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Bounds for Distorted Risk Measures

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Abstract

The aim of this paper is to provide bounds for distorted risk measures when the risk's joint distribution is unspecified but the marginals are known. We will focus on convex distortion functions. A methodology to compute the corresponding bounds is suggested and illustrated by several examples.

MSC: IE43, IM11, IM12

Keywords: algorithm; bounds; copulas; distortion risk measures.

1 Introduction

There are several ways to measure risk. One of the first attempts to provide a coefficient summarizing the risk of a portfolio was based on the expected loss incurred. An economic agent basing his decisions on such measure was called *risk neutral*. The truth is that, in general, people are risk averse and as a consequence, any measure should capture this feature. In accordance to the theory of utilitarianism, this could be done by using a measure based on expected utility. Following this theory, von Neumann and Morgenstern [1] gave the first axiomatic construction of a risk measure. Since then, several principles to compute such measures have been introduced, trying to capture the main aspects of a particular decision-making.

One important class of risk measures is the so-called *distorted risk measure*, $D_g(X)$, defined as follows:

$$D_g(X) = \int_0^\infty g(1 - F(t))dt \quad (1)$$

where X is a continuous random variable with given distribution $F(t) = P(X \leq t)$, (see Wang [2]). The function g is called *distortion function*. It is nondecreasing on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$.

The distorted risk measure can also be defined for negative X by

$$D_g(X) = \int_{-\infty}^0 [g(1 - F(t)) - 1]dt + \int_0^\infty g(1 - F(t))dt$$

(see Reference [3]).

Usually, one is interested in computing $D_g(\cdot)$ for a given function $\Psi(\cdot)$ of interest. Various expressions for $\Psi(\cdot)$ can be used, depending on the situation. The following examples, see Embrechts *et al.* [4], are typical:

- $\Psi(x_1, \dots, x_n) = x_1 + \dots + x_n$. In an insurance situation, one is often interested in the sum of random variables, which represents the aggregated claim of a portfolio in a given period;
- $\Psi(x_1, \dots, x_n) = \sum_{i=1}^n (x_i - k)^+$, $k > 0$, where $a^+ = \max(a, 0)$. This case corresponds to the functional underlying an excess-of-loss treaty in reinsurance for a loss greater than k . The x_i 's could be individual claims or reinsurance losses due to different lines of business;
- $\Psi(x_1, \dots, x_n) = (\sum_{i=1}^n x_i - k)^+$, $k > 0$. This form has an interpretation in derivatives (e.g. Asian options or "stop-loss" reinsurance), for example.

To calculate $D_g(\Psi)$ it is necessary to know the risk's joint distribution which is usually unknown. If we work under the absence of independence, we cannot obtain the joint distribution using only the marginals. Nevertheless, the knowledge of marginals allows us to compute bounds for $D_g(\Psi)$. The decision-making will depend on a limited amount of information. For instance, if one bases his decision on the risk measure's upper bound, then it will have a conservative characteristic following the worst possible scenario.

We will compute bounds for $D_g(\Psi)$ using the approach suggested by Embrechts *et al.* [4] to compute bounds for cumulative distributions. It will be clear that such a methodology is best suited when the distortion function is convex or if the function $\Psi(\cdot)$ is different than the sums given above.

In the next section we briefly present the theory behind distorted risk measures and discuss several of them. In Section 3 we give bounds for typical risk measures, when the risk's joint distribution is unknown, but the marginals are available. Section 4 presents an algorithm and several numerical examples.

2 Yaari's theory

Let X be a non-negative random variable. When $g(\cdot)$ is an increasing function, (1) can be rewritten as follows

$$D_g(X) = \int_0^\infty g(1 - F(t))dt = \int_0^1 F^{-1}(1 - q)dg(q) \quad (2)$$

The class of risk measure defined by (1) was introduced by Yaari [5] who requires independence of direct mixing of risky prospects, instead of the usual expected utility theory acceptance of independence with respect to probability mixtures of risky prospects. In the "Dual Theory of Choice Under Risk" according to [5], the distortion function $g(\cdot)$ applies a principle in the computation of the associated risk measure, and it only depends on the distribution of the risk (i.e., of X). Such class

of measures preserves several desirable economic properties. For instance, if the survival function of a risk X is smaller than the one of Y for each real number, then $D_g(X) \leq D_g(Y)$. This is an important characteristic since it is obvious that Y is riskier than X . The last property tell us that such risk measures preserve the first order stochastic dominance.

Both Value-at-Risk and Tail-Value-at-Risk at level $p \in (0, 1)$ have a representation via distortion function: $g(x) = I(x > 1 - p)$ ($I(\cdot)$ is the indicator function) and $g(x) = \min(\frac{x}{1-p}, 1)$, respectively. Next we comment on two useful distorted risk measures:

The Power Transformation. This risk measure is also known as *Proportional hazard transformation* (see Reference [6]). The distortion function associated to it is $g_p(x) = x^p$, $p \in (0, 1)$. When using this particular function, one incorporates in the premium principle the risk aversion of the economic agent, characterized by the parameter p . The greater its value, the less averse to risk is the economic agent. Therefore, the risk premium demanded will be low for large values of p .

The Gini's Transformation. The name of this risk measure is due to the fact that the Gini's coefficient (a common index for measuring wealthy distribution in populations) is used to compute the risk premium based on this transformation. Its corresponding distortion function is $g_p(x) = (1 + p)x - px^2$, $p \in (0, 1)$. The premium principle behind this measure is the average absolute deviation from median. According to Denneberg [7], to determine the safety loading for insurance premiums, this principle is recommendable. The parameter p gives the mark-up of the insurer in the tail of the distribution. Since $p \in (0, 1)$, it will never exceed 100%.

There are many other distortion functions that may be useful in a given scenario, see Section 3.2 and References [8, 9].

3 Bounds for distribution functions

In this section we obtain bounds for risk measures associated to various distortion functions considered by Wang [2], who gives insurance context interpretations.

3.1 Bounds for VaR and TVaR

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with known marginals $F_{X_i}(x_i)$, $i=1, \dots, n$ defined on $\mathfrak{R} = (-\infty, \infty)$. Let $\Psi : \mathfrak{R}^n \rightarrow \mathfrak{R}$ be an increasing and right-continuous in the last argument function for which we want to compute the risk measure and Ψ_{n-1} is the function Ψ with the first $n - 1$ arguments held fixed. Define the related copula (see Reference [10], for example) by

$$C(u_1, \dots, u_n) = P(U_1 \leq u_1, \dots, U_n \leq u_n)$$

and the corresponding dual copula by

$$C^d(u_1, \dots, u_n) = P(\{U_1 \leq u_1\} \cup \dots \cup \{U_n \leq u_n\})$$

where U_i is uniformly distributed over $(0, 1)$, $i = 1, 2, \dots, n$.

Define the following quantities:

$$\begin{aligned}\tau_{C,\Psi}(s) &= \sup_{x_1, \dots, x_{n-1} \in \mathfrak{R}} C(F_{X_1}(x_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{X_n}(\Psi_{n-1}^{-1}(s))) \\ \sigma_{C,\Psi}(s) &= \int_{\{\Psi(x_1, \dots, x_n) \leq s\}} dC(F_{X_1}(x_1), \dots, F_{X_n}(x_n))\end{aligned}$$

and

$$\rho_{C,\Psi}(s) = \inf_{x_1, \dots, x_{n-1} \in \mathfrak{R}} C^d(F_{X_1}(x_1), \dots, F_{X_{n-1}}(x_{n-1}), F_{X_n}(\Psi_{n-1}^{-1}(s)))$$

where $\Psi_{n-1}^{-1}(s) = \sup\{x \in \mathfrak{R} | \Psi_{n-1}(x) \leq s\}$, $s \in [0, \infty]$, is the generalized inverse of $\Psi_{n-1}(x_1, \dots, x_n)$. Note that the expression $\sigma_{C,\Psi}(s)$ is, in fact, the distribution of Ψ .

Under the above notations the following statement is valid.

Theorem 1 (Embrechts *et al.* [4]).¹ *If the copula $C(\cdot)$ associated to the random vector \mathbf{X} satisfies $C \geq C_0$ and $C^d \leq C_1^d$ for a given pair of n -dimensional copulas C_0 and C_1 , then*

$$\tau_{C_0,\Psi}(s) \leq \sigma_{C,\Psi}(s) \leq \rho_{C_1,\Psi}(s) \quad (3)$$

Besides, the limits are sharp, i.e., for each $s \in [0, \infty]$ there exists a copula $C(\cdot)$ such that the lower bound is attained (and the same holds for the upper bound).

Theorem 1 is used by Embrechts *et al.* [4] to compute bounds of the VaR when the marginal distribution of each risk is known, but the risk manager does not have an indication about their joint distribution. Observe that in the conditions of Theorem 1 it is supposed that a partial information of the copula $C(\cdot)$ is available (since C_0 and C_1^d are given). Here we will work under this hypothesis.

The partial information is a natural assumption if one, starting with marginals, chooses the right copula in order to describe the existing dependence structure. If no information is available at all for C_0 and C_1^d (which is very common in practice), then in the two dimensional case we can use C_0 as Fréchet's lower bound and $C_1^d = C_0^d = \min\{x_1 + x_2, 1\}$ as an upper one. We will refer to such a situation as a "non-informative". For a multidimensional random vector this is not the case, since Fréchet's inferior limit is not always a valid distribution (see Reference [10]).

By applying (2) in the non-informative setting, we may fail to obtain "sharp" bounds. In Embrechts and Pucetti [12] conditions to get a valid distribution from Fréchet's lower bound are given, and they are very restrictive. If $g(\cdot)$ is concave and $\Psi(\cdot)$ is the sum of the risks one could use the bounds given by Dhaene *et al.* [13], in a non-informative situation. Another possibility is to apply the methodology presented by Embrechts and Pucetti [12] which is complicated and it is hard to obtain numerical solutions and even harder to get analytical ones. Only a few expressions for the bounds are known in this case.

If we are unsuccessful in applying any other approach for calculating upper and lower bounds under absence of information, we can still use the inequalities (3). We

¹The original proof of this theorem is wrong. The correction is given in [11].

will not be able to obtain sharp bounds, but at least an idea about the lower and upper bounds will be available.

It is worth stressing that it is very hard to obtain analytical solutions to the bounds given by (3) and in most cases we will have to rely on numerical methods to compute them. Such a methodology is presented by Williamson and Downs [14]. Their method is based on the calculation of the inverses $\rho_{C_1, \Psi}^{-1}(q)$ and $\tau_{C_0, \Psi}^{-1}(q)$ in the bidimensional setting, $q \in [0, 1]$. A more generic approach is suggested by Embrechts *et al.* [4], where the multidimensional case is treated.

By applying theorem 1 and the definitions of VaR and TVaR we can easily obtain the following bounds for each $p \in (0, 1)$:

$$\rho_{C_1, \Psi}^{-1}(p) \leq VaR_p(\Psi) \leq \tau_{C_0, \Psi}^{-1}(p)$$

and

$$\frac{1}{1-p} \int_p^1 \rho_{C_1, \Psi}^{-1}(q) dq \leq TVaR_p(\Psi) \leq \frac{1}{1-p} \int_p^1 \tau_{C_0, \Psi}^{-1}(q) dq$$

3.2 Bounds for various quantile-based measures

Here we apply (3) for obtaining bounds for various types of distortion functions.

For concave distortion functions one can compute upper bounds for $D_g(\Psi)$ whenever Ψ is the sum of the risks or a function like *stop-loss* sum using *stochastic ordering* properties (see Reference [15]). For two risks it is also possible to obtain lower bounds. For a portfolio of two risks the stochastic ordering approach is based on the fact that if a bivariate random vector $\mathbf{X} = (X_1, X_2)$ is riskier than another one $\mathbf{Y} = (Y_1, Y_2)$, i.e., $P(X_1 > s, X_2 > t) \geq P(Y_1 > s, Y_2 > t)$ for all t , than the sum $X_1 + X_2$ is greater than $Y_1 + Y_2$ in *stop-loss* order, see Reference [16]. Since for *concave* distortion functions $D_g(\cdot)$ preserves the *stop-loss* order, one may obtain upper and lower bounds for $D_g(\Psi)$, if Ψ is the sum of the risks. One can extend the result for functions like *stop-loss* sums and for multivariate risks.

If it is possible to compute the bounds following the approach proposed by Goncalves *et al.* [15], and it should be preferred, because the methodology suggested provides sharp bounds and the computations are easier. As a consequence, the following method should be applied in cases where stochastic ordering relations are no longer useful (for instance, in the case of convex distortion function).

We will consider in the next the following measures, displaying the related distortion functions.

- Power Transformation ($PT_p(X)$): $g_p(x) = x^p$
- Square Root Transformation ($SR_p(X)$): $g_p(x) = \frac{\sqrt{1-x \ln p} - 1}{\sqrt{1-\ln p} - 1}$
- Logarithm Transformation ($LT_p(X)$): $g_p(x) = \frac{\ln(1-x \ln p)}{\ln(1-\ln p)}$
- Gini's Transformation ($GT_p(X)$): $g_p(x) = (1+p)x - px^2$

- Dual Power Transformation ($DPT_p(X)$): $g_p(x) = 1 - (1 - x)^{\frac{1}{p}}$
- Exponential Transformation ($ET_p(X)$): $g_p(x) = \frac{1-p^x}{1-p}$

Observation. The possible values of p in the above relations lie between 0 and 1. However, one can always modify the expression of distortion functions considered (if necessary to include values of p outside $(0, 1)$). In such case, it must be taken into account that $g(\cdot)$ may never be negative.

Using (3) for the transformations introduced we obtain the following inequalities:

$$\int_0^\infty [1 - \rho_{C_1, \Psi}(t)]^p dt \leq PT_p(\Psi) \leq \int_0^\infty [1 - \tau_{C_0, \Psi}(t)]^p dt$$

$$\int_0^\infty \frac{\sqrt{1 - (1 - \rho_{C_1, \Psi}(t)) \ln p}}{3(\sqrt{1 - \ln p} - 1) \ln p} dt \leq SR_p(\Psi) \leq \int_0^\infty \frac{\sqrt{1 - (1 - \tau_{C_0, \Psi}(t)) \ln p}}{3(\sqrt{1 - \ln p} - 1) \ln p} dt$$

$$\int_0^\infty \frac{\ln(1 - (1 - \rho_{C_1, \Psi}(t)) \ln p)}{\ln(1 - \ln p)} dt \leq LT_p(\Psi) \leq \int_0^\infty \frac{\ln(1 - (1 - \tau_{C_0, \Psi}(t)) \ln p)}{\ln(1 - \ln p)} dt$$

$$\int_0^\infty \{(1 + p)(1 - \rho_{C_1, \Psi}(t)) - p[1 - \tau_{C_0, \Psi}(t)]^2\}^+ dt \leq GT_p(\Psi) \leq$$

$$\int_0^\infty \{(1 + p)(1 - \tau_{C_0, \Psi}(t)) - p[1 - \rho_{C_1, \Psi}(t)]^2\} dt$$

$$\int_0^\infty (1 - [\rho_{C_1, \Psi}(t)]^{\frac{1}{p}}) dt \leq DPT_p(\Psi) \leq \int_0^\infty (1 - [\tau_{C_0, \Psi}(t)]^{\frac{1}{p}}) dt$$

and

$$\int_0^\infty \frac{1 - p^{[1 - \rho_{C_1, \Psi}(t)]}}{1 - p} dt \leq ET_p(\Psi) \leq \int_0^\infty \frac{1 - p^{[1 - \tau_{C_0, \Psi}(t)]}}{1 - p} dt$$

Example 1. Let X_1 and X_2 be uniformly distributed on $(0, 1)$ and $\Psi(x_1, x_2) = X_1 + X_2$. If $C_0 = \max(x_1 + x_2 - 1, 0)$ and $C_1^d = \min(x_1 + x_2, 1)$, it is possible to obtain explicit analytical bounds for the risk measures listed. By using relations (11), (33) and (34) of Williamson and Downs [14], we obtain

$$\frac{1}{p+1} \leq PT_p(X_1 + X_2) \leq \frac{p+2}{p+1}$$

$$\frac{2 - 3 \ln p - 2[1 - \ln p]^{\frac{3}{2}}}{3[\sqrt{1 - \ln p} - 1] \ln p} \leq SR_p(X_1 + X_2) \leq 1 + \frac{2 - 3 \ln p - 2[1 - \ln p]^{\frac{3}{2}}}{3[\sqrt{1 - \ln p} - 1] \ln p}$$

$$\frac{\ln p - (1 - \ln p) \ln(1 - \ln p)}{\ln[1 - \ln p] \ln p} \leq LT_p(X_1 + X_2) \leq 1 + \frac{\ln p - (1 - \ln p) \ln(1 - \ln p)}{\ln[1 - \ln p] \ln p}$$

$$\frac{2-p}{4} \leq GT_p(X_1 + X_2) \leq \frac{5p+7}{4}$$

and

$$\frac{1}{p+1} \leq DPT_p(X_1 + X_2) \leq \frac{p+2}{p+1}$$

3.3 Bounds computation

Seldom one can obtain the bound's exact expression. In order to use the methodology proposed in [14] we must express the bounds as a function of the inverses $\rho_{C_1, \Psi}^{-1}(p)$ and $\tau_{C_0, \Psi}^{-1}(p)$. To make it possible, we will use relation (2) for the risk measure associated to an increasing distortion function $g(\cdot)$.

Now, proceeding in the same way we did in the previous subsection, we obtain the bounds:

$$\begin{aligned} \int_0^1 [\rho_{C_1, \Psi}^{-1}(1-q)] p q^{p-1} dq &\leq PT_p(\Psi) \leq \int_0^1 [\tau_{C_0, \Psi}^{-1}(1-q)] p q^{p-1} dq \\ &- \int_0^1 \frac{\rho_{C_1, \Psi}^{-1}(1-q) [(1-\ln p)^{-\frac{1}{2}}] \ln p}{2(\sqrt{1-\ln p} - 1)} dq \leq SR_p(\Psi) \leq \\ &- \int_0^1 \frac{\rho_{C_1, \Psi}^{-1}(1-q) [(1-\ln p)^{-\frac{1}{2}}] \ln p}{2(\sqrt{1-\ln p} - 1)} dq \\ \int_0^1 \frac{-\rho_{C_1, \Psi}^{-1}(1-q) \ln p}{(1-q \ln p)(1-\ln p)} dq &\leq LT_p(\Psi) \leq \int_0^1 \frac{-\tau_{C_0, \Psi}^{-1}(1-q) \ln p}{(1-q \ln p)(1-\ln p)} dq \\ \int_0^1 [(1+p) - 2pq] \rho_{C_1, \Psi}^{-1}(1-q) dq &\leq GT_p(\Psi) \leq \int_0^1 [(1+p) - 2pq] \tau_{C_0, \Psi}^{-1}(1-q) dq \end{aligned}$$

and

$$\int_0^1 \frac{\rho_{C_1, \Psi}^{-1}(1-q)(1-q)^{\frac{1-p}{p}}}{p} dq \leq DPT_p(\Psi) \leq \int_0^1 \frac{\tau_{C_0, \Psi}^{-1}(1-q)(1-q)^{\frac{1-p}{p}}}{p} dq$$

As we noted before, if $g(\cdot)$ is concave and $\Psi(\cdot)$ is the sum of the risks, one could use the bounds presented in Dhaene *et al.* [13] under absence of additional information. In fact, such bounds should be preferred, since they are sharp and demand easier computations. Hence, the approach illustrated in the next section should be used for convex distortion functions and (or) functions $\Psi(\cdot)$ which are different than the sum-like expressions.

4 A numerical method and examples

In order to compute the bounds numerically, one has to restrict himself to piecewise constant values of $\rho_{C_1, \Psi}^{-1}(p)$ and $\tau_{C_0, \Psi}^{-1}(p)$ (see Reference [14]). Then, we can state

a numerical method as follows. Consider the expression $\rho_{C_1, \Psi}^{-1}(p)$ and $\tau_{C_0, \Psi}^{-1}(p)$ for increasing $g(\cdot)$ in (2). We could rewrite (2) as

$$D_g(X) = - \int_0^1 F^{-1}(u) dg(1-u) \quad (4)$$

Now, define $p_i = \frac{i}{N}$, with $i = 1, 2, \dots, N-1$, where N denotes the number of points dividing the interval $[0, 1]$. Then, using (4) we obtain the following upper bound for $D_g(\Psi)$

$$D_g(\Psi) \leq - \sum_{i=0}^{N-1} \rho_{C_1, \Psi}^{-1}(p_{i+1}) \int_{p_i}^{p_{i+1}} dg(1-u) = \sum_{i=1}^{N-1} \rho_{C_1, \Psi}^{-1}(p_{i+1}) [g(1-p_i) - g(1-p_{i+1})]$$

One can proceed analogously to compute the lower bound, given by

$$D_g(\Psi) \geq - \sum_{i=0}^{N-1} \tau_{C_0, \Psi}^{-1}(p_i) \int_{p_i}^{p_{i+1}} dg(1-u) = \sum_{i=0}^{N-1} \tau_{C_0, \Psi}^{-1}(p_i) [g(1-p_i) - g(1-p_{i+1})]$$

In summary, for a increasing function $g(\cdot)$ one can compute bounds for $D_g(\Psi)$ numerically using the inequalities

$$\sum_{i=0}^{N-1} \tau_{C_0, \Psi}^{-1}(p_i) [g(1-p_i) - g(1-p_{i+1})] \leq D_g(\Psi) \leq \sum_{i=1}^{N-1} \rho_{C_1, \Psi}^{-1}(p_{i+1}) [g(1-p_i) - g(1-p_{i+1})] \quad (5)$$

In the following table are given the steps of the algorithm we used.

Algorithm 1 Lower and Upper Bounds of $D_g(\Psi)$

Upper Bound := 0

Lower Bound := 0

for $i = 0$ to $N - 1$ **do**

$p_i := i/N$

 Upper Bound := Upper Bound + $\rho_{C_1, \Psi}^{-1}(p_{i+1}) [g(1-p_i) - g(1-p_{i+1})]$

 Lower Bound := Lower Bound + $\tau_{C_0, \Psi}^{-1}(p_i) [g(1-p_i) - g(1-p_{i+1})]$

end for

Note that the quantities $\rho_{C_1, \Psi}^{-1}(p_{i+1})$ and $\tau_{C_0, \Psi}^{-1}(p_i)$ can be thought as weights of the differences $[g(1-p_i) - g(1-p_{i+1})]$ in (5) for $i = 1, 2, \dots, n-1$.

For unbounded random variables X_i , $i = 1, 2, \dots, n$ one can obtain that $\rho_{C_1, \Psi}^{-1}(1) = \infty$. This can cause problems since in this case the upper bound will be ∞ . A possible solution is to define a new random variable Y_i such that

$$Y_i = \begin{cases} X_i & \text{if } X_i \leq M \\ M & \text{if } X_i > M \end{cases} \quad (6)$$

for an arbitrarily large constant M . Such a procedure makes $F_{Y_i}^{-1}(1) = M$ for the new collection of random variables Y_1, \dots, Y_n . A question that might arise is: “Do we lose generality by substituting the X_i ’s by Y_i ’s?”. In most cases, the answer is “no”, because, in general, the claims of insurance companies are bounded. For instance, for a portfolio of car insurances, the insurer will not lose more than the total value of the cars. Even if it is not the case, we can choose M large enough to make the probability

$$P(\{Y_1 \leq x_1, \dots, Y_n \leq x_n\} \neq \{X_1 \leq x_1, \dots, X_n \leq x_n\}), \quad x_i \in \mathfrak{R}, \quad i = 1, \dots, n$$

close to zero, as stated in the following proposition.

Proposition 1. *Let \mathbf{X} be a n -dimensional random vector and \mathbf{Y} be a n -dimensional random vector with elements defined by (6). Then, for each $\epsilon > 0$ it can be found a constant M such that*

$$P(\{Y_1 \leq x_1, \dots, Y_n \leq x_n\} \neq \{X_1 \leq x_1, \dots, X_n \leq x_n\}) \leq \epsilon$$

and the effect of M on $D_g(\Psi(\mathbf{Y}))$ is arbitrarily small.

Proof. Consider the event $A_i = \{X_i > M\}$ for $i = 1, \dots, n$. Obviously, $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$. Since $P(A_i)$ approaches zero as $M \rightarrow \infty$ (it follows from the properties of the cumulative distribution functions) we can select a constant M such that for each i any given $\epsilon > 0$, we have $P(A_i) \leq \frac{\epsilon}{n}$, i.e.

$$P(\{Y_1 \leq x_1, \dots, Y_n \leq x_n\} \neq \{X_1 \leq x_1, \dots, X_n \leq x_n\}) \leq \epsilon$$

By choosing appropriate number of points N , dividing $[0, 1]$ we can make the effect of M arbitrarily small on $D_g(\Psi(Y_1, \dots, Y_n))$. This might be done by selecting an N so that $p_{N-1} \leq P(X_i \leq M) < 1$ for each i . Since $g(x)$ is increasing with $g(0) = 0$, we can choose N so that the last terms $[g(1 - p_i) - g(1 - p_{i+1})]$ in the sums (5) are made arbitrarily small (the last terms are the ones multiplied by M). Hence, the influence of M can be neglected. We shall refer to this risk measure as the *transformed risk measure*.

In fact, the statement of Proposition 1 says that it is always possible to choose a random vector \mathbf{Y} depending on a constant M such that the joint distribution of $\{Y_1, \dots, Y_n\}$ is arbitrarily close to that of $\{X_1, \dots, X_n\}$ (which means that distributions of \mathbf{X} and \mathbf{Y} can be taken the same for practical purposes) and with M having an arbitrarily small influence on the distorted risk measure of $\Psi(\mathbf{Y})$.

Now, to evaluate the bounds, we just have to select the right value of N and use the approach suggested by Williamson and Downs [14] to compute $\rho_{C_1, \Psi}^{-1}(p)$ and $\tau_{C_0, \Psi}^{-1}(p)$. Even when it is possible to obtain analytical expression for these quantities, it is profitable to calculate the bounds numerically, due to the difficulties involved in obtaining the exact expressions for $\rho_{C_1, \Psi}^{-1}(p)$ and $\tau_{C_0, \Psi}^{-1}(p)$.

We finish with two numerical examples. In both we assume that $N = 20$, $\Psi(X_1, X_2) = X_1 + X_2$ where X_1 and X_2 are exponentially distributed random variables with parameter 1. Additionally, C_0 is Fréchet’s lower bound and therefore, $C_1^d = C_0^d = \min(x_1 + x_2, 1)$, i.e., we are in the non-informative case scenario.

Example 2. Consider the convex distortion function given by $g(x) = x^2$. Denote the associated risk measure by PT_2 .² Since $g(x)$ is convex on $[0, 1]$ we cannot use the property that distorted risk measures preserve second stochastic order dominance (see reference [16]). So, we shall use the bounds based on Embrechts *et al.* [4] approach to compute limits for cumulative distribution functions. Using (5) we obtain the following bounds for the transformed PT_2

$$0, 197 \leq PT_2(Y_1 + Y_2) \leq 1, 274 + 0, 125M$$

where we defined the Y_i 's as in (6). Since M is arbitrary, we can choose for instance $M = 10$ to obtain $0, 197 \leq PT_2(Y_1 + Y_2) \leq 2, 524$ and in this case the probability of observing an event $\{Y_1, Y_2\} \neq \{X_1, X_2\}$ is smaller than 0.000091, according to Proposition 1.

Example 3. Let $g(x) = \frac{e^{0.5x}-1}{e^{0.5}-1}$. This function is convex for $p = e^{0.5}$ and by applying (5) to compute bounds for the associated transformed risk measure $ET_{e^{0.5}}$, we get:

$$0, 429 \leq ET_{e^{0.5}}(\Psi(Y_1 + Y_2)) \leq 1, 265 + 0, 118M$$

Letting $M = 10$ it comes out $0, 429 \leq ET_{e^{0.5}}(\Psi(Y_1 + Y_2)) \leq 2, 445$. Analogously, the probability of observing an event $\{Y_1, Y_2\} \neq \{X_1, X_2\}$ is smaller than 0,000079.

5 Conclusions

In this paper we present an approach to compute bounds for distorted risk measures, when for some reason one cannot take advantage of second order stochastic dominance, see References [13, 16]. This is the case when the distortion function $g(\cdot)$ is convex. In such situation, the risk measure may not preserve second order stochastic dominance and the approach based on stochastic ordering to compute bounds will not be useful. The numerical computations involved here are relatively simple in most cases, which makes worthwhile to use the methodology proposed even when we can obtain analytical relations for the bounds. One can compare the bounds obtained in the present study with those in [15], but based on stochastic ordering approach.

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²In Section 3.2 we have assumed that $p \in (0, 1)$ for all the distortion functions listed. But for the Power Transformation, p can be taken to be any number greater than 0. When $p > 1$, in particular for $p = 2$, $PT_2(\cdot)$ will be convex, see the corresponding Observation also.

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