RT-MAE 2001-12

DEPENDENCE STRUCTURES AND A POSTERIORI DISTRIBUTIONS

by

Veronica A. Gonzalez-Lopez
and
Nelson I. Tanaka

Palavras-Chave: Bayesian Association, Dependencies, Negative and Positive, TP2 and RR2 Associations, Positivity.
Classificação AMS: 62C10, 62E10, 60E05.
(AMS Classification)

- Julho de 2001 -

Dependence structures and a posteriori distributions

Veronica A. Gonzalez-Lopez and Nelson I. Tanaka

IME-USP-Estatística
Caixa Postal 66 281
05315-970 São Paulo, SP, Brazil

Abstract:

Initially we extend some relationships known to hold among positive dependence concepts to the case of negative dependence. Secondly, we study situations when it is possible to transform a mixed - positive and negative - structure to an entirely positive one. Ultimately, here, we search when mixed dependent structures can share properties solely exhibited by positive structures, taking advantage of a positive frame . Finally, we study TP_2 and RR_2 type dependencies in the context of predictive and response variables aiming to bring an usual Euclidean order among predictive variable values when an TP_2 order holds for a posteriori densities of the response variables.

Keywords: Bayesian Association, Dependencies, Negative and Positive, TP2 and RR2 Associations, Positivity.

1 Introduction

Generally speaking, if we are to deal with dependence structures we are before two large groups: *positive and negative.* Unless previous specification we adopt the following notation:

If $\underline{X} = (X_1, \cdots, X_n)$ is a random vector then we denote by $F_{\underline{X}}$, $\overline{F}_{\underline{X}}$ and $f_{\underline{X}}$ the joint cumulative distribution function, *c.d.f.*, the joint survival function and the joint probability density function, *p.d.f.*, respectively. The respective marginal distributions are denoted by F_{X_i} and f_{X_i}, $i = 1, \cdots, n$.

Definition 1.1 *We say that a random vector* $\underline{X} = (X_1, X_2, \ldots, X_n)$ *is*

i. *Negative Associated (NA) [Joag-Dev and Proschan(1983)] if for every pair of disjoint subsets* A_1, A_2 *of* $\{1,2,\ldots,n\}$*,where* $A_1 = \{i_1,...,i_k\}, A_2 = \{i_{k+1},...,i_n\}, i_s < i_t$ *for* $s < t$ *and for every pair of increasing functions* f *and* g *from* $\mathbb{R}^k$ *and* $\mathbb{R}^{n-k}$*, respectively, to* $\mathbb{R}$*, we have*

$$Cov(f(X_i, i \in A_1), g(X_j, j \in A_2)) \leq 0, \ \ \forall k \in \{1,2,\ldots,n-1\}.$$

ii. *Pairwise Negative Associated (PNA) if every pair of its distinct coordinates is negative associated.*

Definition 1.2 *We say that a random vector* $\underline{X} = (X_1, X_2, \ldots, X_n)$ *is*

i. *Positive Associated (PA) [Joag-Dev(1983)] if for every pair of increasing (coordinatewise) functions* f *and* g *on* $\mathbb{R}^n$ *we have*

$$Cov(f(\underline{X}), g(\underline{X})) \geq 0.$$

ii. *Pairwise Positive Associated (PPA) if for every pair of its distinct coordinates is positive associated.*

One another notion is of totally positive functions.

Definition 1.3 *(Karlin and Rinott(1980)a, b) Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a non-negative function.*

i. *We say* f *is a multivariate totally positive function of order 2 (*MTP_2*) if*

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y)$$

for any x *and* y *in* $\mathbb{R}^n$ *and*

$$x \vee (\wedge) y =$$
$$(\max(\min)\{x_1, y_1\}, \max(\min)\{x_2, y_2\}, \ldots, \max(\min)\{x_n, y_n\}).$$

When $n = 2$ *we say* f *is totally positive function of order 2 (*TP_2*).*

ii. *We say that* f *is multivariate reverse rule of order 2 (*MRR_2*) if*

$$f(x \vee y) f(x \wedge y) \leq f(x) f(y)$$

for any x *and* y *in* $\mathbb{R}^n$*. When* $n = 2$ *we say that* f *is reverse rule of order 2 (RR2).*

Definition 1.4 *Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with density function $f_{\underline{X}}$. We say that $\underline{X}$ is $MTP_2(MRR_2)$ if $f_{\underline{X}}$ is $MTP_2(MRR_2)$.*

Definition 1.5 *We say that the random variables X_1 and X_2 are $PA(TP_2)$ if the joint probability density function is TP_2. They are $NA(RR_2)$ if the joint probability density function is RR_2.*

Definition 1.6 *(Karlin and Rinott (1980)a, Efron (1964)) A non-negative function $\xi : R \to R$ is said to be order 2 Polya frequency PF_2 if the transformation $f : R^2 \to R$ defined by*

$$f(x_1, y_1) := \xi(x_1 - y_1) \ \ is \ \ TP_2,$$

Definition 1.7 *(Block, Savits, Shaked (1982)) Let $f : R^n \to R$ be a non-negative function.*

(i) We say that f is pairwise RR_2 if f is RR_2 when $n-2$ other coordinates are arbitrarily fixed.

(ii) We say that f is pairwise TP_2 if f is TP_2 when $n-2$ other coordinates are arbitrarily fixed.

Definition 1.8 *(Karlin and Rinott (1980)b) We say an MRR_2 function $f : R^n \to R$ is strongly MRR_2 denoted by "S–MRR_2" if for any set of n PF_2 functions $\{\xi_i\}_{i=1}^n$, and for each $j \le n$, a função:*

$$g_{k-j}(x_{i_{j+1}}, \cdots, x_{i_n}) := \int \cdots \int f(\underline{x}) \prod_{m=1}^{j} \xi_m(x_{i_m}) dx_{i_m}$$

is MRR_2, whenever the integral exists, where, $\{i_1, \cdots, i_n\}$ is any permutation of $\{1, \cdots, n\}$.

Definition 1.9 *(Karlin and Rinott (1980)a) Let f_1 and f_2 be two probability distribution functions in R^n. If for every $\underline{x} \in R^n, \underline{y} \in R^n$, we have*

$$f_2(\underline{x} \vee \underline{y}) f_1(\underline{x} \wedge \underline{y}) \ge f_1(\underline{x}) f_2(\underline{y}),$$

then we say that, $f_2 >_{TP2} f_1$.

In Joag-Dev and Proschan (1983) several examples of NA random vectors are presented also exhibiting the S-MRR_2 property. Some of the those examples are (a): Multinomial, (b): Multivariate Hypergeometric, (c): Dirichlet, (d): Dirichlet composed with the Multinomial and (e): Permutation invariant measures. Other approachs are considered in Block et al.(1982) featuring (a), (b), (c), (d) and (f): Multivariate Normal with special structure of covariance, as examples of pairwise RR_2 densities and consequently exhibiting S-MRR_2 property. Karlin and Rinott (1980)b presented elaborated proofs that (a), (b), (c), (f) and (g): Multivariate Hahn distribution, are S-MRR_2 densities.

With respect to positive dependence the available literature is more extensive and more profound as can be seen in Karlin and Rinott (1980)a. There, a number of MTP_2 densities can be found including the multi-normal distributions (for instance, $\underline{X} \sim N(\underline{0}, \Sigma)$, with the off-diagonal elements of $-\Sigma^{-1}$ being non-negative) as well as the proof that independent random variables have MTP_2 joint probability density functions. Other examples of MTP_2 distributions functions are (i): Multivariate Logistic, (ii): Multivariate gamma, (iii): Multivariate F, (iv): Multivariate absolute value Cauchy, (v): Negative Multinomial.

In order to properly understand what we mean by the advantages of a positive frame over a negative one we refer to Theorem 2.1 of Karlin and Rinott (1980a, p. 475) which states that if four non-negative functions satisfy:

$$f_1(\underline{x})f_2(\underline{y}) \le f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y}),$$

for every $\underline{x}, \underline{y} \in R^n$; then,

$$\int_{-\infty}^{+\infty} f_1(\underline{x})d\underline{x} \int_{-\infty}^{+\infty} f_2(\underline{x})d\underline{x} \le \int_{-\infty}^{+\infty} f_3(\underline{x})d\underline{x} \int_{-\infty}^{+\infty} f_4(\underline{x})d\underline{x}.$$

Such result guarantees that TP_2 positive dependence is preserved after marginalization, i.e., under the hypothesis of the theorem one guarantees: $\forall \underline{x}, \underline{y} \in R^k$,

$$\varphi_1(\underline{x})\varphi_2(\underline{y}) \le \varphi_3(\underline{x} \vee \underline{y})\varphi_4(\underline{x} \wedge \underline{y}),$$

where,

$$\varphi_i(x_{j_1}, \cdots, x_{j_k}) := \int_R \cdots \int_R f_i(\underline{x})dx_{j_{k+1}} \cdots dx_{j_n}, \; i = 1, \cdots, 4,$$

$j_1 \le \cdots \le j_k$, $k < n$, and the set of remaining indices $\{j_1, \cdots, j_k\}$ (with respect to $\{1, \cdots, n\}$) is: $\{j_{k+1}, \cdots, j_n\}$ [Karlin and Rinott(1980)a, Proposition

3.1] which means that the order $>_{TP_2}$ is still preserved after marginalization. Moreover, the last proposition is also true for MTP_2 functions allowing to state that if f is a MTP_2 function on R^n, then the function φ defined on R^k as

$$\varphi(x_1, \cdots, x_k) := \int_R \cdots \int_R f(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n) dx_{k+1} \cdots dx_n,$$

is MTP_2 [Karlin and Rinott (1980)a, Proposition 3.2]. Therefore, every marginal sub-vector from an MTP_2 vector turns out to be MTP_2. This ends up being one of the most transcending properties of the MTP_2 condition. Otherwise, for negative dependence, if we try repeat the sequence of implications aiming to reach the preservation of the MRR_2 condition after a marginalization then a failure is the result. To see this, it is enough to cite the example in Karlin and Rinott (1980)b, namely, set $f(x_1, x_2, x_3) = \exp[-x_3(x_1 + x_2)]$, so that f is MRR_2 for $x_i > 0, i = 1, 2, 3$. However, the margin

$$\int_0^\infty f(x_1, x_2, x_3) dx_3 = \frac{1}{(x_1 + x_2)},$$

is not RR_2.

Under another viewpoint, the difficulties which appear to a negative structure, as well as to positive ones, are produced by terms in agreement and disagreement, respectively, among the random variables. Dealing with agreement among random variables we make use of dependence types which connect them. From this, more appropriate concepts for dealing with such terms appear.

Definition 1.10 *(Joe (1997))A random vector $\underline{X} = (X_1, X_2)$ with c.d.f. F is "Stochastically Decreasing (Increasing)" on X_1, (or the conditional distribution $F_{2|1}$ is "Stochastically Decreasing (Increasing)"), if*

$$P(X_2 > x_2 | X_1 = x_1) = 1 - F_{2|1}(x_2|x_1) \downarrow (\uparrow) \text{ on } x_1 \ \forall x_2,$$

i.e.,

$$F_{2|1}(x_2|x_1) \uparrow (\downarrow) \text{ on } x_1 \ \forall x_2$$

where $F_{2|1}(x_2|x_1) = P(X_2 \leq x_2 | X_1 = x_1)$. Notation: X_2 $SD(SI)$ X_1.

Definition 1.11 *(Joe (1997))Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector with c.d.f. F. We say that, $\underline{X}$ is "Positively dependent through an stochastic order" (PDS) if the conditional distribution of $\{X_i : i \neq j\}$ given $X_j = x$ is stochastically increasing on x for every $j = 1, \cdots, n$.*

Definition 1.12 *(Joe (1997)) Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector with c.d.f. F. We say that, $\underline{X}$ is "Negatively dependent through an stochastic order" (NDS) if the conditional distribution of $\{X_i : i \neq j\}$ given $X_j = x$ is stochastically decreasing on x for every $j = 1, \cdots, n$.*

Definition 1.13 *(Joe (1997)) Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector. We say that $\underline{X}$ is "Conditionally Decreasing (Increasing) in Sequence", which we denote by $CDS(CIS)$, if X_i is Stochastically Decreasing (Increasing) on $X_1, \cdots, X_{i-1}$ for $i = 2, \cdots, n$, i.e.,*

$$P(X_i > x_i | X_j = x_j, j = 1, \cdots, i-1) \downarrow (\uparrow) \text{ on } x_1, \cdots, x_{i-1} \ \forall x_i.$$

During this work - unless previous specification - we assume that the order on R^n, is the *partial* usual order, i.e., $\{X_k \leq x_k, k \in \{i_1, \cdots, i_s\}\}$ means that for every k, $X_k \leq x_k$, $k \in \{i_1, \cdots, i_s\}$.

Definition 1.14 *(Joe (1997)) Let $\underline{X} = (X_1, X_2)$ be a random vector with c.d.f. F and marginal distributions F_i, $i = 1, 2$. We say that, X_2 is "Right Tail Decreasing (Increasing) on X_1", if*

$$P(X_2 > x_2 | X_1 > x_1) = \frac{\overline{F}(x_1, x_2)}{\overline{F}_1(x_1)} \downarrow (\uparrow) \text{ on } x_1 \ \forall x_2,$$

where $\overline{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$.
Notation: X_2 RTD $X_1 (X_2$ RTI $X_1)$.

Definition 1.15 *(Joe (1997))Let $\underline{X} = (X_1, X_2)$ be a random vector with c.d.f. F and marginal distributions F_i, $i = 1, 2$. We say that, X_2 is "Left Tail Increasing (Decreasing) on X_1", if*

$$P(X_2 \leq x_2 | X_1 \leq x_1) = \frac{F(x_1, x_2)}{F_1(x_1)} \uparrow (\downarrow) \text{ on } x_1 \ \forall x_2.$$

Notation: X_2 LTI $X_1 (X_2$ LTD $X_1)$.

Definition 1.16 *(Joe (1997))Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector with c.d.f. F. X_i is $RTD(RTI)$ on X_j, $i \in A^c$, $j \in A$, if*

$$P(X_i > x_i, i \in A^c | X_j > x_j, j \in A) \downarrow (\uparrow) \text{ on } x_j, \ j \in A, \ \forall x_i, \ i \in A^c$$

where, A is a non-empty set of $\{1, \cdots, n\}$.

Definition 1.17 *(Joe (1997)) Let* $\underline{X} = (X_1, \cdots, X_n)$ *be a random vector with c.d.f. F.* X_i *is* $LTI\,(LTD)$ *on* X_j*,* $i \in A^c$*,* $j \in A$*, if*

$$P(X_i \leq x_i, i \in A^c | X_j \leq x_j, j \in A) \uparrow (\downarrow) \text{ on } x_j,\ j \in A,\ \ \forall x_i,\ i \in A^c$$

where, A *is an non-empty subset of* $\{1, \cdots, n\}$.

Joe(1997) shows in the bivariate case a connection between TP_2 $p.d.f.$ and PA random variables, namely, that TP_2 $p.d.f.$ implies PA random variables. Before that, Joag-Dev and Proschan (1983) had already proved that in the bivariate case RR_2 $p.d.f.$ implies NA random variables.

From interpretations and results in Karlin and Rinott (1980)a, b it is possible to notice the reaching and the advantages of a positive structure as compared to a negative one. Among other points, it plays a special role the might of establishing in the multivariate case, what are the conditions, under an appropriate action, to transform a multivariate negative (or mixed, positive and negative) structure into a positive one.

Before going ahead we give some more definitions.

Definition 1.18 *(Lehmann (1966)) Let* X *and* Y *be two random variables. We say that they are negative quadrant dependent (positive quadrant dependent), on short,* $NQD\,(PQD)$*, if for every pair* (x, y) *of real numbers we have*

$$P(X \leq x, Y \leq y) \leq (\geq) P(X \leq x) P(Y \leq y).$$

Definition 1.19 *(Karlin and Rinott (1980)a, b) Let* $X_1, X_2, \cdots, X_n$ *be random variables. We say they are negative upper orthant dependent (positive upper orthant dependent), on short,* $NUOD\,(PUOD)$*, if for every n–ple of real numbers* $(x_1, x_2, \cdots, x_n)$*, we have*

$$P(X_i > x_i, i = 1, \cdots, n) \leq (\geq) \prod_{i=1}^{n} P(X_i > x_i).$$

We say they are negative lower orthant dependent (positive lower orthant dependent), on short, $NLOD\,(PLOD)$*, if for every n–ple* $(x_1, x_2, \cdots, x_n)$*, it holds*

$$P(X_i \leq x_i, i = 1, \cdots, n) \leq (\geq) \prod_{i=1}^{n} P(X_i \leq x_i).$$

Definition 1.20 *(Karlin and Rinott (1980)a, b)* *(i) We say that random variables are negative orthant dependent, denoted by NOD, if they are $NUOD$ and $NLOD$.*

(ii) We say that random variables are positive orthant dependent, POD, if they are $PUOD$ and $PLOD$.

Note that when $n = 2$, as in this case, $NLOD \Rightarrow NUOD$,

$$NOD = NQD \text{ and } POD = PQD.$$

We extend some relationships known to hold among positive dependence concepts to the case of negative dependence. Then, we study situations where it is possible to transform a mixed - positive and negative - structure to an entirely positive one. Ultimately, here, we search when mixed dependent structures can share properties solely exhibited by positive structures, taking advantage of a positive frame. All of those are done in Section 2. Finally, we study TP_2 and RR_2 type dependence in the context of predictive and response variables aiming to bring an usual Euclidean order among predictive variable values when an TP_2 order holds for a posteriori densities of the response variables. This is done in Section 3. The proofs of the results are given in Section 4.

2 Relationships among dependent structures

2.1 Bivariate dependency

It is known [H. Joe (1997), Theorem 2.3] that for bivariate vectors with positive dependence structure,

If (X_1, X_2) has TP_2 p.d.f. then $X_2\, SI\, X_1$. Consequently, $X_2\, LTD\, X_1$ and $X_2\, RTI\, X_1$ hold; any of the last two implies (X_1, X_2) is PA which on its turn implies PQD.

On the other hand, if (X_1, X_2) has TP_2 *p.d.f.* then its *c.d.f.* is TP_2 as well as its survival function. Moreover, TP_2 *p.d.f.* implies $X_2\, LTD\, X_1$ and TP_2 survival function implies $X_2\, RTI\, X_1$.

The SD notion represents a negative dependence condition since X_2 is less probable as X_1 increases. The NDS and CDS notions try to generalize the SD concept. The RTD and LTI concepts can be fitted inside negative dependence. In the first one, X_2 is less probable as much as X_1 takes large values, which means X_2 rarely takes large values in this case. In the second one, X_2 becomes more probable as X_1 increases, but X_2 can only assume a certain value x_2.

In the following theorem we establish relationships to negative dependence equivalent to the ones existing for positive dependence.

Theorem 2.1 *(a)* (X_1, X_2) *with* RR_2 *p.d.f.* $\Rightarrow X_2\, SD\, X_1$;
(b) $(X_1, X_2)\, NA \Leftrightarrow (X_1, X_2)\, NQD$;
(c) RR_2 *p.d.f.* $\Rightarrow$ RR_2 *c.p.f. and* RR_2 *survival function;*
(d_1)RR_2 *c.p.f.* $\Rightarrow X_2\, LTI\, X_1$, *and*
(d_2)RR_2 *survival function implies* $X_2\, RTD\, X_1$.

The following proposition establishes which are the conditions on a density probability function so that the related (X_1, X_2) is NA.

Proposition 2.1 (X_1, X_2) *is* $RR_2 \Rightarrow X_1$ *and* X_2 *are* $NQD \Leftrightarrow X_1$ *and* X_2 *are* NA.

Note that in the bivariate case we have that $NUOD$ and $NLOD$ are equivalent, both together being called NQD and NQD is equivalent to NA, [Joag-Dev and Proscham(1983)]. Moreover, from [Karlin and Rinott(1980)b p. 505], we can conclude that RR_2 *p.d.f.* implies NQD proving the previous proposition. Contrary to the bivariate case, the equivalence between $NUOD$ and $NLOD$ is not true in higher dimensions as can be seen from an example in Ebrahimi and Ghosh (1981).

Diagram 2.1 : *Relationships among positive dependencies, bivariate case*

$$f_{X_1,X_2}\ TP_2 \Longrightarrow \left[\begin{array}{l} F_{X_1,X_2}\ TP_2 \Longrightarrow X_2\ LTD\ X_1 \\ \overline{F}_{X_1,X_2}\ TP_2 \Longrightarrow X_2\ RTI\ X_1 \end{array} \right.$$

$$\Downarrow$$

$$\underline{X_2\ SI\ X_1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$X_2\ LTD\ X_1 \quad X_2\ RTI\ X_1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underline{PA \qquad\qquad PA}$$

$$\Downarrow$$

$$PQD$$

,

Diagram 2.2 : *Relationships among negative dependencies, bivariate case*

$$NQD \Longleftarrow f_{X_1,X_2}\ RR_2 \Longrightarrow \left[\begin{array}{l} F_{X_1,X_2}\ RR_2 \Longrightarrow X_2\ LTI\ X_1 \\ \overline{F}_{X_1,X_2}\ RR_2 \Longrightarrow X_2\ RTD\ X_1 \end{array} \right.$$

$$\begin{array}{cc} \Updownarrow & \Downarrow \\ NA & X_2\ SD\ X_1 \end{array}$$

2.2 Multivariate dependence

[H.Joe (1997)-Theorem 2.4, p.27] brings together $PA, PUOD, PLOD, PDS$, CIS concepts giving a global idea of the strongness of each concept compared to the other ones. From there we know that every sub-vector from a PA vector is PA; if one random vector is PA then it is $PUOD$ and $PLOD$ simultaneously. On the other hand, PDS implies $PUOD$ y $PLOD$ simultaneously. Finally, if a random vector is CIS then it is PA.

Similarly, some of those results are true for negative structures. Every sub-vector from a NA random vector is NA [Joag-Dev and Proschan(1983), p. 288; Kim(1995), p. 852]; NA implies $NUOD$, $NLOD$ simultaneously [Joag-Dev and Proschan (1983), p. 288]; NA does not imply CDS [Joag-Dev and Proschan (1983), p. 291];*p.d.f.* $S\text{-}MRR_2$ implies $NUOD$ and $NLOD$ [Karlin and Rinott(1980)b, p. 505].

In order to get a similar diagram to negative structures we prove the following proposition.

Proposition 2.2 $NDS \Rightarrow NUOD\, and NLOD;$

Diagram 2.3 : *Positive dependency, multivariate case*

$$\begin{array}{c} CIS \\ \Downarrow \\ PA \\ \Downarrow \\ PDS \Longrightarrow \boxed{PUOD\quad PLOD} \end{array}$$

Diagram 2.4 : *Negative dependency, multivariate case*

$$\begin{array}{c} CDS \\ \neg\Uparrow \\ NA \\ \Downarrow \\ NDS \Longrightarrow \boxed{NUOD\ \ NLOD} \Longleftarrow f_{X_1,X_2}\ S-MRR_2 \end{array}$$

2.3 Mixed structures of dependencies

In order to learn under what conditions we can transform a mixed (pairwise PA and NA)structure into a PA or (exclusive) NA one, we have to base in the following facts:

Remark 2.1 *If (X_1, X_2) has TP_2 p.d.f. f_{X_1,X_2} then X_1 and X_2 are PA [Joe(1997)].*

Remark 2.2 *If (X_1, X_2) has RR_2 f.d.p. f_{X_1,X_2} then X_1 and X_2 are NA [Proposition 2.1].*

Theorem 2.2 *Let $\underline{X} = (X_1, X_2)$ be a random vector with RR_2 p.d.f. $f_{\underline{X}}$ and $\underline{Y} = \eta(\underline{X})$, where $\eta(x_1, x_2) = (\eta_1(x_1), \eta_2(x_2))$, whose coordinates η_1, η_2 are differenciable, invertible and disagreeing (i.e., when one of them increases then the other one decreases). Then the $\underline{Y}$ p.d.f. $f_{\underline{Y}}$ is TP_2. Consequently, $\underline{Y}$ is PA.*

Let us examine $(X_1, \cdots, X_k)$, variables satisfying pairwise $PA(TP_2)$ or $NA(RR_2)$ conditions.

Claim 1. If $X_1, X_s \quad s \neq j_0,\ s = 2, \cdots, k$ are $PA(TP_2)$ and X_1, X_{j_0} are $NA(RR_2)$ then we can construct a variable $\eta_{j_0}(X_{j_0})$, so that,

$$X_s, \eta_{j_0}(X_{j_0}),\ 1 \le s < j_0 \text{ are } PA(TP_2) \text{ and}$$
$$\eta_{j_0}(X_{j_0}), X_s,\ \ j_0 + 1 \le s \le k \text{ are } PA(TP_2).$$

Analogously,

Claim 2. If

$$\begin{aligned} X_s, X_{j_0} \;\; NA(RR_2), s &= 1, \cdots, j_0 - 1; \\ X_{j_0}, X_t \;\; NA(RR_2), t &= j_0 + 1, \cdots, k; \\ X_i, X_j \;\; PA(TP_2), i &\neq j_0 \neq j, i \neq j, \end{aligned}$$

then we can construct $k-1$ variables $\eta_i(X_i)$, $i \neq j_0$, so that,

$$\begin{aligned} \eta_i(X_i), \eta_j(X_j) \;\; PA(TP_2), \forall i, j &\neq j_0; \\ \eta_i(X_i), X_{j_0} \;\; PA(TP_2), i &= 1, \cdots, j_0 - 1; \\ X_{j_0}, \eta_j(X_j) \;\; PA(TP_2), j &= j_0 + 1, \cdots, k. \end{aligned}$$

In both cases, X_{j_0} is the variable which brings negativeness to the random vector. Note that the goal in both two previous claims was to get a positive structure. That is not by chance. It is because the positive structure is better known besides of giving some advantages compared to the negative structure.

The idea in the previous results is to conditioning the behaviour of the joint density in order to offer favorable conditions to do the transformation. The required condition admits just one variable in disagreement to the other variables in the random vector.

Theorem 2.3 *Let $(X_1, \cdots, X_{j_0}, \cdots, X_k)$ be a random vector such that the marginal densities $f_{1,j}$ are TP_2 $\forall j \neq j_0$, $j = 1, \cdots, k$; and f_{1,j_0} is RR_2. Moreover, suppose that for $2 \leq s < j_0$, $j_0 + 1 \leq t \leq k$, the margins f_{1,s,j_0} and $f_{1,j_0,t}$ satisfy:*

$(i) f_{1,s,j_0}(x_1, x_s, x_{j_0}) f_{1,s,j_0}(x'_1, x'_s, x'_{j_0})$
$\leq f_{1,s,j_0}(x_1 \vee x'_1, x_s \vee x'_s, x_{j_0} \wedge x'_{j_0}) f_{1,s,j_0}(x_1 \wedge x'_1, x_s \wedge x'_s, x_{j_0} \vee x'_{j_0}),$

$(ii) f_{1,j_0,t}(x_1, x_{j_0}, x_t) f_{1,j_0,t}(x'_1, x'_{j_0}, x'_t)$
$\leq f_{1,j_0,t}(x_1 \vee x'_1, x_{j_0} \wedge x'_{j_0}, x_t \vee x'_t) f_{1,j_0,t}(x_1 \wedge x'_1, x_{j_0} \vee x'_{j_0}, x_t \wedge x'_t),$

respectively. Under such conditions, if $\eta_{j_0} : R \rightarrow R$ is decreasing, invertible and differentiable, then

$$\begin{aligned} &(a)(X_s, \eta_{j_0}(X_{j_0})) \;\; \textit{has } TP_2, \;\; p.d.f. \;\; f_{X_s, \eta_{j_0}(X_{j_0})}, \; 1 \leq s < j_0 \\ &(b)(\eta_{j_0}(X_{j_0}), X_t) \;\; \textit{has } TP_2, \;\; p.d.f. \;\; f_{\eta_{j_0}(X_{j_0}), X_t}, \; j_0 + 1 \leq t \leq k \end{aligned}$$

Notation:

$$f_{i,j,k}(x_i, x_j, x_k) := f_{X_i, X_j, X_k}(x_i, x_j, x_k), \;\; f_{i,j}(x_i, x_j) = f_{X_i, X_j}(x_i, x_j).$$

Corollary 2.1 *Let $(X_1, \cdots, X_{j_0}, \cdots, X_k)$ be a random vector so that the marginal densities $f_{i,j}$ satisfy,*

$$\begin{aligned} f_{s,j_0} &\text{ is } RR_2,\ \forall 1 \le s < j_0, \\ f_{j_0,t} &\text{ is } RR_2,\ \forall j_0 < t \le k, \\ f_{1,j} &\text{ is } TP_2,\ \forall j \ne j_0,\ j = 1, \cdots, k. \end{aligned}$$

Under such conditions, if $\eta_{j_0} : R \to R$ is decreasing, invertible and differentiable, then

$$\begin{aligned} &(a)(X_s, \eta_{j_0}(X_{j_0})) \quad \text{has } TP_2 \text{ p.d.f. } \ f_{X_s,\eta_{j_0}(X_{j_0})},\ \ 1 \le s < j_0 \\ &(b)(\eta_{j_0}(X_{j_0}), X_t) \quad \text{has } TP_2 \text{ p.d.f. } \ f_{\eta_{j_0}(X_{j_0}),X_t},\ \ j_0 + 1 \le t \le k. \end{aligned}$$

Remark 2.3 *When only one variable is responsible for the negativeness of the random vector then we can get a positive random vector which is "equivalent" to the initial vector.*

The point in the following theorem consists in visualizing a decomposition of the density function of the random vector into factors $f = hk$. From that it is possible to visualize the unique component of the vector in disagreement to the others.

Theorem 2.4 *Let $\underline{X} = (X_1, \cdots, X_{J_0}, \cdots, X_k)$ be a random vector with p.d.f. given by*

$$f_{X_1,\cdots,X_{j_0},\cdots,X_k}(x_1, \cdots, x_{j_0}, \cdots, x_k) = h(x_1, \cdots, x_{j_0-1}, x_{j_0+1}, \cdots, x_k) K(x_1, x_{j_0}),$$

where K is RR_2, and h satisfies the following condition:

$$\begin{aligned} (iii) \int h(u, x_{(2)}^{(s-1)}, v, x_{(s+1)}^{(k)}) d(x_{(2)}^{(s-1)}, x_{(s+1)}^{(k)}) \quad &\text{is} \quad TP_2 \text{ on the variables } X_1, X_s, \\ \forall\ 3 \le s < j_0 \text{ or } j_0 + 1 \quad &\le \quad s < k-1 \text{ and} \\ \int h(u, x_{(2)}^{(k-1)}, v) d(x_{(2)}^{(k-1)}) \quad &\text{is} \quad TP_2 \text{ on the variables } X_1, X_k. \end{aligned}$$

Then,

$$f_{X_1,X_s} \text{ is } TP_2,\ \forall s \ne j_0,\ s = 2, \cdots, k \text{ and } f_{X_1,X_{j_0}} \text{ is } RR_2.$$

In such context, if $\eta_{j_0} : R \to R$ is decreasing, invertible and differentiable then,

$$\begin{aligned} &(a)\ (X_s, \eta_{j_0}(X_{j_0})) \text{ has } TP_2,\ \ \text{p.d.f. } f_{X_s,\eta_{j_0}(X_{j_0})},\ \ 1 \le s < j_0, \\ &(b)\ (\eta_{j_0}(X_{j_0}), X_s) \text{ has } TP_2,\ \ \text{p.d.f. } f_{\eta_{j_0}(X_{j_0}),X_s},\ \ j_0 + 1 \le s \le k. \end{aligned}$$

where $x_{(i)}^{(j)} = (x_i, x_{i+1}, \cdots, x_j)$.

2.3.1 Mixture of associations, an example

Example 2.1 *If* (X_1, X_2, X_3, X_4) *has distribution* $N_4(\underline{0}, \Sigma)$, *where*

$$\Sigma = \begin{pmatrix} 0.60 & 0.20 & -0.03 & 0.32 \\ & 1.40 & -0.01 & 0.74 \\ & & 1.24 & -0.02 \\ & & & 1.25 \end{pmatrix},$$

then the distribution of $\underline{X}$ *can be put as a product of a* $N_3(\underline{0}, \Sigma_1^*)$ *and a* $N_2(\underline{0}, \Sigma_2^*)$, *ie,*

$$f_{(X_1,X_2,X_3,X_4)}(x_1, x_2, x_3, x_4) = c_0 f_{(X_1,X_2,X_4)}(x_1, x_2, x_4) f_{(X_1,X_3)}(x_1, x_3)$$

where

$$c_0 \text{ is constant, } \Sigma_1^* = \begin{pmatrix} 1.50 & 0.50 & 0.80 \\ & 1.50 & 0.90 \\ & & 1.50 \end{pmatrix}, \ \Sigma_2^* = \begin{pmatrix} 1.00 & -0.05 \\ & 1.24 \end{pmatrix}$$

and

$$(X_1, X_2, X_4) \text{ has distribution } N_3(\underline{0}, \Sigma_1^*),\ MTP_2$$
$$(X_1, X_3) \text{ has distribution } N_2(\underline{0}, \Sigma_2^*),\ RR_2.$$

The just given example can be shown to satisfy the hypothesis of the coming proposition. We would dare to say that the kind of structure presented in the example would be more frequent in the daily practice than the one involving homogeneous dependencies (totally positive or totally negative).

Proposition 2.3 *Let* $\underline{X} = (X_1, \cdots, X_{j_0}, \cdots, X_k)$ *be a random vector with p.d.f. given by,*

$$f_{X_1,\cdots,X_{j_0},\cdots,X_k}(x_1, \cdots, x_{j_0}, \cdots, x_k) = h(x_1, \cdots, x_{j_0-1}, x_{j_0+1}, \cdots, x_k) K(x_1, x_{j_0}),$$

where K *is* RR_2 *and* h *is* MTP_2; *then,*

$$f_{X_1,X_s} \text{ is } TP_2,\ \forall s \neq j_0,\ s = 1, \cdots, k \text{ and } f_{X_1,X_{j_0}} \text{ is } RR_2.$$

Under such hypotheses, if $\eta_{j_0} : R \to R$ *is decreasing, invertible and differentiable, then,*

$$(a)(X_s, \eta_{j_0}(X_{j_0})) \text{ has } TP_2 \text{ p.d.f. } f_{X_s,\eta_{j_0}(X_{j_0})},\quad 1 \leq s < j_0$$
$$(b)(\eta_{j_0}(X_{j_0}), X_s) \text{ has } TP_2 \text{ p.d.f. } f_{\eta_{j_0}(X_{j_0}),X_s},\quad j_0 + 1 \leq s \leq k.$$

2.4 Positive association invariance

We write down this section using MTP_2 vectors since such condition is stronger compared to the others positive dependence structures. The MTP_2 condition is preserved by agreeing transformations, that is to mean, if $\underline{Y}$ is MTP_2 and f, g are agreeing monotone functions then

$$Cov(f(\underline{Y}), g(\underline{Y})) \geq 0$$

The hypotheses of f and g can be modified through the use of TP_2 order. Such modification is seen in the next proposition.

Proposition 2.4 *Let $\underline{Y}$ be a random vector with TP_2 p.d.f. $f_{\underline{Y}}$ and $f_2 >_{TP_2} f_1$, two functions, f_1 being non-negative, monotonically increasing, then, $Cov(f_1(\underline{Y}), f_2(\underline{Y})) \geq 0$.*

If $(X_1, \cdots, X_k)$, $k \geq 3$, is a PA random vector and $f : R^{k-1} \to R$ is increasing, then X_1 e $f(X_2, \cdots, X_k)$ are PA random variables. The way to see this is to use the following remarks.

Remark 2.4 *$\underline{T} = (T_1, \cdots, T_n)$ (not necessarily binary) is PA if and only if, to every pair of binary, increasing functions Γ and Δ, we have,*

$$Cov\{\Gamma(\underline{T}), \Delta(\underline{T})\} \geq 0.$$

Remark 2.5 *Let S and T be random variables with finite $E(S), E(T)$ and $E(ST)$. Given s and t arbitrary values on the images of S and T respectively, let us consider $X_S(s) := I_{\{S>s\}}$ and $X_T(t) := I_{\{T>t\}}$ as the indicator functions of the sets $\{S > s\}$ and $\{T > t\}$. Then,*

$$Cov\{S, T\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Cov\{X_S(s), X_T(t)\} dsdt.$$

Remark 2.6 *Increasing functions of PA random variables are PA.*

Remarks 2.4, 2.5 and 2.6, can be found in [Barlow e Proschan(1981), p.30]. For the use we will done of them we will prove Remarks 2.4 and 2.6.

Application 2.1 *If (X_1, X_2, X_3) is PA, $f_i : R^3 \to R$, $i = 1, 2$, so that $f_1(u, v, w) = u$, $f_2(u, v, w)$ is a increasing function of v and w, not depending on u, (with abuse of notation, $f_2(u, v, w) = f_2(v, w)$).*

Define,

$$\underline{S} := (S_1, S_2) = (f_1(X_1, X_2, X_3), f_2(X_1, X_2, X_3)) = (X_1, f_2(X_2, X_3)).$$

Then $\underline{S}$ is PA.

Definition 2.1 *Let $h : R^2 \to R$ be an increasing function which takes the pair (u,v) into a real number z. Suppose that, once we fix the first coordinate, it is possible to well define an inverse function $h_u^{-1} : R \to R$ so that for each $z \in R$, $h_u^{-1}(z) = v$, and $h(u,v) = z$. Suppose that $\frac{\partial h}{\partial v}$ exists.*

Proposition 2.5 *Let $h : R^2 \to R$ be increasing, satisfying Definition 2.1 .. Let us consider the random vector (X_1, X_2, X_3) so that (X_1, X_3) is PA. If*

$$f_{X_1,X_3}(x_1, h_{(\cdot)}^{-1}(z)) >_{TP_2(\cdot)} \frac{\partial h}{\partial v}(\cdot, h_{(\cdot)}^{-1}(z)),$$

for every x_1 in the image of X_1 and $(\cdot)$ in the image of X_2, $z \in R$. Then X_1 and $h(X_2, X_3)$ are PA.

Example 2.2 : *Let (X_1, X_2, X_3) be a vector whose coordinates take values on $[a,b], [1,2], [2,3]$ respectively. For sake of simplicity we assume $0 < a < b$. Suppose that the p.d.f. of (X_1, X_3) is given by $f_{X_1,X_3}(u,v) = cve^{uv}$, for $u \in [a,b]$ and $v \in [2,3]$, and otherwise, defined as 0.*

We take $h : R^2_{>0} \to R^+$, as $h(x_2, x_3) := x_2^2 + x_3^2$. Note that

$$\frac{\partial h}{\partial x_3}(x_2, x_3) = 2x_3; \quad h_{x_2}^{-1}(z) = (z - x_2^2)^{\frac{1}{2}}, \ x_2 \text{ fixed.}$$

Moreover, f_{X_1,X_3} is TP_2 and

$$\frac{f_{X_1,X_3}(x, (z - (x_2')^2)^{\frac{1}{2}})}{(z - (x_2')^2)^{\frac{1}{2}}} \geq \frac{f_{X_1,X_3}(x, (z - x_2^2)^{\frac{1}{2}})}{(z - x_2^2)^{\frac{1}{2}}}, \ \forall x_1, \ x_2 < x_2'$$

or equivalently

$$f_{X_1,X_3}(x, h_{(\cdot)}^{-1}(z)) >_{TP_2(\cdot)} \frac{\partial h}{\partial x_3}(\cdot, h_{(\cdot)}^{-1}(z)).$$

3 Pairwise $TP_2(RR_2)$ Condition, Joint Distribution and Bayesian Prior/Posterior Operation

We stick to the following notation:

θ	is the response variable, taking values on R
$\underline{X} = (X_1, \cdots, X_k)$	is the data vector, $\underline{X} \in R^k$, prediction variables
f_i	$p.d.f.$ of X_i, $i = 1, \cdots, k$,
$f_{\theta,i}$	joint $p.d.f.$ of θ and X_i, $i = 1, \cdots, k$,
$f_{\theta,1,\cdots,k}$	$p.d.f.$ of θ and $\underline{X}$
$f(X_i\|\theta)$	is the likelihood function
$f(\underline{X}\|\theta)$	likelihood function
$\xi(\theta)$	a priori $p.d.f.$ of θ
$\xi^*(\theta\|X_i)$	a posteriori $p.d.f.$ of θ
$\xi^*(\theta\|\underline{X})$	a posteriori $p.d.f.$ of θ

$$\xi^*(\theta|X_i) = \frac{f_{\theta,i}(\theta, X_i)}{f_i(X_i)} = H(X_i)f(X_i|\theta)\xi(\theta),$$

where

$$H(X_i) := (\int f(X_i|\theta)\xi(\theta)d\theta)^{-1}.$$

Therefore, if we set $K(X_i) := f_i(X_i)H(X_i)$, then the joint density function can be written as

$$f_{\theta,i}(\theta, X_i) = K(X_i)f(X_i|\theta)\xi(\theta).$$

From there it is easy to conclude that $f_{\theta,i}(\theta, X_i)$ is TP_2 if and only if, $f(X_i|\theta)$ is TP_2, when we see the last f as a function of θ and X_i.

Finally,

$$\xi^*(\theta|X_i) \text{ is } TP_2(\theta, X_i) \Leftrightarrow f_{\theta,i}(\theta, X_i) \text{ is } TP_2(\theta, X_i) \Leftrightarrow$$
$$f(X_i|\theta) \text{ is } TP_2(\theta, X_i),\ i = 1, \cdots, k.$$

Following this line of thinking,

$$\begin{aligned} \xi^*(\theta|\underline{X}) &= \frac{f_{\theta,1,\cdots,k}(\theta, \underline{X})}{f_{1,\cdots,k}(\underline{X})} \\ &= H(\underline{X})f(\underline{X}|\theta)\xi(\theta) \\ H(\underline{X}) &:= (\int f(\underline{X}|\theta)\xi(\theta)d\theta)^{-1} \end{aligned}$$

and we can conclude that

$$\xi^*(\theta|\underline{X}) \text{ is } TP_2(\theta, X_i) \Leftrightarrow f_{\theta,1,\cdots,k}(\theta, \underline{X}) \text{ is } TP_2(\theta, X_i) \Leftrightarrow$$
$$f(\underline{X}|\theta) \text{ is } TP_2(\theta, X_i),\ 1 \leq i \leq k.$$

Analogous conclusions can be obtained for the pairwise RR_2 condition.

In order to study the modifications brought in the a posteriori distribution of the response variable $\underline{\theta}$, when the predictor vector is changed, we have to consider θ as a vector $\underline{\theta} = (\theta_1, \cdots, \theta_n)$ taking values on R^n.

Theorem 2.1 and Proposition 2.1 in Karlin and Rinott (1980)a are the bases for the proofs of the three theorems in this section. The Theorem 2.1 has already been stated. For the sake of completeness we state the proposition.

Let $f(\underline{x})$ be pairwise TP_2 function for $\underline{x} \in R^n$, and suppose that $f(\underline{x})f(\underline{y}) \neq 0$ implies $f(\underline{u})f(\underline{v}) \neq 0 \ \forall \underline{x}, \underline{y} \in R^n, \underline{x} \wedge \underline{y} \leq \underline{u}, \underline{v} \leq \underline{x} \vee \underline{y}$. Then,

$$f(\underline{x})f(\underline{y}) \leq f(\underline{x} \vee \underline{y})f(\underline{x} \wedge \underline{y}), \ \forall \underline{x}, \underline{y} \in R^n.$$

Although the results we present in this section are in spirit close to the line of the following theorem, the tools used in the proofs are different.

Theorem 3.1 *(Fahmy et al.(1982)) If*

$$\xi^*(\underline{\theta}|\underline{X}) = H(\underline{X}) \prod_{i=1}^{k} g_i(X_i, \theta_i) C(\underline{\theta});$$

$$\begin{array}{lll} g_i & TP_2 \ on & (X_i, \theta_i) \ \forall i = 1, \cdots, k; \\ g_i & PF_2(\theta_i) & for \ X_i \ fixed; \\ C(\underline{\theta}) & S - MRR_2 & . \end{array}$$

If $\underline{X} = (X_1, \cdots, X_k)$, $\underline{X}' = (X_1', \cdots, X_k')$, where $X_i' = X_i$, $X_j' \geq X_j$, $j = 1, \cdots, k \ j \neq i$, then

$$\xi_i^*(\theta_i|\underline{X})) >_{TP_2(\theta_i)} \xi_i^*(\theta_i|\underline{X}'),$$

for i fixed, one $i \in \{1, \cdots, k\}$.

The results we have are

Theorem 3.2 *If*

$$\xi^*(\underline{\theta}|\underline{X}) = H(\underline{X}) \prod_{i=1}^{k} g_i(X_i, \theta_i) C(\underline{\theta}),$$

g_i *TP_2 on* (X_i, θ_i), $\forall i = 1, \cdots, k$.

$C(\underline{\theta})$ so that $\forall j$ *and for every pair of vectors* $\underline{\theta}$ *and* $\underline{\theta}'$

where $(\underline{\theta})_u = \theta_u$ *and* $(\underline{\theta}')_u = \theta'_u$, $u = 1, \cdots, k$ *satisfy*

$$\text{(C)}\ C(\underline{\theta})C(\underline{\theta}') \leq C(\theta_{(1)}^{(j-1)} \vee \theta'^{(j-1)}_{(1)}, \theta_j \wedge \theta'_j, \theta_{(j+1)}^{(k)} \vee \theta'^{(k)}_{(j+1)})$$

$$C(\theta_{(1)}^{(j-1)} \wedge \theta'^{(j-1)}_{1}, \theta_j \vee \theta'_j, \theta_{(j+1)}^{(k)} \wedge \theta'^{(k)}_{(j+1)}).$$

Then, if $\underline{X}$ *and* $\underline{X}'$, $(\underline{X})_u = X_u$, $(\underline{X}')_u = X'_u$, $u = 1, \cdots, k$, *are so that*

$$X'_u = X_u \text{ for } u < j,\ \ X'_j > X_j \text{ and } X'_u \leq X_u \text{ for } u > j;\ j \in \{1, \cdots, k\}$$

then it follows that,

(A.1)

$$\xi^*_{s,t}(\theta_s, \theta_t | \underline{X}) >_{TP_2(\theta_s, \theta_t)} \xi^*_{s,t}(\theta_s, \theta_t | \underline{X}')\ t \neq s, \quad t, s < j;$$

(A.2)

$$\xi^*_{1,\cdots,j-1}(\theta_1, \cdots, \theta_{j-1} | \underline{X}) >_{TP_2(\theta_1, \cdots, \theta_{j-1})}$$

$$\xi^*_{1,\cdots,j-1}(\theta_1, \cdots, \theta_{j-1} | \underline{X}'),\ \ j \in \{2, \cdots, k\};$$

(A.3) $\xi^*_{1,\cdots,j-1,j+1,\cdots,k}(\theta_{(1)}^{(j-1)}, \theta_{(j+1)}^{(k)}) | \underline{X}) >_{TP_2(\theta_{(1)}^{(j-1)}, \theta_{(j+1)}^{(k)})}$

$$\xi^*_{1,\cdots,j-1,j+1,\cdots,k}(\theta_{(1)}^{(j-1)}, \theta_{(j+1)}^{(k)} | \underline{X}'),\ \ j \in \{2, \cdots, k-1\}.$$

In the previous theorem we have in the involved vectors $\underline{X}$ and $\underline{X}'$ a disagreeing relationship on the ordering of their components, with the j–th coordinate presenting the disagreeing behaviour among the coordinates in the set $\{j+1, \cdots, k\}$. Shortly speaking, the coordinates in this set determine the ordering shown by the a posteriori density, as can be seen in $A.1$, $A.2$, $A.3$. However, we cannot forget that the involved likelihood is ***essentially positive*** (g_i being TP_2 and C counteract the disagreeing influence of the j–th coordinate.).

Theorem 3.3 *Set*

$$\xi^*(\underline{\theta} | \underline{X}) = H(\underline{X}) \prod_{i=1}^{k} g_i(X_i, \theta_i) C(\underline{\theta}).$$

Suppose that $\exists j$, so that

$$g_j \text{ is } RR_2(X_j, \theta_j),\ \ g_i \text{ is } TP_2(X_i, \theta_i)\ \forall i \neq j, i = 1, \cdots, k$$

and let C to be $MTP_2(\underline{\theta})$. Then, if

$$\underline{X} = (X_1, \cdots, X_{j-1}, X_j, X_{j+1}, \cdots, X_k),$$

and

$$\underline{X'} = (X'_1, \cdots, X'_{j-1}, X'_j, X'_{j+1}, \cdots, X'_k)$$

where $X'_i \leq X_i\ \forall i \neq j,\ X'_j \geq X_j$, one concludes that

$$\xi^*(\underline{\theta}|\underline{X}) >_{TP_2(\underline{\theta})} \xi^*(\underline{\theta}|\underline{X'})$$

In the previous result the vectors present the following relationships

$$X'_j \geq X_j \text{ y } X'_u \leq X_u \text{ for } u \neq j, \quad j \text{ fixed}$$

which means every coordinates, but the j–th one, are on agreement, this fact being portrayed by their a posteriori density being TP_2. The disagreeing action of the j–th coordinate is counteracted by the agreeing behavior of the other coordinates plus the fact of g_j being RR_2. This way, the action of C can be seen to end up in the result of the previous theorem. Next, we present a direct generalization of such result.

Theorem 3.4

$$\xi^*(\underline{\theta}|\underline{X}) = H(\underline{X}) \prod_{i=1}^{k} g_i(X_i, \theta_i) C(\underline{\theta})$$

g_i is $RR_2(X_i, \theta_i)\ \forall i = 1, \cdots, j,\ \ g_i$ is $TP_2(X_i, \theta_i)\ \forall i = j+1, \cdots, k,$
C is $MTP_2(\underline{\theta})$.

Then, if

$$\underline{X} = (X_1, \cdots, X_{j-1}, X_j, X_{j+1}, \cdots, X_k)$$

and

$$\underline{X'} = (X'_1, \cdots, X'_{j-1}, X'_j, X'_{j+1}, \cdots, X'_k)$$

with $X'_i \geq X_i\ \forall i = 1, \cdots, j$ and $X'_i \leq X_i\ \ \forall i = j+1, \cdots, k$, we have that

$$\xi^*(\underline{\theta}|\underline{X}) >_{TP_2(\underline{\theta})} \xi^*(\underline{\theta}|\underline{X'}).$$

Example 3.1 (Multivariate Normal Distribution) *Assume that $\underline{X}$ is a k-dimensional random vector, with independent X_i, $i = 1, \cdots, k$ coordinates. Suppose that for every i, $\theta_i := E(X_i)$ is unknown, while $\sigma_i^2 := Var(X_i)$ is known. We have,*

$$\begin{aligned}
f(\underline{X}|\underline{\theta}) &= |\Sigma|^{-\frac{1}{2}} exp\left(\frac{-(\underline{X}-\underline{\theta})^t |\Sigma|^{-1} (\underline{X}-\underline{\theta})}{2}\right) \\
&= h(\underline{X}) g(\underline{X}, \underline{\theta}) \\
h(\underline{X}) &:= \prod_{i=1}^{k} exp \frac{-X_i^2}{2\sigma_i^2} \\
g(\underline{X}, \underline{\theta}) &:= \prod_{i=1}^{k} exp \frac{X_i \theta_i}{\sigma_i^2} \prod_{i=1}^{k} exp \frac{-\theta_i^2}{2\sigma_i^2}
\end{aligned}$$

Claim: *h is $MTP_2(\underline{X})$ and g is $MTP_2(\underline{X}, \underline{\theta})$.*

The ingredients in the proof that g is $MTP_2(\underline{X}, \underline{\theta})$ are given next,

$$\begin{aligned}
g(\underline{X}, \underline{\theta}) &= \prod_{i=1}^{k} g_i(X_i, \theta_i) D(\underline{\theta}), \\
g_i(X_i, \theta_i) &= exp \frac{X_i \theta_i}{\sigma_i^2} \\
D(\underline{\theta}) &:= \prod_{i=1}^{k} D_i(\theta_i), \\
D_i(\theta_i) &:= exp \frac{-\theta_i^2}{2\sigma_i^2}.
\end{aligned}$$

It goes like: $D(\underline{\theta})$ is pairwise TP_2, while $\prod_{i=1}^{k} g_i(X_i, \theta_i)$ is TP_2 on the pairs

$$(X_s, X_j), (\theta_s, \theta_j), (X_s, \theta_j), \; j \neq s.$$

Hence, $g(\underline{X}, \underline{\theta})$ is pairwise TP_2, but as in our case $g(\underline{X}, \underline{\theta}) \neq 0$, by Proposition 2.1 in Karlin and Rinott (1980)a it results that g, is MTP_2. Using the same proposition we can prove that h is MTP_2.

Application 3.1 *of Theorem 3.2*

The a posteriori distribution is given by,

$$\begin{aligned}
\xi^*(\underline{\theta}|\underline{X}) &= H(\underline{X})\prod_{i=1}^{k} g_i(X_i,\theta_i)C(\underline{\theta}) \\
\textit{where } C(\underline{\theta}) &:= \xi(\underline{\theta})\prod_{i=1}^{k} exp\frac{-\theta_i^2}{2\sigma_i^2} \\
\textit{and } H(\underline{X}) &:= h(\underline{X})(\int f(\underline{X}|\underline{\theta})\xi(\underline{\theta})d\underline{\theta})^{-1}
\end{aligned}$$

Hence, in order to verify **(C)** *in Theorem 3.2 on C is equivalent to verify* **(C)** *on ξ. We assume as the a priori distribution for $\underline{\theta}$ a Normal vector with means $\underline{0}$ and the covariance matrix given by the identity I. We get*

$$\xi(\underline{\theta}) \propto \prod_{i=1}^{k} \exp\{\frac{-\theta_i^2}{2}\}$$

which satisfies **(C)**.

4 Demonstrations

4.1 Proof of Theorem 2.1

(a): Given $x < x'$, to check that $P(X_2 > y|X_1 = x) \geq P(X_2 > y|X_1 = x')$ is equivalent to prove that

$$\int_y^{\infty} f(x,z)dz \int_{-\infty}^{+\infty} f(x',w)dw \geq \int_y^{\infty} f(x',z)dz \int_{-\infty}^{+\infty} f(x,w)dw$$

or

$$\int_y^{\infty}\int_{-\infty}^{+\infty}[f(x,z)f(x',w) - f(x',z)f(x,w)]dwdz \geq 0,$$

where f is the joint *p.d.f.*. Now, we know that, if $w \leq z$, then

$$f(x,z)f(x',w) - f(x',z)f(x,w) \geq 0,$$

therefore, it follows that

$$f(x,z)f(x',w) - f(x',z)f(x,w) \geq 0, \quad \forall(z,w) \in (y,\infty] \times (-\infty, y],$$

and then,

$$\int_y^{\infty}\int_{-\infty}^{y}[f(x,z)f(x',w)-f(x',z)f(x,w)]dwdz \geq 0.$$

Notice that

$$\int_y^{\infty}\int_{y}^{+\infty}[f(x,z)f(x',w)-f(x',z)f(x,w)]dwdz \geq 0.$$

(b): $(\Rightarrow)$: it is right away from the definitions.

$(\Leftarrow)$: we prove the statement for indicator functions: Let A and B be Borelian sets on the real line. Whenever $I_A(x)$ and $I_B(y)$ are agreeing functions, as X and Y are NQD, we have

$$E\left\{I_A(X)I_B(Y)\right\} \leq E\left\{I_A(X)\right\}E\left\{I_B(Y)\right\}.$$

As a direct consequence, given two collections of values $\{x_i\}_{i=1}^m$, $\{y_j\}_{j=1}^k$ and Borelian sets on the line, $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^k$, respectively, it follows that

$$E\left\{\sum_{i=1}^m x_i I_{A_i}(X)\sum_{j=1}^k y_j I_{B_j}(Y)\right\} \leq E\left\{\sum_{i=1}^m x_i I_{A_i}(X)\right\}E\left\{\sum_{j=1}^k y_j I_{B_j}(Y)\right\}.$$

Hence, the statement holds on the space of simple functions.

Consider now two Borel-measurable, monotonically increasing, non-negative functions, f and g. The usual Monotone Convergence Theorem together with [Ash(1972), Theorem 1.5.5, p. 38] completes the proof of the statement.

Finally, for general Borel-measurable functions, the decompositions $f = f^+ - f^-$ and $g = g^+ - g^-$ where f^+, f^-, g^+, g^- are non-negative, monotone, Borel-measurable functions, takes care of the rest, since

$$E\left\{f^+(X)g^+(Y)\right\} \leq E\left\{f^+(X)\right\}E\left\{g^+(Y)\right\},$$

$$E\left\{f^-(X)g^-(Y)\right\} \leq E\left\{f^-(X)\right\}E\left\{g^-(Y)\right\},$$

for the involved functions, $f^+ = f \vee 0$, $g^+ = g \vee 0$, are agreeing and

$$E\left\{f^+(X)g^-(Y)\right\} \geq E\left\{f^+(X)\right\}E\left\{g^-(Y)\right\},$$

$$E\left\{f^-(X)g^+(Y)\right\} \geq E\left\{f^-(X)\right\}E\left\{g^+(Y)\right\},$$

for the two involved functions, $f^- = (-f) \vee 0$, $g^- = (-g) \vee 0$ are in disagreement. From there, we conclude

$$E\{f(X)g(Y)\} \leq E\{f(X)\}\, E\{g(Y)\}.$$

Notice that, in general, $NUOD$ and/or $NLOD$ does not imply NA. [Joag-Dev and Proschan(1983), p.289].

(c): Let us consider first the equivalence,
$\forall x_1 < y_1, \forall x_2 < y_2$; F, *c.d.f.* is RR_2 if and only if

$$F(x_1, x_2)F(y_1, y_2) - F(x_1, y_2)F(y_1, x_2) \leq 0$$

if and only if $F(x_1, x_2)[F(y_1, y_2) - F(y_1, x_2) - F(x_1, y_2) + F(x_1, x_2)]$

$$-[F(x_1, y_2) - F(x_1, x_2)][F(y_1, x_2) - F(x_1, x_2)] \leq 0.$$

Now, by hypothesis, we have that,

$$f(s_1, s_2)f(t_1, t_2) - f(s_1, t_2)f(t_1, s_2) \leq 0, \quad s_2 < t_2, s_1 < t_1;\ f,\ p.d.f.$$

Given $x_1 < y_1, x_2 < y_2$ and considering, $-\infty \leq s_2 < x_2 < t_2 \leq y_2$ and $-\infty \leq s_1 < x_1 < t_1 \leq y_1$, one can check that

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{x_1}^{y_1}\int_{x_2}^{y_2}[f(s_1, s_2)f(t_1, t_2) - f(s_1, t_2)f(t_1, s_2)]dt_2dt_1ds_2ds_1 \leq 0,$$

or equivalently,

$$F(x_1, x_2)[F(y_1, y_2) - F(y_1, x_2) - F(x_1, y_2) + F(x_1, x_2)]$$
$$\leq [F(x_1, y_2) - F(x_1, x_2)][F(y_1, x_2) - F(x_1, x_2)],$$

from where, F is RR_2.

(d):RR_2 *c.d.f.* $\Rightarrow$ LTI:

$$F \text{ is } RR_2, \text{ if and only if } \frac{F(x_1, x_2)}{F(x_1, y_2)} \leq \frac{F(y_1, x_2)}{F(y_1, y_2)},\ x_1 < y_1, x_2 < y_2.$$

If we let $y_2 \to +\infty$ then the previous inequality becomes

$$\frac{F(x_1, x_2)}{F_1(x_1)} \leq \frac{F(y_1, x_2)}{F_1(y_1)},$$

so that, $\frac{F(\cdot, x_2)}{F_1(\cdot)}$ is increasing.

♣

4.2 Proof of Proposition 2.2

(c): $NDS \Rightarrow NUOD$:

Given x_j, x'_j, so that $x_j > x'_j$, by hypothesis we have that

$$P(X_i > x_i, i \neq j | X_j = x_j) \leq P(X_i > x_i, i \neq j | X_j = x'_j).$$

Consequently,

$$\frac{\int_{x_j}^{\infty} P(X_i > x_i, i \neq j | X_j = x) dF_j(x)}{\int_{x_j}^{\infty} dF_j(x)} \leq \frac{\int_{x'_j}^{\infty} P(X_i > x_i, i \neq j | X_j = x) dF_j(x)}{\int_{x'_j}^{\infty} dF_j(x)}.$$

Consider $j = 1$ and $x'_j \to -\infty$. Then, the previous inequality becomes ,

$$P(X_i > x_i, i = 1, \cdots, n) \leq P(X_1 > x_1) P(X_i > x_i, i = 2, \cdots, n).$$

Moreover, $(X_2, \cdots, X_n)$ is NDS and we can apply the same reasoning for the iterated process, i.e.,

$$P(X_i > x_i, i = 2, \cdots, n) \leq P(X_2 > x_2) P(X_i > x_i, i = 3, \cdots, n).$$

This way, by iteration we get $NUOD$.

A similar procedure proves that $NDS \Rightarrow NLOD$.

♣

4.3 Proof of Theorem 2.2

Let us consider $y'_1 \leq y_1$, $y_2 \leq y'_2$. We have to check that

$$f_{\underline{Y}}(y_1, y_2) f_{\underline{Y}}(y'_1, y'_2) \leq f_{\underline{Y}}(y'_1, y_2) f_{\underline{Y}}(y_1, y'_2)$$

where $y_i = \eta_i(x_i), y'_i = \eta_i(x'_i),\ i = 1, 2$. Checking that is the same as to verify

$$f_{\underline{X}}(\eta_1^{-1}(y_1), \eta_2^{-1}(y'_2)) f_{\underline{X}}(\eta_1^{-1}(y'_1), \eta_2^{-1}(y_2)) \geq f_{\underline{X}}(\eta_1^{-1}(y_1), \eta_2^{-1}(y_2)) f_{\underline{X}}(\eta_1^{-1}(y'_1), \eta_2^{-1}(y'_2))$$

Without loss of generality we suppose that η_1 is monotonically decreasing, invertible with inverse function being monotonically decreasing. Also, suppose η_2 is monotonically increasing, invertible and hence, with monotonically increasing inverse function. This way, as $y'_1 \leq y_1$ and $y_2 \leq y'_2$, we have that

$$\begin{aligned} \eta_1^{-1}(y_1) &\leq \eta_1^{-1}(y'_1), \\ \eta_2^{-1}(y_2) &\leq \eta_2^{-1}(y'_2), \end{aligned}$$

and finally, as $f_{\underline{X}}$ is RR_2, we have the inequality.

♣

4.4 Proof of Proposition 2.4

Recall that

(1)

$$f_{\underline{Y}}\, MTP_2 \Leftrightarrow f_{\underline{Y}}(\underline{u})f_{\underline{Y}}(\underline{v}) \leq f_{\underline{Y}}(\underline{u} \vee \underline{v})f_{\underline{Y}}(\underline{u} \wedge \underline{v})$$

(2)

$$f_2 >_{TP_2} f_1 \Leftrightarrow f_1(\underline{u})f_2(\underline{v}) \leq f_2(\underline{u} \vee \underline{v})f_1(\underline{u} \wedge \underline{v})$$

(3)

$$f_1 \text{ monotonically increasing} \Rightarrow f_1(\underline{u} \wedge \underline{v}) \leq f_1(\underline{u} \vee \underline{v})$$

From (1)(2) and (3), it follows that,

$$f_1(\underline{u})f_{\underline{Y}}(\underline{u})f_2(\underline{v})f_{\underline{Y}}(\underline{v}) \leq f_1(\underline{u} \vee \underline{v})f_2(\underline{u} \vee \underline{v})f_{\underline{Y}}(\underline{u} \vee \underline{v})f_{\underline{Y}}(\underline{u} \wedge \underline{v}).$$

Define,

$$\begin{aligned} g_i(\cdot) &:= f_i(\cdot)f_{\underline{Y}}(\cdot) \;\; i = 1, 2, \\ g_3(\cdot) &:= f_{\underline{Y}}(\cdot), \\ g_4(\cdot) &:= f_1(\cdot)f_2(\cdot)f_{\underline{Y}}(\cdot). \end{aligned}$$

We have that,

$$g_1(\underline{u})g_2(\underline{v}) \leq g_3(\underline{u} \wedge \underline{v})g_4(\underline{u} \vee \underline{v}),$$

so that, [Karlim and Rinott(1980)a, Theorem 2.1]allows us to guarantee the following inequality,

$$\int g_1(\underline{y})d\underline{y} \int g_2(\underline{y})d\underline{y} \leq \int g_3(\underline{y})d\underline{y} \int g_4(\underline{y})d\underline{y},$$

or equivalently,

$$E_{\underline{Y}}(f_1)E_{\underline{Y}}(f_2) \leq E_{\underline{Y}}(f_1 f_2),$$

i.e.,

$$Cov(f_1(\underline{Y}), f_2(\underline{Y})) \geq 0. \qquad \clubsuit$$

4.5 Proof of Theorem 2.3

Let us consider, without loss of generality, $k = 4,\ j_0 = 3$.

The proof of (a) follows from Theorem 2.2.

(b) : $f_{X_2,\eta_3(X_3)}$ is TP_2 if and only if,

$f_{X_2,\eta_3(X_3)}(x_2, \eta_3(x_3)) f_{X_2,\eta_3(X_3)}(x_2', \eta_3(x_3'))$

$$\leq f_{X_2,\eta_3(X_3)}(x_2 \vee x_2', \eta_3(x_3) \vee \eta_3(x_3')) f_{X_2,\eta_3(X_3)}(x_2 \wedge x_2', \eta_3(x_3) \wedge \eta_3(x_3')).$$

The interesting case is when $x_2 \leq x_2',\ \ \eta_3(x_3') \leq \eta_3(x_3)$ (i.e., $x_3 \leq x_3'$) and in this case we should verify,

$$f_{X_2,\eta_3(X_3)}(x_2, \eta_3(x_3)) f_{X_2,\eta_3(X_3)}(x_2', \eta_3(x_3')) \leq f_{X_2,\eta_3(X_3)}(x_2', \eta_3(x_3)) f_{X_2,\eta_3(X_3)}(x_2, \eta_3(x_3')),$$

or equivalently,

$$f_{X_2,X_3}(x_2, x_3) f_{X_2,X_3}(x_2', x_3') \leq f_{X_2,X_3}(x_2', x_3) f_{X_2,X_3}(x_2, x_3')$$

or

$$\int f_{X_1,X_2,X_3}(x_1, x_2, x_3) dx_1 \int f_{X_1,X_2,X_3}(x_1, x_2', x_3') dx_1$$
$$\leq \int f_{X_1,X_2,X_3}(x_1, x_2', x_3) dx_1 \int f_{X_1,X_2,X_3}(x_1, x_2, x_3') dx_1,$$

By hypothesis, we have,

$(i) f_{X_1,X_2,X_3}(u, x_2, x_3) f_{X_1,X_2,X_3}(v, x_2', x_3')$

$$\leq f_{X_1,X_2,X_3}(u \vee v, x_2 \vee x_2', x_3 \wedge x_3') f_{X_1,X_2,X_3}(u \wedge v, x_2 \wedge x_2', x_3 \vee x_3'),$$

or analogously,

$$f_{X_1,X_2,X_3}(u, x_2, x_3) f_{X_1,X_2,X_3}(v, x_2', x_3') \leq f_{X_1,X_2,X_3}(u \vee v, x_2', x_3) f_{X_1,X_2,X_3}(u \wedge v, x_2, x_3').$$

Therefore, by Theorem 2.1 in Karlin and Rinott (1980)a the inequality involving the integral holds. Hence, it results that (b) : $f_{X_2,\eta_3(X_3)}$ is TP_2. ♣

4.6 Proof of Theorem 2.4

We need to check the conditions (i) and (ii) in Theorem 2.3, and that f_{X_1,X_s} is TP_2 $\forall s \neq j_0$, $s = 1, \cdots, k$; and $f_{X_1,X_{j_0}}$ is RR_2.

For the last part it is enough to recall that step by step through the values of s we should fix all the others $k-2$ remaining variables and also, that h does not dependent on X_{j_0}.

From that, recalling that η_{j_0} is decreasing, invertible and differentiable, we can conclude that (a) in the theorem holds. Similar procedure can be used to prove (b).

In order to prove the condition (i) in Theorem 2.3, note that ,

$$f_{X_1,X_s,X_{j_0}}(x_1,x_s,x_{j_0}) = K(x_1,x_{j_0}) \int h(x_1,x_{(2)}^{(s-1)},x_s,x_{(s+1)}^{(k)})d(x_{(2)}^{(s-1)},x_{(s+1)}^{(k)}).$$

Since K is RR_2,

$$K(x_1,x_{j_0})K(x_1',x_{j_0}') \le K(x_1',x_{j_0})K(x_1,x_{j_0}'), \text{ when } x_1 \le x_1', x_{j_0} \le x_{j_0}'.$$

Equivalently,

$$(1)\ K(x_1,x_{j_0})K(x_1',x_{j_0}') \le K(x_1 \vee x_1', x_{j_0} \wedge x_{j_0}')K(x_1 \wedge x_1', x_{j_0} \vee x_{j_0}').$$

The hypothesis (iii) allows us to finish the proof since,

$$(2) \int h(x_1,x_{(2)}^{(s-1)},x_s,x_{(s+1)}^{(k)})d(x_{(2)}^{(s-1)},x_{(s+1)}^{(k)}) \int h(x_1',x_{(2)}^{(s-1)},x_s',x_{(s+1)}^{(k)})d(x_{(2)}^{(s-1)},x_{(s+1)}^{(k)})$$

$$\le \int h(x_1 \vee x_1',x_{(2)}^{(s-1)},x_s \vee x_s',x_{(s+1)}^{(k)})d(x_{(2)}^{(s-1)},x_{(s+1)}^{(k)})$$
$$\int h(x_1 \wedge x_1',x_{(2)}^{(s-1)},x_s \wedge x_s',x_{(s+1)}^{(k)})d(x_{(2)}^{(s-1)},x_{(s+1)}^{(k)}).$$

Doing the product between (1) and (2) we get that $f_{X_1,X_s,X_{j_0}}$ satisfies the condition (i) of Theorem 2.3, namely,

$$(i) f_{X_1,X_s,X_{j_0}}(x_1,x_s,x_{j_0}) f_{X_1,X_s,X_{j_0}}(x_1',x_s',x_{j_0}')$$

$$\le f_{X_1,X_s,X_{j_0}}(x_1 \vee x_1', x_s \vee x_s', x_{j_0} \wedge x_{j_0}') f_{X_1,X_s,X_{j_0}}(x_1 \wedge x_1', x_s \wedge x_s', x_{j_0} \vee x_{j_0}').$$

This completes the proof for $2 \le s < j_0$. By an analogous procedure, taking $j_0 + 1 \le t \le k$, we can get that f_{X_1,X_{j_0},X_t} satisfies the condition (ii), of Theorem 2.3.

♣

4.7 Proof of Proposition 2.3

It is enough to notice that h is MTP_2, so that, condition (iii) of Theorem 2.4 follows.

♣

4.8 Proof of Remark 2.4

($\Rightarrow$) : clear.
($\Leftarrow$) : Note that if f and g are arbitrary increasing functions then $f(\underline{T})$ and $g(\underline{T})$ are random vectors, whose images can be all the R or a subset of it. $X_{f(\underline{T})}(s)$ and $X_{g(\underline{T})}(t)$, are increasing, binary functions of $\underline{T}$. Defining,

$$S := f(\underline{T}),\ T := g(\underline{T}),\ \Gamma_s(\underline{T}) := X_{f(\underline{T})}(s) \text{ and } \Delta_t(\underline{T}) := X_{g(\underline{T})}(t),$$

since, for each pair of fixed values s and t, the functions Γ_s and Δ_t are increasing and binary, we have that,

$$Cov\{\Gamma_s(\underline{T}), \Delta_t(\underline{T})\} \geq 0,$$

by the hypothesis. Finally, by Remark 2.5 it results

$$Cov\{f(\underline{T}), g(\underline{T})\} \geq 0,$$

for f and g increasing and arbitrary. It follows that $\underline{T}$ is PA.

♣

4.9 Proof of Remark 2.6

Let us consider $\underline{T} = (T_1, \cdots, T_n)$ a *PA* vector and a sequence of m increasing functions $f_i : R^n \to R$, $i = 1, \cdots, m$ where $m \neq n$ in general. Define the random vectors $S_i := f_i(\underline{T})$, $i = 1, \cdots, m$ and then the vector $\underline{S} := (S_1, \cdots, S_m)$ whose coordinates are combinations of all *PA* random variables $\{T_i\}_{i=1}^n$. Let $\Gamma, \Delta : R^m \to R$ be arbitrary, binary and increasing functions. Then,

$$\Gamma(f(\underline{T})) := \Gamma(f_1(\underline{T}), \cdots, f_m(\underline{T})) \quad \text{and}$$
$$\Delta(f(\underline{T})) := \Delta(f_1(\underline{T}), \cdots, f_m(\underline{T}))$$

are binary, increasing functions of $\underline{T}$. Therefore,

$$Cov_{\underline{S}}\{\Gamma(\underline{S}), \Delta(\underline{S})\} = Cov_{\underline{T}}\{\Gamma(f(\underline{T})), \Delta(f(\underline{T}))\} \geq 0,$$

since, $\underline{T}$ is *PA* and $\Gamma \circ f$, $\Delta \circ f$ are binary, increasing. Finally, by this last result and the arbitrariness of the Γ and Δ functions we can conclude that $\underline{S}$ is *PA*.

♣

4.10 Proof of Proposition 2.5

We use two well known properties of the random variables. First, if U_1 and U_2 are random variables and $\underline{Z}$ is a random vector then

$$Cov\{U_1, U_2\} = E\{Cov(U_1, U_2|\underline{Z})\} + Cov\{E(U_1|\underline{Z}), E(U_2|\underline{Z})\}. \quad (4.1)$$

Second, that ,

$$Cov\{\psi_1(X), \psi_2(X)\} \geq 0, \quad (4.2)$$

where X is a $r.v.$ and ψ_i, $i = 1, 2$ are two agreeing monotone functions.

Consider Γ, $\Delta : R^2 \to R$ binary increasing functions. We need to prove that,

$$Cov\{\Gamma(X_1, h(X_2, X_3)), \Delta(X_1, h(X_2, X_3))\} \geq 0,$$

so that, using Remark 2.4, we have X_1 and $h(X_2, X_3)$ are PA.

Let us consider in the Property 4.1,

$$U_1 := \Gamma(X_1, h(X_2, X_3)), \;\; U_2 := \Delta(X_1, h(X_2, X_3)), \;\; \underline{Z} := X_2.$$

Note that, for each fixed $X_2 = x_2$, the functions

$$\Gamma(X_1, h(x_2, X_3)) \text{ and } \Delta(X_1, h(x_2, X_3))$$

are binary, increasing functions of X_1 and X_3, for h is increasing and Γ, Δ are binary increasing. Since(X_1, X_3) is PA, then,

$$Cov\{\Gamma(X_1, h(x_2, X_3)), \Delta(X_1, h(x_2, X_3))\} \geq 0, \;\; \forall x_2 \in R. \quad (4.3)$$

Since by hypothesis,

$$f_{X_1,X_3}(x_1, h_{(\cdot)}^{-1}(z)) >_{TP_2(\cdot)} \frac{\partial h}{\partial v}(\cdot, h_{(\cdot)}^{-1}(z)),$$

let x_2 and x_2' be two values on the image of X_2 so that $x_2 < x_2'$ and we have,

$$f_{X_1,X_3}(x_1, h_{x_2}^{-1}(u))\frac{\partial h}{\partial x_3}(x_2', h_{x_2'}^{-1}(u)) \leq f_{X_1,X_3}(x_1, h_{x_2'}^{-1}(u))\frac{\partial h}{\partial x_3}(x_2, h_{x_2}^{-1}(u)).$$

On the other hand, for each fixed x_2,

$$E(\quad \Gamma(X_1, h(x_2, X_3)) \quad) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma(x_1, h(x_2, x_3)) f_{X_1,X_3}(x_1, x_3) dx_3 dx_1$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma(x_1, u) f_{X_1,X_3}(x_1, h_{x_2}^{-1}(u)) \frac{du}{\frac{\partial h(x_2, h_{x_2}^{-1}(u))}{\partial x_3}} dx_1$$

where $u = h(x_2, x_3)$, allowing to see that, $x_3 = h_{x_2}^{-1}(u)$ and $dx_3 = \frac{du}{\frac{\partial h(x_2, h_{x_2}^{-1}(u))}{\partial x_3}}$; by the previous inequality it is possible to prove that, for $x_2 < x_2'$,

$$E(\Gamma(X_1, h(x_2, X_3))) \leq E(\Gamma(X_1, h(x_2', X_3))).$$

Therefore, $E(\Gamma(X_1, h(\cdot, X_3)))$ is an increasing function on the image of X_2. Similarly, we can show that $E(\Delta(X_1, h(\cdot, X_3)))$ is an increasing function on the image of X_2. Finally, by the Property in 4.2 we have,

$$Cov\{E(\Gamma(X_1, h(X_2, X_3))|X_2), E(\Delta(X_1, h(X_2, X_3))|X_2)\} \geq 0 \ a.s.. \quad (4.4)$$

Now, from the results in 4.3 and in 4.4 we conclude that

$$Cov\{\Gamma(X_1, h(X_2, X_3)), \Delta(X_1, h(X_2, X_3))\} \geq 0.$$

♣

4.11 Proof of Theorem 3.2

(A.1): it is enough to show it for $t = 1, s = 2, j = 3$,

- $\underline{X} = (X_1, X_2, X_3, X_{(4)}^{(k)})$, $\underline{X}' = (X_1, X_2, X_3', X_{(4)}^{(k)})$, $X_3' > X_3$.

∘ If $k > 3$ we have that
$\xi_{1,2}^*(\theta_1, \theta_2|\underline{X})$

$$= H(\underline{X}) g_1(X_1, \theta_1) g_2(X_2, \theta_2) \int g_3(X_3, \theta_3) \left[\int \cdots \int C(\underline{\theta}) \prod_{i=4}^{k} g_i(X_i, \theta_i) d\theta_i\right] d\theta_3.$$

Consider now two arbitrary pair of points (θ_1, θ_2) and (θ_1', θ_2'). One has

$$(1)\ \xi_{1,2}^*(\cdot, \cdot|\underline{X}) >_{TP_2(\cdot,\cdot)} \xi_{1,2}^*(\cdot, \cdot|\underline{X}')$$

if and only if,

$$(2)\ \xi_{1,2}^*(\theta_1 \vee \theta_1', \theta_2 \vee \theta_2'|\underline{X}) \xi_{1,2}^*(\theta_1 \wedge \theta_1', \theta_2 \wedge \theta_2'|\underline{X}') \geq \xi_{1,2}^*(\theta_1', \theta_2'|\underline{X}) \xi_{1,2}^*(\theta_1, \theta_2|\underline{X}')$$

if and only if,

$$g_1(x_1,\theta_1\vee\theta_1')g_2(x_2,\theta_2\vee\theta_2')\int g_3(x_3,\theta_3)\left[\int\cdots\int C(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_{(3)}^{(k)})\right.$$

$$\left.\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3 g_1(x_1,\theta_1\wedge\theta_1')g_2(x_2,\theta_2\wedge\theta_2')\int g_3(x_3',\theta_3)$$

$$\left[\int\cdots\int C(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3$$

$$\geq g_1(x_1,\theta_1')g_2(x_2,\theta_2')\int g_3(x_3,\theta_3)\left[\int\cdots\int C(\theta_1',\theta_2',\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3$$

$$g_1(x_1,\theta_1)g_2(x_2,\theta_2)\int g_3(x_3',\theta_3)\left[\int\cdots\int C(\theta_1,\theta_2,\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3.$$

But, as g_i is TP_2 on (X_i,θ_i) $\forall i=1,\cdots,k$, the previous inequality is equivalent to

$$\text{(3)}\quad \int g_3(x_3,\theta_3)\left[\int\cdots\int C(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3$$

$$\int g_3(x_3',\theta_3)\left[\int\cdots\int C(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3$$

$$\geq\int g_3(x_3,\theta_3)\left[\int\cdots\int C(\theta_1',\theta_2',\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3$$

$$\int g_3(x_3',\theta_3)\left[\int\cdots\int C(\theta_1,\theta_2,\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i\right]d\theta_3.$$

Let us denote by $G(\theta_1,\theta_2,\theta_3,x_{(4)}^{(k)})$ the

$$\int\cdots\int C(\theta_1,\theta_2,\theta_{(3)}^{(k)})\prod_{i=4}^{k}g_i(x_i,\theta_i)d\theta_i.$$

Claim:

$$\text{(4)}\ G(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_3\wedge\theta_3',x_{(4)}^{(k)})G(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_3\vee\theta_3',x_{(4)}^{(k)})$$

$$\geq G(\theta_1,\theta_2,\theta_3,x_{(4)}^{(k)})G(\theta_1',\theta_2',\theta_3',x_{(4)}^{(k)})$$

Proof of (4):

Defining

$$
\begin{aligned}
f_1(\theta_{(4)}^{(k)}) &:= C(\theta_1, \theta_2, \theta_3, \theta_{(4)}^{(k)}) \prod_{i=4}^{k} g_i(x_i, \theta_i) \\
f_2(\theta_{(4)}^{(k)}) &:= C(\theta'_1, \theta'_2, \theta'_3, \theta_{(4)}^{(k)}) \prod_{i=4}^{k} g_i(x_i, \theta_i) \\
f_3(\theta_{(4)}^{(k)}) &:= C(\theta_1 \wedge \theta'_1, \theta_2 \wedge \theta'_2, \theta_3 \vee \theta'_3, \theta_{(4)}^{(k)}) \prod_{i=4}^{k} g_i(x_i, \theta_i) \\
f_4(\theta_{(4)}^{(k)}) &:= C(\theta_1 \vee \theta'_1, \theta_2 \vee \theta'_2, \theta_3 \wedge \theta'_3, \theta_{(4)}^{(k)}) \prod_{i=4}^{k} g_i(x_i, \theta_i)
\end{aligned}
$$

it is possible to show that for

$$\underline{v} = (\theta_4, \theta_5, \cdots, \theta_k) \text{ and } \underline{w} = (\theta'_4, \theta'_5, \cdots, \theta'_k),$$

one has

$$f_1(\underline{v}) f_2(\underline{w}) \leq f_3(\underline{v} \wedge \underline{w}) f_4(\underline{v} \vee \underline{w}),$$

or, equivalently, (for the g_is are TP_2)

$$C(\underline{\theta})C(\underline{\theta}') \leq C(\theta_{(1)}^{(j-1)} \vee \theta_{(1)}^{\prime(j-1)}, \theta_j \wedge \theta'_j, \theta_{(j+1)}^{(k)} \vee \theta_{j+1}^{\prime(k)})$$

$$C(\theta_{(1)}^{(j-1)} \wedge \theta_1^{\prime(j-1)}, \theta_j \vee \theta'_j, \theta_{(j+1)}^{(k)} \wedge \theta_{(j+1)}^{\prime(k)}).$$

The last inequality comes from hypothesis. Now, by Karlin and Rinott (1980a, Theorem 2.1) we have that,

$$\int f_1(\theta_{(4)}^{(k)}) d\theta_{(4)}^{(k)} \int f_2(\theta_{(4)}^{(k)}) d\theta_{(4)}^{(k)} \leq \int f_3(\theta_{(4)}^{(k)}) d\theta_{(4)}^{(k)} \int f_4(\theta_{(4)}^{(k)}) d\theta_{(4)}^{(k)},$$

so that we have proven (4).

(3) is equivalent to

(5)

$$
\begin{aligned}
&\int g_3(x_3, \theta_3) G(\theta_1 \vee \theta'_1, \theta_2 \vee \theta'_2, \theta_3, x_{(4)}^{(k)}) d\theta_3 \times \\
&\int g_3(x'_3, \theta_3) G(\theta_1 \wedge \theta'_1, \theta_2 \wedge \theta'_2, \theta_3, x_{(4)}^{(k)}) d\theta_3 \\
&\geq \int g_3(x_3, \theta_3) G(\theta'_1, \theta'_2, \theta_3, x_{(4)}^{(k)}) d\theta_3 \int g_3(x'_3, \theta_3) G(\theta_1, \theta_2, \theta_3, x_{(4)}^{(k)}) d\theta_3
\end{aligned}
$$

In order to prove (5), we define

$$\begin{aligned} f_1^*(\theta_3) &:= g_3(x_3,\theta_3)G(\theta_1',\theta_2',\theta_3,x_{(4)}^{(k)}) \\ f_2^*(\theta_3) &:= g_3(x_3',\theta_3)G(\theta_1,\theta_2,\theta_3,x_{(4)}^{(k)}) \\ f_3^*(\theta_3) &:= g_3(x_3,\theta_3)G(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_3,x_{(4)}^{(k)}) \\ f_4^*(\theta_3) &:= g_3(x_3',\theta_3)G(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_3,x_{(4)}^{(k)}), \end{aligned}$$

verifying that

$$f_1^*(\theta_3')f_2^*(\theta_3) \le f_3^*(\theta_3\wedge\theta_3')f_4^*(\theta_3\vee\theta_3'),$$

because of g_3 being TP_2; then, by Karlin and Rinott (1980a, Theorem 2.1) we have just proved (5), and equivalently (1).

- If $k=3$ then

$$\xi_{1,2}^*(\theta_1,\theta_2|\underline{X}) = H(\underline{X})g_1(X_1,\theta_1)g_2(X_2,\theta_2)\int g_3(X_3,\theta_3)C(\theta_1,\theta_2,\theta_3)d\theta_3.$$

Hence, as g_1 and g_2 are TP_2, (1) can be checked through

(6) $$g_3(x_3,\theta_3)C(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_3)g_3(x_3',\theta_3)C(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_3)$$

$$\ge g_3(x_3',\theta_3)C(\theta_1,\theta_2,\theta_3)g_3(x_3,\theta_3)C(\theta_1',\theta_2',\theta_3)$$

and (6) is verified from the hypotheses on C, g_3. Karlin and Rinott (1980a, Theorem 2.1) implies (1).

- $\underline{X} = (X_1,X_2,X_3,X_4,\cdots,X_k)$, $\underline{X}' = (X_1,X_2,X_3',X_4',\cdots,X_k)$ with $X_3' > X_3$, $X_4' < X_4$.

In this case (3) becomes

(a3)

$$\int g_3(x_3,\theta_3)\left[\int g_4(x_4,\theta_4)\left(\int\cdots\int C(\theta_1\vee\theta_1',\theta_2\vee\theta_2',\theta_{(3)}^{(k)})\prod_{i=5}^{k} g_i(x_i,\theta_i)d\theta_i\right)d\theta_4\right]d\theta_3$$

$$\int g_3(x_3',\theta_3)\left[\int g_4(x_4',\theta_4)\left(\int\cdots\int C(\theta_1\wedge\theta_1',\theta_2\wedge\theta_2',\theta_{(3)}^{(k)})\prod_{i=5}^{k} g_i(x_i,\theta_i)d\theta_i\right)d\theta_4\right]d\theta_3$$

$$\ge \int g_3(x_3,\theta_3)\left[\int g_4(x_4,\theta_4)\left(\int\cdots\int C(\theta_1',\theta_2',\theta_{(3)}^{(k)})\prod_{i=5}^{k} g_i(x_i,\theta_i)d\theta_i\right)d\theta_4\right]d\theta_3$$

$$\int g_3(x_3',\theta_3)\left[\int g_4(x_4',\theta_4)\left(\int\cdots\int C(\theta_1,\theta_2,\theta_{(3)}^{(k)})\prod_{i=5}^{k} g_i(x_i,\theta_i)d\theta_i\right)d\theta_4\right]d\theta_3$$

With $(x_5, \cdots, x_k)$ fixed, let us define

$$H(\theta_1, \theta_2, \theta_3, x_4) := \int g_4(x_4, \theta_4) \left(\int \cdots \int C(\theta_1, \theta_2, \theta_3, \cdots, \theta_k) \prod_{i=5}^{k} g_i(x_i, \theta_i) d\theta_i \right) d\theta_4$$

so that **(a3)** becomes
(a3∗)

$$\int g_3(x_3, \theta_3) H(\theta_1 \vee \theta_1', \theta_2 \vee \theta_2', \theta_3, x_4) d\theta_3 \int g_3(x_3', \theta_3) H(\theta_1 \wedge \theta_1', \theta_2 \wedge \theta_2', \theta_3, x_4') d\theta_3$$
$$\geq \int g_3(x_3, \theta_3) H(\theta_1', \theta_2', \theta_3, x_4) d\theta_3 \int g_3(x_3', \theta_3) H(\theta_1, \theta_2, \theta_3, x_4') d\theta_3.$$

Claim: For θ_3' and θ_3 both arbitrary,

$$g_3(x_3, \theta_3 \wedge \theta_3') H(\theta_1 \vee \theta_1', \theta_2 \vee \theta_2', \theta_3 \wedge \theta_3', x_4)$$
$$g_3(x_3', \theta_3 \vee \theta_3') H(\theta_1 \wedge \theta_1', \theta_2 \wedge \theta_2', \theta_3 \vee \theta_3', x_4')$$
$$\geq g_3(x_3, \theta_3') H(\theta_1', \theta_2', \theta_3', x_4) g_3(x_3', \theta_3) H(\theta_1, \theta_2, \theta_3, x_4').$$

Using the claim and Karlin and Rinott (1980a, Theorem 2.1)we conclude the proof of **(a3)**.

The proof of the last claim follows using Karlin and Rinott (1980a, Theorem 2.1), the fact that g_4 is TP_2 and that $x_4 > x_4'$.

Proof of (A.2):

(7) $\xi^*_{1,\cdots,j-1}(\theta_1, \theta_2, \cdots, \theta_{j-1} | \underline{X})$

$$= H(\underline{X}) g_1(X_1, \theta_1) \cdots g_{j-1}(X_{j-1}, \theta_{j-1}) \int \cdots \int C(\underline{\theta}) \prod_{i=j}^{k} g_i(X_i, \theta_i) d\theta_i$$

Take two pair of arbitrary points $(\theta_1, \cdots, \theta_{j-1})$ e $(\theta_1', \cdots, \theta_{j-1}')$. Supposing that

$$\underline{X} = (X_1, \cdots, X_{j-1}, X_j, X_{j+1}, \cdots, X_k) \text{ and } \underline{X'} = (X_1, \cdots, X_{j-1}, X_j', X_{j+1}', \cdots, X_k')$$

with $X_j' > X_j$ $X_s' \leq X_s$ $s > j$ note that (A.2) is equivalent to
(8)

$$g_1(x_1, \theta_1 \vee \theta_1') \cdots g_{j-1}(x_{j-1}, \theta_{j-1} \vee \theta_{j-1}') \left[\int \cdots \int C(\theta_1 \vee \theta_1', \cdots, \theta_{j-1} \vee \theta_{j-1}', \theta_{(j)}^{(k)}) \right.$$
$$\left. \prod_{i=j}^{k} g_i(x_i, \theta_i) d\theta_i \right] g_1(x_1, \theta_1 \wedge \theta_1') \cdots g_{j-1}(x_{j-1}, \theta_{j-1} \wedge \theta_{j-1}')$$

$$\left[\int\cdots\int C(\theta_1\wedge\theta'_1,\cdots,\theta_{j-1}\wedge\theta'_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x'_i,\theta_i)d\theta_i\right]$$

$$\geq g_1(x_1,\theta'_1)\cdots g_{j-1}(x_{j-1},\theta'_{j-1})\left[\int\cdots\int C(\theta'_1,\cdots,\theta'_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x_i,\theta_i)d\theta_i\right]$$

$$g_1(x_1,\theta_1)\cdots g_{j-1}(x_{j-1},\theta_{j-1})\left[\int\cdots\int C(\theta_1,\cdots,\theta_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x'_i,\theta_i)d\theta_i\right].$$

But, since g_i is TP_2 on (X_i,θ_i) $\forall i=1,\cdots,k$, **(8)** is equivalent to

$$\text{(9)}\quad \int\cdots\int C(\theta_1\vee\theta'_1,\cdots,\theta_{j-1}\vee\theta'_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x_i,\theta_i)d\theta_i$$

$$\int\cdots\int C(\theta_1\wedge\theta'_1,\cdots,\theta_{j-1}\wedge\theta'_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x'_i,\theta_i)d\theta_i$$

$$\geq\int\cdots\int C(\theta'_1,\cdots,\theta'_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x_i,\theta_i)d\theta_i$$

$$\int\cdots\int C(\theta_1,\cdots,\theta_{j-1},\theta^{(k)}_{(j)})\prod_{i=j}^{k}g_i(x'_i,\theta_i)d\theta_i.$$

∘ Consider the case $X'_s=X_s$ $\forall s>j$.

Denoting by $G(\theta_1,\cdots,\theta_{j-1},\theta_j,x^{(k)}_{(j+1)})$ the expression

$$\int\cdots\int C(\theta_1,\cdots,\theta_{j-1},\theta^{(k)}_{(j)})\prod_{i=j+1}^{k}g_i(x_i,\theta_i)d\theta_i$$

we can claim that

$$\text{(10)}\ G(\theta_1\vee\theta'_1,\cdots,\theta_{j-1}\vee\theta'_{j-1},\theta_j\wedge\theta'_j,x^{(k)}_{(j+1)})\times G(\theta_1\wedge\theta'_1,\cdots,\theta_{j-1}\wedge\theta'_{j-1},\theta_j\vee\theta'_j,x^{(k)}_{(j+1)})$$

$$\geq G(\theta_1,\cdots,\theta_{j-1},\theta_j,x^{(k)}_{(j+1)})G(\theta'_1,\cdots,\theta'_{j-1},\theta'_j,x^{(k)}_{(j+1)}),$$

just by adapting the proof done for **(4)**. Therefore, **(9)** becomes

$$\text{(11)}\quad \int g_j(x_j,\theta_j)G(\theta_1\vee\theta'_1,\cdots,\theta_{j-1}\vee\theta'_{j-1},\theta_j,x^{(k)}_{(j+1)})d\theta_j$$

$$\int g_j(x'_j,\theta_j)G(\theta_1\wedge\theta'_1,\cdots,\theta_{j-1}\wedge\theta'_{j-1},\theta_j,x^{(k)}_{(j+1)})d\theta_j$$

$$\geq \int g_j(x_j,\theta_j)G(\theta'_1,\cdots,\theta'_{j-1},\theta_j,x^{(k)}_{(j+1)})d\theta_j$$

$$\int g_j(x'_j,\theta_j)G(\theta_1,,\cdots,\theta_{j-1},\theta_j,x^{(k)}_{(j+1)})d\theta_j,$$

and by a proof similar to done for **(5)**, using **(10)**, we get a proof for **(11)**. Hence, we get that (A.2) is verified.

o The general case ($X'_s \leq X_s\ \forall s > j$) is treated as in the proof of (A.1).

Proof of (A.3):

(12) $\xi^*_{1,\cdots,j-1,j+1,\cdots,k}(\theta^{(j-1)}_{(1)},\theta^{(k)}_{(j+1)}|\underline{X})$

$$= H(\underline{X}) \prod_{\substack{i=1 \\ i\neq j}}^{k} g_i(X_i,\theta_i)\int g_j(X_j,\theta_j)C(\underline{\theta})d\theta_j.$$

Consider now two pairs of arbitrary points

$$(\theta_1,\cdots,\theta_{j-1},\theta_{j+1},\cdots,\theta_k),\ (\theta'_1,\cdots,\theta'_{j-1},\theta'_{j+1},\cdots,\theta'_k).$$

Supposing $\underline{X} = (X_1,\cdots,X_{j-1},X_j,X_{j+1},\cdots,X_k)$ and $\underline{X'} = (X_1,\cdots,X_{j-1},X'_j,X'_{j+1},\cdots,X'_k)$ where $X'_j > X_j$ and $X'_s \leq X_s$ for $s > j$ let us consider the two cases below:

o $X'_s = X_s\ s > j$ case: note that (A.3) is equivalent to,

$$\prod_{\substack{i=1 \\ i\neq j}}^{k} g_i(x_i,\theta_i\vee\theta'_i) \prod_{\substack{i=1 \\ i\neq j}}^{k} g_i(x'_i,\theta_i\wedge\theta'_i)$$

$$\int g_j(x_j,\theta_j)C(\theta^{(j-1)}_{(1)}\vee\theta'^{(j-1)}_{(1)},\theta_j,\theta^{(k)}_{(j+1)}\vee\theta'^{(k)}_{(j+1)})d\theta_j$$

$$\int g_j(x'_j,\theta_j)C(\theta^{(j-1)}_{(1)}\wedge\theta'^{(j-1)}_{(1)},\theta_j,\theta^{(k)}_{(j+1)}\wedge\theta'^{(k)}_{(j+1)})d\theta_j$$

$$\geq \prod_{\substack{i=1 \\ i\neq j}}^{k} g_i(x_i,\theta'_i)\int g_j(x_j,\theta_j)C(\theta'^{(j-1)}_{(1)},\theta_j,\theta'^{(k)}_{(j+1)})d\theta_j$$

$$\prod_{\substack{i=1\\ i\neq j}}^{k} g_i(x'_i,\theta_i)\int g_j(x'_j,\theta_j)C(\theta_{(1)}^{(k)})d\theta_j.$$

But, since $X'_s = X_s$, $s > j$ and g_i is TP_2 the previous inequality is equivalent to

$$(13)\quad \int g_j(x_j,\theta_j)C(\theta_{(1)}^{(j-1)}\vee\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\vee\theta_{(j+1)}^{\prime(k)})d\theta_j$$
$$\int g_j(x'_j,\theta_j)C(\theta_{(1)}^{(j-1)}\wedge\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\wedge\theta_{(j+1)}^{\prime(k)})d\theta_j$$
$$\geq \int g_j(x_j,\theta_j)C(\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{\prime(k)})d\theta_j \int g_j(x'_j,\theta_j)C(\theta_{(1)}^{(k)})d\theta_j,$$

if,

$$\begin{aligned}
f_1(\theta_j) &:= C(\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{\prime(k)})g_j(x_j,\theta_j)\\
f_2(\theta_j) &:= C(\theta_{(1)}^{(k)})g_j(x'_j,\theta_j)\\
f_3(\theta_j) &:= C(\theta_{(1)}^{(j-1)}\wedge\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\wedge\theta_{(j+1)}^{\prime(k)})g_j(x'_j,\theta_j)\\
f_4(\theta_j) &:= C(\theta_{(1)}^{(j-1)}\vee\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\vee\theta_{(j+1)}^{\prime(k)})g_j(x_j,\theta_j)
\end{aligned}$$

and since $f_1(u)f_2(v) \leq f_3(u\vee v)f_4(u\wedge v)$ if and only if

$$C(\theta_{(1)}^{\prime(j-1)},u,\theta_{(j+1)}^{\prime(k)})g_j(x_j,u)C(\theta_{(1)}^{(j-1)},v,\theta_{(j+1)}^{(k)})g_j(x'_j,v)$$
$$\leq C(\theta_{(1)}^{(j-1)}\wedge\theta_{(1)}^{\prime(j-1)},u\vee v,\theta_{(j+1)}^{(k)}\wedge\theta_{(j+1)}^{\prime(k)})g_j(x'_j,u\vee v)$$
$$C(\theta_{(1)}^{(j-1)}\vee\theta_{(1)}^{\prime(j-1)},u\wedge v,\theta_{(j+1)}^{(k)}\vee\theta_{(j+1)}^{\prime(k)})g_j(x_j,u\wedge v),$$

comes from the hypotheses. By using arguments already used in similar proofs we verify **(13)**; immediately (A.3) follows.

∘ $X'_s = X_s$, $s > j+1$, $X'_{j+1} < X_{j+1}$ case:

(A.3) is equivalent to

$$g_{j+1}(x_{j+1},\theta_{j+1}\vee\theta'_{j+1})g_{j+1}(x'_{j+1},\theta_{j+1}\wedge\theta'_{j+1})$$
$$\int g_j(x_j,\theta_j)C(\theta_{(1)}^{j-1}\vee\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\vee\theta_{(j+1)}^{\prime(k)})d\theta_j$$
$$\int g_j(x'_j,\theta_j)C(\theta_{(1)}^{(j-1)}\wedge\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{(k)}\wedge\theta_{(j+1)}^{\prime(k)})d\theta_j$$
$$\geq g_{j+1}(x_{j+1},\theta'_{j+1})\int g_j(x_j,\theta_j)C(\theta_{(1)}^{\prime(j-1)},\theta_j,\theta_{(j+1)}^{\prime(k)})d\theta_j$$
$$g_{j+1}(x'_{j+1},\theta_{j+1})\int g_j(x'_j,\theta_j)C(\theta_{(1)}^{(k)})d\theta_j,$$

for the g_i,s are TP_2. But, since $x'_{j+1} \leq x_{j+1}$ and g_{j+1} is TP_2 the problem is reduced to the verification of **(13)**, already done. ♣

4.12 Proof of Theorem 3.3

Note that

$$(B.1) \quad \xi^*(\underline{\theta}|\underline{X}) >_{TP_2(\underline{\theta})} \xi^*(\underline{\theta}|\underline{X'})$$

is equivalent to

(B.2)

$$\prod_{\substack{i=1\\ i\neq j}}^{k} g_i(x_i, \theta_i \vee \theta'_i) \prod_{\substack{i=1\\ i\neq j}}^{k} g_i(x'_i, \theta_i \wedge \theta'_i)$$
$$g_j(x_j, \theta_j \vee \theta'_j) g_j(x'_j, \theta_j \wedge \theta'_j) C(\underline{\theta} \vee \underline{\theta}') C(\underline{\theta} \wedge \underline{\theta}')$$

$$\geq g_j(x_j, \theta'_j) \prod_{\substack{i=1\\ i\neq j}}^{k} g_i(x_i, \theta'_i) g_j(x'_j; \theta_j) \prod_{\substack{i=1\\ i\neq j}}^{k} g_i(x'_i, \theta_i) C(\underline{\theta}') C(\underline{\theta}).$$

As g_i is $TP_2(X_i, \theta_i)$ and $x'_i \leq x_i$, $\forall i \neq j$, (B.2) is equivalent to $g_j(x_j, \theta_j \vee \theta'_j) C(\underline{\theta} \vee \underline{\theta}') g_j(x'_j, \theta_j \wedge \theta'_j) C(\underline{\theta} \wedge \underline{\theta}') \geq g_j(x_j, \theta'_j) C(\underline{\theta}') g_j(x'_j, \theta_j) C(\underline{\theta})$, which results directly from the hypotheses, when $x'_j \geq x_j$, g_j is RR_2 and C is MTP_2.

♣

4.13 Proof of Theorem 3.4

Since g_i is $RR_2(x_i, \theta_i)$ and $x'_i \geq x_i$, $i = 1, \cdots, j$,

$$g_i(x_i, \theta_i \vee \theta'_i) g_i(x'_i, \theta_i \wedge \theta'_i) \geq g_i(x_i, \theta'_i) g_i(x'_i, \theta_i)$$

following that,

$$(a) \prod_{i=1}^{j} g_i(x_i, \theta_i \vee \theta'_i) \prod_{i=1}^{j} g_i(x'_i, \theta_i \wedge \theta'_i) \geq \prod_{i=1}^{j} g_i(x_i, \theta'_i) \prod_{i=1}^{j} g_i(x'_i, \theta_i).$$

Since g_i is $TP_2(x_i, \theta_i)$ and $x'_i \leq x_i$, $i = j+1, \cdots, k$, it follows that,

$$(b) \prod_{i=j+1}^{k} g_i(x_i, \theta_i \vee \theta'_i) \prod_{i=j+1}^{k} g_i(x'_i, \theta_i \wedge \theta'_i) \geq \prod_{i=j+1}^{k} g_i(x_i, \theta'_i) \prod_{i=j+1}^{k} g_i(x'_i, \theta_i)$$

and

$$(c) C(\underline{\theta} \wedge \underline{\theta}')C(\underline{\theta} \vee \underline{\theta}') \geq C(\underline{\theta})C(\underline{\theta}')$$

$(a), (b), (c)$ besides of the form of the posterior distribution allow us to finish the proof.

♣

Acknowledgements

It is a pleasure to thank Carlos A.B. Pereira, Sergio Wechsler, Jorge Aragona and Luis G. Esteves for many fruitful discussions and suggestions. VAGL acknowledges support from CAPES/Brazil. NIT acknowledges partial support from FAPESP/Brazil-Projeto Temático 99/10611-8.

VAGL current address: U.B.A. Universidad de Buenos Aires, Pabellon II, 2do piso , Ciudad Universitaria, Buenos Aires, Argentina
e-mails: VAGL: paprica30@yahoo.es **NIT:** nitanaka@ime.usp.br

References

Ash, R.B. (1972). *Real Analysis and Probability.* Academic Press. New York and London.

Barlow, R. E.; Proschan, F. (1975). *Statistical Theory of Reliability and Life Test.* Holt, Rinehart and Winston. New York and London.

Block H.W.; Savits T.H.; Shaked M. (1982). Some concepts of negative dependence, *Ann. Probab.* **10**, 765-772.

Boland, P. J.; Hollander, M.; Joag-Dev, K.; Kochar, S. (1996). Bivariate dependence properties of order statistics. *J. Multivar. Anal.* Vol. 56, **1**, 75-89.

Ebrahimi, N.; Ghosh, M. (1981). Multivariate Negative Dependence. *Commun. in Stat.* (Theory & Meth.) A10, **4**, 307-337.

Efron P. (1965). Increasing properties of Pólya frequency functions. *Ann. Math. Stat.* **36**, 272-279.

Fahmy, S.; C.A.de B.Pereira; Proschan, F.; Shaked, M. (1982). The influence of the sample on the posterior distribution. *Commun. in Stat.* (Theory & Meth.) Vol.11, **16**, 1757-1768.

Joag-Dev, K. (1983). Independence via uncorrelatedness under certain dependence structures. *Ann. Probab.* Vol. 11, **4**, 1037-1041.

Joag-Dev, K.; Perlman, M. D.; Pitt, L. D.(1983). Association of normal random variables and Slepian's inequality. *Ann. Probab.* Vol. 11, **2**, 451-455.

Joag-Dev, K.; Proschan, F. (1983). Negative association of random variables, with applications. *Ann. Stat.*Vol. 11, **1**, 286-295.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Monogra. Stat. & Appl. Probab. 73 (Chapman and Hall).

Karlin, S. (1968). *Total Positivity.* Stanford Uni. Press. Stanford, California.

Karlin, S.; Rinott, Y. (1980)a. Classes of ordering of measures and related correlation inequalities.I. Multivariate totally positive distributions. *J. Multivar. Anal.* Vol.10, **4**, 467-498.

Karlin, S.; Rinott, Y. (1980)b. Classes of ordering of measures and related correlation inequalities.II. Multivariate reverse rule distributions. *J. Multivar. Anal.* Vol.10, **4**, 499-518.

Kim, J. S.; Proschan, F.(1995). A review: the arrangement increasing partial ordering. *Comput. Oper. Res.* Vol 22, **4**, 357-371.

Lehmann, E.L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **43**, 1137-1153.

Li H.; Scarsini M.; Shaked M. (1996). A tool for the construction of multivariate distributions with given non-overlapping multivariate marginals. *J. Multivar. Anal.* Vol 56, **1**, 20-41.

ÚLTIMOS RELATÓRIOS TÉCNICOS PUBLICADOS

2001-01 - KOTTAS, A., BRANCO, M.D., GELFAND, A.E., A Nonparametric bayesian modeling approach for cytogenetic dosimetry. 2001. 19p. (RT-MAE-2001-01)

2001-02 - AOKI, R., BOLFARINE, H., SINGER, J.M., Null Intercept measurement error regression models. 2001. 18p.(RT-MAE-2001-02)

2001-03 - AOKI, R., BOLFARINE, H., SINGER, J.M., Asymptotic Efficiency of null intercept measurement error regression models. 2001. 11p. (RT-MAE-2001-03).

2001-04 - ALMEIDA, S.S., LIMA, C.R.O.P., SANDOVAL, M.C., Linear Calibration in functional models without the normality assumption. 2001. 18p. (RT-MAE-2001-04).

2001-05 - GARCIA-ALFARO, K.E., BOLFARINE, H. Comparative calibration with subgroups. 2001. 18p. (RT-MAE-2001-05)

2001-06 - DUNLOP, F.M., FERRARI, P.A., FONTES, L.R.G. A dynamic one-dimensional interface interacting with a wall. 2001. 21p. (RT-MAE-2001-06).

2001-07 - FONTES, L.R., ISOPI, M., NEWMAN, C.M., STEIN, D.L. 1*D* Aging. 2001. 10p. (RT-MAE-2001-07).

2001-08 - BARROS, S.R.M., FERRARI, P.A., GARCIA, N.L., MARTÍNEZ, S. Asymptotic behavior of a stationary silo with absorbing walls. 2001. 20p. (RT-MAE-2001-08).

2001-09 - GONZALEZ-LOPEZ, V.A., TANAKA, N.Y. Characterization of copula and its relationships with TP_2 (RR_2) association. 2001. 24p. (RT-MAE-2001-09).

2001-10 - ORTEGA, E.M.M., BOLFARINE, H., PAULA, G.A. Influence Diagnostics in Generalized Log-Gamma Regression Models. 2001. 24p. (RT-MAE-2001-10).

2001-11 - M.D, BRANCO, BOLFARINE, H., IGLESIAS, P., ARELLANO-VALLE, R.B. Bayesian and classical solutions for binomial cytogenetic dosimetry problem. 2001. 16p. (RT-MAE-2001-11).

The complete list of "Relatórios do Departamento de Estatística", IME-USP, will be sent upon request.

Departamento de Estatística
IME-USP
Caixa Postal 66.281
05315-970 - São Paulo, Brasil