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NON-ORIENTABLE MINIMAL SURFACES IN $\mathbb{R}^n$

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Introduction

The problem of existence of non-orientable minimal surfaces has been an interesting problem in minimal surfaces'theory, since the discovery of the classical Henneberg's surface (1875) ([4],[5],[14]). Very recently, Meeks (1981) gave the first example of a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of total curvature -6π([10]).

In this paper, a general theory has been set up through the orientable double coverings and the orientable minimal surface theory. The Chern-Osserman's theorem ([2]), the Gackstatter's formula ([3]) and the Hoffman-Osserman's representation theorem for genus zero surfaces ([6]) have been extended and developed to non-orientable cases.

Through this general set up, some existence conditions have been proved and quite a few examples of complete regular non-orientable minimal surfaces in both $\mathbb{R}^3$ and $\mathbb{R}^n$ have been constructed. Further, genus one complete regular non-orientable minimal surfaces with higher total curvatures have also been studied in details.

The results in this paper are based on the author's doctoral dissertation under the direction of Professor Chi Cheng Chen. It is a pleasure to express my gratitude to him for his inspiring guidance.

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§1. Orientable minimal surfaces in $\mathbb{R}^n$.

In this section we review some fundamental results for the orientable minimal surfaces in $\mathbb{R}^n$ which are applied in this work. Details can be found in [1], [6], [12].

We consider surfaces in $\mathbb{R}^n$ defined by maps $X: M \longrightarrow \mathbb{R}^n$ where M is a two-dimensional manifold. Locally if $X(\xi,\eta) = (x_1(\xi,\eta),\dots,x_n(\xi,\eta))$ is a local parametrization and $\phi_k(\zeta) = 2\partial X = \partial x_k/\partial\xi - i\partial x_k/\partial\eta$, $\xi = \xi + i\eta$, then:

(1.1) $\sum_{k=1}^{n} \phi_k^2 = E-G-2iF$,

$E = \langle\partial X/\partial\xi,\partial X/\partial\xi\rangle$, $F = \langle\partial X/\partial\eta,\partial X/\partial\xi\rangle$

$G = \langle\partial X/\partial\eta,\partial X/\partial\eta\rangle$, $\langle,\rangle$ inner product in $\mathbb{R}^n$.

(1.2) $\sum_{k=1}^{n} |\phi_k|^2 = E+G$

(1.3) ϕ_k is analytic in ξ and only if x_k is harmonic in (ξ,η)

(1.4) (ξ,η) are isothermal parameters if and only if $\sum_{k=1}^{n} |\phi_k|^2 \equiv 0$

(that is, $E = G = \lambda^2$, $F \equiv 0$)

Let (ξ,η) be isothermal parameters; the surface is regular if and only if

(1.5) $\sum_{k=1}^{n} |\phi_k|^2 = 2\lambda^2 \neq 0$

and the induced metric is given by

(1.6) $ds^2 = \lambda^2(d\xi^2 + d\eta^2) = \lambda^2|d\zeta|^2$

The laplacian

(1.7) $\Delta X = 2\lambda^2 \vec{H}$, $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$, $\Delta X = (\Delta x_1, \dots, \Delta x_n)$, $\vec{H}$ the mean curvature vector.

A regular surface S in $\mathbb{R}^n$ is said <u>minimal</u> if $\vec{H} \equiv 0$ and from (1.7), S is a minimal surface if and only if the coordinate functions are harmonic.

An orientable, regular, connected minimal surface in $\mathbb{R}^n$ admits an integral representation

(1.8) $X(p) = (x_1(p), \dots, x_n(p))$, $x_k(p) = \mathrm{Re}\int_{p_0}^{p} \alpha_k$, $1 \leq k \leq n$, p_0, $p \in M$

where the 1-forms α_k are analytic, globally defined on M and can be expressed locally, in isothermal parameters, as $\alpha_k = \phi_k(\zeta)d\zeta$, $\phi_k(\zeta) = \partial x_k/\partial\xi - i\partial x_k/\partial\eta$. The forms α_k are called the <u>Weierstrass forms</u>.

The <u>generalized Gauss map</u> of an orientable minimal surface in $\mathbb{R}^n$ is the map $G\colon M \longrightarrow G_{2,n}$ defined by $G(p) = T_pM$, where $G_{2,m}$ is the Grassmannian of oriented planes in $\mathbb{R}^n$ and T_pM is the oriented tangent plane to M at p.

The Grassmannian $G_{2,n}$ can be identified with the hyperquadric Q_{n-2} in the complex projective space $\mathbb{C}P^{n-1}$ defined by

$Q_{n-2} = \{[z_1,\dots,z_n] \quad \mathbb{C}P^{n-1} / \sum_{k=1}^{n} z_k^2 \equiv 0\}$, associating a positive orthogonal base $\{\vec{u},\vec{v}\}$, $|\vec{u}|=|\vec{v}|$ of a plane $\pi \in G_{2-n}$ to the element $[\vec{u}+i\vec{v}]$ of Q_{n-2}. Then, the generalized Gauss map can be viewed as

$$G(\zeta) = [\bar{\phi}_1(\zeta),\dots,\bar{\phi}_n(\zeta)]$$

and G is antiholomorphic if and only if M is minimal.

Further, a generalized Weierstrass representation formula has been given by Hoffman-Osserman [6]:

$$(1.9) \quad X(p) = \mathrm{Re}\int_{p_0}^{p} \frac{1}{2}\Big(1 - \sum_{k=1}^{n-2} \zeta_k^2, i\big(1 + \sum_{k=1}^{n-2} \zeta_k^2\big), 2\zeta_1,\dots,2\zeta_{n-2}\Big)w,\ p_0,\ p \in M$$

where $\zeta_1,\dots,\zeta_{n-2}$ are meromorphic functions, $w = fd\zeta$ analytic 1-form and the regularity condition is $\nu_p = \mu_p$, μ_p the order of zero of w at p, ν_p the maximum order of pole at p of $\zeta_1,\dots,\zeta_{n-2}$, $\sum_{k=1}^{n-2} \zeta_k^2$ for each p in M.

When n=3, the generalized Weierstrass representation coincides with the classical Weierstrass' representation

$$(1.10) \qquad X(p) = \mathrm{Re}\int_{p_0}^{p} \frac{f(\zeta)}{2}\big(1-g^2(\zeta)\ ,\ i(1+g^2(\zeta))\ ,\ 2g(\zeta)\big)d\zeta$$

where the function g is a meromorphic function with the property that $\sigma^{-1} \circ g = N$, with σ: $S^2(1) \longrightarrow \mathbb{C}$ the stereographic projetion and N: $M^2 \longrightarrow S^2(1)$ the classical Gauss map.

When n=4 there is an alternative representation:

$$(1.11) \quad X(p) = \mathrm{Re}\int_{p_0}^{p} (1+g_1g_2\ ,\ i(1-g_1g_2),\ g_1-g_2,\ -i(g_1+g_2)w\ ,\ p_0\ ,\ p \in M\ ,$$

(ii) the differentials $\alpha_k = \phi_k(\zeta)d\zeta$ are either regular or have a pole at each p_j

(iii) the Gauss map $\bar{G} = [(\phi_1, \dots, \phi_n)]$ has a pole of order $m_j \geqq 2$ at each p_j, with m_j the maximum order of pole of $\phi_1 \dots, \phi_n$ at p_j.

(iv) the Gauss map extends analytically to $\overline{M}$ and its image, counting the multiplicities has area $-C(S)$

Proposition 1.2. - Let S be a complete regular minimal surface in $\mathbb{R}^n$. The total curvature of S is $C(S) = -2\pi m$, $m = 0, 1, \dots, \infty$, and

(1.15) $\quad C(S) \leqq 2\Pi(\chi(M) - r)$ (Chern-Osserman's inequality),

where $\chi(M)$ is the Euler characteristic of M and r the number of boundary components.

If the total curvature is finite, then

(i) when $n=3$, m is even, and $m/2 = \text{degree}(g)$, g the function in (1.10).

(ii) when $n=4$, $m = n_1 + n_2$, $n_j = \text{degree}(g_j)$, $j = 1, 2$, g_j the functions in (1.11).

(iii) with the notations of theorem 1.1, and γ the genus of $\overline{M}$

(1.16) $$m - \sum_{j=1}^{r} m_j = 2\gamma - 2$$

Gackstatter [3] has proved a relation between the total curvature of a complete minimal surface, its topological strucuture and the dimension of the smallest affine subspace containing S, which is called the dimension of S:

where g_1, g_2 are meromorphic functions, $w=fd\zeta$ an analytic 1-form. The functions g_1 and g_2 are related to ζ_1, ζ_2 in (1.9) by $g_1 = \zeta_1 + i\zeta_2$, $g_2 = -\zeta_1 + i\zeta_2$; we can still obtain g_1 and g_2 by $g_j = F_j o \tilde{G}$, $j=1,2$, with G the generalised Gauss map and $F=(F_1,F_2)$, $F: Q_2 \longrightarrow \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ the biholomorphic and isometric equivalence between Q_2 and $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ which extends the maps

$$(1.12) \quad \begin{cases} F([z_1,z_2,z_3,z_4]) = \left(\dfrac{z_3+iz_4}{z_1-iz_2}, \dfrac{-z_3+iz_4}{z_1-iz_2}\right), \; z_1-iz_2 \neq 0 . \\ \\ F^{-1}(w_1,w_2) = [(1+w_1w_2), i(1-w_1w_2), w_1-w_2, -i(w_1+w_2)] \end{cases}$$

The <u>total curvature</u> of a regular, orientable minimal surface S defined by $X: M \longrightarrow \mathbb{R}^n$ is given by

$$(1.13) \qquad C(S) = \int_M KdA ,$$

where K is the gaussian curvature of the surface. It's known ([2]) that $C(S) = -A(S)$, where $A(S)$ is the area of the Gaussian image in $\mathbb{C}P^{n-1}$ with respect to the Fubini-Study metric

$$(1.14) \qquad d\hat{s}^2 = 2 \frac{|z \wedge dz|^2}{|z|^4}$$

Some important theorems about complete regular minimal surfaces have been proved by Chern-Osserman ([2]):

<u>Theorem 1.1</u>: - If the total curvature of a complete regular minimal surface is finite, then:

(i) M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points $p_1, \dots, p_r$.

<u>Theorem 1.3.</u> ([3]) - Let S be a regular complete minimal surface of finite total curvature $-2\pi m$ which lies fully in $\mathbb{R}^n$. Then

$$(1.17) \qquad 2m \geq 4\gamma + r + n - 3$$

A technical instrument to construct example of complete regular minimal surfaces of finite total curvature, with genus zero has been provided by Hoffman-Osserman ([6]):

<u>Theorem 1.4.</u> - Let M denote the complex plane minus (r-1) points $\{z_0, z_1, \dots, z_{r-2}\}$. Let $X: M \longrightarrow \mathbb{R}^n$ be defined by

$$(1.17) \qquad X(p) = \mathrm{Re}\int_{p_0}^{p} \phi(\zeta)d\zeta \quad , \; p_0, p \in M$$

If the complex vector ϕ is of the form

$$\phi(\zeta) = F(\zeta)(p_1(\zeta), \dots, p_n(\zeta))$$

with $F(\zeta) = 1/\prod_{k=0}^{r-2} (\zeta - z_k)^{\nu_k}$, $p_j(\zeta)$ satisfying

(i) each p_j is a polynomial,
(ii) the maximum degree of the p_j is m_j,
(iii) the p_j's have no common factor,
(iv) $\sum_{j=1}^{n} p_j^2(\zeta) \equiv 0$,

the ν_k's satisfy $\nu_k \geq 2$, $\sum_{k=0}^{r-2} \nu_k \leq m+1$, and, finally, given any closed curve γ in M, $\mathrm{Re}\int_\gamma \phi(\zeta)d\zeta = 0$, then, (1.17) defines a complete regular minimal surface in $\mathbb{R}^n$ of genus zero, connectivity r and total curvature $-2\pi m$.

Conversely, given a complete regular minimal surface S in $\mathbb{R}^n$ of genus zero, connectivity r and total curvature $-2\pi m$, there exist points $z_0,\dots,z_{r-2}$ and functions F, p_j satisfying the conditions above such that S is given by (1.17).

§2. Non-orientable minimal surfaces in $\mathbb{R}^n$

For the non-orientable surfaces we always consider the two-sheeted orientable covering of the surface, that is, if M is a non-orientable connected surface, let $\Pi: \tilde{M} \longrightarrow M$ be the oriented two-sheeted covering of M and $\tilde{I}: \tilde{M} \longrightarrow \tilde{M}$ the correspondent involution (i.e., $\tilde{I} \neq Id$, $\tilde{I}^2 = Id$) The existence of isothermal parameters on $\tilde{M}$ gives $\tilde{M}$ a conformal structure such that $\tilde{I}$ is antiholomorphic.

We call the <u>double surface associated</u> to a non-orientable surface S given by $X: M \longrightarrow \mathbb{R}^n$ the surface $\tilde{S}$ given by $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^n$ such that $X \circ \Pi = \tilde{X}$, with $\Pi: \tilde{M} \longrightarrow M$ the two-sheeted covering of M.

For the non-orientable minimal surfaces we establish a representation theorem:

<u>Theorem 2.1.</u> - Let S be a non-orientable regular connected minimal surface in $\mathbb{R}^n$. The double surface $\tilde{S}$ is a minimal surface in $\mathbb{R}^n$ defined by $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^n$,

$$\tilde{X}(p) = \mathrm{Re}\int_{p_0}^{p} \tilde{\phi}(\zeta)d\zeta \ , \ p_0, \ p \in \tilde{M} \tag{2.1}$$

Conversely, given a complete regular minimal surface S in $\mathbb{R}^n$ of genus zero, connectivity r and total curvature $-2\pi m$, there exist points $z_0, \dots, z_{r-2}$ and functions F, p_j satisfying the conditions above such that S is given by (1.17).

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$$(2.1) \qquad \tilde{X}(p) = Re\int_{p_0}^{p} \tilde{\phi}(\zeta)d\zeta \ , \ p_0, \ p \in \tilde{M}$$

with $\alpha = \tilde{\phi}(\zeta)d\zeta$ such that $\tilde{I}^*\alpha = \bar{\alpha}$ (that is, $\tilde{I}^*\alpha_k = \bar{\alpha}_k$, $1 \leq k \leq n$).

Reciprocally, if $\tilde{S}$ is a regular orientable connected minimal surface in $\mathbb{R}^n$ given by (2.1) and if there exists an anti-holomorphic involution $\tilde{I}$: $\tilde{M} \longrightarrow \tilde{M}$ without fixed points such that $\tilde{I}^*\alpha = \bar{\alpha}$, then S is the double surface of a regular non--orientable minimal surface in $\mathbb{R}^n$.

<u>Proof</u>: - Initially, we know that the local properties of $\tilde{S}$ are those of S. The map $\tilde{X}$: $\tilde{M} \longrightarrow \mathbb{R}^n$ gives a double surface, therefore, $\tilde{X}(\tilde{I}(p)) = (X \circ \Pi)(\tilde{I}(p)) = X(\Pi(p)) = \tilde{X}(p)$, $\forall p \in \tilde{M}$.
Comparing the integrals:

$$\tilde{X}(\tilde{I}(p)) = Re\int_{p_0}^{\tilde{I}(p)} \alpha = Re\int_{p_0}^{p} \alpha = \tilde{X}(p) \quad , \forall p \in \tilde{M}$$

We can rewrite the integrals:

$$Re\int_{p_0}^{p} \alpha = Re[\int_{p_0}^{\tilde{I}(p_0)} \alpha + \int_{I(p_0)}^{\tilde{I}(p)} \alpha] = \tilde{X}(\tilde{I}(p_0)) + Re\int_{p_0}^{p} \tilde{I}^*\alpha$$

and the last equality, from the fact of $\tilde{X}(\tilde{I}(p_0)) = \tilde{X}(p_0)) = 0$ gives:

$$(2.2) \qquad Re\int_{p_0}^{p} \alpha = Re\int_{p_0}^{p} \tilde{I}^*\alpha$$

The involution $\tilde{I}$ is anti-holomorphic, thus,

$$\tilde{I}^*\alpha_k = \tilde{I}^*(\tilde{\phi}_k(\zeta)d\zeta) = \tilde{\phi}_k(\tilde{I}(\zeta))\cdot\bar{\partial}(\tilde{I}(\zeta))d\bar{\zeta} \quad , \quad 1 \leq k \leq n$$

If, locally, $\tilde{\phi}_k(\zeta)d\zeta = u_k(\zeta) + i\, v_k(\zeta)$, $\alpha_k = f_k(\zeta)d\zeta$, $f_k(\zeta) = r_k(\zeta) + is_k(\zeta)$,

$d\zeta = d\xi + i d\eta$, from (1.3), in a neighborhood of p_0,

$$\int_{p_0}^{p} u_k d\xi - r_k d\eta = \int_{p_0}^{p} r_k d\xi + s_k d\eta$$

and in this neighborhood $u_k = r_k$, $s_k = -v_k$, that is, $f_k = \bar{\phi}_k$ and $\tilde{I}^*\alpha_k = \bar{\alpha}_k$. Hence, $\tilde{I}^*\alpha = \bar{\alpha}$.

For the reciprocal, given a regular minimal surface defined by (2.1) and the antiholomorphic involution $\tilde{I}$ such that $\tilde{I}^*\alpha = \bar{\alpha}$, we have:

$$\tilde{X}(\tilde{I}(p)) = \mathrm{Re}\int_{p_0}^{\tilde{I}(p)} \alpha = \mathrm{Re}\left[\int_{p_0}^{\tilde{I}(p_0)} \alpha + \int_{\tilde{I}(p_0)}^{\tilde{I}(p)} \alpha\right] = X_0 + \tilde{X}(p) \ , \ p \in \tilde{M}$$

Taking $p = \tilde{I}(q)$ in $\tilde{X}(\tilde{I}(p)) = X_0 + \tilde{X}(p)$,

$\tilde{X}(q) = \tilde{X}(\tilde{I}(\tilde{I}(q)) = X_0 + \tilde{X}(\tilde{I}(q)) = 2X_0 + \tilde{X}(q)$.

Thus, $X_0 = 0$, $\tilde{X}(\tilde{I}(p)) = \tilde{X}(p)$ and $\tilde{X}\colon \tilde{M} \longrightarrow \mathbb{R}^n$ is the double surface associated to a non-orientable minimal surface S given by $X\colon \tilde{M}/\sim \longrightarrow \mathbb{R}^n$. ▲

We now investigate the consequences of the condition $\tilde{I}^*\alpha = \bar{\alpha}$ with respect to the various forms of representations for minimal surfaces in $\mathbb{R}^n$. We have:

<u>Corolary 2.2.</u> (Meeks, [10]). Let f,g be the functions of the Weierstrass' representation (1.10) of an orientable regular connected minimal surface $\tilde{S}$ in $\mathbb{R}^3$. The surface $\tilde{S}$ is the double surface of a non-orientable minimal surface in $\mathbb{R}^3$ if and only if

$$(2.3)\quad \begin{cases} \text{(i)} & g(\tilde{I}(p)) = -1/\overline{g(p)}\ ,\ \forall p \in \tilde{M} \\ \text{(ii)} & \tilde{I}^*(w) = -g^2 w\ ,\ w = f(z)dz \end{cases}$$

for some antiholomorphic involution $\tilde{I}\colon \tilde{M} \longrightarrow \tilde{M}$ without fixed points.

<u>Proof</u>: - (i) follows from $\sigma o \vec{N} = g$, $\vec{N}$ the classical Gauss map and σ the stereographic projection, and from the geometrical property: $N(\tilde{I}(p)) = -N(p)$

(ii) follows immediately from (i) and $\tilde{I}^* w_3 = \bar{w}_3$;

The reciprocal can be easily verified. ▲

<u>Corolary 2.3.</u> - Let $\tilde{S}$ be an orientable regular connected minimal surface in $\mathbb{R}^4$ defined by $\tilde{X}\colon M \longrightarrow \mathbb{R}^4$ in (1.11). Then, $\tilde{S}$ is the double surface of a non-orientable minimal surface in $\mathbb{R}^4$ if and only if for some anti-holomorphic involution $\tilde{I}$ on $\tilde{M}$ without fixed points,

$$(2.4)\quad \begin{cases} \text{(i)} & g_k(I(p)) = -1/\overline{g_k(p)}\ ,\ k= 1,2 \\ \text{(ii)} & \tilde{I}^* w = g_1 . g_2 . w\ ,\ w = f(z)dz \end{cases}$$

Proof: - The generalized Gauss map of $\tilde{S}$, $G: \tilde{M} \longrightarrow Q_2$ satisfies, in homogeneous coordinates, $[G(\tilde{I}(p))] = [\overline{G(p)}]$. The functions g_1, g_2 are obtained by $F_j o \overline{G}$, $j=1,2$, $F = (F_1, F_2)$ the function of (1.12); then (i) follows from the fact that $F_j o \tau = -1/\overline{F}_j$, $\tau: Q_2 \longrightarrow Q_2$, $\tau([z_1,z_2,z_3,z_4]) = [\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4]$. From $\tilde{I}^*\alpha_k = \bar{\alpha}_k$, $k=1,2$ we have (ii). The reciprocal, can also be easily verified. Δ

Corolary 2.4. - Let $\tilde{S}$ be an orientable regular connected minimal surface in $\mathbb{R}^n$ defined by $X: M \longrightarrow \mathbb{R}^n$ in (1.9). The surface $\tilde{S}$ is the double surface of a non-orientable minimal surface if and only if

(i) $\quad \tilde{I}^*(\zeta_k w) = \overline{\zeta_k w}$, $1 \leq k \leq n-2$, $w=fdz$

(ii) $\quad \sum_{k=1}^{n-2} \overline{\zeta_k(\tilde{I}(p))}.\zeta_k(p) = -1$, $p \in \tilde{M}$, $\tilde{I}: \tilde{M} \longrightarrow \tilde{M}$ an anti-holomorphic involution without fixed points.

Proof: - From $I^*(\alpha_{k+2}) = \overline{\alpha_{k+2}}$, $1 \leq k \leq n-2$, (i) follows immediately; (ii) follows from $\tilde{I}^*\alpha_j = \bar{\alpha}_j$, $j=1,2$. Δ

We next prove that some special minimal surfaces in $\mathbb{R}^n$ can not be double surfaces; these special surfaces are surfaces in $\mathbb{R}^{2m}$ which are holomorphic curves in $\mathbb{C}^m$ with respect to some orthogonal complex structure in $\mathbb{R}^{2m}$([6], p. 36) and the associated surfaces to a double surface.

<u>Proposition 2.5.</u> - If $\tilde{S}$ is a regular minimal surface in $\mathbb{R}^{2m}$ which is a complex analytic curve with respect to some orthogonal complex structure on $\mathbb{R}^{2m}$, then $\tilde{S}$ cannot be a double surface of a non-orientable minimal surface in $\mathbb{R}^{2m}$.

<u>Proof:</u> - If $\tilde{S}$ is a complex analytic curve, $\tilde{S}$ can be given by $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^{2m}$, $\tilde{X} = (x_1,\dots,x_{2m})$ such that $(x_1+ix_2,\dots,x_{2m-1}+ix_{2m})$ is analytic. By the Cauchy-Riemann equations, the generalized Gauss map is $\phi = (\phi_1,-i\phi_1,\dots,\phi_{2m-1},-i\phi_{2m-1})$ and $\tilde{S}$ will be a double surface if and only if $\tilde{I}^*\phi = \overline{\phi}$, that is,
$\tilde{I}^*(\phi_{2j-1}(\zeta)d\zeta) = \overline{\phi_{2j-1}(\zeta)d\zeta}$ and $\tilde{I}^*(-i\phi_{2j-1}(\zeta)d\zeta) = \overline{i.\phi_{2j-1}(\zeta)d\zeta}$, $1 \leq j \leq m$; then, $\phi_{2j-1} \equiv 0$, contradicting the regularity condition. Δ

<u>Proposition 2.6.</u> - The associated surfaces of an orientable double minimal surface can not be double surfaces.

<u>Proof:</u> - If the double surface is given by $Y: M^2 \longrightarrow \mathbb{R}^n$, the associated surfaces are given by

$$Y_\alpha = Y\cos\alpha + \hat{Y}\sin\alpha \qquad 0 \leq \alpha < \Pi ,$$

with $\hat{Y}$ the harmonic conjugate of Y.

Using the Cauchy-Riemann equations for $Y + i\hat{Y}$, the generalized Gauss map of the associated surfaces can be written:

$$\phi_\alpha = \phi\cos\alpha - i\,\phi\sin\alpha = e^{-i\alpha}\phi$$

The surfaces Y_α will be double surfaces if and only if $\tilde{I}^*(\phi_\alpha d\zeta) = \overline{\phi_\alpha d\zeta}$, that is, $e^{-i\alpha}\tilde{I}^*(\phi(\zeta)d\zeta) = e^{i\alpha}\overline{\phi(\zeta)d\zeta}$. With the hypothesis $I^*(\phi(\zeta)d\zeta) = \overline{\phi(\zeta)d\zeta}$, this last equation implies $e^{2i\alpha} = 1$, that is, $\alpha \equiv 0 \pmod{\pi}$. ▲

In the following study, the double surface $\tilde{S}$ associated to a non-orientable minimal surface S will always be connected, complete and of finite total curvature.

<u>Proposition 2.7.</u> - Let $\tilde{S}$ be a complete regular minimal double surface of finite total curvature defined by $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^n$. Then:

(i) $\tilde{M}$ is conformally equivalent to a compact Riemann surface $\overline{M}$ of genus $\tilde{\gamma}$, punctured at a finite number of points $\{p_1, \dots, p_r, q_1, \dots q_r\}$, with $\tilde{I}(p_j) = q_j$, $1 \leq j \leq r$, $\tilde{I}(p_j)$ the extension of $\tilde{I}$ to p_j

(ii) If m_{p_j} and m_{q_j} are the orders of pole of $\phi = (\phi_1, \dots, \phi_n)$, at p_j and q_j respectively, then, $m_{p_j} = m_{q_j}$, $1 \leq j \leq r$

(iii) If p and q are ends of $\tilde{M}$ related by $\tilde{I}(\tilde{I}(p) = q)$, and locally,

$$\phi_p(z) = \sum_{j=1}^{m} \frac{a_j}{z^j} + \sum_{k=0}^{\infty} b_k z^k \ , \ a_j, b_k \in \mathbb{C}^n$$

$$\phi_q(w) = \sum_{\ell=1}^{m} \frac{A_\ell}{w^\ell} + \sum_{k=0}^{\infty} B_k w^k \ , \ A_\ell, B_k \in \mathbb{C}^n \ , \text{ then,}$$

the complex subspaces generated by $\{a_1, \dots, a_m\}$ and $\{\overline{A}_1, \dots, \overline{A}_m\}$ are the same.

(iv) $a_1, A_1 \in \mathbb{R}^n$, $a_1 = A_1$

(v) If we denote a_1^j the corresponding a_1 in the development of ϕ_{p_j} , $1 \leq j \leq r$, then $\sum_{j=1}^{r} a_1^j = 0$.

<u>Proof:</u> - From Chern-Osserman's theorem we have $\tilde{M}$ conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at $\tilde{r}$ points; let $p \in \overline{M} - \tilde{M}$ be one of these points, $q=\tilde{I}(p)$, and $V \subset \mathbb{C}$ a coordinate neighborhood with $z(p) = 0$, where ϕ_p has a pole of order m_p , and $\phi_p(z) = 2\partial X/\partial z$, $z \in V$.

If $W \subset \mathbb{C}$ is a neighborhood of $w=0$, $f\colon W \longrightarrow V$ anti--holomorphic, $f(0)=0$, $\bar{f}'(0) \neq 0$, then Xof parametrizes locally $q=\tilde{I}(p)$, and

$$(2.5) \qquad \phi_q(w) = 2\,\frac{\partial X}{\partial w} = 2\,\frac{\partial}{\partial w}(X\mathrm{o}f)(w) = 2\,\overline{\frac{\partial X}{\partial z}} \cdot \overline{\frac{\partial f}{\partial \bar{w}}} = \overline{\phi_p(f(w))} \cdot \overline{\frac{\partial f}{\partial \bar{w}}} \quad .$$

Therefore, ϕ_q has a pole of order m_p at $w=0$, since $\overline{\partial f/\partial \bar{w}} = \partial \bar{f}/\partial w$ is holomorphic and does not vanish in W.

(iii) follows immediately from (2.5).

To verify (iv) and (v), we recall that $\mathrm{Re}\int_{\gamma}\phi(\zeta)d\zeta = 0$ for γ closed curve in M. Let γ_p, γ_q be simple closed oriented curves around p and q; respectively, with $\tilde{I}(\gamma_p) = -\gamma_q$. Then $\int_{\gamma_p}\phi(\zeta)d\zeta = 2\pi i a_1$, $\int_{\gamma_q}\phi(\zeta)d\zeta = 2\pi i A_1$ and $a_1, A_1 \in \mathbb{R}^n$; the equality $a_1 = A_1$ follows from the non-orientable condition $\tilde{I}^*(\phi d\zeta) = \overline{\phi d\zeta}$.

We now take simple closed curves β_j, γ_j around $p_j, q_j = \tilde{I}(p_j)$, respectively, for $1 \leq j \leq r$ such that $\tilde{I}(\beta_j) = -\gamma_j$, $1 \leq j \leq r$ and $\{\beta_j, \gamma_j, 1 \leq j \leq r\}$ being the oriented boundary of M' $\tilde{M}$; thus:

$$\sum_{j=1}^{r}\int_{\beta_j}\phi d\zeta + \int_{\gamma_j}\phi d\zeta = \int_{\partial M'}\phi d\zeta = \int_{M'} d(\phi d\zeta) = 0 .$$

Calculating the integrals: $\sum_j 2\pi i . a_1^j = -2\pi i . \sum_j a_1^j$, and (v) follows. Δ

The <u>total curvature of a non-orientable minimal</u> surface S may be defined as

$$C(S) = \frac{1}{2}C(\tilde{S}) \tag{2.6}$$

With this definition, we have:

<u>Proposition 2.8.</u> - The total curvature of a non-orientable regular complete minimal surface in $\mathbb{R}^n$ is

$$C(S) = -2\pi m , \ m = 0, 1, \dots, \infty .$$

Proof: - From the proposition 2.7., if $C(\tilde{S}) > -\infty$ the double surface $\tilde{S}$ has $\tilde{r} = 2r$ ends, $p_j = \tilde{I}(q_j)$, $1 \leq j \leq r$, p_j and q_j with the some order of pole m_j.

From propositions 1.2., the double surface $\tilde{S}$ has total curvature $\tilde{c} = -2\pi\tilde{m}$, with $\tilde{m}$ satisfying

$$\tilde{m} - 2 \sum_{j=1}^{r} m_j = 2\tilde{\gamma}-2$$

Thus, $\tilde{m}$ is even and $C(S) = C(\tilde{S})/2 = -2\pi.m$. If the total curvature is infinite, $C(S) = C(\tilde{S}) = -\infty$. ▲

As a consequence of proposition 2.7 we can prove that Chern-Osserman's inequality for non-orientable minimal surfaces is analogous to the orientable case.

Proposition 2.9: - Let S be a non-orientable regular complete minimal surface in $\mathbb{R}^n$, with r ends given by $X: M^2 \longrightarrow \mathbb{R}^n$; then

$$C(S) \leq 2\pi(\chi(M)-r), \tag{2.8}$$

with $\chi(M)$ the Euler characteristic of M.

Proof: - The total curvature of the double surface $\tilde{S}$ associated to S is given by $C(\tilde{S}) \leq 2\pi(\chi(\tilde{M})-\tilde{r})$.

From $\tilde{r} = 2r$ (proposition 1.7), $\chi(\tilde{M}) = 2.\chi(M)$ and $C(\tilde{S}) = 2C(S)$ we have $C(S) \leq 2\pi(\chi(M)-r)$ ▲

The <u>dimension</u> of a <u>non-orientable minimal surface</u> S is the dimension of the double surface $\tilde{S}$ associated to S.

The non-orientable version of Gackstatter's theorem (theorem 1.3) is:

<u>Theorem 2.10.</u> - The dimension of a non-orientable regular complete minimal surface in $\mathbb{R}^n$, with finite total curvature $c = -2\pi m$, r ends and genus γ satisfies:

$$\dim S \leq 2m - 2\gamma - r+3 \tag{2.10}$$

<u>Proof</u>. - From Gackstatter's theorem the dimension of $\tilde{S}$ is the dimension of the real subspace generated by

$\{a_1^j, A_1^j, \mathrm{Re}\, a_{\nu_j}^j, \mathrm{Im}\, a_{\nu_j}^j, \mathrm{Re}\, A_{\nu_j}^j, \mathrm{Im}\, A_{\nu_j}^j, \ 2 \leq \nu_j \leq m_j, \ 1 \leq j \leq r\}$, with

$$\phi_{p_j}(z) = \frac{a_{m_j}^j}{z^{m_j}} + \dots + \frac{a_1^j}{z} + \sum_{k=0}^{\infty} b_k z^k ,$$

$$\phi_{q_j}(z) = \frac{A_{m_j}^j}{z^{m_j}} + \dots + \frac{A_1^j}{z} + \sum_{k=0}^{\infty} B_k z^k \qquad 1 \leq j \leq r .$$

From proposition 2.7., (iii) to (v) , we have:

$$\dim S \leq (r-1) + 2 \sum_{j=1}^{r} (m_j - 1)$$

Using (2.7), with $\tilde{m}=2m$, and the fact that $\tilde{\gamma} = \gamma - 1$, we have $\dim S \leq 2m-2\gamma-r+3$. ▲

An upper bound for the total curvature is obtained from the following proposition:

Proposition 2.11. - The total curvature of a non-orientable regular complete minimal in $\mathbb{R}^n$ is at most -4π.

Proof: - From (2.10) and the fact that $\dim S \geq 3$, $\gamma \geq 1, r \geq 1$, then, $2m \geq 2\gamma + r \geq 3$. Therefore, $2m \geq 4$ and $c = -2\pi m \leq -4\pi$. ▲

A characterization of the complete minimal surfaces of finite total curvature and genus one can be set up combining theorem 2.1 and the Hoffman-Osserman's theorem (theorem 1.4).

Theorem 2.12. - Let $\tilde{M}$ be the complex plane minus 2r-1 points $\{0, z_1, \dots z_{r-1}, -1/\bar{z}_1, \dots, -1/\bar{z}_{r-1}\}$. If the complex vector $\phi(\zeta)$ is of the form $\phi(\zeta) = F(\zeta)(p_1(\zeta), \dots, p_n(\zeta))$, with

$$F(\zeta) = \frac{1}{\zeta^{\nu_0} \prod_{k=1}^{r-1} (\zeta - z_k)^{\nu_k} (\zeta + \frac{1}{\bar{z}_k})^{\nu_k}}$$

$p_j(\zeta)$ satisfying

(i) $p_j(\zeta)$ is a polynomial

(ii) the maximum degree of p_j is 2m

(iii) the p_j's have no common factor

(iv) $\sum_{j=1}^{n} p_j^2 \equiv 0$

(v) $(-1)^{m+1} \bar{\zeta}^{2m} \ p_j(-1/\bar{\zeta}) = \overline{p_j(\zeta)}$, $1 \leq j \leq m$, and ν_k's satisfy $\sum_{k=0}^{r-1} \nu_k = m+1$, $\nu_k \geqq 2$, $0 \leqq k \leqq r-1$ and

$\operatorname{Re}\int_{\gamma} \phi = 0$, $\forall\gamma$: simple closed curve em $\tilde{M}$, then $\tilde{X}(p) = \operatorname{Re}\int_{p_0}^{p} \phi(\zeta)d\zeta$

defines a double minimal surface associated to a non-orientable regular complete minimal surface in $\mathbb{R}^n$ of genus one, r ends and total curvature $-2\pi m$.

Conversely, every genus one non-orientable minimal surface in $\mathbb{R}^n$ admits a representation mentioned above.

<u>Proof</u>: - The double surface associated to a genus one non-orientable minimal surface is conformally equivalent to the complex plane minus 2r points with the involution $\tilde{I}: \mathbb{C} \longrightarrow \mathbb{C}$ given by $\tilde{I}(z) = -1/\bar{z}$, that is, $\tilde{M} = \mathbb{C} - \{0, z_1, \dots, z_{r-1}, -1/\bar{z}_1, \dots, -1/\bar{z}_{r-1}\}$. From Hoffman-Osserman's result, $\phi = F(p_1, \dots, p_n)$, p_j polynomials satisfying (i) to (iv).

The map $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^n$ defines a double surface if and only if $\tilde{I}^*(\phi_j(\zeta)d\zeta) = \overline{\phi_j(\zeta)d\zeta}$, $1 \leqq j \leqq n$; from this fact and (2.7), we have $\sum_{k=0}^{r-1} \nu_k = m+1$ and (v). $\blacktriangle$

§3. Genus one non-orientable complete minimal surfaces in $\mathbb{R}^n$

First, we observe that

<u>Theorem 3.1.</u>- Let S be a non orientable regular complete minimal surface of finite total curvature -4π.

Then, S is a minimal immersion of the projective plane minus one point and lies fully in $\mathbb{R}^4$; and any two such minimal surfaces are similar

<u>Proof</u>: - From Chern-Osserman's inequality (2.8), using $\chi(M) = 2-\gamma-r$, we have $-4\pi \leq 2\pi(2-\gamma-2r)$ and $\gamma+2r \leq 4$. We then have the possibilities:

(i) $\gamma=1$, $r=1$, which, from (2.10), implies dim S ≤ 4

(ii) $\gamma=2$, $r=1$, which, from (2.10), implies dim S ≤ 2.

Excluding (ii), if $\gamma=1$, $r=1$, the order of pole at the end is, by (2.7), $m_1=3$.

The double surface of (i) has genus zero and two ends, thus, by theorem 2.12, it can be given by

$$\tilde{X}: C - \{0\} \longrightarrow \mathbb{R}^n , \ \tilde{X} = \mathrm{Re}\int_{p_0}^{p} \phi(\zeta)d\zeta, \ p_0, p \in \tilde{M}$$

with the functions $\phi_j(\zeta) = p_j(\zeta)/\zeta^3$, $p_j(\zeta)$ polynomials satisfying:

(i) the maximum degree of the $p_j's$ is 4.

(ii) $\sum_{j=1}^{n} p_j^2(\zeta) \equiv 0$

(iii) $\mathrm{Re}\int_\gamma \phi_j(\zeta) = 0$, γ simple closed curve in $\mathbb{C}-\{0\}$

(iv) $-\bar\zeta^4 p_j(-1/\bar\zeta) = \overline{p_j(\zeta)}$, $1 \leq j \leq n$

Setting $p_j(\zeta) = a_j\zeta^4+b_j\zeta^3+c_j\zeta^2+d_j\zeta+e_j$, from (iv) we have $a_j=-\bar e_j$, $b_j = \bar d_j$, $c_j=-\bar c_j$. Calculating

$\int_\gamma \phi_j(\zeta) = 2\pi i.c_j$, for any simple closed curve γ around $z=0$ and the condition (iii) is equivalent to $\mathrm{Im}\, c_j = 0$, $1 \leq j \leq r$.

Therefore, $p_j(\zeta) = a_j\zeta^4 + b_j\zeta^3 + \bar b_j\zeta^2 - \bar a_j$, $1 \leq j \leq r$.

The polinomials satisfy (ii) if and only if $\sum_j a_j^2=\sum_j b_j^2 = 0$, $\sum_{j=1}^{n} a_j b_j = 0$, $\sum_j |a_j|^2 = \sum_j |b_j|^2$. Calling $A_1 = (\mathrm{Re}a_1,\dots,\mathrm{Re}a_n)$, $A_2=(\mathrm{Im}\, a_1,\dots,\mathrm{Im}\, a_n)$, $B_1 = (\mathrm{Re}b_1,\dots,\mathrm{Re}b_n)$, $B_2 = (\mathrm{Im}\, b_1,\dots,\mathrm{Im}b_n)$, these last conditions are equivalent to $\{A_1,A_2,B_1,B_2\}$orthogonal, $\| A_1\| = \|A_2\| = \| B_1\| = \| B_2\|$, and $A = (1,i,0,0)$, $B = (0,0,1,i)$ give a solution.

Any two immersions $\tilde X,\tilde Y$ may be obtained by considering polinomials $p_j(\zeta) = a_j\zeta^4 + b_j\zeta^3 + \bar b_j\zeta - \bar a_j$ and $q_j(\zeta)=a_j'\zeta^4+b_j'\zeta^3+ + \bar b_j'\zeta - \bar a_j'$ respectively such that $\{A_1,A_2,B_1,B_2\}$, $\{A_1',A_2',B_1',B_2'\}$ are orthogonal vectors in $\mathbb{R}^n$ with $\| A_1\| = \| A_2\| = \| B_1\| = \| B_2\| = \lambda$, $\| A_1'\| = \| A_2'\| = \| B_1'\| = B_2'\| = \mu$.

Therefore, there exists a real orthogonal matrix M such that $A_j' = \frac{\mu}{\lambda} MA_j$, $B_j' = \frac{\mu}{\lambda} MB_j$, $j=1,2$, and $= \frac{\mu}{\lambda} M\tilde X + X_0$. ∆

Immediately, we have Meeks's theorem:

Corollary 3.2. - The total curvature of non-orientable regular complete minimal surfaces in $\mathbb{R}^3$ is at most -6π.

The total curvature of an orientable minimal surface in $\mathbb{R}^3$ is of the form $c(\tilde{S}) = -4\pi \times .(\text{degree } g)$, with g the meromorphic function of the Weierstrass representation (1.10); thus, for non-orientable minimal surfaces,

$$C(S) = -2\pi(\text{degree } g) \tag{3.1}$$

with $g: \tilde{M} \longrightarrow \mathbb{C}$ the function of the Weierstrass representation of the double surface $\tilde{S}$, which is a rational function when it has finite degree and the genus of $\tilde{M}$ is zero.

The next result is very important for the construction of examples:

Proposition 3.3. - Let g be a rational complex function satisfying $g(-1/\bar{z}) = -1/\overline{g(z)}$. Then, the degree of g is odd and g is of the form

$$g(z) = c\, z^{\alpha} \frac{\prod_{j=1}^{m}(z - a_j)}{\prod_{j=1}^{m}(z + \frac{1}{\bar{a}_j})}, \text{ with } |c|\prod_{j=1}^{m}|a_j| = 1 .$$

roof: - Being a rational function, let

$$g(z) = c\, z^{\alpha} \frac{\prod_{j=1}^{m}(z-a_j)}{\prod_{k=1}^{n}(z-b_k)}, \ a_j \neq b_k \text{ for any } j,k, a_j \neq 0, \ b_k \neq 0,$$

where the factors appears with multiplicities.

The condition $g(-1/\bar{z}) = -1/\overline{g(z)}$ is equivalent to

$$(3.2)\quad (-1)^{\alpha+m+n}|c|^2\bar{z}^{n-m}\prod_{j=1}^{m}(a_j\bar{z}+1)(\bar{z}-\bar{a}_j) = -\prod_{k=1}^{n}(b_k\bar{z}+1)(\bar{z}-\bar{b}_k)$$

We can easily conclude that m=n, and (3.2) can be rewritten as:

$$(-1)^{\alpha}|c|^2\prod_{j=1}^{m}(a_j\bar{z}+1)(\bar{z}-\bar{a}_j) = -\prod_{k=1}^{m}(b_k\bar{z}+1)(\bar{z}-\bar{b}_k)\ .$$

The polynomial $\prod_{k=1}^{m}(b_k\bar{z}+1)(\bar{z}-b_k)$ vanish at $z=b_k$; then, there exists j, $1 \leq j \leq m$ such that $a_j\bar{b}_k+1 = 0$, since $a_j \neq b_k$ for any j, k. Therefore, $a_j = -1/\bar{b}_k$ and this relation occurs with the same multiplicity, that is:

$$(3.3)\qquad (-1)^{\alpha}|c|^2\prod_{j=1}^{m}\left(\frac{-1}{\bar{b}_j}\bar{z}+1\right)\left(\bar{z}+\frac{1}{\bar{b}_j}\right) = -\prod_{j=1}^{m}(\bar{b}_j\bar{z}_j+1)(\bar{z}-\bar{b}_j).$$

From (3.3), $(-1)^{\alpha+m}|c|^2\prod_{j=1}^{m}|a_j|^2 = -1$, and degree g is odd. ▲

<u>Proposition 3.4.</u> - The total curvature of a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of genus one is of the form $c(S) = -2\pi\, m$, m odd, $m \geq 3$.

<u>Proof</u>: - This follows immediately from proposition 3.3. ▲

Meeks [6] has proved that there exists unique non--orientable régular complete minimal surface in $\mathbb{R}^3$ of total curvature -6π. We extend Meeks's surface in the sense of

Theorem 3.5. - There exists a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of genus one, one end and total curvature $c = -2\pi m$, for any m odd, $m \geqq 3$.

Proof: - With straightforward calculations, one can show that the functions $f(z) = \dfrac{i(z+1)^2}{z^{m+1}}$, $g(z) = \dfrac{z^{m-1}(z-1)}{z+1}$ are the functions of the Weierstrass's Representation of a double minimal surface $\tilde{S}$, that is, the map

$$X(p) = \mathrm{Re}\int_{p_0}^{p} \frac{f}{2}(1-g^2\ ,\ i(1+g^2),2g)d\zeta,\ p_0,p \in \mathbb{C} - \{0\}$$

satisfies (i) to (v) of theorem 2.12. The surfaces obtained are regular and complete. $\triangle$

Remarks:

1. If n=3 the surface of theor. 3.5. is exactly the Meeks'surface.

2. The surface is an infinite Moebius'band with (m-1)/2 twists and the image of $|z|=1$ is a circle centered at $(0,-2/(m-1),0)$ with radius $2/(m-1)$ covered $(m-1)$ times.

3. The normals to the surface along $|z|=1$ have the third component $N_3\Big|_{|z|=1} = \mathrm{Re}(z)\Big|_{|z|=1}$.

The Chern-Osserman's inequality for non-orientable minimal surfaces of total curvature -6π gives us the relation $\gamma+2r \leq 5$; hence, if $\gamma=1$, we may have $r=1$ or $r=2$. Meeks has proved that the case $r=2$, $c=-6\pi$ cannot occur. We will give an alternative proof of this fact in theor. 3.11. Using Meeks' result, we have:

Proposition 3.6. - The total curvature of a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of genus one and two ends is at most -10π.

For $c=-10\pi$, we have:

Theorem 3.7. - There exists a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of genus one, two ends and total curvature -10π.

Proof: - The double surface associated to such a surface will be an orientable surface of genus zero, four ends with total curvature -20π, and, by theorem 2.12, it may be given by

$$\tilde{X}: \mathbb{C} - \{0,1,-1\} \longrightarrow \mathbb{R}^3 \ , \ \tilde{I}: \mathbb{C} \longrightarrow \mathbb{C}, \tilde{I}(z)=-1/\bar{z}$$

Let $f(z) = a(\bar{b}^2z^2-1)^2/z^2(z-1)^4(z+1)^4$ and $g(z) = \alpha\bar{b}^2z^3(z^2-b^2)/(\bar{b}^2z^2-1)$; f and g are the functions of the Weierstrass'representation of a double minimal surface if and only if (corol 2.2):

(3.4) $|\alpha|\ |b|^2 = 1$ and $a = -\bar{a}\bar{\alpha}b^4$ or $a\alpha\bar{b}^2 = -\bar{a}\bar{\alpha}b^2$.

The completeness and regularity can be easily verified.

To choose α, a and b such that

$$\mathrm{Re}\int_\gamma \frac{f(z)}{2}\,(1-g^2(z),\ i(1+g^2(z)),\ 2g(z))dz = \vec{0}$$

for any closed curve γ in $\mathbb{C} - \{0,1,-1\}$, we observe that, by Cauchy's Integral Formula,

$$(3.5)\quad \int_{\gamma_0} \frac{p_j(\zeta)d\zeta}{\zeta^2(\zeta-1)^4(\zeta+1)^4} = 2\pi i p_j'(0),$$

γ_0 simple closed curve around z=0 , and

$$(3.6)\quad \int_{\gamma_1} \frac{p_j(\zeta)d\zeta}{\zeta^2(\zeta-1)^4(\zeta+1)^4} = \frac{2\pi i}{3!}\cdot\frac{1}{16}\,[p_j'''(1)-12p_j''(1)+57p_j'(1)-105p_j(1)]$$

γ_1 simple closed curve around z=1.

The polynomials $p_j(\zeta)$, j=1,2,3, obtained from the choice of f and g are:

$$p_1(z) = \bar{a}z^{10}-2\bar{a}b^2z^8+\bar{a}b^4z^6+a\bar{b}^4z^4+2a\bar{b}^2z^2+a$$
$$p_2(z) = i[-\bar{a}z^{10}+2\bar{a}b^2z^8-\bar{a}b^4z^6+a\bar{b}^4z^4-2a\bar{b}^2z^2+a]$$
$$p_3(z) = a\alpha\bar{b}^2[\bar{b}^2z^7-(1+|b|^2)z^5+b^2z^3]$$

and the integrals (3.5) are null for j=1,2,3.

Calling $R_j = p_j'''(1)-12p_j''(1)+57p_j'(1)-105p_j(1)$, j=1,2,3, we have:

$R_1 = -105(a-\bar{a}) + 30(a\bar{b}^2-\bar{a}b^2) + 3(a\bar{b}^4-\bar{a}b^4)$

$R_2 = i[-105(a+\bar{a}) + 30(a\bar{b}^2+\bar{a}b^2) + 3(a\bar{b}^4+\bar{a}b^4)]$

$R_3 = 0.$

Thus, $Re[2\pi i R_j/3!.16] = 0$, $j=1,2$ if

$-105a+30a\bar{b}^2+3a\bar{b}^4 = 0$ or $3b^4+30b^2-105 = 0$

For each b, $\alpha= -1/\bar{b}^2$, $a=i$ satisfy (3.4) and

$$\tilde{X}(p) = Re\int_{p_0}^{p} \frac{f}{2}(1-g^2,\ i(1+g^2), 2g)$$

gives the double surface. $\triangle$

We investigate the problem of the regular minimal surfaces in $\mathbb{R}^3$ of finite total curvature, genus one and three ends, and, we obtain:

<u>Theorem 3.8.</u> - There exists a non-orientable regular, complete minimal surface in $\mathbb{R}^3$ of genus one, three ends and total curvature -14π.

<u>Proof:</u> - We construct the double surface $\tilde{S}$ defined by $\tilde{X}: \mathbb{C}-\{0,1,-1,i,-i\} \longrightarrow \mathbb{R}^3$ with the functions f and g of the Weierstrass'representation (1.10) given by $g(z)=a\bar{b}^4z^3(z^4-b^4)/(\bar{b}^4z^4-1)$ and $f(z)=\alpha(\bar{b}^4z^4-1)^2/z^2(z^2-1)^3(z^2+1)^3$. The functions f and g satisfy (2.3) if and only if

(3.7) $$|a|\ |b|^4 = 1\ ,\ \alpha = \bar{\alpha}\bar{a}^2b^8$$

With some calculations, we have:

$$(3.8)\quad \int_{\gamma_o} \frac{P_j(z)dz}{z^2(z^4-1)^3} = 2\pi i.\left[\frac{P_j(z)}{(z^4-1)^3}\right]'_{z=0} = 2\pi i P_j'(0),$$

Where γ_o is a simple closed curve around $z=0$;

$$(3.9)\quad \int_{\gamma_1} \frac{P_j(z)}{z^2(z^4-1)^3}\, dz = \frac{2\pi i}{2}\left[\frac{P_j(z)}{z^2(z+1)^3(z^2+1)^3}\right]''_{z=1} =$$

$$= \frac{2\pi i}{128}\,[P_j''(1) - 13P_j'(1) + 45P_j(1)],$$

γ_1 : simple closed curve around $z=1$;

$$(3.10)\quad \int_{\gamma_i} \frac{P_j(z)dz}{z^2(z^4+1)^3} = \frac{2\pi i}{2}\left[\frac{P_j(z)}{z^2(z+i)^3(z^2-1)^3}\right]''_{z=i} =$$

$$= \frac{2\pi i}{128}\,.\, i\,[P_j''(i) + 13iP_j'(i) - 45P_j(i)],$$

γ_i simple closed curve around $z=i$.

The polynomials P_j obtained from f and g are, using (3.7),

$$P_1(z) = -\bar{\alpha}z^{14} + 2\bar{\alpha}b^4z^{10} + \bar{\alpha}b^8z^8 - \bar{\alpha}b^8z^6 - 2\alpha\bar{b}^4z^4 + \alpha$$

$$P_2(z) = \bar{\alpha}z^{14} - 2\bar{\alpha}b^4z^{10} + \alpha\bar{b}^8z^8 + \bar{\alpha}b^8z^6 - 2\alpha\bar{b}^4z^4 + \alpha$$

$$P_3(z) = \alpha a\bar{b}^4(\bar{b}^4z^{11} - (1+|b|^8)z^7 + b^4z^3)$$

We can easily verify that the real part of the integrals (3.8) are null for any j, j=1,2,3.

Calling $R_j = P_j''(1) - 13P_j'(1) + 45$, $R_j^i = i[P_j''(i) + 13iP_j'(i) - 45P_j(i)]$

thus,

$R_1 = 45(\alpha-\bar{\alpha}) - 10(\alpha\bar{b}^4 - \bar{\alpha}b^4) - 3(\alpha\bar{b}^8 - \bar{\alpha}b^8)$

$R_2 = i[45(\alpha+\bar{\alpha}) - 10(\alpha\bar{b}^4 + \bar{\alpha}b^4) - 3(\alpha\bar{b}^8 + \bar{\alpha}b^8)]$

$R_3 = \alpha a\bar{b}^4[12(b^4+\bar{b}^4) + 4(1+|b|^8)]$

$R_1^i = -i[45(\alpha+\bar{\alpha}) - 10(\bar{\alpha}b^4 + \alpha\bar{b}^4) - 3(\alpha\bar{b}^8 + \bar{\alpha}b^8)]$

$R_2^i = [45(\alpha-\bar{\alpha}) - 10(\alpha\bar{b}^4 - \bar{\alpha}b^4) - 3(\alpha\bar{b}^8 - \bar{\alpha}b^8)]$

$R_3^i = -\alpha a\bar{b}^4[12(b^4+\bar{b}^4) + 4(1+|b|^8)]$

and the R_j and R_j^i are real if b satisfies:

$$3b^8 + 10b^4 - 45 = 0 \implies b^4 \in \mathbb{R}$$

Choosing $a = 1/b^4$ and $\alpha=1$ the conditions (3.7) are verified. The completeness and regularity can be easily seen. ▲

In $\mathbb{R}^4$, exploring the properties of the functions g_1, g_2 of the representation (1.11), we have

<u>Theorem 3.9</u>: - For any integer m, $m \geq 2$, there exists a non-orientable regular complete minimal surface S in $\mathbb{R}^4$ of genus one, one end and total curvature $-2\pi m$.

<u>Proof:</u> - The double surface $\tilde{S}$ associated to S has total curvature $-4\pi m = -2\pi\tilde{m}$, $\tilde{m} = 2m$, genus zero and two ends. We set the functions f, g_1, g_2 in the representation (1.11) by $f(z) = z^{m-1}$, $g_1(z) = \varepsilon_1/z^{k_1}$, $g_2(z) = \varepsilon_2/z^{k_2}$, with $\tilde{m} = 2m = k_1 + k_2$, k_1, k_2 odd, $k_1, k_2 \geq 1$, and $\tilde{M} = \mathbb{C} - \{0\}$.

The functions $f, g_1 g_2$ are the functions in the representation (1.11) of a complete regular double surface if and only if they satisfy (i) and (ii) (corolary 2.3), that is,

$$|\varepsilon_1| = |\varepsilon_2| = 1 \ , \ \varepsilon_1\varepsilon_2 = (-1)^{m-1} \tag{3.11}$$

To verify that $\mathrm{Re}\int_\gamma \phi(\zeta)d\zeta = \vec{0}$, for any closed curve in $\mathbb{C} - \{0\}$, we observe that:

$$\phi_1(\zeta) = f(1 + g_1 g_2) = z^{m-1} + \varepsilon_1\varepsilon_2/z^{m+1}$$

$$\phi_2(\zeta) = if(1 - g_1 g_2) = i(z^{m-1} - \varepsilon_1\varepsilon_2/z^{m+1})$$

$$\phi_3(\zeta) = f(g_1 - g_2) = (\varepsilon_1 z^{k_2} - \varepsilon_2 z^{k_1})/z^{m+1}$$

$$\phi_4(\zeta) = -if(g_1 + g_2) = -i(\varepsilon_1 z^{k_2} + \varepsilon_2 z^{k_1})/z^{m+1}$$

and rewriting $k_1 = m-n$, $k_2 = m+n$, $n \geq 1$, we have $\int_\gamma \phi_j = 0$, $j = 1, \ldots, 4$; thus, the double surface $\tilde{S}$ is defined by

$$\tilde{X}(p) = \mathrm{Re}\int_{p_o}^{p} \phi(\zeta)d\zeta \ , \quad p_o, p \quad \mathbb{C} - \{0\}.$$

▲

The dimension of a non-orientable regular complete minimal surface in $\mathbb{R}^n$ of finite total curvature has an upper bound $\dim S \leq 2m-2\gamma-r+3$, given by Theorem 2.10, with γ: genus of S, r: number of ends, $c=-2\pi m$ the total curvature. As $\gamma \geqq 1$, $r \geqq 1$, then: $\dim S \leqq 2m$. We will show that this upper bound is sharp for $\gamma=1$, $r=1$. In fact, we have:

<u>Theorem 3.10.</u> - For any integer m, $m \geqq 2$, there exists a non-orientable regular complete minimal surface S in $\mathbb{R}^{2m}$ of genus one, one end and total curvature $-2\pi m$ that lies fully in $\mathbb{R}^{2m}$.

<u>Proof:</u> - The double surface $\tilde{S}$ associated to S, by theorem 2.12 is given by $\tilde{X}: \mathbb{C}-\{0\} \longrightarrow \mathbb{R}^n$, $\tilde{X}(p) = \mathrm{Re}\int_{p_0}^{p} \phi(\zeta)d\zeta$, with $\phi_j(\zeta) = \dfrac{p_j(\zeta)}{\zeta^{m+1}}$, $p_j(\zeta)$ satisfying (i) to (v).

Particularly, condition (v) is equivalent to:

$$(3.12) \qquad (-1)^{m+1}\bar{\zeta}^{2m} p_j(-1/\bar{\zeta}) = \overline{p_j(\zeta)}, \quad 1 \leq j \leq n.$$

Taking $p_j(\zeta) = a_0^j + a_1^j\,\zeta + \dots + a_{2m-1}^j\zeta^{2m-1} + a_{2m}^j\zeta^{2m}$ in

$$(3.12): \quad (-1)^{m+1}(a_0^j\bar{\zeta}^{2m} - a_1^j\bar{\zeta}^{2m-1} + \dots + a_{2m}^j) = (\bar{a}_{2m}^j\bar{\zeta}^{2m} + \bar{a}_{2m-1}^j\bar{\zeta}^{2m-1} + \dots + \bar{a}_0^j)$$

Then, there are two possibilities:

(a) (m+1) even

$$a_0^j = \bar{a}_{2m}^j$$
$$a_1^j = \bar{a}_{2m-1}^j$$
$$\vdots$$
$$a_{m-1}^j = \bar{a}_{m+1}^j$$
$$a_m^j = \bar{a}_m^j$$

(b) (m+1) odd

$$a_0^j = -\bar{a}_{2m}^j$$
$$a_1^j = \bar{a}_{2m-1}^j$$
$$\vdots$$
$$a_{m-1}^j = \overline{a_{m+1}^j}$$
$$a_m^j = -\overline{a_m^j}$$

The integrals $\int_\gamma \phi_j(\zeta)d\zeta$, γ closed curve in $\mathbb{C}-\{0\}$ should have null real part; therefore, $\mathrm{Re}\{2\pi i.a_m^j\} = 0$, $1 \leq j \leq n$, that is $\mathrm{Im}a_m^j = 0$, $1 \leq j \leq n$. Comparing with the last condition in (a) or (b), we have $a_m^j = 0$, $1 \leq j \leq n$.

To determine solutions in $\mathbb{R}^n$, we should find the complex vectors $A_k = (a_k^1, \dots, a_k^n)$, $0 \leq k \leq m-1$ such that the corresponding polynomials satisfy $\sum_{j=1}^n p_j^2(\zeta) \equiv 0$.

It is easy to verify that $A_0 = (1,i,0,\dots,0)$, $A_1 = (0,0,1,i,0,\dots,0) \dots, A_{m-1} = (0,\dots,0,1,i)$ give a solution in $\mathbb{C}^{2m}$ if m is even and $A_0 = (1,i,0,\dots,0)$, $A_2 = (0,0,1,i,0,\dots,0)$, $A_{m-2} = (0,0,\dots,0,\sqrt{2},i\sqrt{2},0,0)$, $A_{m-1} = (0,0,\dots,0,1,i)$ give a solution if m is odd. $\blacktriangle$

For the problem with two ends, we have dim S $\leq$ 2m-1; for m=3 , we obtain

Theorem 3.11: - There does not exist a non-orientable regular complete minimal surface in $\mathbb{R}^3$ of total curvature -6π, with genus one and two ends, but there are examples in $\mathbb{R}^4$ and $\mathbb{R}^5$.

Proof: - The double surface can be defined by $\tilde{X}: \tilde{M} \longrightarrow \mathbb{R}^n$, with $\tilde{M} = \mathbb{C}-\{0,1,-1\}$, $X(p) = \mathrm{Re}\int_{p_0}^{p} \phi(\zeta)d\zeta$, $\phi_j(\zeta) = p_j(\zeta)/\zeta^2(\zeta-1)^2(\zeta+1)^2$

$p_j(\zeta)$ satisfying (i) to (v) in theorem 2.12.

Let $p_j(\zeta) = a_j\zeta^6+b_j\zeta^5+c_j\zeta^4+d_j\zeta^3+e_j\zeta^2+f_j\zeta+g_j$; the condition (v) is equivalent to $a_j=\bar{g}_j$, $b_j=-\bar{f}_j$, $c_j=\bar{e}_j$, $d_j=-d_j$. On the other hand $\sum_{j=1}^{n} p_j^2(\zeta) \equiv 0$ is equivalent to:

$$(3.13)\quad \begin{cases} \sum_j a_j^2=0 \ , \ \sum_j a_jb_j=0 \ , \ \sum_j(b_j^2+2a_jc_j)=0 \ , \ \sum_j(a_jd_j+b_jc_j)=0 \ , \\ \sum_j(c_j^2+2b_jd_j+2a_j\bar{c}_j)=0 \ , \ \sum_j(-a_j\bar{b}_j+b_j\bar{c}_j+c_jd_j)=0 \\ \sum_j(-|d_j|^2+2|a_j|^2+2|b_j|^2+2|c_j|^2) = 0 \end{cases}$$

Calculating:

$$\int_{\gamma_0} \phi_j(\zeta)d\zeta = 2\pi i\, p_j'(0) = -2\pi i\bar{b}_j \ , \ \gamma_0 \text{ simple closed curve around } z=0$$

γ_0 simple closed curve around $z=0$

$$\int_{\gamma_1} \phi_j(\zeta)d\zeta = 2\pi i(p_j'(1)-3p_j(1)/4 = \pi i[3(a_j-\bar{a}_j)+4b_j+(c_j-\bar{c}_j)]/2 \ ,$$

γ_1 simple closed curve around $z=1$.

We have $\mathrm{Re}\int_{\gamma}\phi_j(\zeta)d\zeta = 0$ for any closed curve γ in $\mathbb{C}-\{0,1,-1\}$ if and only if $b_j \quad \mathbb{R}, (c_j-\bar{c}_j) = -3(a_j-\bar{a}_j)$, $1 \le j \le n$.

Let $A = (a_1,\dots,a_n)$, $B = (b_1,\dots,b_n)$, $C = (c_1,\dots,c_n)$, $D = (d_1,\dots,d_n)$. In terms of the real and imaginary parts let $A = A_1+iA_2$, $B=B_1$, $C = C_1-3iA_2$, $D = iD_2$. Using these decompositions, the relations (3.13) can be rewritten as:

$$(3.14)\begin{cases} |A_1| = |A_2| , \quad <A_1,A_2> = 0 \\ <B_1,A_1> = 0 , \quad <B_1,A_2> = 0 \\ |B_1|^2 = 2[-3|A_2|^2 - <A_1,C_1>] \\ <C_1,A_2> = 0 \\ <A_2,D_2> = <B_1,C_1> \\ <A_1,D_2> = 0 \\ |C_1|^2 - 9|A_2|^2 = -2<A_1,C_1>+ 6.|A_2|^2 \\ <C_1,C_2> = - <B_1,D_2> = 0 \\ <B_1,C_1> = -3<A_2,D_2> \\ <C_1,D_2> = 0 \\ 2[2|A_1|^2 + |C_1|^2 + 9|A_2|^2] = |D_2|^2 + 2|B_1|^2 \end{cases}$$

Comparing, in (3.14): $|B_1|^2 = -6|A_2|^2 - 2<A_1,C_1>$, $|C_1|^2 = 15|A_2|^2 - 2<A_1,C_1>$, we have

$$(3.15) \qquad |C_1|^2 = |B_1|^2 + 21|A_2|^2$$

From $<A_2,D_2> = <B_1,C_1> = -3<A_2,D_2>$,

$$(3.16) \qquad <A_2,D_2> = <B_1,C_1> = 0$$

and, from last equation of (3.14),

$$(3.17) \qquad |D_2|^2 = 64|A_2|^2 \quad ;$$

sumarizing (3.14) to (3.17) we obtain

$$\begin{cases} |A_1| = |A_2| \quad , \quad <A_1,A_2> = 0 \\ <B_1,A_1> = <B_1,A_2> = 0 \\ |B_1|^2 = -6|A_1|^2 - 2<A_1,C_1> \\ <C_1,A_2> = <C_1,B_1> = <C_1,D_2> = 0 \\ |C_1|^2 = |B_1|^2 + 21|A_2|^2 \\ <D_2,A_1> = <D_2,A_2> = <D_2,B_1> = 0 \\ |D_2| = 8|A_2| \end{cases}$$

The vectors B_1 and D_2 are orthogonal to the subspace generated by $\{A_1,A_2\}$; if we are looking for solutions in $\mathbb{R}^3$, then $B_1 = \lambda D_2$ $(D_2 \neq \vec{0})$ and $<B_1,D_2> = 0$ implies $B_1 = \vec{0}$. Then, $<C_1,A_1> = -3|A_2|^2$, $<C_1,A_2> = <C_1,D_2> = 0$, $|C_1|^2 = 21|A_2|^2$ and $C_1 = \alpha A_1$, with $\alpha=-3$; this is incompatible with $|C_1|^2 = 21|A_2|^2$. Therefore, there does not exist example in $\mathbb{R}^3$.

Em $\mathbb{C}^4$ we have the solution $A = (1,i,0,0)$, $B = \vec{0}$, $C = (-3,-3i,0,2\sqrt{3})$, $D = (0,0,8i,0)$ that gives a double surface in $\mathbb{R}^4$.

Em $\mathbb{C}^5$, $A = (1,i,0,0,0)$, $B = (0,0,\sqrt{2},0,0)$, $C = (-4,-3i,0,0,\sqrt{7})$, $D = (0,0,0,8i,0)$ are solutions that give a double surface in $\mathbb{R}^5$. ▲

We have already prove that there exists a non-orientable minimal surface in $\mathbb{R}^3$ of genus one, two ends and total curvature -10π. Using the some technique in the last theorem, we have

<u>Theorem 3.12.</u> - There exists a complete minimal immersion of the projective plane punctured at two points in $\mathbb{R}^9$ with total curvature -10π.

<u>Proof:</u> - By theorem 2.12, the double surface is given by

$$\tilde{X}(p) = \mathrm{Re}\int_{p_o}^{p} \phi(\zeta)d\zeta \ , \text{ where } \phi_j(\zeta) = \frac{p_j(\zeta)}{\zeta^3(\zeta-1)^3(\zeta+1)^3} \ , \text{ if}$$

we choose the order of the poles of $\phi(\zeta)d\zeta$ at the ends to be 3.

The polynomials $p_j(\zeta)$ satisfy the condition (v) in theo. 2.12 if and only if $p_j(\zeta) = a_j\zeta^{10}+b_j\zeta^9+c_j\zeta^8+d_j\zeta^7+e_j\zeta^6+f_j\zeta^5+e_j\zeta^4 -\bar{d}_j\zeta^3+\bar{c}_j\zeta^2-\bar{b}_j\zeta+\bar{a}_j$, $\mathrm{Re}\, f_j = 0$

Calculating: $\int_{\gamma_o}\phi_j(\zeta)d\zeta=-2\pi i(3\bar{a}_j+\bar{c}_j)$, γ_o simple closed curve around $z=0$; $\int_{\gamma_1}\phi_j(\zeta)d\zeta=\pi i[24(a_j+\bar{a}_j)+15(b_j-\bar{b}_j)+8(c_j+\bar{c}_j)+3(d_j-\bar{d}_j)-f_j]/8$, γ_1: simple closed curve around $z=1$. Then, the integrals have null real part if and only if

$$\begin{cases} c_j+\bar{c}_j = -3(a_j+\bar{a}_j) \\ f_j = 15(b_j-\bar{b}_j)+3(d_j-\bar{d}_j) \end{cases}$$

In terms of the real and imaginary parts, we write:

$A = (a_1,\dots,a_n) = A_1+iA_2 = (\alpha, i\alpha, 0,\dots,0)$

$B = (b_1,\dots,b_n) = B_1+iB_2 = (0,0,\beta_1, i\beta_2,\dots,0)$

$C = (c_1,\dots,c_n) = C_1-3iA_2 = (0,-3i\alpha,0,0,\gamma,0,\dots,0)$

$D = (d_1,\dots,d_n) = D_1+iD_2 = (0,0,0,0,0,\delta_1, i\delta_2,0,0)$

$E = (e_1,\dots,e_n) = E_1+iE_2 = (0,\dots,0,0,\mu_1, i\mu_2)$

$F = (f_1,\dots,f_n) = i(30B_2+6D_2) = (0,0,0,30i\beta_2,0,6i\delta_2,0,0)$

and A,B,C,D,E e F satisfy the condition $\sum_j p_j^2 \equiv 0$, if and only if

(1) $\sum a_j^2 = 0$

(2) $\sum a_j b_j = 0$

(3) $\sum (b_j^2+2a_j c_j) = 0$

(4) $\sum (a_j d_j + b_j c_j) = 0$

(5) $\sum (c_j^2+2a_j e_j+2b_j d_j) = 0$

(6) $\sum (a_j f_j + b_j e_j + c_j d_j) = 0$

(7) $\sum (d_j^2+2a_j\bar{e}_j+2b_j f_j+2c_j e_j) = 0$

(8) $\sum (-a_j\bar{d}_j+b_j\bar{e}_j+c_j f_j+d_j e_j) = 0$

(9) $\sum (e_j^2+2a_j\bar{c}_j-2b_j\bar{d}_j+2c_j\bar{e}_j+2d_j f_j) = 0$

(10) $\sum (-a_j\bar{b}_j+b_j\bar{c}_j-c_j\bar{d}_j+d_j\bar{e}_j-e_j f_j) = 0$

(11) $\sum (2|a_j|^2-2|b_j|^2+2|c_j|^2 -2|d_j|^2+2|e_j|^2-|f_j|^2) = 0$

i.e.

$$6\alpha^2 = \beta_2^2-\beta_1^2$$

$$\gamma^2 = 9\alpha^2$$

$$\delta_1^2 = \delta_2^2+60\beta_2^2$$

$$\mu_1^2 = \mu_2^2+6\alpha^2+12\delta_2^2$$

$$11\alpha^2+\gamma^2+\mu_1^2+\mu_2^2 = \beta_1^2+451\beta_2^2+\delta_1^2+19\delta_2^2$$

Choosing $\alpha=1$, $\beta_1=\sqrt{10}$, $\delta_2=2$, $\mu_2=2^7$, $\beta_2=4$, $\gamma=3$, $\delta_1=\sqrt{964}$, $\mu_1=\sqrt{5050}$, we have a solution in $\mathbb{R}^9$. ▲

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