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On asymptotic stability in impulsive semidynamical systems

K. A. G. Azevedo* and E. M. Bonotto†

Abstract

In the present paper, we study results about asymptotic stability for semidynamical systems with impulses at variable times. By considering an impulsive semidynamical system $(X, \pi; M, I)$, we state conditions for a closed subset A of X to be asymptotically stable in the impulsive system. In order to obtain the results we make use of Lyapunov functionals. In conclusion, we show that the continuous time three species prey-predator population controlled by a nonlinear feedback control input still globally asymptotically stable if we consider such system with impulses perturbation.

1 Introduction

The theory of impulsive differential equations has been used to model real-world problems in science and technology. This theory of impulsive systems has been attracting the attention of many mathematicians and the interest in the subject is still growing. In the last years, the action of impulses on dynamical systems has been intensively investigated. We refer to the papers [4, 12], [14, 18] and the references therein for instance.

In this paper, we consider a class of impulsive semidynamical systems where the impulses vary on time. We study sufficient conditions in order to obtain results about asymptotic stability. We start by presenting a summary of the basis of semidynamical systems with impulse effect. For details, see Refs. [4], [5], [10] and [14]. Then we present the main results of this paper. First, we consider an impulsive semidynamical system defined on a metric space X and we generalize some results of asymptotic stability for closed sets studied in [2, 3] to the impulsive case. Second, we consider impulsive semidynamical systems defined in $\mathbb{R}^n$ and we prove results of stability by using Lyapunov functionals of class C^1. We finish the paper by presenting a model of three species prey-predator population controlled by a nonlinear feedback control input with impulsive condition, that is, we prove that the equilibrium of this system is globally asymptotically stable.

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2 Preliminaries

In this section we present the basic definitions and notations of the theory of impulsive semidynamical systems. We also include some fundamental results which are necessary for understanding the basis of the theory.

2.1 Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x,t), s) = \pi(x, t+s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system simply by (X, π). For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x,t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x,t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in X$, we define $F(x,t) = \{y \in X : \pi(y,t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \cup\{F(x,t) : x \in D \text{ and } t \in \Delta\}.$$

Then a point $x \in X$ is called an *initial point*, if $F(x,t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The points of M are isolated in every trajectory of system (X, π). The set M is called the *impulsive set*, the function I is called *impulse function*. We also define

$$M^+(x) = (\pi^+(x) \cap M) \setminus \{x\}.$$

Another property of the impulsive set M is that M is a meager set in X, see Lemma 2.1 in [9].

Given an impulsive semidynamical system $(X, \pi; M, I)$ and $x \in X$ such that $M^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it can be found in [4].

Lemma 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Then for every $x \in X$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x,t) \notin M$, whenever $0 < t < s$, and $\pi(x,s) \in M$ if $M^+(x) \neq \emptyset$.*

Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. By means of Lemma 2.1, it is possible to define a function $\phi : X \to (0, +\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x,s) \in M \text{ and } \pi(x,t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $M^+(x) \neq \emptyset$, it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x, t) \notin M$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t - s_0) \notin M$, for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leq t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$, $n = 1, 2, 3, \ldots$, and if $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x, t) : t \in \mathbb{R}_+\}.$$

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies standard properties which follow straightforwardly from the definition. See the next proposition and [5] for a proof of it.

Proposition 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. The following properties hold:*

i) $\widetilde{\pi}(x, 0) = x$,

ii) $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t + s)$, *for all* $t, s \in [0, T(x))$ *such that* $t + s \in [0, T(x))$.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [4, 12] and [14, 15].

2.2 Semicontinuity and continuity of ϕ

The result of this section is borrowed from [10]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system. Such result is applied sometimes intrinsically in the proofs of the main theorems of the next section.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;

(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;

(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [10], Theorem 3.8.

Theorem 2.1. *Consider an impulsive semidynamical system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition (TC). Then ϕ is continuous at x if and only if $x \notin M$.*

2.3 Additional definitions

Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and ratio δ. Let $B(A, \delta) = \{x \in X : \rho_A(x) < \delta\}$ where $\rho_A(x) = \inf\{\rho(x, y) : y \in A\}$.

In what follows, $(X, \pi; M, I)$ is an impulsive semidynamical system and $x \in X$.

We define the *limit set* of x in $(X, \pi; M, I)$ by

$$\widetilde{L}^+(x) = \{y \in X : \widetilde{\pi}(x, t_n) \overset{n \to +\infty}{\longrightarrow} y, \text{ for some } t_n \overset{n \to +\infty}{\longrightarrow} +\infty\}$$

and the *prolongation set* of x in $(X, \pi; M, I)$ by

$$\widetilde{D}^+(x) = \{y \in X : \widetilde{\pi}(x_n, t_n) \overset{n \to +\infty}{\longrightarrow} y, \text{ for some } x_n \overset{n \to +\infty}{\longrightarrow} x \text{ and } t_n \in [0, +\infty)\}.$$

For a set $A \subset X$ we consider $\widetilde{D}^+(A) = \cup\{\widetilde{D}^+(x) : x \in A\}$.

If $\widetilde{\pi}^+(A) \subset A$, we say that A is positively $\widetilde{\pi}-$*invariant.*

A point $x \in X$ is called *stationary* or *rest point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for all $t \geq 0$.

Let $A \subset X$. The set A is orbitally $\widetilde{\pi}-$stable if for every neighborhood U of A, there is a positively $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset U$. We define the set

$$\widetilde{P}_W^+(A) = \{x \in X : \text{ for every neighborhood } U \text{ of } A, \text{ there is a sequence}$$

$$\{t_n\}_{n\geq 1} \subset \mathbb{R}_+,\ t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ such that } \widetilde{\pi}(x, t_n) \in U\}.$$

The set $\widetilde{P}_W^+(A)$ is called *region of weak attraction* of A with respect to $\widetilde{\pi}$. If $x \in \widetilde{P}_W^+(A)$, then we say that x is $\widetilde{\pi}-$*weakly attracted* to A. A subset $A \subset X$ is called a *weak* $\widetilde{\pi}-$*attractor*, if $\widetilde{P}_W^+(A)$ is a neighborhood of A. A set $A \subset X$ is called *asymptotically* $\widetilde{\pi}-$*stable*, if it is both a weak $\widetilde{\pi}-$attractor and orbitally $\widetilde{\pi}-$stable.

3 The main results

In this section, we shall present sufficient conditions to characterize asymptotic stability of closed sets. We are going to make use of a non-negative scalar function defined on a neighborhood of the given set and decreasing along its trajectory to get the results. We divide this section in two parts. In the first one, we consider impulsive semidynamical systems defined on a metric space X and in the second part we consider impulsive systems defined in $\mathbb{R}^n$.

3.1 Asymptotic Stability

Throughout this section we shall consider an impulsive semidynamical system $(X, \pi;\ M, I)$, where (X, ρ) is a locally compact metric space. Moreover, we shall assume the following additional hypotheses:

(H1) no initial point in $(X,\ \pi)$ belongs to the impulsive set M, that is, given $x \in M$ there are $y \in X$ and $t \in \mathbb{R}_+$ such that $\pi(y,\ t) = x$.

(H2) each element of M satisfies the condition (STC) (*consequently, ϕ is continuous on* $X \setminus M$).

(H3) $M \cap I(M) = \emptyset$.

(H4) For each $x \in X$, the motion $\widetilde{\pi}(x, t)$ is defined for every $t \geq 0$, i.e. $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$. By following [14], the impulsive systems where the motion $\widetilde{\pi}(x, t)$ is defined for all $t \geq 0$ are the most important and interesting, and, moreover, in many cases we may restrict ourselves to such systems (because of the existence of suitable isomorphisms), due to the paper [12].

The first lemma is proved in [7], Lemma 3.1. We note that X does not need to be locally compact to obtain Lemma 3.1.

Lemma 3.1. *Let $(X, \pi;\ M, I)$ be an impulsive semidynamical system where X is a metric space. Let $\psi : X \to \mathbb{R}_+$ be a functional satisfying:*

a) $\psi(\pi(x,t)) \leq \psi(x)$ *for* $x \in X$ *and* $t \geq 0$*;*

b) $\psi(I(x)) \leq \psi(x)$ *for* $x \in M$*.*

Then $\psi(\tilde{\pi}(x,t)) \leq \psi(x)$ *for all* $x \in X$ *and* $t \geq 0$.

Let A be a non-empty closed positively $\tilde{\pi}$−invariant subset of X with boundary ∂A compact. Our aim is establish sufficient conditions to guarantee asymptotic $\tilde{\pi}$−stability of the set A. We start by presenting some auxiliary results.

Lemma 3.2. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* A *be a non-empty closed positively* $\tilde{\pi}$*−invariant subset of* X *with* ∂A *compact. Suppose* G *is a neighborhood of* A*,* $I((G \setminus A) \cap M) \subset G \setminus A$ *and let* $\psi : G \to \mathbb{R}_+$ *be a non-negative scalar function satisfying:*

a) $\psi(\pi(x,t)) \leq \psi(x)$ *whenever* $\pi(x,[0,t]) \subset G$*,* $t \geq 0$*;*

b) $\psi(I(x)) \leq \psi(x)$ *for all* $x \in M \cap G$*;*

c) Given $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $\psi(x) < \varepsilon$ *whenever* $\rho(x, A) < \delta$*. If* $\{x_n\}_{n \geq 1} \subset G$ *and* $\psi(x_n) \overset{n \to +\infty}{\longrightarrow} 0$ *then* $\rho(x_n, A) \overset{n \to +\infty}{\longrightarrow} 0$*.*

For each $\alpha > 0$*, let* $V_\alpha = \{x \in G : \psi(x) < \alpha\}$*. Then there exist* $\alpha_0 > 0$ *and a neighborhood* U *of* A *such that* $U \cap V_\alpha$ *is a positively* $\tilde{\pi}$*−invariant neighborhood of* A *for* $\alpha \leq \alpha_0$*.*

Proof. By hypothesis of X and A, there exists a closed neighborhood U of A such that

$$A \subset U = \overline{U} \subset G \quad \text{and} \quad \partial U \subset \overline{U - A} \quad \text{is compact.}$$

By item c), given $\alpha > 0$ there exists $\delta_\alpha > 0$ such that $\psi(x) < \alpha$ whenever $x \in B(A, \delta_\alpha)$. Thus V_α is a neighborhood of A and consequently $U \cap V_\alpha$ is a neighborhood of A.

Let $\alpha_0 = \inf\{\psi(x) : x \in G \setminus U\}$. From item c) we have $\alpha_0 > 0$. We claim that $U \cap V_\alpha$ is positively $\tilde{\pi}$−invariant if $\alpha \leq \alpha_0$. In fact, take $0 < \alpha \leq \alpha_0$. By the proof of Theorem 10.10, [2], we can assure that

$$\pi(x,t) \in U \cap V_\alpha \quad \text{for all} \quad x \in U \cap V_\alpha \quad \text{and for all} \quad t \geq 0. \tag{3.1}$$

Now, let $z \in (U \cap V_\alpha) \cap M$. By item b) we have

$$\psi(I(z)) \leq \psi(z) < \alpha,$$

which implies $I(z) \in V_\alpha$. Suppose $I(z) \notin U$, then

$$\alpha_0 \leq \psi(I(z)) \leq \psi(z) < \alpha,$$

which is a contradiction by the choice of α. Hence,

$$I(z) \in U \cap V_\alpha \quad \text{for all} \quad z \in (U \cap V_\alpha) \cap M. \tag{3.2}$$

By (3.1) and (3.2) we get $\tilde{\pi}(x,t) \in U \cap V_\alpha$ for all $x \in U \cap V_\alpha$ and for all $t \geq 0$. □

Definition 3.1. Let $x \in M$ be given and suppose there exists a sequence $\{w_n\}_{n\geq 1} \subset X$ such that $w_n \stackrel{n\to+\infty}{\longrightarrow} x$. We say that $x \in M_l$ if the sequence $\{w_n\}_{n\geq 1}$ admits a subsequence $\{w_{n_k}\}_{k\geq 1}$ such that $w_{n_k} \notin M$ for all natural n_k and $\pi(w_{n_k}, \phi(w_{n_k})) \stackrel{k\to+\infty}{\longrightarrow} x$, see Figure 1. We say that $x \in M_c$ if the sequence $\{w_n\}_{n\geq 1}$ admits a subsequence $\{w_{n_k}\}_{k\geq 1}$ such that $w_{n_k} \in M$ for all n_k, see Figure 2. We say that $x \in M_r$ if the sequence $\{w_n\}_{n\geq 1}$ admits a subsequence $\{w_{n_k}\}_{k\geq 1}$ such that $w_{n_k} \notin M$ for all natural n_k and $\pi(w_{n_k}, \lambda) \stackrel{k\to+\infty}{\longrightarrow} \pi(x, \lambda)$ for $0 < \lambda < \phi(x)$, see Figure 3.

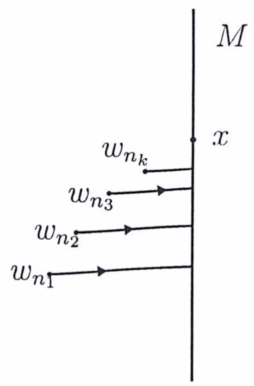

Figure 1: $x \in M_l$

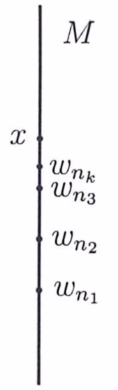

Figure 2: $x \in M_c$

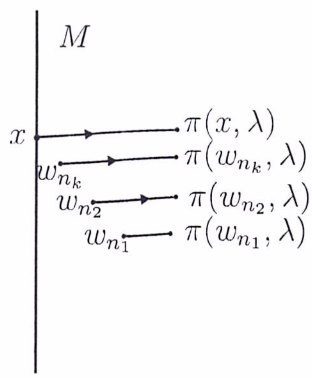

Figure 3: $x \in M_r$

Lemma 3.3 below deals with the orbital $\widetilde{\pi}-$stability of the set A defined in Lemma 3.2.

Lemma 3.3. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a non-empty closed positively $\widetilde{\pi}-$invariant subset of X with ∂A compact. Consider G and $\psi : G \to \mathbb{R}_+$ satisfying the hypotheses of Lemma 3.2. Assume $\psi : G \to \mathbb{R}_+$ is a continuous function on $G \setminus M$. Then A is orbitally $\widetilde{\pi}-$stable.*

Proof. Let us prove that every neighborhood V of A contains some neighborhood $U \cap V_\alpha$, $\alpha \leq \alpha_0$, where V_α, $\alpha_0 = \inf\{\psi(x) : x \in G \setminus U\} > 0$ and U are constructed in Lemma 3.2 and its proof. Suppose the contrary, then there are sequences $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}_+$ and $\{x_{\lambda_n}\}_{n\geq 1} \subset X$ such that $\lambda_n \stackrel{n\to+\infty}{\longrightarrow} 0$, $\lambda_n \leq \alpha_0$ and

$$x_{\lambda_n} \in U \cap V_{\lambda_n} - V,$$

for each $n \in \mathbb{N}$. Set $W = V_{\alpha_0} \cap U$. Since $\overline{W - A}$ is compact, we can assume without loss of generality that

$$x_{\lambda_n} \stackrel{n\to+\infty}{\longrightarrow} p.$$

Note that $p \in \overline{W} \subset G$ and

$$p \notin A \tag{3.3}$$

because $x_{\lambda_n} \notin V$ for each $n \in \mathbb{N}$. We have two cases to consider: when $p \in M$ and $p \notin M$.

First, we consider the case when $p \notin M$. From the continuity of ψ on $G \setminus M$ we have

$$\psi(x_{\lambda_n}) \overset{n\to+\infty}{\longrightarrow} \psi(p). \tag{3.4}$$

Since $\psi(x_{\lambda_n}) < \lambda_n$, $n \in \mathbb{N}$, and $\lambda_n \overset{n\to+\infty}{\longrightarrow} 0$, it follows by (3.4) that

$$\psi(p) = 0.$$

By property $c)$ of ψ we have $p \in A$ and it contradicts (3.3).

Now, we consider the case when $p \in M$. We need to study three subcases: when $p \in M_c$, when $p \in M_l$ and when $p \in M_r$. First, suppose $p \in M_c$. We can assume without loss of generality that $x_{\lambda_n} \in M$, for each $n = 1, 2, \ldots$. By continuity of I, we have

$$I(x_{\lambda_n}) \overset{n\to+\infty}{\longrightarrow} I(p) \notin M.$$

Then

$$\psi(I(x_{\lambda_n})) \overset{n\to+\infty}{\longrightarrow} \psi(I(p)).$$

Since $\psi(I(x_{\lambda_n})) \leq \psi(x_{\lambda_n}) < \lambda_n$ and $\lambda_n \overset{n\to+\infty}{\longrightarrow} 0$ we have

$$\psi(I(p)) = 0,$$

that is, $I(p) \in A$ (by condition $c)$ of ψ) and it is a contradiction since $I((G \setminus A) \cap M) \subset G \setminus A$.

Second, we consider the case $p \in M_l$. We can assume without loss of generality that $\{x_{\lambda_n}\}_{n\geq 1} \subset X \setminus M$ and $\pi(x_{\lambda_n}, \phi(x_{\lambda_n})) \overset{n\to+\infty}{\longrightarrow} p$. Then

$$\psi(I(\pi(x_{\lambda_n}, \phi(x_{\lambda_n})))) \overset{n\to+\infty}{\longrightarrow} \psi(I(p)).$$

Since $\psi(I(\pi(x_{\lambda_n}, \phi(x_{\lambda_n})))) \leq \psi(\pi(x_{\lambda_n}, \phi(x_{\lambda_n}))) \leq \psi(x_{\lambda_n}) < \lambda_n$ and $\lambda_n \overset{n\to+\infty}{\longrightarrow} 0$, we have

$$\psi(I(p)) = 0,$$

and $I(p) \in A$ which is a contradiction.

The last situation occurs if $p \in M_r$, that is, there exists a subsequence $\{x_{\lambda_n}\}_{n\geq 1}$ in $X \setminus M$ (we also denote this subsequence by $\{x_{\lambda_n}\}_{n\geq 1}$) such that $\tilde{\pi}(x_{\lambda_n}, \epsilon) = \pi(x_{\lambda_n}, \epsilon) \overset{n\to+\infty}{\longrightarrow} \pi(p, \epsilon) = \tilde{\pi}(p, \epsilon)$, with $0 < \epsilon < \phi(p)$ and $\pi(p, \epsilon) \notin A \cup M$. Thus $\psi(\pi(x_{\lambda_n}, \epsilon)) \overset{n\to+\infty}{\longrightarrow} \psi(\pi(p, \epsilon))$. Since $\psi(\pi(x_{\lambda_n}, \epsilon)) \leq \psi(x_{\lambda_n}) < \lambda_n$ and $\lambda_n \overset{n\to+\infty}{\longrightarrow} 0$, we have $\psi(\pi(p, \epsilon)) = 0$ and thus $\pi(p, \epsilon) \in A$ which is a contradiction.

Therefore, every neighborhood V of A admits a positively $\tilde{\pi}-$invariant neighborhood $U \cap V_\alpha$ of A, $\alpha \leq \alpha_0$, and A is orbitally $\tilde{\pi}-$stable. □

Now, we mention an important lemma that will be very useful in the next result. The reader may consult [5] for a proof.

Lemma 3.4. *Given an impulsive semidynamical system $(X, \pi;\, M, I)$, where X is a metric space, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geq 1}$ is a sequence in X which converges to the point w. Then, for any $t \geq 0$, there exists a sequence of real numbers $\{\varepsilon_n\}_{n\geq 1}$, with $\varepsilon_n \overset{n\to+\infty}{\longrightarrow} 0$, such that $\tilde{\pi}(z_n, t + \varepsilon_n) \overset{n\to+\infty}{\longrightarrow} \tilde{\pi}(w, t)$.*

In Lemma 3.4, when $\widetilde{\pi}(w,\, t) \neq w_j^+ = I(w_j)$ for every $j = 1, 2, 3, \ldots$, the convergence $\widetilde{\pi}(z_n,\, t+\varepsilon_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$ does not depend on the sequence $\{\varepsilon_n\}_{n\geq 1}$, that is, $\widetilde{\pi}(z_n,\, t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$, whenever $t \neq \sum_{j=0}^{k} \phi(w_j^+)$ for every $k = 0, 1, 2, \ldots$. We present this fact in the next lemma whose the proof is in [8], Lemma 3.3.

Lemma 3.5. *Given an impulsive semidynamical system $(X,\, \pi;\, M,\, I)$, where X is a metric space, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geq 1}$ is a sequence in X which converges to w. Then, for any $t \geq 0$ such that $t \neq \sum_{j=0}^{k} \phi(w_j^+)$, $k = 0, 1, 2, \ldots$, we have $\widetilde{\pi}(z_n,\, t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$.*

The attraction of set A is presented next.

Lemma 3.6. *Let $(X, \pi;\, M, I)$ be an impulsive semidynamical system. Consider A, G, I and ψ as defined in Lemma 3.2. Suppose in addition that ψ satisfies the following properties:*

1. *ψ is continuous on $G \setminus M$;*
2. *If $\widetilde{\pi}(x, [0, s)) \subset G \setminus A$ for $x \in G \setminus A$ and $s > 0$ (s may be $+\infty$), then ψ is not constant in $\widetilde{\pi}(x, [0, s))$.*

Then A is weak $\widetilde{\pi}-$attractor.

Proof. Let U, V_α and α_0 as in Lemma 3.2. If we prove that $U \cap V_{\frac{\alpha_0}{2}} \subset \widetilde{P}_W^+(A)$, the result follows. Suppose the contrary, that is, there exist $x \in U \cap V_{\frac{\alpha_0}{2}}$ and neighborhood $\mathcal{V}$ of A such that for every sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ we have

$$\widetilde{\pi}(x, t_n) \notin \mathcal{V},$$

for each $n \in \mathbb{N}$. Since $U \cap V_{\frac{\alpha_0}{2}}$ is positively $\widetilde{\pi}-$invariant we have $\widetilde{\pi}(x, t_n) \in U \cap V_{\frac{\alpha_0}{2}} - \mathcal{V}$, for all $n \in \mathbb{N}$. By the compactness of $\overline{U \cap V_{\frac{\alpha_0}{2}} - \mathcal{V}}$ we may assume that

$$\widetilde{\pi}(x, t_n) \overset{n\to+\infty}{\longrightarrow} y \in \overline{U \cap V_{\frac{\alpha_0}{2}} - \mathcal{V}}. \tag{3.5}$$

We have two cases to consider: when $y \in M$ and when $y \notin M$.

First, let us consider the case when $y \notin M$. By the continuity of ψ on $G \setminus M$ and (3.5) we get

$$\psi(\widetilde{\pi}(x, t_n)) \overset{n\to+\infty}{\longrightarrow} \psi(y).$$

Since $\psi(\widetilde{\pi}(x, t_n)) \leq \psi(x) < \dfrac{\alpha_0}{2}$ we have $\psi(y) < \dfrac{\alpha_0}{2} < \alpha_0$ and $y \in V_{\alpha_0}$. Now, fixe $t \geq 0$ such that $t \neq \sum_{i=0}^{n} \phi(y_i^+)$, for each $n = 0, 1, 2, \ldots$. Then, there exists a subsequence $\{t_{n_k}\}_{k\geq 1}$ of $\{t_n\}_{n\geq 1}$ such that $t_{n_k} > t_{n_{k-1}} + t$, $k \geq 2$. Note that

$$\psi(\widetilde{\pi}(x, t_{n_k})) \leq \psi(\widetilde{\pi}(x, t_{n_{k-1}} + t)) = \psi(\widetilde{\pi}(\widetilde{\pi}(x, t_{n_{k-1}}), t)), \tag{3.6}$$

for $k \geq 2$. By Lemma 3.5, we have the convergency $\tilde{\pi}(\tilde{\pi}(x, t_{n_{k-1}}), t) \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}(y, t) \notin M$. By passing the limit in (3.6) as $k \to +\infty$ we have

$$\psi(y) \leq \psi(\tilde{\pi}(y, t)).$$

Since $y \in U \cap V_{\alpha_0}$ and $\psi(\tilde{\pi}(y, t)) \leq \psi(y)$ for all $t \geq 0$, it follows that $\psi(\tilde{\pi}(y, t)) = \psi(y)$ for all $t \neq \sum_{i=0}^{n} \phi(y_i^+)$, $n = 0, 1, 2, 3, \ldots$. Consequently, if $t = \sum_{i=0}^{n} \phi(y_i^+)$ for any $n = 0, 1, 2, \ldots$, and $0 < \epsilon < \phi(y_{n+1}^+)$ we have $\psi(y) = \psi(\tilde{\pi}(y, t + \epsilon))$. Thus

$$\psi(y) = \psi(\tilde{\pi}(y, t + \epsilon)) \leq \psi(\tilde{\pi}(y, t)) \leq \psi(y).$$

Hence, $\psi(\tilde{\pi}(y, t)) = \psi(y)$ for all $t \geq 0$. By (3.5) it implies $y \notin A$ and since $\psi(\tilde{\pi}(y, t)) = \psi(y)$ for all $t \geq 0$ we have $\tilde{\pi}^+(y) \in V_{\alpha_0} \setminus A$, which contradicts the hypothesis 2).

Now, we consider the case when $y \in M$. Since $\tilde{\pi}(x, t_n) \notin M$ because $I(M) \cap M = \emptyset$ (hypothesis (H3)) and M satisfies the $STC-$condition, we may consider just the two possibilities: either $\tilde{\pi}(x, t_n) \in M_l$ or $\tilde{\pi}(x, t_n) \in M_r$.

If $\tilde{\pi}(x, t_n) \in M_l$, let $z_n = \tilde{\pi}(x, t_n)$, $n = 1, 2, \ldots$. Then we can suppose $\pi(z_n, \phi(z_n)) \overset{n\to+\infty}{\longrightarrow} y$, with $\phi(z_n) \overset{n\to+\infty}{\longrightarrow} 0$. Let $t > 0$ be fixed (arbitrary) such that $t \neq \sum_{i=0}^{n} \phi(y_i^+)$, for each $n = 0, 1, 2, \ldots$. Let $\{t_{n_k}\}_{k\geq 1}$ be a subsequence of $\{t_n\}_{n\geq 1}$ such that $t_{n_k} > t_{n_{k-1}} + t$, $k \geq 2$, and $N > 0$ be an integer such that $\phi(z_n) < t$, for all $n > N$. Then

$$\begin{aligned}\psi(\tilde{\pi}(x, t_{n_k} + \phi(z_{n_k}))) &\leq \psi(\tilde{\pi}(x, t_{n_{k-1}} + t + \phi(z_{n_k}))) = \\ &= \psi(\tilde{\pi}(x, t_{n_{k-1}} + t + \phi(z_{n_k}) + \phi(z_{n_{k-1}}) - \phi(z_{n_{k-1}}))) = \\ &= \psi(\tilde{\pi}(\tilde{\pi}(x, t_{n_{k-1}} + \phi(z_{n_{k-1}})), t + \phi(z_{n_k}) - \phi(z_{n_{k-1}}))),\end{aligned}$$

that is,

$$\psi(I(\pi(z_{n_k}, \phi(z_{n_k})))) \leq \psi(\tilde{\pi}(\tilde{\pi}(x, t_{n_{k-1}} + \phi(z_{n_{k-1}})), t + \phi(z_{n_k}) - \phi(z_{n_{k-1}}))),$$

that is,

$$\psi(I(\pi(z_{n_k}, \phi(z_{n_k})))) \leq \psi(\tilde{\pi}(I(\pi(z_{n_{k-1}}, \phi(z_{n_{k-1}}))), t + \phi(z_{n_k}) - \phi(z_{n_{k-1}}))).$$

Since

$$I(\pi(z_{n_k}, \phi(z_{n_k})) \overset{k\to+\infty}{\longrightarrow} I(y) \text{ and } \phi(z_{n_k}) - \phi(z_{n_{k-1}}) \overset{n\to+\infty}{\longrightarrow} 0,$$

we get by Lemma 3.5 the following inequality

$$\psi(I(y)) \leq \psi(\tilde{\pi}(I(y), t)) \leq \psi(I(y)),$$

Then we have

$$\psi(\tilde{\pi}(I(y), t)) = \psi(I(y)) \quad \text{for all} \quad t \geq 0 \quad \text{such that} \quad t \neq \sum_{i=0}^{n} \phi(y_i^+), \tag{3.7}$$

for each $n = 0, 1, 2, \ldots$. On the other hand, if $t = \sum_{i=0}^{n} \phi(y_i^+)$ for any $n = 0, 1, 2, \ldots$, and $0 < \epsilon < \phi(y_{n+1}^+)$ it follows by equation (3.7) that $\psi(I(y)) = \psi(\tilde{\pi}(I(y), t + \epsilon))$. Thus

$$\psi(I(y)) = \psi(\tilde{\pi}(I(y), t + \epsilon)) \leq \psi(\tilde{\pi}(I(y), t)) \leq \psi(I(y)).$$

Therefore, $\psi(I(y)) = \psi(\tilde{\pi}(I(y), t))$ for all $t \geq 0$. Note that $\tilde{\pi}^+(I(y)) \in V_{\alpha_0} \setminus A$ since $y \notin A$ and $I((G \setminus A) \cap M) \subset G \setminus A$. It is a contradiction.

But if $\tilde{\pi}(x, t_n) \in M_r$. Choose $0 < \epsilon < \phi(y)$ such that $\tilde{\pi}(y, \epsilon) = \pi(y, \epsilon) \notin A$. We have

$$\tilde{\pi}(\tilde{\pi}(x, t_n), \epsilon) \overset{n \to +\infty}{\longrightarrow} \tilde{\pi}(y, \epsilon).$$

Let $t > 0$ be fixed such that $t \neq \sum_{i=0}^{n} \phi(y_i^+)$, for each $n = 0, 1, 2, \ldots$. Let $\{t_{n_k}\}_{k \geq 1}$ be a subsequence of $\{t_n\}_{n \geq 1}$ such that $t_{n_k} > t_{n_{k-1}} + t$, $k \geq 2$. Then

$$\psi(\tilde{\pi}(x, t_{n_k} + \epsilon)) \leq \psi(\tilde{\pi}(x, t_{n_{k-1}} + t + \epsilon)),$$

for all $k \geq 2$. By Lemma 3.5 and by passing the limit in the previous inequality as $k \to +\infty$, we have

$$\psi(\tilde{\pi}(y, \epsilon)) \leq \psi(\tilde{\pi}(y, \epsilon + t)) \leq \psi(\tilde{\pi}(y, \epsilon)).$$

Thus $\psi(\tilde{\pi}(y, \epsilon)) = \psi(\tilde{\pi}(y, \epsilon + t))$ for all $t \geq 0$ such that $t \neq \sum_{i=0}^{n} \phi(y_i^+)$, for each $n = 0, 1, 2, \ldots$. If $t = \sum_{i=0}^{n} \phi(y_i^+)$ for any $n = 0, 1, 2, \ldots$, we proceed as before. Then $\psi(\tilde{\pi}(y, \epsilon)) = \psi(\tilde{\pi}(y, \epsilon + t))$ for all $t \geq 0$. Since $\tilde{\pi}^+(\tilde{\pi}(y, \epsilon)) \in V_{\alpha_0} \setminus A$, we have a contradiction.

Therefore, A is weak $\tilde{\pi}-$attractor. $\square$

According to Lemmas 3.3 and 3.6, we can state the asymptotic stability result to the set A as show the next theorem.

Theorem 3.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system, where X is a locally compact metric space. Let A be a non-empty closed positively $\tilde{\pi}-$invariant subset of X with ∂A compact. Suppose G is a neighborhood of A and $I((G \setminus A) \cap M) \subset G \setminus A$. Assume $\psi : G \to \mathbb{R}_+$ is a continuous functional on $G \setminus M$ satisfying the following properties:*

P1) $\psi(\pi(x, t)) \leq \psi(x)$ whenever $\pi(x, [0, t]) \subset G$ and $t \geq 0$;

P2) $\psi(I(x)) \leq \psi(x)$ for $x \in M \cap G$;

P3) Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\psi(x) < \varepsilon$ whenever $\rho(x, A) < \delta$. If $\{x_n\}_{n \geq 1} \subset G$ and $\psi(x_n) \overset{n \to +\infty}{\longrightarrow} 0$ then $\rho(x_n, A) \overset{n \to +\infty}{\longrightarrow} 0$;

P4) If $\tilde{\pi}(x, [0, s)) \subset G \setminus A$ for $x \in G \setminus A$ and $s > 0$ (s may be $+\infty$), then ψ is not constant in $\tilde{\pi}(x, [0, s))$.

Then, A is asymptotically $\tilde{\pi}-$stable.

Example 3.1. Consider the space $X = \mathbb{R}^2 \times \{0,1\}$ and the dynamical system

$$\begin{aligned} \dot{x} &= -x + \epsilon y, \\ \dot{y} &= -y + \sigma x, \end{aligned} \tag{3.8}$$

on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times 1$, independently. The constants ϵ and σ are positive such that $\epsilon < 1$ and $\sigma < 1$. Now let $M_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\ z = 0\}$, $M_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1/4,\ z = 1\}$ and $M = M_0 \cup M_1$. We define $I(x,y,0) = (x,y,1)$ for $(x,y,0) \in M_0$ and $I(x,y,1) = (x,y,0)$ for $(x,y,1) \in M_1$. Take $A_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \times \{0\}$, $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \times \{1\}$ and $A = A_0 \cup A_1$. We claim that the set A is asymptotically $\tilde{\pi}-$stable. We are going to use Theorem 3.1 to prove it. It is easy to check that A is a closed positively $\tilde{\pi}-$invariant subset of X with ∂A compact. Let $G = X$ be the neighborhood of A. Since $(G \setminus A) \cap M = \emptyset$ it follows that the hypothesis $I((G \setminus A) \cap M) \subset G \setminus A$ is satisfied. Consider the function $\psi : X \to \mathbb{R}_+$ given by

$$\psi(x,y,z) = \begin{cases} \dfrac{\sqrt{x^2+y^2}-1}{\sqrt{x^2+y^2}}, & if\ \sqrt{x^2+y^2} > 1 \text{ and } z \in \{0,1\} \\ 0, & if\ \sqrt{x^2+y^2} \leq 1 \text{ and } z \in \{0,1\}. \end{cases}$$

It is clear that the function ψ is continuous on X. Now, let us start by verifying the four conditions of Theorem 3.1:

(P1) Given $(x_0,y_0) \in \mathbb{R}^2$, we consider the two flows

$$\varphi_1((x_0,y_0,0),t) = (x(x_0,t),y(y_0,t),0) \quad \text{and} \quad \varphi_2((x_0,y_0,1),t) = (x(x_0,t),y(y_0,t),1)$$

such that $(x(t),y(t)) = (x(x_0,t),y(y_0,t))$ satisfies system (3.8) and $(x(0),y(0)) = (x_0,y_0)$. Let $z_0 = (x_0,y_0,0)$ and $w_0 = (x_0,y_0,1)$.

If $\sqrt{x_0^2+y_0^2} > 1$, we have

$$\dot{\psi}(\varphi_1(z_0,t)) = \frac{\partial\psi}{\partial x}\dot{x}(x_0,t) + \frac{\partial\psi}{\partial y}\dot{y}(y_0,t) + \frac{\partial\psi}{\partial z}\dot{z}(z_0,t) =$$

$$= \frac{1}{\sqrt{x^2(x_0,t)+y^2(y_0,t)}}\left(-1+\frac{\epsilon+\sigma}{2}\right) < 0,$$

for all $t \geq 0$ such that $\varphi_1(z_0,t) \in X \setminus A$. If $\sqrt{x_0^2+y_0^2} \leq 1$ then $\dot{\psi}(\varphi_1(z_0,t)) = 0$ for $t \geq 0$. Hence, $\dot{\psi}(\varphi_1(z_0,t)) \leq 0$ whenever $(x_0,y_0) \in \mathbb{R}^2$ and $t \geq 0$. Analogously, $\dot{\psi}(\varphi_2(w_0,t)) \leq 0$ for all $(x_0,y_0) \in \mathbb{R}^2$ and $t \geq 0$. Then, $\psi(\varphi_1(z_0,t)) \leq \psi(z_0)$ and $\psi(\varphi_2(w_0,t)) \leq \psi(w_0)$ whenever $(x_0,y_0) \in \mathbb{R}^2$ and $t \geq 0$.

(P2) Since $\psi(x,y,z) = 0$ for each $(x,y,z) \in A$ and $I(x,y,z) \subset$ A for $(x,y,z) \in M$, we have $\psi(I(x,y,z)) = \psi(x,y,z) = 0$ if $(x,y,z) \in M$.

(P3) Given $0 < \varepsilon < 1$, if $\rho((x,y,0),A_0) < \frac{\varepsilon}{1-\varepsilon}$ and $x^2+y^2 > 1$ then

$$\sqrt{x^2+y^2}-1 \leq \frac{\varepsilon}{1-\varepsilon} \Rightarrow 1 - \frac{1}{\sqrt{x^2+y^2}} \leq \varepsilon.$$

Thus, $\psi(x,y,0) < \varepsilon$. If $x^2+y^2 \leq 1$ it is clear. Analogously, if $\rho((x,y,1),A_1) < \frac{\varepsilon}{1-\varepsilon}$ then $\psi(x,y,1) < \varepsilon$. Hence, given $\varepsilon > 0$, there is $\delta > 0$ such that $\psi(x,y,z) \leq \varepsilon$ whenever $\rho((x,y,z),A) \leq \delta$.

It is clear that if $\{(x_n,y_n,z_n)\}_{n\geq 1} \subset G$ and $\psi(x_n,y_n,z_n) \stackrel{n\to+\infty}{\longrightarrow} 0$ then $\rho((x_n,y_n,z_n),A) \stackrel{n\to+\infty}{\longrightarrow} 0$;

(P4) Note that $M \subset A$. By the property (P1), $\dot{\psi}(\varphi_i((x_0,y_0,z_i),t)) < 0$ for $t \geq 0$ such that $\varphi_i((x_0,y_0,z_i),t) \in X \setminus A$, $i=1,2$. Then, $\psi(\varphi_i((x_0,y_0,z_i),t)) < \psi((x_0,y_0,z_i))$ for all $(x_0,y_0,z_i) \in X\setminus A$ and $t \geq 0$ with $\varphi_i((x_0,y_0,z_i),t) \in X \setminus A$, $i=1,2$. As the inequality is strict we get the result.

By Theorem 3.1, A is asymptotically $\widetilde{\pi}-$stable.

3.2 C^1-Lyapunov functionals and asymptotic stability in $\mathbb{R}^n$

This section concerns about impulsive semidynamical systems on $X \subseteq \mathbb{R}^n$. Throughout the results, we will assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies sufficient conditions such that the ordinary differential equation

$$x' = f(x), \quad x \in \mathbb{R}^n, \tag{3.9}$$

define a dynamical system in $\mathbb{R}^n$. Thus we consider that equation (3.9) satisfies conditions for the existence, uniqueness and extendability to the whole real line of its solutions for all points in $\mathbb{R}^n$.

Let $M \subset \mathbb{R}^n$ be an impulsive set and $I : M \to \mathbb{R}^n$ be the impulse operator as presented in subsection 2.1. Then, we consider the impulsive semidynamical system $(\mathbb{R}^n, \pi; M, I)$.

We also assume that hypotheses (H1) $-$ (H4) from the last subsection hold.

We present the first asymptotic stability result for impulsive semidynamical systems in $\mathbb{R}^n$. We recall that a set A is positively $\pi-$invariant if $\pi(A,t) \subset A$ for all $t \geq 0$, and it will be said $I-$invariant if $I(A\cap M) \subset A$, see [10].

Theorem 3.2. *Let $A \subset \mathbb{R}^n$ be a non-empty closed subset of $\mathbb{R}^n$ with ∂A compact. Assume A positively $\pi-$invariant and $I-$invariant. Let G be a positively $\pi-$invariant neighborhood of A, $I((G\setminus A)\cap M) \subset G \setminus A$ and $\psi : G \to \mathbb{R}_+$ be a continuously differentiable real-valued function defined on G such that:*

a) $\psi(x) = 0$ if $x \in A$ and if $\{x_n\}_{n\geq 1} \subset G$ with $\psi(x_n) \stackrel{n\to+\infty}{\longrightarrow} 0$ then $\rho(x_n, A) \stackrel{n\to+\infty}{\longrightarrow} 0$;

b) $\langle \nabla\psi(x), f(x)\rangle < 0$ if $x \notin A$;

c) $\psi(I(x)) < \psi(x)$, if $x \in M \cap G$.

Then, A is asymptotically $\widetilde{\pi}$-stable in $(\mathbb{R}^n, \pi; M, I)$.

Proof. From item $b)$ we have $\psi(\pi(x,t)) < \psi(x)$ for all $x \in G \setminus A$ and for all $t > 0$. Since A is positively $\pi-$invariant, we have $\psi(\pi(x,t)) \leq \psi(x)$ for all $x \in G$ and for all $t \geq 0$.

On the other hand, given $\epsilon > 0$ and $x \in \partial A$, there exists $\delta_x > 0$ such that $\psi(y) < \epsilon$ whenever $\rho(y,x) < \delta_x$. Let $\{B(x,\delta_x) : x \in \partial A\}$ be a cover of ∂A. By the compactness of ∂A there are $x_1, \ldots, x_k$ such that $\partial A \subset B(x_1,\delta_{x_1}) \cup \ldots \cup B(x_k, \delta_{x_k})$. Take

$\delta < \min\{\delta_{x_1}, ..., \delta_{x_k}\}$. Then provided $\rho(x, A) < \delta$ we have $\psi(x) < \epsilon$ ($\psi(w) = 0$ if $w \in A$). Moreover, if $\psi(x) = 0$ it follows by item a) that $x \in A$.

Suppose $\widetilde{\pi}(x, [0, s)) \subset G \setminus A$ for some $s > 0$ and $x \in G \setminus A$. Since $\psi(\pi(x, t)) < \psi(x)$ for $t \in (0, \phi(x)) \cap (0, s)$ and $\psi(I(x)) \leq \psi(x)$ it follows that ψ is not constant on the whole trajectory $\widetilde{\pi}(x, [0, s))$.

Therefore, by Theorem 3.1 the set A is asymptotically $\widetilde{\pi}-$stable. □

Lemma 3.7. *Let X be a non-empty subset of $\mathbb{R}^n$ and $(X, \pi; M, I)$ be an impulsive semidynamical system. Let G be an open positively $\pi-$invariant neighborhood of a compact set $A \subset X$ in X. Suppose $I((G \setminus A) \cap M) \subset G \setminus A$ and $\widetilde{L}^+(x)$ is a non-empty set such that $\widetilde{L}^+(x) \subset G \setminus M$ for all $x \in G$. Let $\psi : G \to \mathbb{R}_+$ be a functional satisfying:*

a) $\psi \in C^1$;

b) $\langle \nabla\psi(x), f(x) \rangle \leq 0$, for $x \in G$;

c) $\psi(I(x)) \leq \psi(x)$, if $x \in M \cap G$.

Then, for any $x \in G$, we have $\langle \nabla\psi(y), f(y) \rangle = 0$ for all $y \in \widetilde{L}^+(x)$.

Proof. By items b), c) and Lemma 3.1, we have $\psi(\widetilde{\pi}(x, t)) \leq \psi(x)$ for all $x \in G$ and $t \geq 0$. Let $x \in G$. If $\widetilde{L}^+(x)$ is singleton, that is, $\widetilde{L}^+(x) = \{s_0\}$ for some $s_0 \in G$ then s_0 is a stationary point (because $s_0 \notin M$) and $f(s_0) = 0$. Otherwise, take $z_1, z_2 \in \widetilde{L}^+(x)$. Then, there exist sequences $\{t_n\}$ and $\{\tau_n\}$ in $\mathbb{R}_+$ such that $t_n \longrightarrow +\infty$, $\tau_n \longrightarrow +\infty$,

$$\widetilde{\pi}(x, t_n) \overset{n \to +\infty}{\longrightarrow} z_1 \quad \text{and} \quad \widetilde{\pi}(x, \tau_n) \overset{n \to +\infty}{\longrightarrow} z_2.$$

We can assume without loss of generality that $\tau_n > t_n$ for each natural n. Then,

$$\psi(\widetilde{\pi}(x, \tau_n)) \leq \psi(\widetilde{\pi}(x, t_n)).$$

Since $\psi \in C^1$, we may take the limit in the previous inequality as $n \longrightarrow +\infty$ and we obtain

$$\psi(z_2) \leq \psi(z_1).$$

Analogously, we can assume $t_n > \tau_n$ and we get $\psi(z_1) \leq \psi(z_2)$. Hence ψ is constant on $\widetilde{L}^+(x)$. Consequently, $\langle \nabla\psi(y), f(y) \rangle = 0$ for all $y \in \widetilde{L}^+(x)$, $x \in G$. □

In order to prove Theorem 3.4, we need the following theorem from [11].

Theorem 3.3. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Assume that X is locally compact and A is a compact subset of X. Then the following conditions are equivalent:*

a) A is $\widetilde{\pi}$-stable.

b) A is orbitally $\widetilde{\pi}$-stable.

c) $\widetilde{\mathrm{D}}^+(A) = A$.

From subsets of $\mathbb{R}^n$ we have the following theorem about asymptotic stability.

Theorem 3.4. *Let X be a non-empty subset of $\mathbb{R}^n$ and $(X, \pi; M, I)$ be an impulsive semidynamical system. Let A be a compact subset of X and G be an open positively $\pi-$invariant neighborhood of A in X. Suppose $I((G\setminus A)\cap M) \subset G\setminus A$ and let $\psi : G \to \mathbb{R}_+$ satisfying:*

a) $\psi \in C^1$;

b) $\psi(x) = 0$ *for all* $x \in A$;

c) $\langle \nabla\psi(x), f(x)\rangle < 0$ *for all* $x \in G \setminus A$;

d) $\psi(I(x)) \leq \psi(x)$ *if* $x \in M \cap G$.

Let $\eta > 0$ such that $B(A, \eta) \subset G$. If $\widetilde{L}^+(x) \subset B(A, \eta) \setminus M$ for all $x \in G$, then A is asymptotically $\widetilde{\pi}-$stable.

Proof. Items $b)$, $c)$, $d)$ and Lemma 3.1 imply that $\psi(\widetilde{\pi}(x, t)) \leq \psi(x)$ for all $x \in G$ and $t \geq 0$. By Lemma 3.7, for any $x \in G$, we have

$$\langle \nabla\psi(y), f(y)\rangle = 0 \quad \text{for all} \quad y \in \widetilde{L}^+(x). \tag{3.10}$$

Suppose A is not $\widetilde{\pi}$-attractor, then there exists $y_0 \in G$, $\widetilde{L}^+(y_0) \neq \emptyset$ such that $\widetilde{L}^+(y_0)$ is not contained in A. Now, let $z \in \widetilde{L}^+(y_0)$ then $\langle \nabla\psi(z), f(z)\rangle = 0$ since we have (3.10). Consequently, by $b)$ and $c)$ it implies that $z \in A$. Hence, $\widetilde{L}^+(y_0) \subset A$ and it is a contradiction. In conclusion, A is $\widetilde{\pi}$-attractor.

Now let us show that A is orbitally $\widetilde{\pi}$-stable. Suppose there exists $x \in \widetilde{D}^+(A) - A$. Then, there are $a \in A$, $\{w_n\}_{n\geq 1} \subset G$ and $\{t_n\}_{n\geq 1} \subset [0, +\infty)$, such that

$$w_n \overset{n\to+\infty}{\longrightarrow} a, \quad \text{and} \quad \widetilde{\pi}(w_n, t_n) \overset{n\to+\infty}{\longrightarrow} x.$$

Since

$$\psi(\widetilde{\pi}(w_n, t_n)) \leq \psi(w_n)$$

and $\psi \in C^1$, by taking the limit as $n \to +\infty$, we obtain

$$\psi(x) \leq \psi(a) = 0.$$

It is a contradiction because $x \notin A$. Thus $\widetilde{D}^+(A) = A$ and by Theorem 3.3 the set A is orbitally $\widetilde{\pi}$-stable. Therefore, A is asymptotically $\widetilde{\pi}$-stable. □

Example 3.2. The system which describe the three species prey-predator that consists of two competing preys and one predator can be described by

$$\begin{cases} \dfrac{dx}{dT} &= r_1 x\left(1 - k_1^{-1}x - k_1^{-1}c_2 y\right) - \Phi_1(x, y)z \\ \dfrac{dy}{dT} &= r_2 y(1 - k_2^{-1}c_1 x - k_2^{-1}y) - \Phi_2(x, y)z \\ \dfrac{dz}{dT} &= e_1\Phi_1(x, y)z + e_2\Phi_2(x, y)z - \alpha z, \end{cases} \tag{3.11}$$

where α, k_1, k_2, r_1, r_2, c_1, c_2, e_1 and e_2 assume positive values. The parameters e_1 and e_2 represent the conversion rates of the preys x, y to predator z. The functions Φ_1 and Φ_2 represent a functional response of predator to preys. The variable x and y represent the densities of the two prey species and the variable z represents the density of the predator species. It is known that the predator z consumes the preys x, y according to the response functions

$$\Phi_1(x,y) = \frac{a_1 x}{1 + b_1 x + b_2 y} \quad \text{and} \quad \Phi_2(x,y) = \frac{a_2 y}{1 + b_1 x + b_2 y},$$

where a_1 and a_2 are the search rates of a predator for the preys x, y respectively, $b_1 = h_1 a_1$ and $b_2 = h_2 a_2$ with h_1 and h_2 denote the expected handing times spent with the preys x, y, respectively. See [16] for details.

By using the transformations

$$x_1 = k_1^{-1} x,\ x_2 = k_2^{-1} y,\ x_3 = k_2^{-1} z \text{ and } t = r_1 T$$

we can rewrite system (3.11) in the non-dimensional form:

$$\begin{cases} \dfrac{dx_1}{dt} &= x_1(1 - x_1 - \alpha_2 x_2) - \dfrac{\alpha_1 x_1 x_3}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_2}{dt} &= \gamma_2 x_2 (1 - x_2 - \gamma_1 x_1) - \dfrac{\nu_2 x_2 x_3}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_3}{dt} &= \dfrac{\nu_1 x_1 x_3 + \mu_2 x_2 x_3}{1 + \beta_1 x_1 + \beta_2 x_2} - \mu_1 x_3, \end{cases} \tag{3.12}$$

where $\alpha_1 = a_1 k_1 r_1^{-1}$, $\alpha_2 = c_1 k_2 k_1^{-1}$, $\beta_1 = b_1 k_1$, $\beta_2 = b_2 k_2$, $\gamma_1 = c_2 k_1 k_2^{-1}$, $\gamma_2 = r_2 r_1^{-1}$, $\nu_1 = \alpha_1 e_1$, $\nu_2 = \alpha_2 k_1 r_1^{-1}$, $\mu_1 = \alpha r_1^{-1}$ and $\mu_2 = e_2 k_2 \nu_2^{-1} k_1^{-1}$.

In [1], the authors study the adaptive control of the three species prey-predator system by using nonlinear feedback control approach. They consider system (3.12) with control inputs as present the following system:

$$\begin{cases} \dfrac{dx_1}{dt} &= \dfrac{x_1(1 - x_1 - \alpha_2 x_2)(1 + \beta_1 x_1 + \beta_2 x_2) - \alpha_1 x_1 x_3 + u_1}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_2}{dt} &= \dfrac{\gamma_2 x_2 (1 - \gamma_1 x_1 - x_2)(1 + \beta_1 x_1 + \beta_2 x_2) - \nu_2 x_2 x_3 + u_2}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_3}{dt} &= \dfrac{x_3(\nu_1 x_1 + \mu_2 x_2) - \mu_1 x_3 (1 + \beta_1 x_1 + \beta_2 x_2) + u_3}{1 + \beta_1 x_1 + \beta_2 x_2}, \end{cases} \tag{3.13}$$

where u_1, u_2 and u_3 are the control inputs. By using the nonlinear feedback controllers

$$\begin{cases} u_1 &= x_1[(\beta_1 x_1 + \beta_2 x_2)(\alpha_2 x_2 + x_1 - 1) + \alpha_2 x_2 - 1] \\ u_2 &= \gamma_2 x_2 [(\beta_1 x_1 + \beta_2 x_2)(x_2 + \gamma_1 x_1 - 1) + \gamma_1 x_1 - 1] \\ u_3 &= \mu_1 (\beta_1 x_1 + \beta_2 x_2) x_3 - (\nu_1 x_1 + \mu_2 x_2) x_3, \end{cases} \tag{3.14}$$

system (3.13) becomes

$$\begin{cases} \dfrac{dx_1}{dt} &= \dfrac{-(x_1 + \alpha_1 x_3) x_1}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_2}{dt} &= \dfrac{-(\gamma_2 x_2 + \nu_2 x_3) x_2}{1 + \beta_1 x_1 + \beta_2 x_2} \\ \dfrac{dx_3}{dt} &= \dfrac{-\mu_1 x_3}{1 + \beta_1 x_1 + \beta_2 x_2}. \end{cases} \tag{3.15}$$

In [1], the authors show that the equilibrium point of (3.15) is asymptotically stable. If we change the number of population of predators, for instance, by imposing impulse conditions, we are going to show that system (3.15) with this impulse condition still asymptotically stable.

Since the densities $x_1(t)$, $x_2(t)$ and $x_3(t)$ are non-negative, we consider the phase space $X = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ which is invariant. Thus (3.15) define a dynamical system (X, π) on X.

By Theorem 4.1 from [1], the equilibrium $(0, 0, 0)$ of system (3.15) is asymptotically stable. Actually, by Corollary 1.2 (Chapter X) from [13], the equilibrium $(0, 0, 0)$ is globally asymptotically stable. In fact, the Lyapunov function $V(x_1, x_2, x_3) = \dfrac{x_1^2 + x_2^2 + x_3^2}{2}$ satisfies the conditions to get the result. Then (3.15) is stable and for each $x_0 \in X$ we have

$$\lim_{t \to +\infty} x(t, 0, x_0) = 0, \tag{3.16}$$

where $x(t) := x(t, 0, x_0) = (x_1(t), x_2(t), x_3(t))$ is solution of (3.15) satisfying the initial condition $x(0) = x_0$.

Now, let $M = \bigcup_{i=1}^{p} \{(x_1, x_2, x_3) \in X : x_1 + x_2 + x_3 = \beta_i\}$ be an impulsive set in X, where $p \in \mathbb{N}$ and β_i are positive numbers for $i = 1, \ldots, p$. We define the impulsive operator $I : M \to X$ by

$$I(x_1, x_2, x_3) = (x_1, x_2, I_3(x_1, x_2, x_3))$$

where $0 < I_3(x_1, x_2, x_3) < x_3$ and $I(M) \cap M = \emptyset$.

Let $(X, \pi; M, I)$ be the impulsive semidynamical system defined by system (3.15) and by the impulse operator I above. Set $A = \{(0, 0, 0)\}$ and let $G = X$ be the neighborhood of the equilibrium $(0, 0, 0)$ in X. Note that $I((G \setminus A) \cap M) \subset G \setminus A$. By definition of I and by (3.16), we have $\widetilde{L}^+(x_1, x_2, x_3) = \{(0, 0, 0)\}$ for all $(x_1, x_2, x_3) \in G$.

The impulsive condition says that if the solution of (3.15) meets surface M then the quantity of predators decrease. Even we have loss of predators, the equilibrium $(0, 0, 0)$ still asymptotically stable. Indeed, consider the Lyapunov functional

$$\psi(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{2}$$

defined on X. We have $\psi(0, 0, 0) = 0$. If $w = (x_1, x_2, x_3) \in G \setminus A$ then

$$\langle \nabla\psi(w), f(w) \rangle = -\frac{(x_1 + \alpha_1 x_3)x_1^2}{1 + \beta_1 x_1 + \beta_2 x_2} - \frac{(\gamma_2 x_2 + \nu_2 x_3)x_2^2}{1 + \beta_1 x_1 + \beta_2 x_2} - \frac{\mu_1 x_3^2}{1 + \beta_1 x_1 + \beta_2 x_2} < 0,$$

and if $(x_1, x_2, x_3) \in M$ then

$$\psi(I(x_1, x_2, x_3)) = \frac{x_1^2 + x_2^2 + I_3^2(x_1, x_2, x_3)}{2} < \frac{x_1^2 + x_2^2 + x_3^2}{2} = \psi(x_1, x_2, x_3).$$

By Theorem 3.4, A is asymptotically $\widetilde{\pi}$−stable. Actually, A is globally asymptotically $\widetilde{\pi}$−stable because $\widetilde{L}^+(x_1, x_2, x_3) = \{(0, 0, 0)\}$ for all $(x_1, x_2, x_3) \in X$.

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