{j,n},\varphi\right\rangle \geq -Cr_nR\left(R^{N-2}+r_nR^{-1}\|\varphi\|_1+r_n^2\|\varphi\|_2^2+\right.$$
$$\left.+\|\nabla\varphi\|_2^2+r_n^N\|\varphi\|_1\|\nabla\varphi\|_2^2+r_n^N\|\varphi\|_2^2\|\nabla\varphi\|_2^2\right), \tag{3.5}$$

in the distributional sense, where the constant C comes from Proposition 3.2 and does not depend on n.

Proof. Set $\varphi(x)=\tilde\varphi(x_0+r_nx)$, so that $supp(\tilde\varphi)\subset B_{r_nR}(x_0)$, and

$$\|\tilde\varphi\|_1=r_n^N\|\varphi\|_1,\quad \|\tilde\varphi\|_2^2=r_n^N\|\varphi\|_2^2,\quad \|\nabla\tilde\varphi\|_2^2=r_n^{N-2}\|\nabla\varphi\|_2^2.$$

Now, for $y=x_0+r_nx$ we have,

$$\begin{aligned}\int_{B_R}\nabla\hat u_{i,n}(x)\cdot\nabla\varphi(x)dx&=r_n^2\int_{B_R}\nabla\hat u_i(x_0+r_nx)\cdot\nabla\tilde\varphi(x_0+r_nx)dx\\&=\frac{1}{r_n^{N-2}}\int_{B_{r_nR}(x_0)}\nabla\hat u_i(y)\cdot\nabla\tilde\varphi(y)dy.\end{aligned}$$

Hence, from Proposition 3.2 we obtain

$$\begin{aligned}\int_{B_R}\nabla\hat u_{i,n}(x)\cdot\nabla\varphi(x)dx\geq{}&\frac{1}{r_n^{N-2}}\int_{B_{r_nR}(x_0)}\lambda_{\bar u_i}\bar u_i\tilde\varphi dy-\frac{1}{r_n^{N-2}}\sum_{j\neq i}\int_{B_{r_nR}dy(x_0)}\lambda_{\bar u_j}\bar u_j\tilde\varphi\\&-\frac{C}{r_n^{N-2}}\left[(r_nR)^{N-1}+\|\tilde\varphi\|_1+r_nR\|\tilde\varphi\|_2^2+r_nR\|\nabla\tilde\varphi\|_2^2\right]\\&-\frac{C}{r_n^{N-2}}\left[r_nR\|\tilde\varphi\|_1\|\nabla\tilde\varphi\|_2^2+r_nR\|\tilde\varphi\|_2^2\|\nabla\tilde\varphi\|_2^2\right]\\={}&\frac{1}{r_n^{N-2}}\int_{B_{r_nR}(x_0)}\lambda_{\bar u_i}\bar u_{i,n}\left(\frac{x-x_0}{r_n}\right)\varphi\left(\frac{x-x_0}{r_n}\right)dy\\&-\frac{1}{r_n^{N-2}}\sum_{j\neq i}\int_{B_{r_nR}(x_0)}\lambda_{\bar u_j}\bar u_{j,n}\left(\frac{x-x_0}{r_n}\right)\varphi\left(\frac{x-x_0}{r_n}\right)dy\\&-Cr_nR^{N-1}-\frac{C}{r_n^{N-2}}\left(r_n^N\|\varphi\|_1+r_n^{N+1}R\|\varphi\|_2^2+r_n^{N-1}R\|\nabla\varphi\|_2^2\right)\\&-\frac{C}{r_n^{N-2}}\left(r_n^{2N-1}R\|\varphi\|_1\|\nabla\varphi\|_2^2+r_n^{2N-1}R\|\varphi\|_2^2\|\nabla\varphi\|_2^2\right)\\={}&r_n^2\int_{B_R}\lambda_{\bar u_i}\bar u_{i,n}\varphi dx-r_n^2\sum_{j\neq i}\int_{B_R}\lambda_{\bar u_j}\bar u_{j,n}\varphi dx-Cr_nR^{N-1}-Cr_n^2\|\varphi\|_1\\&-Cr_n^3R\|\varphi\|_2^2-Cr_nR\|\nabla\varphi\|_2^2-Cr_n^{N+1}R\|\varphi\|_1\|\nabla\varphi\|_2^2-Cr_n^{N+1}R\|\varphi\|_2^2\|\nabla\varphi\|_2^2,\end{aligned}$$

which implies (3.5). □

The next proposition shows the intuitive fact that the information provided by deformations is stronger when the test functions φ do not alter the measure of the positive sets.

Proposition 3.4. *Let $i\in\{1,\dots,k\}$ and $B\subset\Omega$ be an open ball such that $|B\cap\{\hat u_i=0\}|=0$. Then*

$$-\Delta\hat u_i\geq\lambda_{\bar u_i}\bar u_i-\sum_{j=1,j\neq i}^k\lambda_{\bar u_j}\bar u_j\quad in\quad B. \tag{3.6}$$

Proof. Without loss of generality, we prove (3.6) for $i=1$. Let $\varphi\in C_c^\infty(B)$ be a nonnegative function and define, once again, the auxiliary deformations

$$\tilde u_t=(\tilde u_{1,t},\dots,\tilde u_{k,t})=\left(\frac{(\hat u_1+t\varphi)^+}{\left\|(\hat u_1+t\varphi)^+\right\|_2},\frac{(\hat u_2-t\varphi)^+}{\left\|(\hat u_2-t\varphi)^+\right\|_2},\dots,\frac{(\hat u_k-t\varphi)^+}{\left\|(\hat u_k-t\varphi)^+\right\|_2}\right),$$

with $t \in (0,1)$ sufficiently small. Unless stated, all the L^2 norms are taken in Ω. From Lemma A.1, we obtain

$$(3.7) \qquad \frac{1}{\|(\hat{u}_1 + t\varphi)^+\|_2^2} = 1 - 2t \int_B \bar{u}_1 \varphi + o(t) \quad \text{and} \quad \frac{1}{\|(\hat{u}_i - t\varphi)^+\|_2^2} = 1 + 2t \int_B \bar{u}_i \varphi + o(t).$$

By Lemma A.3, it follows $\tilde{u}_{i,t} \cdot \tilde{u}_{j,t} \equiv 0$ for all $i \neq j$ and, by the assumption $|B \cap \{\hat{u}_i = 0\}| = 0$, we have $\sum_{i=1}^k |\Omega_{\bar{u}_i} \cap B| = |B|$. Therefore

$$\begin{aligned}\sum_{i=1}^k |\Omega_{\tilde{u}_{i,t}}| &= \sum_{i=1}^k |\Omega_{\tilde{u}_{i,t}} \cap B| + \sum_{i=1}^k |\Omega_{\tilde{u}_{i,t}} \cap (\Omega \setminus B)| \leq |B| + \sum_{i=1}^k |\Omega_{\bar{u}_i} \cap (\Omega \setminus B)| \\ &= \sum_{i=1}^k |\Omega_{\bar{u}_i} \cap B| + \sum_{i=1}^k |\Omega_{\bar{u}_i} \cap (\Omega \setminus B)| = \sum_{i=1}^k |\Omega_{\bar{u}_i}| \leq a.\end{aligned}$$

By combining (3.7), the identity $|\nabla(\hat{u}_1 + t\varphi)^+|^2 + \sum_{i=2}^k |\nabla(\hat{u}_i - t\varphi)^+|^2 = |\nabla(\hat{u}_1 + t\varphi)|^2$, the fact that φ is supported in B and $(\hat{u}_1 + t\varphi)^+ = \bar{u}_1$ in $\Omega \setminus B$, $(\hat{u}_i - t\varphi)^+ = \bar{u}_i$ in $\Omega \setminus B$ for $i > 1$, we obtain

$$\begin{aligned} J(\tilde{u}_{1,t}, \dots, \tilde{u}_{k,t}) = \int_B |\nabla(\hat{u}_1 + t\varphi)|^2 + \sum_{i=1}^k \int_{\Omega \setminus B} |\nabla \bar{u}_i|^2 - 2t \int_B \bar{u}_1 \varphi \int_B |\nabla(\hat{u}_1 + t\varphi)^+|^2 \\ - 2t \int_B \bar{u}_1 \varphi \int_{\Omega \setminus B} |\nabla \bar{u}_1|^2 + 2t \sum_{i=2}^k \int_B \bar{u}_i \varphi \int_B |\nabla(\hat{u}_i - t\varphi)^+|^2 + 2t \sum_{i=2}^k \int_B \bar{u}_i \varphi \int_{\Omega \setminus B} |\nabla \bar{u}_i|^2 + o(t). \end{aligned}$$

Since

$$\int_B |\nabla(\hat{u}_1 + t\varphi)^+|^2 \to \int_B |\nabla \bar{u}_1|^2 \quad \text{and} \quad \int_B |\nabla(\hat{u}_i - t\varphi)^+|^2 \to \int_B |\nabla \bar{u}_i|^2 \text{ for all } i > 1$$

as t approaches 0, we deduce that

$$\begin{aligned} J(\tilde{u}_{1,t}, \dots, \tilde{u}_{k,t}) =& 2t \int_B \nabla \hat{u}_1 \cdot \nabla \varphi + \sum_{i=1}^k \int_B |\nabla \bar{u}_i|^2 + \sum_{i=1}^k \int_{\Omega \setminus B} |\nabla \bar{u}_i|^2 - 2t \int_B \bar{u}_1 \varphi \int_B |\nabla \bar{u}_1|^2 \\ & - 2t \int_B \bar{u}_1 \varphi \int_{\Omega \setminus B} |\nabla \bar{u}_1|^2 + 2t \sum_{i=2}^k \int_B \bar{u}_i \varphi \int_B |\nabla \bar{u}_i|^2 + 2t \sum_{i=2}^k \int_B \bar{u}_i \varphi \int_{\Omega \setminus B} |\nabla \bar{u}_i|^2 + o(t) \\ =& 2t \int_B \nabla \hat{u}_1 \cdot \nabla \varphi + \sum_{i=1}^k \int_\Omega |\nabla \bar{u}_i|^2 - 2t \int_B \bar{u}_1 \varphi \int_\Omega |\nabla \bar{u}_1|^2 \\ & + 2t \sum_{i=2}^k \int_B \bar{u}_i \varphi \int_\Omega |\nabla \bar{u}_i|^2 + o(t). \end{aligned}$$

At this point, we use the fact that $(\bar{u}_1, \dots, \bar{u}_k)$ is a minimizer of J; in addition, we denote $\lambda_{\bar{u}_i} := \int_\Omega |\nabla \bar{u}_i|^2$. By passing to the limit as $t \to 0$, we obtain the following inequality

$$\int_B \nabla \hat{u}_1 \cdot \nabla \varphi - \lambda_{\bar{u}_1} \int_B \bar{u}_1 \varphi + \sum_{i=2}^k \lambda_{\bar{u}_i} \int_B \bar{u}_i \varphi \geq 0,$$

which finishes the proof. ∎

At this point, we introduce some auxiliary results. For $w \in H_0^1(\Omega)$, let $\lambda_w := \displaystyle\int_\Omega |\nabla w|^2$. Generalizing the notation given before, for $(u_1, \dots, u_k) \in H_0^1(\Omega; \mathbb{R}^k)$ define

$$\hat{u}_i := u_i - \sum_{j \neq i} u_j, \qquad i = 1, \dots, k.$$

Then, as in [18, 19, 20], given an open set $\mathcal{A} \subset \Omega$ and $\lambda_1, \dots, \lambda_k > 0$, set

$$(3.8) \quad \begin{aligned} \mathcal{S}_{\lambda_1, \dots, \lambda_k}(\mathcal{A}) := \Big\{ & (w_1, \dots, w_k) \in H^1(\mathcal{A}; \mathbb{R}^k) : w_i \geq 0, \ w_i \cdot w_j = 0 \text{ if } i \neq j \text{ in } \mathcal{A} \\ & -\Delta w_i \leq \lambda_i w_i, \quad -\Delta \hat{w}_i \geq \lambda_i w_i - \sum_{j \neq i} \lambda_j w_j \text{ in } \mathcal{A} \text{ in the distributional sense} \Big\}. \end{aligned}$$

Lemma 3.5. *Let $\mathcal{A} \subset \Omega$ be an open set and $\lambda_1, \dots, \lambda_k > 0$. Take $(u_1, \dots, u_k) \in \mathcal{S}_{\lambda_1,\dots,\lambda_k}(\mathcal{A}) \setminus \{(0, \dots, 0)\}$. Then $u_i \in C^{0,1}_{loc}(\mathcal{A})$, and*

$$-\Delta u_i = \lambda_i u_i \text{ in the open set } \{u_i > 0\}.$$

In addition,

$$|\{x \in \Omega : \ u_i(x) = 0 \ \text{ for } i = 1, \dots, k\}| = 0.$$

Proof. The first conclusion follows from [20, Theorem 8.3]. The last sentence is a consequence of [45, Corollary 8.5], taking therein $f_i(s) := \lambda_i s$. □

Remark 3.6. *For the problem without measure constraint (1.7) (i.e., where the partition exhausts the whole Ω), minimizers of the associated weak formulation belong to the class $\mathcal{S}_{\lambda_1,\dots,\lambda_k}(\Omega)$ for some $\lambda_1, \dots, \lambda_k > 0$, see [19, Lemma 2.1]. Therefore, Proposition 3.4 shows that, in a region where the zero set has null measure, we are in the same situation, whereas Proposition 3.2 covers the general case. The right hand side in (3.3) can be seen as an error term, and in some sense allows to capture the transition from the positivity set $\{u_i > 0\}$ to an empty region where $u_i \equiv 0$.*

The following is a Liouville type result.

Lemma 3.7. *Let $(u_1, \dots, u_k) \in H^1_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ nonnegative functions such that $u_i \cdot u_j \equiv 0$ for all $i \neq j$ and*

$$-\Delta u_j \leq 0, \qquad -\Delta \hat{u}_j \geq 0 \qquad \text{in the distributional sense in } \mathbb{R}^N, \quad \forall\, j.$$

Then there exists $c \in \mathbb{R}$ and $i \in \{1, \dots, k\}$ such that $u_i \equiv c$ and $u_j \equiv 0$ for $j \neq i$.

Proof. First of all, observe that $(u_1, \dots, u_k) \in \mathcal{S}_{(0,\dots,0)}(B_R(0))$ for every $R > 0$. Then, by Lemma 3.5, each u_i is a continuous function. By Proposition A.4 in the appendix, since all components are continuous, belong to $H^1_{loc}(\mathbb{R}^N)$, and are bounded, we have that all components except possibly one are trivial. Without loss of generality, assume that $u_2 \equiv \dots \equiv u_k \equiv 0$. Then, from the assumptions,

$$-\Delta u_1 \leq 0, \qquad 0 \leq -\Delta \hat{u}_1 = -\Delta u_1,$$

hence u_1 is harmonic and bounded in $\mathbb{R}^N$, thus it is constant. □

We are ready to prove that minimizers of (1.9) are continuous functions. In particular, this shows that $\Omega_{\bar{u}_i} = \{\bar{u}_i > 0\}$, $i = 1, \dots, k$, are open sets.

Proposition 3.8. *Let $U = (\bar{u}_1, \dots \bar{u}_k)$ be a minimizer of (1.9). Then each $\bar{u}_i$ is a continuous function in Ω.*

Proof. We recall that, by Proposition 3.1, each component $\bar{u}_i$ is defined at *every* point. Given $x_0 \in \Omega$, we are going to prove the continuity of each $\bar{u}_i$ at x_0. Take a sequence $(x_n)_{n\in\mathbb{N}} \subset \Omega$ such that $x_n \to x_0$ and set $r_n := |x_0 - x_n| \to 0$. We split the proof into two cases:

Case 1: Suppose that, for some n, we have $|B_{r_n}(x_0) \cap \{U = 0\}| = 0$. Then, from Propositions 3.1 and 3.4,

$$(\bar{u}_1, \dots, \bar{u}_k) \in \mathcal{S}_{\lambda_{\bar{u}_1},\dots,\lambda_{\bar{u}_k}}(B_{r_n}(x_0)).$$

Then, by Lemma 3.5, we have $\bar{u}_i \in C^{0,1}_{loc}(B_{r_n}(x_0))$. In particular, $\bar{u}_i$ is continuous at x_0.

Case 2: Suppose that, for all n, $|B_{r_n}(x_0) \cap \{U = 0\}| > 0$. We introduce the auxiliary functions, for $i = 1, \dots, k$,

$$\bar{u}_{i,n}(x) = \bar{u}_i(x_0 + r_n x), \quad \text{with} \quad x \in \mathbb{R}^N,$$

where we are considering the extension of $\bar{u}_i$ by zero to $\mathbb{R}^N \setminus \Omega$. In particular, from Proposition 3.1 and since $\bar{u}_i \in H^1_0(\Omega)$ is nonnegative in Ω,

$$-\Delta \bar{u}_{i,n} \leq \lambda_{\bar{u}_i} r_n^2 \bar{u}_{i,n} \quad \text{in} \quad \mathbb{R}^N \tag{3.9}$$

in the distributional sense. Note that, since $\bar{u}_i$ is bounded, we have that $\bar{u}_{i,n}$ is uniformly bounded in i and n. Our aim is to show that $\bar{u}_{i,n} \to 0$ in $L^\infty_{loc}(\mathbb{R}^N)$, which proves the continuity of $\bar{u}_i$ (and shows that $\bar{u}_i(x_0) = 0$). We split the proof of this in several steps.

Step 1. We show that, for each $i = 1, \dots, k$, there exist constants $c_1, \dots, c_k \in \mathbb{R}$, where at most one is nonzero, such that

$$\bar{u}_{i,n} \rightharpoonup c_i \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N), \text{ strongly in } L^2_{loc}(\mathbb{R}^N) \text{ for each } i = 1, \dots, k. \tag{3.10}$$

Given $r < R' < R$, take $0 \le \varphi \in C_c^\infty(B_R)$ such that $\varphi \equiv 1$ in B_r and $\varphi \equiv 0$ outside $B_{R'}$. From the definition of weak solution (with the test function $\bar u_{i,n}\varphi^2 \ge 0$) we get (since $\bar u_{i,n} \in H^1(\mathbb{R}^N)$)

$$\int_{B_R} \nabla \bar u_{i,n} \cdot \nabla(\bar u_{i,n}\varphi^2) \le \int_{B_R} \lambda_{\bar u_i} r_n^2 \bar u_{i,n}^2 \varphi^2 \le C.$$

Thus,

$$\int_{B_R} |\nabla \bar u_{i,n}|^2 \varphi^2 \le C - 2\int_{B_R} \bar u_{i,n}\varphi \nabla \bar u_{i,n} \cdot \nabla\varphi,$$

which implies

$$\int_{B_R} |\nabla \bar u_{i,n}|^2\varphi^2 \le C + 2\int_{B_R} \bar u_{i,n}^2 |\nabla\varphi|^2 + \frac{1}{2}\int_{B_R} |\nabla \bar u_{i,n}|^2\varphi^2 \le C + \frac{1}{2}\int_{B_R} |\nabla \bar u_{i,n}|^2\varphi^2$$

and, therefore,

$$\int_{B_r} |\nabla \bar u_{i,n}|^2 \le C.$$

From the bound above and the uniformly boundedness of $\bar u_{i,n}$, since r is arbitrary there exists $\bar u_\infty = (\bar u_{1,\infty}, \dots, \bar u_{k,\infty}) \in H^1_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\bar u_{i,n} \rightharpoonup \bar u_{i,\infty} \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N), \text{ strongly in } L^2_{loc}(\mathbb{R}^N) \text{ for each } i = 1, \dots, k.$$

Fix $R > 0$, and let n be large such that $B_{r_n R}(x_0) \Subset \Omega$. Applying Proposition 3.3 to the functions $\bar u_{i,n}$, and by letting $n \to \infty$ in (3.5), we conclude that $\hat u_{i,\infty}$ solves

$$-\Delta \hat u_{i,\infty} \ge 0 \quad \text{in} \quad B_R.$$

From the inequality above and (3.9) (by passing the limit as $r_n \to 0$), we can infer that $(\bar u_{1,\infty}, \dots, \bar u_{k,\infty}) \in \mathcal{S}_{0,\dots,0}(B_R)$. Since $R > 0$ is arbitrary, we have that $(\bar u_{1,\infty}, \dots, \bar u_{k,\infty})$ satisfies the assumptions in Lemma 3.7. Therefore, there exist constants $c_1, \dots, c_k$ such that $\bar u_{i,\infty} \equiv c_i$, at most one constant is nonzero and (3.10) holds true.

Step 2. We now claim that

$$\bar u_{i,n} \to c_i \quad \text{strongly in} \quad H^1_{loc}(B_R), \quad \forall\, i = 1, \dots, k. \tag{3.11}$$

In fact, by setting φ as above and using $\bar u_{i,n}\varphi^2$ as a test function, we conclude that

$$\int_{B_R} |\nabla \bar u_{i,n}|^2\varphi^2 + 2\int_{B_R} \bar u_{i,n}\varphi\nabla \bar u_{i,n} \cdot \nabla\varphi \le \int_{B_R} \lambda_{\bar u_i} r_n^2 \bar u_{i,n}^2 \varphi^2 \to 0.$$

On the other hand,

$$\int_{B_R} \bar u_{i,n}\varphi\nabla \bar u_{i,n} \cdot \nabla\varphi = \int_{B_R} (\bar u_{i,n} - c_i)\varphi\nabla \bar u_{i,n} \cdot \nabla\varphi + \frac{1}{2}\int_{B_R} c_i \nabla \bar u_{i,n} \cdot \nabla(\varphi^2) \to 0$$

by the weak convergence $\bar u_{i,n} \rightharpoonup c_i$ in $H^1(B_R)$ (which is strong in $L^2(B_R)$) and

$$\left|\int_{B_R} (\bar u_{i,n} - c_i)\varphi\nabla \bar u_{i,n} \cdot \nabla\varphi\right| \le \left(\frac{1}{2}\|\varphi\|_\infty^2 \|\nabla\varphi\|_2^2 + \frac{1}{2}\|\nabla \bar u_{i,n}\|^2_{L^2(B_R)}\right) \|\bar u_{i,n} - \bar u_{i,\infty}\|_{L^2(B_R)}$$

yields, by the definition of φ, to

$$\int_{B_r} |\nabla \bar u_{i,n}|^2 \to 0, \quad \forall\, 0 < r < R, \tag{3.12}$$

and the claim (3.11) is proved.

Step 3. Suppose, without loss of generality, that $c_2 = \dots = c_k = 0$. We show in this step that also $c_1 = 0$. In particular, $\bar u_{i,n} \to 0$ in $L^\infty_{loc}(\mathbb{R}^N)$ for every $i = 1, \dots, k$.

From the assumptions, we have $\bar u_{i,n} \to 0$ in $H^1_{loc}(B_R)$ for $i > 1$ and, since $\bar u_{i,n}$ also satisfies (3.9),then $\bar u_{i,n} \to 0$ in $L^\infty_{loc}(\mathbb{R}^N)$ by [27, Theorem 8.17]. On the other hand, since $|B_{r_n}(x_0) \cap \{\hat u_1 = 0\}| > 0$, we can take $y_n \in B_{r_n}(x_0)$ such that $\hat u_1(y_n) = 0$. Write $y_n = x_0 + r_n z_n \in B_{r_n}(x_0)$, for some $z_n \in B_1$.

Now, for each large fixed n, take $r \le r_n$. Consider a test function $\varphi \in C_c^\infty(B_{2r}(y_n))$, such that $0 \le \varphi \le 1$ with $\varphi \equiv 1$ on $B_r(y_n)$ and $\|\nabla\varphi\|_{L^\infty(B_{2r}(y_n))} \le C/r$. By using (3.1) once again, we see that each $\sigma_i := \Delta \bar u_i + \lambda_{\bar u_i} \bar u_i$ defines a positive measure. By Proposition 3.2 we infer that

$$\sigma_1(B_r(y_n)) \le \langle \sigma_1, \varphi\rangle = \langle \sigma_1 - \sum_{i>1} \sigma_i, \varphi\rangle + \sum_{i>1} \langle \sigma_i, \varphi\rangle \le C r^{N-1} + \sum_{i>1} \sigma_i(B_{2r}(y_n)).$$

Therefore, since $\int_{B_r(y_n)} \lambda_{\bar{u}_i}\bar{u}_i \le Cr^N$, for all $i = 1, \dots, k$ and $r \le 1$,

$$\begin{aligned}\Delta\bar{u}_1(B_r(y_n)) &= (\Delta\bar{u}_1 + \lambda_{\bar{u}_1}\bar{u}_1 - \lambda_{\bar{u}_1}\bar{u}_1)(B_r(y_n)) \\ &\le Cr^{N-1} + \sum_{i>1}\sigma_i(B_{2r}(y_n)) \le C'r^{N-1} + \sum_{i>1}\Delta\bar{u}_i(B_{2r}(y_n)).\end{aligned}$$

By multiplying the inequality above by r^{1-N} and integrating from 0 to r_n, we obtain

$$\int_0^{r_n} r^{1-N}\Delta\bar{u}_1(B_r(y_n))dr \le Cr_n + \sum_{i>1}\int_0^{r_n} r^{1-N}\Delta\bar{u}_i(B_{2r}(y_n))dr.$$

Now, we apply (A.4) with $x_0 = y_n$ and $r = r_n$ to obtain (recall that $\hat{u}(y_n) = 0$)

$$\begin{aligned}C(N)\fint_{\partial B_{r_n}(y_n)} \bar{u}_1 = \int_0^{r_n} r^{1-N}\Delta\bar{u}_1(B_r(y_n))dr &\le Cr_n + \sum_{i>1}\int_0^{r_n} r^{1-N}\Delta\bar{u}_i(B_{2r}(y_n))dr \\ &\le Cr_n + C(N)\sum_{i>1}\fint_{\partial B_{2r_n}(y_n)} \bar{u}_i,\end{aligned}$$

which leads to

$$\fint_{\partial B_1} \bar{u}_{1,n}(z_n + x) \le Cr_n + C\sum_{i>1}\fint_{\partial B_2} \bar{u}_{i,n}(z_n + x) \to 0, \tag{3.13}$$

as n goes to infinity (recall that $\bar{u}_{i,n} \to 0$ for all $i > 1$). Up to a subsequence, we have $z_n \to z_\infty \in \overline{B}_1$ and $\bar{u}_{1,n}(z_n + x) \to c_1$ in $H^1(B_1)$ which implies strong convergence in $L^1(\partial B_1)$, and then

$$\fint_{\partial B_1} \bar{u}_{1,n}(z_n + x) \to c_1,$$

and hence $c_1 = 0$, as wanted. The fact that also $\bar{u}_{1,n} \to 0$ in $L^\infty_{loc}(\mathbb{R}^N)$ is, again, a consequence of [27, Theorem 8.17].

Finally, from the the convergence $\bar{u}_{i,n} \to 0$ in $L^\infty_{loc}(\mathbb{R}^N)$, we obtain the continuity of each $\bar{u}_i$ at x_0, since $|\bar{u}_1(x) - \bar{u}_1(x_0)| \le 2\|\bar{u}_{1,n}\|_{L^\infty(B_2)} \to 0$ for all $x \in B_{2r_n}(x_0)$ (and also $U(x_0) = 0$)). ☐

4. Lipschitz regularity of minimizers

Let $U := (\bar{u}_1, \dots, \bar{u}_k) \in H_a$ be a minimizer of (1.9), extended by zero in $\mathbb{R}^N \setminus \Omega$. Now, we introduce two quantities related to whether a point x belongs or not to $\partial\Omega_{\bar{u}_i}$, for some $i = 1, \dots, k$. Define the multiplicity of a point $x \in \Omega$ as being

$$m(x) := \#\{i\ ;\ |\Omega_{\bar{u}_i} \cap B_r(x)| > 0,\ \text{for all } r > 0\},$$

and

$$Z_\ell(U) = \{x \in \Omega\ ;\ m(x) \ge \ell\}.$$

Consider the function $\Sigma : \bar{\Omega} \times (0, \infty) \to \mathbb{R}$ defined as

$$\Sigma(x, r) := \frac{1}{r^N}\int_{B_r(x)} |\nabla U|^2, \quad \text{for} \quad (x, r) \in \bar{\Omega} \times (0, \infty),$$

where $|\nabla U|^2 = \sum_{i=1}^k |\nabla\bar{u}_i|^2$. In order to prove the interior local Lipschitz regularity, it is enough to show that Σ is bounded over $\Omega' \times (0, \infty)$, for every Ω' compactly contained in Ω. So, fix such a set Ω' and suppose, by contradiction, that Σ is unbounded in $\Omega' \times (0, \infty)$. Then, there exist sequences $(x_n)_{n\in\mathbb{N}} \subset \Omega'$ and $r_n \to 0$ such that $B_{r_n}(x_n) \subset \Omega$ and

$$\lim_{n\to\infty}\frac{1}{r_n^N}\int_{B_{r_n}(x_n)} |\nabla U|^2 = +\infty. \tag{4.1}$$

In what follows, we present two technical lemmas that will be applied mainly to the sets Z_ℓ. Their proofs can be found in [20] but, for completeness, we also include them here.

Lemma 4.1. *Let (x_n, r_n) be as in (4.1). Then, there exists a sequence $r'_n \to 0$ as $n \to \infty$, with (x_n, r'_n) satisfying (4.1) and such that*

$$\int_{\partial B_{r'_n}(x_n)} |\nabla U|^2 \le \frac{N}{r'_n}\int_{B_{r'_n}(x_n)} |\nabla U|^2 \quad \textit{for all} \quad n \in \mathbb{N}. \tag{4.2}$$

Proof. First, notice that

$$\frac{d}{dr}\left(\frac{1}{r^N}\int_{B_r(x_n)}|\nabla U|^2\right)=\frac{1}{r^N}\left(\int_{\partial B_r(x_n)}|\nabla U|^2-\frac{N}{r}\int_{B_r(x_n)}|\nabla U|^2\right)=:\frac{1}{r^N}f(x_n,r). \tag{4.3}$$

Hence, it is enough to find a sequence $(r_n')_{n\in\mathbb{N}}$ such that (4.1) and $f(x_n,r_n')\leq 0$ holds true. Define $r_n':=\inf\{r\geq r_n : f(x_n,r)\leq 0\}$. Since $\Sigma(x_n,r)\to 0$ as $r\to\infty$ and $\Sigma(x_n,r_n)>0$, we infer that $\frac{d}{dr}\Sigma(x_n,r)\leq 0$ for some r sufficiently large, hence $r_n'<\infty$ for all n. Moreover, since $\Sigma(x_n,r_n')\geq\Sigma(x_n,r_n)\to\infty$, we then infer that $r_n'\to 0$. Then, up to a finite number of indices, $B_{r_n'}(x_n)\subset\Omega$. Moreover, from the definition of r_n', $f(x_n,r_n')\leq 0$, that is, (4.2) is verified. This finishes the proof. □

From the last lemma, since Σ is not bounded over $\Omega'\times(0,\infty)$, then from now on we may assume the existence of sequences $(x_n)_{n\in\mathbb{N}}\subset\Omega'$, $r_n\to 0$ satisfying (4.1) and (4.2).

Lemma 4.2. *Let $A\subset\bar\Omega$ be such that $dist(x_n,A)\leq C r_n$, for all n, and assume that (4.1) holds true. Then, there exist sequences $(x_n')_{n\in\mathbb{N}}\subset A$ and $r_n'\to 0$ such that $B_{r_n'}(x_n')\subset\Omega$ and (x_n',r_n') satisfies (4.1) and (4.2).*

Proof. By assumption, we can find $x_n'\in A$ such that $\mathrm{dist}(x_n,x_n')\leq 2Cr_n$. Now, set $r_n':=(2C+1)r_n$ and observe that $B_{r_n}(x_n)\subset B_{r_n'}(x_n')$ and, for all n sufficiently large, $B_{r_n'}(x_n')\subset\Omega$. Hence

$$\frac{(2C+1)^{-N}}{r_n^N}\int_{B_{r_n}(x_n)}|\nabla U|^2\leq\frac{1}{(r_n')^N}\int_{B_{r_n'}(x_n')}|\nabla U|^2=\Sigma(x_n',r_n').$$

Since the left-hand side in the inequality above goes to infinity, the same holds for the right-hand side, hence (4.1) is satisfied. By eventually changing the radii r_n' (recall Lemma 4.1) we may assume without loss of generality that (4.2) is also true. □

Remark 4.3. *Notice that, for $\ell\geq 0$, if $m(x_n)=\ell$ and $dist(x_n,Z_{\ell+1})<r_n$ for all n, then by Lemma 4.2, we can find sequences $(x_n')_{n\in\mathbb{N}}\subset Z_{\ell+1}$ and $r_n'\to 0$ such that $B_{r_n'}(x_n')\subset\Omega$ for all n, with (x_n',r_n') satisfying (4.1) and (4.2). In particular, $m(x_n')\geq\ell+1$ for all n and $\Omega'\cup\left(\bigcup_{n\in\mathbb{N}}B_{r_n'}(x_n')\right)\cup\left(\bigcup_{n\in\mathbb{N}}B_{r_n}(x_n)\right)$ is compactly contained in Ω.*

Now, we aim to get a contradiction from with (4.1). We achieve this by splitting the proof into cases, based on the quantity $m(x_n)$. We first treat the case $m(x_n)\geq 2$.

Proposition 4.4. *Under the conditions above, suppose that, up to a subsequence, $m(x_n)\geq 2$ for all n. Then (4.1) cannot hold true.*

Proof. We argue by contradiction by assuming that (4.1) is satisfied. It follows from hypothesis $m(x_n)\geq 2$, combined with the continuity of $\bar u_i$, that $\bar u_i(x_n)=0$, for every $i=1,\ldots,k$, and for all n. Set $v_i:=\bar u_i$ and $w_i:=\sum_{j\neq i}^k\bar u_j$. Applying Proposition 3.1, we have that

$$-\Delta v_i\leq\gamma\quad\text{in}\quad B_{r_n}(x_n)\quad\text{and}\quad-\Delta w_i\leq\gamma\quad\text{in}\quad B_{r_n}(x_n),$$

in the sense of distributions, where

$$\gamma:=\max\left\{\lambda_{\bar u_i}\|\bar u_i\|_{L^\infty(\Omega)},\sum_{j\neq i}^k\lambda_{\bar u_j}\|\bar u_j\|_{L^\infty(\Omega)}\right\}.$$

Now, we employ Lemma A.6 in $B_{r_n}(x_n)\subset\Omega$ to the pair (v_i,w_i) to conclude that

$$\left(\frac{1}{r_n^2}\int_{B_{r_n}(x_n)}\frac{|\nabla v_i|^2}{|x-x_n|^{N-2}}\right)\left(\frac{1}{r_n^2}\int_{B_{r_n}(x_n)}\frac{|\nabla w_i|^2}{|x-x_n|^{N-2}}\right)\leq C,$$

where $C>0$ is independent of n. Since $|x-x_n|\leq r_n$ for $x\in B_{r_n}(x_n)$, we obtain

$$\left(\frac{1}{r_n^N}\int_{B_{r_n}(x_n)}|\nabla v_i|^2\right)\left(\frac{1}{r_n^N}\int_{B_{r_n}(x_n)}|\nabla w_i|^2\right)\leq C.$$

From the inequality above and (4.1), we infer the existence of only one component, say $\bar u_1$, such that

$$\frac{1}{r_n^N}\int_{B_{r_n}(x_n)}|\nabla\bar u_1|^2\to\infty, \tag{4.4}$$

and

$$\frac{1}{r_n^N}\int_{B_{r_n}(x_n)}|\nabla \bar u_i|^2 \to 0, \tag{4.5}$$

for every $i = 2, \dots, k$. For the sake of clarity, we split the proof into five steps.

We introduce the blow-up sequence

$$U_n(x) = \frac{1}{L_n r_n} U(x_n + r_n x), \quad \text{for} \quad x \in B_1,$$

where

$$L_n^2 := \frac{1}{r_n^N}\int_{B_{r_n}(x_n)} |\nabla U|^2 \to \infty.$$

Let us denote $U_n = (\bar u_{1,n}, \dots, \bar u_{k,n})$. As a consequence of the definition of L_n,

$$\int_{B_1} |\nabla U_n|^2 = \frac{1}{L_n^2}\int_{B_1} |\nabla U(x_n + r_n x)|^2 = \frac{1}{L_n^2 r_n^N}\int_{B_{r_n}(x_n)} |\nabla U|^2 = 1. \tag{4.6}$$

Hence, from (4.2), we conclude that $\int_{\partial B_1} |\nabla U_n|^2$ is also bounded.

From here, we split the proof into two cases:

Case 1: Suppose that there exists a positive constant C, independent of n such that $\|U_n\|_{L^2(B_1)} \leq C$. In this case, using (4.6), there exists $U_\infty \in H^1(B_1)$ such that $U_n \rightharpoonup U_\infty$ in $H^1(B_1)$.

Step 1.1 ($U_\infty \not\equiv 0$): We denote $U_\infty = (\bar u_{1,\infty}, \dots, \bar u_{k,\infty})$. From (4.5), we get that

$$\int_{B_1} |\nabla \bar u_{i,n}|^2 \to 0, \quad \text{for every} \ \ i = 2, \dots, k,$$

which implies $\|\nabla \bar u_{i,\infty}\|_{L^2(B_1)} = 0$ for every $i = 2, \dots, k$. Since $\bar u_{i,n} \cdot \bar u_{j,n} = 0$ for $i \neq j$, the a.e. convergence implies that at most one component of U_∞ is nonzero. In view of the definition of U_n and Proposition 3.1, we obtain

$$-\Delta \bar u_{1,n}(x) = \frac{r_n}{L_n}(-\Delta \bar u_1(x_n + r_n x)) \leq \frac{r_n}{L_n}\lambda_{\bar u_1} \bar u_1(x_n + r_n x) = r_n^2 \lambda_{\bar u_1} \bar u_{1,n}(x), \quad \text{for } x \in B_1.$$

Multiplying the inequality above by $\bar u_{1,n}$ and integrating by parts, we conclude

$$\int_{B_1} |\nabla \bar u_{1,n}|^2 \leq \int_{\partial B_1} \bar u_{1,n}\frac{\partial \bar u_{1,n}}{\partial \nu} + r_n^2 \lambda_{\bar u_1}\int_{B_1} \bar u_{1,n}^2,$$

where ν is an outward unit normal vector. Now, suppose that $\bar u_{1,\infty} \equiv 0$. Employing the compacts embeddings of $H^1(B_1)$ in $L^2(B_1)$ and in $L^2(\partial B_1)$, and the fact that $\|\nabla U_n\|_{L^2(\partial B_1)}$ is bounded, we obtain that

$$\int_{\partial B_1} \bar u_{1,n}\frac{\partial \bar u_{1,n}}{\partial \nu} + r_n^2 \lambda_{\bar u_1}\int_{B_1} \bar u_{1,n}^2 \to 0,$$

which implies $\|\nabla \bar u_{1,n}\|_{L^2(B_1)} \to 0$, and so $\|\nabla U_n\|_{L^2(B_1)} \to 0$, a contradiction with $\|\nabla U_n\|_{L^2(B_1)} = 1$. Therefore, $\bar u_{1,\infty} \not\equiv 0$, which implies $U_\infty = (\bar u_{1,\infty}, 0, \dots, 0) \not\equiv 0$.

Step 1.2 ($\bar u_{1,\infty}$ is a harmonic function): First, we recall that, for every $i = 1, \dots, k$,

$$-\Delta \bar u_{i,n}(x) \leq r_n^2 \lambda_{\bar u_i} \bar u_{i,n}(x) \quad \text{in} \ \ B_1.$$

Hence, by passing the limit as $n \to \infty$, we see in particular that

$$-\Delta \bar u_{1,\infty}(x) \leq 0 \quad \text{in} \ \ B_1. \tag{4.7}$$

Now, reasoning as Proposition 3.3 we can infer that for any nonnegative $\varphi \in H_0^1(B_1)$ with $\text{supp}(\varphi) \subseteq B_{2s}$, $2s \in (0, 1]$, we have

$$\left\langle -\Delta \hat u_{i,n} - r_n^2 \lambda_{\bar u_i} \bar u_{i,n} + r_n^2 \sum_{j\neq i} \lambda_{\bar u_j} \bar u_{j,n}, \varphi \right\rangle \geq -\frac{Cs}{L_n}\left(s^{N-2} + r_n s^{-1}\|\varphi\|_1 + r_n^2\|\varphi\|_2^2 + \right. \tag{4.8}$$

$$\left. + \|\nabla\varphi\|_2^2 + r_n^N\|\varphi\|_1\|\nabla\varphi\|_2^2 + r_n^N\|\varphi\|_2^2\|\nabla\varphi\|_2^2\right),$$

in the sense of distributions. Now, taking $\varphi \in C_c^\infty(B_1)$, and recalling that $r_n \to 0$, $L_n \to \infty$ and $\bar u_{j,n} \to 0$ in $H^1(B_1)$ for every $j = 2, \dots, k$, we conclude

$$-\Delta \bar u_{1,\infty} = -\Delta \hat u_{1,\infty} \geq 0 \quad \text{in} \ \ B_1,$$

and combined with (4.7), we infer that

$$\Delta \bar{u}_{1,\infty} = 0 \quad \text{in} \quad B_1.$$

According to the maximum principle, we have $\bar{u}_{1,\infty} > 0$ in B_1 (recall that $\bar{u}_{1,\infty} \geq 0$ and $\bar{u}_{1,\infty} \not\equiv 0$).

Step 1.3 (Contradiction): At this point, we apply Proposition A.5 to the functions $\bar{u}_{i,n}$. Denote $\sigma_{i,n} := \Delta \bar{u}_{i,n} + r_n^2 \lambda_{\bar{u}_i} \bar{u}_{i,n}$, and recall that $\sigma_{i,n} \geq 0$, for every $i = 1, 2, \dots, k$. Notice that, for n sufficiently large, we have $\Delta \bar{u}_{i,n} \geq -1$. Moreover, by Proposition 3.1, we have $\bar{u}_{i,n} \in L^\infty(B_{1/2})$. Therefore, by applying Proposition A.5, we obtain, for all $r \in (0, 1/2)$, that

$$\begin{aligned} \fint_{\partial B_r} [\bar{u}_{1,n} - \bar{u}_{1,n}(0)] &= C(N) \int_0^r s^{1-N} \left[\int_{B_s} d(\Delta \bar{u}_{1,n}) \right] ds = C(N) \int_0^r s^{1-N} \Delta \bar{u}_{1,n}(B_s) ds \\ &= C(N) \int_0^r s^{1-N} \sigma_{1,n}(B_s) ds - C(N) \int_0^r s^{1-N} (r_n^2 \lambda_{\bar{u}_1} \bar{u}_{1,n})(B_s) ds \\ &\leq C(N) \int_0^r s^{1-N} \sigma_{1,n}(B_s) ds. \end{aligned}$$

Now, for each fixed $s \in (0, r)$, we take a test function $\varphi \in C_c^\infty(B_{2s})$ such that $0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ in B_s and $\|\nabla \varphi\|_{L^\infty(B_{2s})} \leq C/s$. Hence,

$$\begin{aligned} \sigma_{1,n}(B_s) \leq \langle \sigma_{1,n}, \varphi \rangle &= \left\langle \sigma_{1,n} - \sum_{i=2}^k \sigma_{i,n}, \varphi \right\rangle + \sum_{i=2}^k \langle \sigma_{i,n}, \varphi \rangle \\ &\leq \left\langle \Delta \hat{u}_{1,n} + r_n^2 \lambda_{\bar{u}_1} \bar{u}_{1,n} - r_n^2 \sum_{i=2}^k \lambda_{\bar{u}_i} \bar{u}_{i,n}, \varphi \right\rangle + \sum_{i=2}^k \sigma_{i,n}(B_{2s}) \leq \frac{C s^{N-1}}{L_n} + \sum_{i=2}^k \sigma_{i,n}(B_{2s}), \end{aligned}$$

where in the last inequality we applied (4.8) and used the fact that $r_n \leq 1$ and $|B_{2s}| \leq C(N) s^N$ (and hence we have $\|\varphi\|_1 \leq C(N) s^N$, $\|\varphi\|_2^2 \leq C(N) s^N$ and $\|\nabla \varphi\|_2^2 \leq C(N) s^{N-2}$). Therefore,

$$s^{1-N} \sigma_{1,n}(B_s) \leq \frac{C}{L_n} + s^{1-N} \sum_{i=2}^k \sigma_{i,n}(B_{2s}), \quad \text{for all } s \in (0, r).$$

Plugging the inequalities above, we get (recall that $\bar{u}_{1,n}(0) = 0$, as observed in the first line of the proof)

$$\fint_{\partial B_r} \bar{u}_{1,n} \leq C(N) \int_0^r \left[\frac{C}{L_n} + s^{1-N} \sum_{i=2}^k \sigma_{i,n}(B_{2s}) \right] ds. \tag{4.9}$$

Now, we use Proposition A.5 to estimate the last term in (4.9):

$$\begin{aligned} \int_0^r s^{1-N} \sum_{i=2}^k \sigma_{i,n}(B_{2s}) ds &= \sum_{i=2}^k \int_0^r s^{1-N} (\Delta \bar{u}_{i,n} + r_n^2 \lambda_{\bar{u}_i} \bar{u}_{i,n})(B_{2s}) ds \\ &= \sum_{i=2}^k \int_0^r s^{1-N} \Delta \bar{u}_{i,n}(B_{2s}) ds + \sum_{i=2}^k r_n^2 \lambda_{\bar{u}_i} \int_0^r s^{1-N} \int_{B_{2s}} \bar{u}_{i,n} \, dx ds \\ &\leq C(N) \sum_{i=2}^k \fint_{\partial B_r} \bar{u}_{i,n} + \sum_{i=2}^k \int_0^r \frac{C(N) s r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)}}{L_n} ds \\ &= C(N) \sum_{i=2}^k \fint_{\partial B_r} \bar{u}_{i,n} + \sum_{i=2}^k \frac{C(N) r^2 r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)}}{2 L_n}. \end{aligned}$$

Plugging the inequality above into (4.9) and multiplying it by $|\partial B_r|$ yields to

$$\int_{\partial B_r} \bar{u}_{1,n} \leq \frac{C(N) r^N}{L_n} + C(N) \sum_{i=2}^k \int_{\partial B_r} \bar{u}_{i,n} + \sum_{i=2}^k \frac{C(N) r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)} r^{N+1}}{L_n}, \quad \text{for all } r \in (0, 1/2).$$

Now, we integrate the inequality above with respect to r to obtain

$$\int_{B_{1/2}} \bar{u}_{1,n} \leq \frac{C(N)}{L_n} + C(N) \sum_{i=2}^k \int_{B_{1/2}} \bar{u}_{i,n} + \sum_{i=2}^k \frac{C(N) r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)}}{L_n}.$$

Finally, we pass the limit as $n \to 0$, and recalling that $L_n \to \infty$, $r_n \to 0$, $\bar{u}_{1,n} \to \bar{u}_{1,\infty} > 0$ and $\|\bar{u}_{i,n}\|_{L^1(B_{1/2})} \to 0$, we get

$$0 < \int_{B_{1/2}} \bar{u}_{1,\infty} \leq 0,$$

which is a contradiction, and so the result is proved under the assumptions of Case 1.

Case 2: Assume now that $\|U_n\|_{L^2(B_1)} \to \infty$, and set $V_n := U_n \cdot \|U_n\|_{L^2(B_1)}^{-1} = (v_{1,n}, \dots, v_{k,n})$. Hence,

$$\|V_n\|_{L^2(B_1)} = 1 \quad \text{and} \quad \|\nabla V_n\|_{L^2(B_1)} \to 0.$$

Therefore, there exists $V_\infty \in H^1(B_1)$ such that $V_n \rightharpoonup V_\infty = (v_{1,\infty}, \dots, v_{k,\infty})$ locally in $H^1(B_1)$ and $\|V_\infty\|_{L^2(B_1)} = 1$. Since, for every $\varphi \in H^1(B_1)$,

$$\int_{B_1} \nabla v_{i,n} \cdot \nabla \varphi \to 0 \quad \text{and} \quad \int \nabla v_{i,n} \cdot \nabla \varphi \to \int \nabla v_{i,\infty} \cdot \nabla \varphi,$$

for every $i = 1, \dots, k$, we have that $\|\nabla V_\infty\|_{L^2(B_1)} = 0$, consequently $V_\infty = (c_1, \dots, c_k)$, where $0 \leq c_i \in \mathbb{R}$, for every $i = 1, \dots, k$. Moreover, since $v_{i,n} \cdot v_{j,n} = 0$ for $i \neq j$, the a.e. convergence implies that only one component of V_∞ is nonzero; without loss of generality, we can say $V_\infty = (c_1, 0, \dots, 0)$. In particular, we have $v_{1,\infty} > 0$, and $\Delta v_{1,\infty} = 0$.

Following the general lines of *Step 1.3* of Case 1, we define $\tilde{\sigma}_{i,n} := \Delta v_{i,n} + r_n^2 \lambda_{\bar{u}_i} v_{i,n} \geq 0$, and apply Proposition A.5 to v_n and we obtain that for all $r \in (0, 1/2)$,

$$\begin{aligned}
\fint_{\partial B_r} [v_{1,n} - v_{1,n}(0)] &= C(N) \int_0^r s^{1-N} \left[\int_{B_s} d(\Delta v_{1,n}) \right] ds = C(N) \int_0^r s^{1-N} \Delta \bar{u}_{1,n}(B_s) ds \\
&\leq C(N) \int_0^r s^{1-N} \tilde{\sigma}_{1,n}(B_s) ds.
\end{aligned}$$

Once again, for each fixed $s \in (0, r)$, we take a test function $\varphi \in C_c^\infty(B_{2s})$ such that $0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ in B_s and $\|\nabla \varphi\|_{L^\infty(B_{2s})} \leq C/s$. Hence,

$$\begin{aligned}
\tilde{\sigma}_{1,n}(B_s) \leq \langle \tilde{\sigma}_{1,n}, \varphi \rangle &= \left\langle \tilde{\sigma}_{1,n} - \sum_{i=2}^k \tilde{\sigma}_{i,n}, \varphi \right\rangle + \sum_{i=2}^k \langle \tilde{\sigma}_{i,n}, \varphi \rangle \\
&\leq \left\langle \Delta v_{1,n} + r_n^2 \lambda_{\bar{u}_1} v_{1,n} - r_n^2 \sum_{i=2}^k \lambda_{\bar{u}_i} v_{i,n}, \varphi \right\rangle + \sum_{i=2}^k \tilde{\sigma}_{i,n}(B_{2s}) \leq \frac{C s^{N-1}}{\|U_n\|_{L^2(B_1)} L_n} + \sum_{i=2}^k \tilde{\sigma}_{i,n}(B_{2s}),
\end{aligned}$$

where in the last inequality we applied (4.8) (just multiply the inequality (4.8) by $\|U_n\|_{L^2(B_1)}^{-1}$). Therefore,

$$s^{1-N} \sigma_{1,n}(B_s) \leq \frac{C}{\|U_n\|_{L^2(B_1)} L_n} + s^{1-N} \sum_{i=2}^k \tilde{\sigma}_{i,n}(B_{2s}),$$

for all $s \in (0, r)$. Plugging the inequalities above, we get (recall that $v_{1,n}(0) = 0$)

$$\fint_{\partial B_r} v_{1,n} \leq C(N) \int_0^r \left[\frac{C}{\|U_n\|_{L^2(B_1)} L_n} + s^{1-N} \sum_{i=2}^k \tilde{\sigma}_{i,n}(B_{2s}) \right] ds. \tag{4.10}$$

Now, we use again the Proposition A.5 to estimate the last term in the inequality above:

$$\begin{aligned}
\int_0^r s^{1-N} \sum_{i=2}^k \tilde{\sigma}_{i,n}(B_{2s}) ds &= \sum_{i=2}^k \int_0^r s^{1-N} (\Delta v_{i,n} + r_n^2 \lambda_{\bar{u}_i} v_{i,n})(B_{2s}) ds \\
&= \sum_{i=2}^k \int_0^r s^{1-N} \Delta v_{i,n}(B_{2s}) ds + \sum_{i=2}^k r_n^2 \lambda_{\bar{u}_i} \int_0^r s^{1-N} v_{i,n}(B_{2s}) ds \\
&\leq C(N) \sum_{i=2}^k \fint_{\partial B_r} v_{i,n} + \sum_{i=2}^k \int_0^r \frac{C(N) s r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)}}{\|U_n\|_{L^2(B_1)} L_n} ds \\
&= C(N) \sum_{i=2}^k \fint_{\partial B_r} v_{i,n} + \sum_{i=2}^k \frac{C(N) r^2 r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)}}{2 \|U_n\|_{L^2(B_1)} L_n}.
\end{aligned}$$

Plugging the inequality above into (4.10), and multiplying it by $|\partial B_r|$ yields to

$$\int_{\partial B_r} v_{1,n} \leq \frac{C(N)r^N}{\|U_n\|_{L^2(B_1)}L_n} + C(N)\int_{\partial B_r} v_{i,n} + \sum_{i=2}^{k} \frac{C(N)r_n\lambda_{\bar{u}_i}\|\bar{u}_i\|_{L^\infty(B_1)}r^{N+1}}{\|U_n\|_{L^2(B_1)}L_n},$$

for all $r \in (0,1/2)$. Now, we integrate the inequality above with respect to r to obtain

$$\int_{B_{1/2}} v_{1,n} \leq \frac{C(N)}{\|U_n\|_{L^2(B_1)}L_n} + C(N)\int_{B_{1/2}} v_{i,n} + \sum_{i=2}^{k} \frac{C(N)r_n\lambda_{\bar{u}_i}\|\bar{u}_i\|_{L^\infty(B_1)}}{\|U_n\|_{L^2(B_1)}L_n}.$$

Finally, we pass the limit as $n \to 0$, and recalling that $\|U_n\|_{L^2(B_1)} \to \infty$, $L_n \to \infty$, $r_n \to 0$, $v_{1,n} \to c_1 > 0$ and $\|v_{i,n}\|_{L^1(B_{1/2})} \to 0$, for $i = 2, \dots, k$, we get,

$$0 < \int_{B_{1/2}} c_1 \leq 0,$$

which is a contradiction. □

It remains to deal with the case $m(x_n) = 1$. This is the content of the next proposition.

Proposition 4.5. *Under the conditions above, suppose that, up to a subsequence, $m(x_n) = 1$ for all n. Then (4.1) cannot hold true.*

Proof. Define $\Gamma := \{x \in \Omega \ : \ \bar{u}_1(x) = 0\}$. We analyse two cases:

Case 1: Suppose that $\frac{\operatorname{dist}(x_n,\Gamma)}{r_n}$ is unbounded. Then, up to a subsequence $\frac{\operatorname{dist}(x_n,\Gamma)}{r_n} \geq 1$ for large n. In this scenario, we can conclude that $B_{r_n}(x_n) \subset \Omega_{\bar{u}_1}$, for all n. Hence, by Proposition 2.1 and Proposition 3.8 (or by combining Propositions 3.1 and 3.4), $\bar{u}_1$ solves

$$-\Delta \bar{u}_1 = \lambda_{\bar{u}_1}\bar{u}_1 \quad \text{in} \quad B_{r_n}(x_n).$$

Hence, by applying Proposition 3.1 and by elliptic regularity theory, we get that $\bar{u}_1 \in C^\infty(B_{r_n})$, which is a contradiction with (4.1).

Case 2: On the other hand, if $\frac{\operatorname{dist}(x_n,\Gamma)}{r_n} < C$, for some $C > 0$, it follows from Lemma 4.2 that we can also assume without loss of generality that $x_n \in \Gamma$, for all n. In view of Proposition 4.4 and Remark 4.3, we can assume also that $\frac{\operatorname{dist}(x_n,Z_2)}{r_n} \geq 1$. In this case, only one component, say $\bar{u}_1$, is nonidentically zero in $B_{r_n}(x_n)$.

The proof then follows the general lines of Proposition 4.4, but we do not need to use the Lemma A.6, since (4.1) is equivalent to

$$\frac{1}{r_n^N}\int_{B_{r_n}(x_n)} |\nabla \bar{u}_1|^2 \to \infty. \tag{4.11}$$

We perform again the blow-up analysis. Define

$$w_n(x) = \frac{1}{L_n r_n}\bar{u}_1(x_n + r_n x), \quad \text{for} \quad x \in B_1,$$

where

$$L_n^2 := \frac{1}{r_n^N}\int_{B_{r_n}(x_n)} |\nabla \bar{u}_1|^2.$$

As before, we have

$$\int_{B_1} |\nabla w_n|^2 = 1,$$

which implies the boundness of $\int_{\partial B_1} |\nabla w_n|^2$ by Lemma 4.1.

For simplicity, we divide the proof of this case into four steps:

Step 1 (Convergence of w_n): As in Proposition 4.4, either there exists $w_\infty \in H^1(B_1)$, such that up to a subsequence, $w_n \rightharpoonup w_\infty$ locally in $H^1(B_1)$, or the sequence $v_n := w_n \cdot \|w_n\|_{L^2(B_1)}$ such that $v_n \to c > 0$ locally in $H^1(B_1)$.

Step 2 ($w_\infty \not\equiv 0$): It follows from the definition of w_n and Proposition 3.1 that

$$-\Delta w_n(x) \leq r_n^2 \lambda_{\bar{u}_1} w_n(x),$$

for $x \in B_1$. Multiplying the inequality above by w_n and integrating by parts, we conclude

$$\int_{B_1} |\nabla w_n|^2 \leq \int_{\partial B_1} w_n \frac{\partial w_n}{\partial \nu} + r_n^2 \lambda_{\bar{u}_1} \int_{B_1} w_n^2.$$

Assume that $w_\infty \equiv 0$. Since $\|\nabla w_n\|_{L^2(\partial B_1)}$ is bounded, the right-hand side of the inequality above goes to zero, which is a contradiction.

Step 3 (w_∞ is a harmonic function): As in Proposition 4.4, we can show that

$$\left\langle -\Delta w_n - r_n^2 \lambda_{\bar{u}_1} w_n, \varphi \right\rangle \geq -\frac{C(\varphi)}{L_n}. \tag{4.12}$$

By taking the limit as $n \to \infty$ yields $-\Delta w_\infty \geq 0$ in B_1. Moreover, recall that $-\Delta w_n \leq r_n^2 \lambda_{\bar{u}_1} w_n$ in B_1, we obtain $-\Delta w_\infty = 0$. Therefore, by the maximum principle, $w_\infty > 0$ in B_1.

Step 4 (Contradiction): Now, we argue exactly as in Proposition 4.4 to conclude that

$$\int_{\partial B_r} w_n \leq \frac{C(N) r^N}{L_n} + C(N) \int_{\partial B_r} w_n + \sum_{i=2}^{k} \frac{C(N) r_n \lambda_{\bar{u}_i} \|\bar{u}_i\|_{L^\infty(B_1)} r^{N+1}}{L_n},$$

for all $r \in (0, 1/2)$. Finally, we integrate the inequality above and pass to the limit as $n \to \infty$ to conclude that

$$0 < \int_{B_{1/2}} w_\infty \leq 0,$$

which is a contradiction. This finishes the proof. $\square$

Proposition 4.6. *Under the conditions above, suppose that, up to a subsequence, $m(x_n) = 0$ for all n. Then (4.1) cannot hold true.*

Proof. From Remark 4.3, Propositions 4.5 and 4.4, we can assume that $\text{dist}(x_n, Z_1) \geq r_n$. In this case, it follows from the definition of $m(x_n)$ that $\bar{u}_i \equiv 0$ in $B_{r_n}(x_n)$ for all $i = 1, \dots, k$, which is a contradiction with (4.1). $\square$

5. Conclusion of the proof of the main results

Lemma 5.1. *Assume that problem (1.8) is achieved by an optimal partition $(\omega_1, \dots, \omega_k) \in \mathcal{P}_a(\Omega)$. Then*

$$\sum_{i=1}^{k} |\omega_i| = a.$$

Proof. Let $(\omega_1, \dots, \omega_k) \in \mathcal{P}_a(\Omega)$ be an optimal partition, and let $(u_1, \dots, u_k)$ be an L^2-normalized sequence of associated positive first eigenfunctions, which minimizes J in H_a by Proposition 2.1. Assume by contradiction that $\sum_{i=1}^k |\omega_i| < a$.

Claim. $(u_1, \dots, u_k) \in \mathcal{S}_{\lambda_{u_1}, \dots, \lambda_{u_k}}(\Omega)$.

From (3.1), we have

$$-\Delta u_i \leq \lambda_{u_i} u_i \text{ in } \Omega \qquad \text{for every } i.$$

Now we prove that

$$-\Delta \hat{u}_1 \geq \lambda_{u_1} u_1 - \sum_{j \neq 1} \lambda_{u_j} u_j \text{ in } \Omega.$$

The latter inequality has analogous proof for $i = 2, \dots, k$. Let $\bar{\varepsilon}$ be such that

$$|B_{\bar{\varepsilon}}| < a - \sum_{i=1}^{k} |\omega_i|. \tag{5.1}$$

Take $x_0 \in \Omega$ and $\varepsilon < \bar{\varepsilon}$ such that $B_\varepsilon(x_0) \subset \Omega$. We now check that $-\Delta \hat{u}_1 \geq \lambda_{u_1} u_1 - \sum_{j \neq 1} \lambda_{u_j} u_j$ in $B_\varepsilon(x_0)$. Given $\varphi \in C_c^\infty(B_\varepsilon(x_0))$ nonnegative, for small $t > 0$, we consider the deformation

$$\tilde{u}_t = (\tilde{u}_{1,t}, \dots, \tilde{u}_{k,t}) = \left(\frac{(\hat{u}_1 + t\varphi)^+}{\left\| (\hat{u}_1 + t\varphi)^+ \right\|_2}, \frac{(\hat{u}_2 - t\varphi)^+}{\left\| (\hat{u}_2 - t\varphi)^+ \right\|_2}, \dots, \frac{(\hat{u}_k - t\varphi)^+}{\left\| (\hat{u}_k - t\varphi)^+ \right\|_2} \right).$$

Take $\varepsilon > 0$ small such that:

$$\sum_{j=1}^{k} |\omega_j| + |B_\varepsilon| < a$$

(recall the contradiction assumption (5.1) and that $\omega_j = \Omega_{u_j}$, for all $j = 1, \dots, k$). Then, by Lemma A.3, we have $\tilde{u}_t \in H_a$. We now argue exactly as in the proof of [19, Lemma 2.1]:

$$\sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 = J(u_1, \dots, u_k) \le J(\tilde{u}_{1,t}, \dots, \tilde{u}_{k,t}) = \sum_{i=1}^{k} \frac{\int_\Omega |\nabla \tilde{u}_{i,t}|^2}{\int_\Omega \tilde{u}_{i,t}^2}$$

$$= \frac{\int_\Omega |\nabla(\hat{u}_1 + t\varphi)^+|^2}{\int_\Omega [(\hat{u}_1 + t\varphi)^+]^2} + \sum_{j\neq 1} \frac{\int_\Omega |\nabla(\hat{u}_j - t\varphi)^+|^2}{\int_\Omega [(\hat{u}_j - t\varphi)^+]^2}.$$

As $t \to 0^+$, we have by Lemma A.1

$$\frac{\int_\Omega |\nabla(\hat{u}_1 + t\varphi)^+|^2}{\int_\Omega [(\hat{u}_1 + t\varphi)^+]^2} = \int_\Omega \left|\nabla\left(\hat{u}_1 + t\varphi\right)^+\right|^2 \left(1 - 2t\int_\Omega u_1\varphi + \mathrm{o}(t)\right)$$

$$= \int_\Omega \left|\nabla\left(\hat{u}_1 + t\varphi\right)^+\right|^2 - 2t\int_\Omega u_1\varphi \int_\Omega |\nabla u_1|^2 + \mathrm{o}(t)$$

and, for $j \ge 2$,

$$\frac{\int_\Omega |\nabla(\hat{u}_i - t\varphi)^+|^2}{\int_\Omega [(\hat{u}_i - t\varphi)^+]^2} = \int_\Omega \left|\nabla\left(\hat{u}_j - t\varphi\right)^+\right|^2 \left(1 + 2t\int_\Omega u_j\varphi + \mathrm{o}(t)\right)$$

$$= \int_\Omega \left|\nabla\left(\hat{u}_j - t\varphi\right)^+\right|^2 + 2t\int_\Omega u_j\varphi \int_\Omega |\nabla u_j|^2 + \mathrm{o}(t).$$

Therefore, using the fact that $u_i u_j \equiv 0$ for $i \neq j$, and $(\hat{u}_1 + t\varphi)^+ + \sum_{j\geqslant 2} (\hat{u}_j - t\varphi)^+ = |\hat{u}_1 + t\varphi|$, we have

$$\sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 \le \sum_{i=1}^{k} \int_\Omega |\nabla(\hat{u}_i + t\varphi)^+|^2 - 2t\int_\Omega \lambda_{u_1} u_1\varphi + 2t\sum_{j\geqslant 2}\int_\Omega \lambda_{u_j} u_j\varphi + \mathrm{o}(t)$$

$$= \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 + 2t\int_\Omega \left(\nabla\hat{u}_1 \cdot \nabla\varphi - \left(\lambda_{u_1} u_1 - \sum_{j\geqslant 2} \lambda_{u_j} u_j\right)\varphi\right) + \mathrm{o}(t)$$

as $t \to 0^+$, and hence

$$\int_\Omega \left(\nabla\hat{u}_1 \cdot \nabla\varphi - \left(\lambda_{u_1} u_1 - \sum_{j\geqslant 2} \lambda_{u_j} u_j\right)\varphi\right) \ge 0.$$

Therefore the claim holds true.

Conclusion of the proof. Since $(u_1, \dots, u_k) \in \mathcal{S}_{\lambda_{u_1},\dots,\lambda_{u_k}}(\Omega)$, then $\Gamma_U := \{x \in \Omega \ : \ u_i(x) = 0 \text{ for } i = 1, \dots, k\}$ has zero measure by Lemma 3.5. Therefore $|\Omega| = |\Omega \setminus \Gamma_u| = |\cup_{i=1}^k \Omega_{u_i}| = \sum_{i=1}^k |\Omega_{u_i}| \le a < |\Omega|$, a contradiction. □

Remark 5.2. *An alternative proof of Lemma 5.3 would be to:*

- *first, prove the existence of a one phase point, that is, the existence of i, $x_0 \in \partial\Omega_{u_i}$ and $\delta > 0$, such that $B_\delta(x_0) \cap \Omega_{u_j} = \emptyset$ for every $j \neq i$;*
- *then, argue by contradiction and consider the partition $(\Omega_{u_1}, \dots, \Omega_{u_i} \cup B_\delta(x_0), \dots, \Omega_{u_k})$, which lowers the shape functional.*

Lemma 5.3. *Assume that problem (1.8) is achieved by an optimal partition $(\omega_1, \dots, \omega_k) \in \mathcal{P}_a(\Omega)$. Then:*

$$\omega_i \text{ is connected for every } i = 1, \dots, k.$$

Proof. Assume by contradiction that ω_1 is not connected, and let A_1 be a connected component of ω_1 such that $\lambda_1(\omega_1) = \lambda_1(A_1)$. Observe that $\omega_1 \setminus A_1$ is open and nonempty. Then $(A_1, \omega_2, \dots, \omega_k) \in \mathcal{P}_a(\Omega)$ satisfies

$$\lambda_1(A_1) + \sum_{i=2}^{k} \lambda_1(\omega_i) = c_a,$$

i.e. it is an optimal partition. On the other hand,

$$|A_1| + \sum_{i=2}^{k} |\omega_i| < \sum_{i=1}^{k} |\omega_i| = a,$$

which contradicts the previous lemma. □

Proof of Theorem 1.2 completed. By Proposition 2.2, we have the existence of $(u_1, \dots, u_k) \in H_a$ which minimizes J over the set H_a, that is, the level $\tilde{c}_a$ is achieved. As it is proved in Section 4, each $\bar{u}_i$ is locally Lipschitz continuous. Therefore, by Proposition 2.1, (1.8) has a solution, and problems (1.8) and (1.9) are equivalent. Moreover, from Lemma 5.1, we have that solutions to (1.8) are also minimizers to (1.1), and these levels coincide. □

Proof of Theorem 1.3. The proof is a consequence of the Faber-Krahn inequality: given an open set $\omega \subset \mathbb{R}^N$, then $\lambda_1(\omega) \geq \lambda_1(\omega^*)$, where ω^* is an open ball such that $|\omega^*| = |\omega|$; moreover, equality is achieved if and only if ω is a ball.

Let $(\omega_1 \dots, \omega_k) \in \mathcal{P}_a(\Omega)$ be an optimal partition for problem (1.8) (which exists, by Theorem 1.2). Let B_{r_i} be an open ball such that $|B_{r_i}| = |\omega_i|$, for each i. If a is sufficiently small, then we can assume that

$$B_{r_i} \cap B_{r_j} = \emptyset \quad \forall i \neq j, \qquad \text{and} \qquad \cup_{i=1}^k B_{r_i} \subset \Omega.$$

By the Faber-Krahn inequality we have that $(B_{r_1}, \dots, B_{r_k})$ is an optimal partition and, up to translation of the center, we may assume that $\omega_i = B_{r_i}$.

We now claim that $r_1 = \ldots = r_k$, which finishes the proof. But this is a consequence of the fact that the function:

$$(r_1, \dots, r_k) \in (\mathbb{R}^+)^k \mapsto \sum_{i=1}^{k} \lambda_1(B_{r_i}) = \lambda_1(B_1) \sum_{i=1}^{k} \frac{1}{r_i^2}$$

admits a unique minimizer on the set

$$\left\{ (r_1, \dots, r_k) \in \mathbb{R}^k : \sum_{i=1}^{k} |B_{r_i}| = a \right\} = \left\{ (r_1, \dots, r_k) \in \mathbb{R}^k : |B_1| \sum_{i=1}^{k} {r_i}^N = a \right\}$$

precisely at a point where $r_1 = \ldots = r_k$, by the Lagrange multipliers rule. □

In what follows, we present the proof of Theorem 1.4. Recall that a function $u \in C(\Omega)$ is foliated Schwarz symmetric with respect to $p \in \mathbb{S}^{N-1}$, if u is axially symmetric with respect to the axis $p\mathbb{R}$ and nonincreasing in the polar angle $\theta := \arccos\left(\frac{x}{|x|} \cdot p\right) \in [0, \pi]$.

Proof of Theorem 1.4. Let (ω_1, ω_2) be a solution of (1.1) with corresponding eigenfunctions u_1, u_2. Then, consider ω_1^* and ω_2^* the cap symmetrization of ω_1 and ω_2, with respect e and $-e$, respectively. Then $(\omega_1^*, \omega_2^*) \in \mathcal{P}_a(\Omega)$ and, since $\lambda_1(\omega_i^*) \leq \lambda_1(\omega_i)$, for $i = 1, 2$, see [3, Section 7.5], then (ω_1^*, ω_2^*) solves (1.1). Moreover, the positive first eigenfunctions in ω_1^* and ω_2^* are foliated Schwarz symmetric with respect to e and $-e$, respectively. □

A. Auxiliary results

A.1. **Deformations.** Here we collect some results regarding the deformations used in the paper. This type of deformation appears in the context of multiphase optimal shape problems without the volume constraints. We refer the reader to [18, 19, 20]. Recall the notation $\Omega_u := \{x \in \Omega \mid u(x) \neq 0\}$.

Lemma A.1. *Let $u \in L^2(\Omega)$ with $u^+ \not\equiv 0$. Then, for all $\varphi \in L^2(\Omega)$,*

$$\frac{1}{\|(u \pm t\varphi)^+\|_2^2} = \frac{1}{\|u^+\|_2^2} \mp \frac{2t}{\|u^+\|_2^4} \int_\Omega u^+\varphi + ct^2 \|\varphi\|_2^2 \qquad \text{as } t \to 0^+, \tag{A.1}$$

where $c > 0$ depends only on $\|u^+\|_2$, as $t \to 0^+$.

Proof. Observe that

$$\begin{aligned}
\frac{1}{\|(u\pm t\varphi)^+\|_2^2}-\frac{1}{\|u^+\|_2^2} &= \frac{\int_{\{u>0\}\cap\Omega_\varphi} u^2-\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi} u^2\pm 2tu\varphi+t^2\varphi^2}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}\\
&= \frac{\int_{\{u>0\}\cap\Omega_\varphi} u^2-\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi} u^2\mp 2t\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi} u\varphi - t^2\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi}\varphi^2}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}\\
&= \frac{\int_{\{0<u\le\mp t\varphi\}\cap\{\pm\varphi<0\}} u^2-\int_{\{\mp t\varphi<u<0\}\cap\{\pm\varphi>0\}} u^2 - t^2\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi}\varphi^2}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}\\
&\quad \mp 2t\frac{\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi} u\varphi}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}\\
&= \mp 2t\frac{\int_{\{u>0\}\cap\Omega_\varphi} u\varphi}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}\mp 2t\frac{\int_{\{u\pm t\varphi>0\}\cap\Omega_\varphi} u\varphi-\int_{\{u>0\}\cap\Omega_\varphi} u\varphi}{\|(u\pm t\varphi)^+\|_2^2\,\|u^+\|_2^2}+ct^2\|\varphi\|_2^2\\
&= \mp\frac{2t}{\|u^+\|_2^4}\int_\Omega u^+\varphi+ct^2\|\varphi\|_2^2 \qquad \text{as } t\to 0^+.
\end{aligned}$$

□

Lemma A.2. *Let $u_1,\dots,u_k\in L^2(\Omega)$ be nonnegative functions such that $u_i\cdot u_j\equiv 0$ for all $i\neq j$, $\|u_i\|_{L^2}=1$ for all $i=1,\dots,k$, and $\varphi\in C_c^\infty(\Omega)$ be a nonnegative function. Consider, for $t>0$ small, the deformation*

$$\check{u}_t=(\check{u}_{1,t},\check{u}_{2,t},\dots,\check{u}_{k,t})=\left(\frac{(u_1-t\varphi)^+}{\|(u_1-t\varphi)^+\|_2},u_2,\dots,u_k\right). \tag{A.2}$$

Then:

i) $\int_\Omega \check{u}_{i,t}^2=1$ *for every* i*;*
ii) $\Omega_{\check{u}_{i,t}}\subseteq\Omega_{u_i}$ *for all* $i\ge 1$*;*
iii) $\check{u}_{i,t}\cdot\check{u}_{j,t}\equiv 0$ *for* $i\neq j$*.*

Proof. It is obvious that i) holds and iii) is a consequence of ii). Regarding ii), observe that

$$x\notin\Omega_{u_1}\implies \check{u}_{1,t}(x)=0.$$

□

Recall that, for $u_1,\dots,u_k$ such that $u_i\cdot u_j\equiv 0$ for all $i\neq j$, we denote:

$$\hat{u}_i=u_i-\sum_{j\neq i}u_j.$$

Lemma A.3. *Let $u_1,\dots,u_k\in L^2(\Omega)$ be such that $u_i\cdot u_j\equiv 0$ for all $i\neq j$, and $u_i\ge 0$ and $\|u_i\|_{L^2}=1$ for every i. Take $\mathcal{A}\Subset\Omega$, and let $\varphi\in C_c^\infty(\mathcal{A})$ be a nonnegative function. Consider, for $t>0$ small, the deformation*

$$\tilde{u}_t=(\tilde{u}_{1,t},\dots,\tilde{u}_{k,t})=\left(\frac{(\hat{u}_1+t\varphi)^+}{\left\|(\hat{u}_1+t\varphi)^+\right\|_2},\frac{(\hat{u}_2-t\varphi)^+}{\left\|(\hat{u}_2-t\varphi)^+\right\|_2},\dots,\frac{(\hat{u}_k-t\varphi)^+}{\left\|(\hat{u}_k-t\varphi)^+\right\|_2}\right).$$

Then:

i) $\int_\Omega\tilde{u}_{i,t}^2=1$ *for every* i*;*
ii) $\Omega_{\tilde{u}_{1,t}}\subseteq\Omega_{u_1}\cup\mathcal{A}$*, and* $\Omega_{\tilde{u}_{i,t}}\subseteq\Omega_{u_i}$ *for* $i>1$*.*
iii) $\tilde{u}_{i,t}\cdot\tilde{u}_{j,t}\equiv 0$ *for* $i\neq j$*.*

Proof. The statement i) is obviously true. Regarding ii), we have

$$\hat{u}_1(x)+t\varphi(x)>0\implies u_1(x)+t\varphi(x)>\sum_{j\neq 1}u_j(x)\ge 0\implies u_1(x)>0 \text{ or } \varphi(x)>0.$$

For $i>1$,

$$\hat{u}_i(x)-t\varphi(x)>0\implies u_i(x)>\sum_{j\neq i}u_j(x)+t\varphi(x)\ge 0\implies u_i(x)>0.$$

Finally, for iii), since $u_i\cdot u_j\equiv 0$ for $i\neq j$, it is obvious that $\tilde{u}_{i,t}\cdot\tilde{u}_{j,t}\equiv 0$ for $i\neq j$ with $i,j\ge 2$. Now, by contradiction, suppose that there exists $x\in\Omega_{\tilde{u}_{1,t}}\cap\Omega_{\tilde{u}_{i,t}}\neq\emptyset$, for some $i>1$. Then

$$\hat{u}_1(x)+t\varphi(x)>0 \text{ and } \hat{u}_i(x)-t\varphi(x)>0 \iff \sum_{j\neq 1} u_j(x) < u_1(x)+t\varphi(x) < u_i(x) - \sum_{j\neq 1,i} u_j(x)$$
$$\implies u_i(x) > u_i(x) + 2\sum_{j\neq 1,i} u_j(x) = u_i(x),$$

a contradiction. Notice that for the last equality we have used that $u_i \cdot u_j \equiv 0$ for $i \neq j$ and $u_i \geq 0$ for every i.

□

A.2. **Auxiliary lemmas.** We recall the following Liouville type theorem for subharmonic functions.

Proposition A.4. *Assume that $u_1, \ldots, u_k \in H^1_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ are nonnegative subharmonic functions such that $u_i \cdot u_j \equiv 0$ in $\mathbb{R}^N$. Assume moreover that $u_1, \ldots, u_k$ are bounded. Then all functions but possibly one function are trivial.*

Proof. The result follows directly from [38, Proposition 2.2] applied with $\alpha = 0$. We observe that, even though this case is not stated in the proposition, the proof is exactly the same. □

The next useful inequality is used to prove the continuity of the solutions and its proof can be found in [27, 28].

Proposition A.5. *Let $B_{r_0}(x_0) \subset \Omega$, $u \in H^1(B_{r_0}(x_0))$ and suppose Δu is a measure satisfying*

$$\int_0^r s^{1-N} \left[\int_{B_s(x_0)} d|\Delta u| \right] ds < +\infty, \tag{A.3}$$

for all $r \in (0, r_0)$. Then, the limit $\lim_\rho \fint_{\partial B_\rho(x_0)} u$ exists, and we can define

$$u(x_0) = \lim_\rho \fint_{\partial B_\rho(x_0)} u.$$

In addition, for all $r \in (0, r_0)$

$$\fint_{\partial B_r(x_0)} [u - u(x_0)] = C(N) \int_0^r s^{1-N} \left[\int_{B_s(x_0)} d\Delta u \right] ds. \tag{A.4}$$

The inequality in (A.3) is also true in the case where $u \in L^\infty(B_{r_0}(x_0))$, and there exists $f \in L^\infty(B_{r_0}(x_0))$ such that $-\Delta u^+ \leq f$ and $-\Delta u^- \leq f$.

In the proof of Lipschitz regularity, we make use of the Caffarelli-Jerison-Kenig Monotonicity Lemma that we state next. For a proof of this result, we refer the reader to [16] (see also [20, 46]).

Lemma A.6 (Caffarelli-Jerison-Kenig monotonicity lemma)**.** *Let $u_1, u_2 \in H^1_0(B_{r_0}(x_0)) \cap L^\infty(B_{r_0}(x_0))$ with $u_1 \cdot u_2 = 0$ a.e. in Ω. Suppose that, for some constant $\gamma \geq 0$,*

$$\Delta u_1 \geq -\gamma \quad \text{and} \quad \Delta u_2 \geq -\gamma \quad \text{in} \quad B_{r_0}(x_0).$$

Set

$$\psi(r) := \left(\frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla u_1|^2}{|x - x_0|^{N-2}} \right) \left(\frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla u_2|^2}{|x - x_0|^{N-2}} \right).$$

Then, there exists a constant $C > 0$ such that $\psi(r) \leq C$ for all $r \in (0, r_0/2)$.

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