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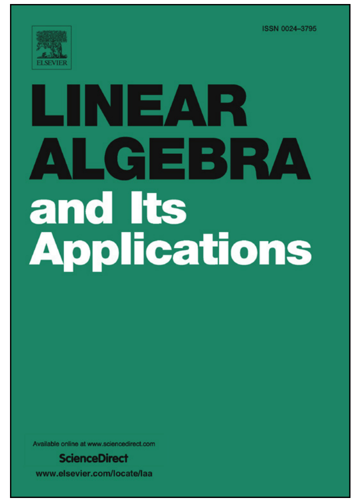

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DE CONCINI-KAC FILTRATION AND GELFAND-TSETLIN GENERATORS FOR QUANTUM $\mathfrak{gl}_N$

VYACHESLAV FUTORNY AND JONAS T. HARTWIG

Abstract. In this note we compute the leading term with respect to the De Concini-Kac filtration of $U_q(\mathfrak{gl}_n)$ of a generating set for the quantum Gelfand-Tsetlin subalgebra.

1. Introduction

An important class of associative algebras, called *Galois rings* was introduced in [FO1]. This class of algebras includes for example Generalized Weyl algebras over integral domains with infinite order automorphisms (in particular, the n-th Weyl algebra, the quantum plane, q-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra [B], [BO]); the universal enveloping algebra of $\mathfrak{gl}_N$ over the Gelfand-Tsetlin subalgebra [DFO1], [DFO2], associated shifted Yangians and finite W-algebras [FMO2], [FMO1].

These algebras contain a special commutative subalgebra Γ which allows one to embed the algebra into a certain invariant subalgebra of some skew group algebra. In particular, such an embedding enables the computation of the skew field of fractions [FMO2],[FH].

A natural choice of a commutative subalgebra in many associative algebras is a so-called Gelfand-Tsetlin subalgebra. Classical Gelfand-Tsetlin subalgebras of the universal enveloping algebras of a simple Lie algebras were considered in [FM], [Vi], [KW1], [KW2], [G1], [G2] among the others.

In this paper we study the quantized enveloping algebra $U_q(\mathfrak{gl}_N)$. This algebra contains a quantum analog of the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_N)$, which we denote by Γ_q. Based on the properties of so called generic Gelfand-Tsetlin modules constructed in [MT], it was shown in [FH] that $U_q(\mathfrak{gl}_N)$ is a Galois ring with respect to Γ_q. This allowed us to prove the quantum Gelfand-Kirillov conjecture for $U_q(\mathfrak{gl}_N)$ [FH],[F]. Unlike all the examples listed above, $U_q(\mathfrak{gl}_N)$ is a Galois rings with respect to a subalgebra which is not a polynomial algebra.

Our main result is the calculation of the leading terms of a set of generators d_{rs} for the quantum Gelfand-Tsetlin subalgebra.

Theorem 1.1. *The leading term of d_{rs} (see (2.13)), with respect to the De Concini-Kac filtration using (2.1) as decomposition of the longest Weyl group element, is*

$$\mathrm{lt}(d_{rs}) = \lambda \cdot t^{(0)}_{1+s,1} t^{(0)}_{2+s,2} \cdots t^{(0)}_{r,r-s} \cdot t^{(1)}_{1,r-s+1} t^{(1)}_{2,r-s+2} \cdots t^{(1)}_{s,r} \tag{1.1}$$

for some nonzero $\lambda \in \mathbb{C}$.

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Notation. $[\![a,b]\!]$ denotes the set $\{x \in \mathbb{Z} \mid a \le x \le b\}$. The cardinality of a set S is denoted $\#S$. Throughout this paper, the ground field is $\mathbb{C}$ and $q \in \mathbb{C}$ is nonzero and not a root of unity. We put $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

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2. The algebra $U_q(\mathfrak{gl}_N)$

In this section we recall some facts about the quantized enveloping algebra $U_q(\mathfrak{gl}_N)$ which will be used.

2.1. **Definition.** For positive integers N we let $U_N = U_q(\mathfrak{gl}_N)$ denote the unital associative $\mathbb{C}$-algebra with generators $E_i^\pm$, K_j, K_j^{-1}, $i \in [\![1, N-1]\!]$, $j \in [\![1,N]\!]$ and relations [KS, p.163]

$$\begin{aligned}
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \quad [K_i, K_j] = 0, \quad \forall i,j \in [\![1,N]\!],\\
K_iE_j^\pm K_i^{-1} &= q^{\pm(\delta_{ij}-\delta_{i,j+1})}E_j^\pm, \quad \forall i \in [\![1,N]\!], \forall j \in [\![1,N-1]\!],\\
[E_i^+, E_j^-] &= \delta_{ij}\frac{K_iK_{i+1}^{-1} - K_{i+1}K_i^{-1}}{q-q^{-1}}, \quad \forall i,j \in [\![1,N-1]\!],\\
[E_i^\pm, E_j^\pm] &= 0, \quad |i-j|>1,\\
(E_i^\pm)^2E_j^\pm - (q+q^{-1})E_i^\pm E_j^\pm E_i^\pm + E_j^\pm(E_i^\pm)^2 &= 0, \quad |i-j| = 1.
\end{aligned}$$

2.2. **De Concini-Kac filtration.** [BG, Section I.6.11] Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i \in [\![1, N-1]\!]$ be the standard simple roots of $\mathfrak{gl}_N$ where $\varepsilon_i(\mathrm{diag}(a_1,\ldots,a_N)) = a_i$. Fix the following decomposition of the longest Weyl group element:

$$w_0 = s_{i_1}\cdots s_{i_M} = (s_1s_2\cdots s_{N-1})(s_1s_2\cdots s_{N-2})\cdots(s_1s_2)s_1, \tag{2.1}$$

where $s_i = (i\ i+1) \in S_N$, and $M = N(N-1)/2$. Let $\{\beta_j = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})\}_{j=1}^M$ be the corresponding enumeration of positive roots of $\mathfrak{gl}_N$. One checks that

$$(\beta_1,\beta_2,\ldots,\beta_M) = (\beta_{12},\beta_{13},\ldots,\beta_{1N},\ \beta_{23},\beta_{24},\ldots,\beta_{2N},\ \ldots,\ \beta_{N-1,N}), \tag{2.2}$$

where $\beta_{ij} = \varepsilon_i - \varepsilon_j$ for all $i,j \in [\![1,N]\!]$, $i<j$. Let E_{β_i}, $F_{\beta_i} \in U_q(\mathfrak{gl}_N)$ be the corresponding positive and negative root vectors (see e.g. [BG, Section I.6.8]). The following PBW theorem for $U_q(\mathfrak{gl}_N)$ is well-known:

Theorem 2.1. *The set of ordered monomials*

$$F^rK_\lambda E^k := F_{\beta_1}^{r_1}\cdots F_{\beta_M}^{r_M}\cdot K_1^{\lambda_1}\cdots K_N^{\lambda_N}\cdot E_{\beta_1}^{k_1}\cdots E_{\beta_M}^{k_M} \tag{2.3}$$

where $r,k \in \mathbb{Z}_{\ge 0}^M$ *and* $\lambda \in \mathbb{Z}^N$, *form a basis for* $U_q(\mathfrak{gl}_N)$.

Define the *total degree* of a monomial $F^rK_\lambda E^k$ to be

$$d(F^rK_\lambda E^k) = \big(k_M,\ldots,k_1,r_1,\ldots,r_M,\mathrm{ht}(F^rK_\lambda E^k)\big) \in \mathbb{Z}_{\ge 0}^{2M+1}, \tag{2.4}$$

where

$$\mathrm{ht}(F^rK_\lambda E^k) = \sum_{j=1}^M (k_j+r_j)\mathrm{ht}(\beta_j) \tag{2.5}$$

and $\mathrm{ht}(\beta) = \sum_{i=1}^{N-1} a_i$ if $\beta = \sum_{i=1}^{N-1} a_i\alpha_i$. Equip the monoid $\mathbb{Z}_{\geq 0}^{2M+1}$ with the reverse lexicographical order uniquely determined by the inequalities

$$u_1 < u_2 < \cdots < u_M$$

where $u_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the i:th position.

Theorem 2.2 (De Concini-Kac)**.** *The total degree function d defined above equips $U = U_q(\mathfrak{gl}_N)$ with a $\mathbb{Z}_{\geq 0}^{2M+1}$-filtration $\{U_{(k)}\}_{k\in\mathbb{Z}_{\geq 0}^{2M+1}}$. The associated graded algebra $\operatorname{gr} U$ is the $\mathbb{C}$-algebra on the generators*

$$\bar{E}_{\beta_i}, \bar{F}_{\beta_j}, \bar{K}_\alpha$$

$i = 1, \ldots, M$, $\alpha \in \mathbb{Z}^N$ subject to the following defining relations:

$$\begin{aligned} &\bar{K}_\alpha \bar{K}_\beta = \bar{K}_{\alpha+\beta} && \bar{K}_0 = 1 \\ &\bar{K}_\alpha \bar{E}_{\beta_i} = q^{(\alpha,\beta_i)}\bar{E}_{\beta_i}\bar{K}_\alpha && \bar{K}_\alpha \bar{F}_{\beta_i} = q^{-(\alpha,\beta_i)}\bar{F}_{\beta_i}\bar{K}_\alpha \\ &\bar{E}_{\beta_i}\bar{F}_{\beta_j} = \bar{F}_{\beta_j}\bar{E}_{\beta_i} && \\ &\bar{E}_{\beta_i}\bar{E}_{\beta_j} = q^{(\beta_i,\beta_j)}\bar{E}_{\beta_j}\bar{E}_{\beta_i} && \bar{F}_{\beta_i}\bar{F}_{\beta_j} = q^{(\beta_i,\beta_j)}\bar{F}_{\beta_j}\bar{F}_{\beta i} \end{aligned} \tag{2.6}$$

for $\alpha, \beta \in \mathbb{Z}^N$ and $1 \leq i, j \leq M$.

Proof. That d actually defines a filtration follows from the commutation relation known as the *Levendorskiĭ-Soibelman straightening rule* [LS, Proposition 5.5.2]. See [DK, Proposition 1.7] for details. □

Observe that the root vectors E_α, F_α, hence the De Concini-Kac filtration, depend on the choice of decomposition of the longest Weyl group element.

A simple but important corollary which will be used implicitly throughout is that

$$d(ab) = d(a) + d(b) = d(ba) \tag{2.7}$$

for all $a, b \in U_q(\mathfrak{gl}_N)$, where now $d(a)$ denotes the smallest $k \in \mathbb{Z}_{\geq 0}^{2M+1}$ such that $a \in U_{(k)}$. This follows from the fact that the associated graded algebra is a domain.

2.3. **RTT presentation.** $U_q(\mathfrak{gl}_N)$ has an alternative presentation. It is isomorphic to the algebra with generators t_{ij}, $\bar{t}_{ij}$, $i, j \in [\![1, N]\!]$ and relations

$$t_{ij} = 0 = \bar{t}_{ji}, \quad \forall i < j, \tag{2.8a}$$

$$t_{ii}\bar{t}_{ii} = 1 = \bar{t}_{ii}t_{ii}, \quad \forall i, \tag{2.8b}$$

$$q^{\delta_{ij}} t_{ia}t_{jb} - q^{\delta_{ab}} t_{jb}t_{ia} = (q - q^{-1})(\delta_{b<a} - \delta_{i<j})t_{ja}t_{ib} \tag{2.8c}$$

$$q^{\delta_{ij}} \bar{t}_{ia}\bar{t}_{jb} - q^{\delta_{ab}} \bar{t}_{jb}\bar{t}_{ia} = (q - q^{-1})(\delta_{b<a} - \delta_{i<j})\bar{t}_{ja}\bar{t}_{ib} \tag{2.8d}$$

$$q^{\delta_{ij}} \bar{t}_{ia}t_{jb} - q^{\delta_{ab}} t_{jb}\bar{t}_{ia} = (q - q^{-1})(\delta_{b<a}t_{ja}\bar{t}_{ib} - \delta_{i<j}\bar{t}_{ja}t_{ib}) \tag{2.8e}$$

for all $i, a, j, b \in [\![1, N]\!]$, where δ_S equals 1 is S is true and 0 if S is false. An identification of the two sets of generators is given by [KS, Section 8.5.4]:

$$\begin{aligned} \bar{t}_{ii} &= K_i^{-1} & t_{ii} &= K_i \\ \bar{t}_{i,i+1} &= (q - q^{-1})K_i^{-1}E_i & t_{i+1,i} &= -(q - q^{-1})F_iK_i \\ \bar{t}_{ij} &= (q - q^{-1})(-1)^{i-j+1}K_i^{-1}E_{\beta_{ij}} & t_{ji} &= -(q - q^{-1})F_{\beta_{ij}}K_i \end{aligned} \tag{2.9}$$

for $j > i + 1$, where $E_{\beta_{ij}}, F_{\beta_{ij}}$ are the root vectors, defined in Section 2.2.

2.4. **Gelfand-Tsetlin subalgebra.** Let $U_q = U_q(\mathfrak{gl}_N)$. It is immediate by the defining relations that, for each $r \in [\![1, N]\!]$, the subalgebra $U_q^{(r)}$ of U_q generated by E_i, F_i, K_j for $i \in [\![1, r-1]\!]$, $j \in [\![1, r]\!]$ (or equivalently, by $t_{ij}, \bar{t}_{ij}$ for $i, j \in [\![1, r]\!]$) can be identified with $U_q(\mathfrak{gl}_r)$. Thus we have a chain of subalgebras

$$U_q^{(1)} \subset U_q^{(2)} \subset \cdots \subset U_q^{(N)} = U_q.$$

Let Z_r denote the center of $U_q^{(r)}$. The subalgebra of U_q generated by $Z_1, \ldots, Z_N$ is called the *Gelfand-Tsetlin subalgebra* and will be denoted by Γ_q. It is immediate that Γ_q is commutative.

The discussion in [HM, Section 5] below formula (5.4) shows that Z_r is generated by the coefficients of the following polynomial in $U_q^{(r)}[u^{-1}]$:

$$z_r(u) = \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} \prod_{j=1}^{r} \big(t_{\sigma(j)j} - \bar{t}_{\sigma(j)j} q^{2(j-1)} u^{-1} \big). \tag{2.10}$$

It will be useful to rewrite this polynomial in a different way. For this purpose it will be convenient to use the notation

$$t_{ij}^{(k)} = \begin{cases} t_{ij}, & k = 0, \\ \bar{t}_{ij}, & k = 1. \end{cases} \tag{2.11}$$

A direct computation gives that

$$z_r(u) = \sum_{s=0}^{r} (-1)^r d_{rs} \cdot (q^2 u)^{-s}, \tag{2.12}$$

where

$$d_{rs} = \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} \sum_{k \in \{0,1\}^r : \sum k_i = s} q^{2(k_1 + 2k_2 + \cdots + rk_r)} t_{\sigma(1)1}^{(k_1)} \cdots t_{\sigma(r)r}^{(k_r)}. \tag{2.13}$$

Notice that $d_{r0} = t_{11} \cdots t_{rr}$ and $d_{rr} = \bar{t}_{11} \cdots \bar{t}_{rr}$, so $d_{r0} = d_{rr}^{-1}$. Therefore, the (commuting) elements d_{rs}, $1 \le s \le r \le N$, generate Γ_q, provided we allow taking negative powers of d_{rr}. We show that they are algebraically independent.

Lemma 2.3.

$$\Gamma_q \simeq \mathbb{C}[d_{rs} \mid 1 \le s \le r \le N][d_{rr}^{-1} \mid 1 \le r \le N]. \tag{2.14}$$

Moreover, the generators d_{rs} are algebraically independent.

Proof. By applying the quantum Harish-Chandra isomorphism $h_r : Z_r \to (U_r^0)^{W_r}$ (see [FH, Lemma 5.3]) to the polynomial $z_r(u)$ from (2.10) (as in [HM, Section 5]) we get

$$\begin{aligned} h_r(z_r(u)) =& (K_1 - K_1^{-1} u^{-1})(K_2 - q^2 K_2^{-1} u^{-1}) \cdots (K_r - q^{2(r-1)} K_r^{-1} u^{-1}) \\ =& q^{r(r+1)} (K_1 \cdots K_r)^{-1} \prod_{j=1}^{r} (q^{-2j} K_j^2 - (q^2 u)^{-1}) \end{aligned}$$

So

$$h_r(d_{rs}) = q^{r(r+1)/2} (\widetilde{K}_1 \cdots \widetilde{K}_r)^{-1} \cdot e_{rs}(\widetilde{K}_1^2, \ldots, \widetilde{K}_r^2), \quad r \in [\![1, N]\!], s \in [\![0, r]\!]$$

where $\widetilde{K}_i = q^{-i} K_i$, and e_{rs} is the elementary symmetric polynomial in r variables of degree s. By the proof of [FH, Lemma 5.3], this shows that

$$Z_r \simeq \mathbb{C}[d_{rs} \mid s = 1, 2, \ldots, r][d_{rr}^{-1}]. \tag{2.15}$$

Let $\Lambda_k = \mathbb{C}[x_{ki}^{\pm 1} \mid i \in [\![1,k]\!]]$ be a Laurent polynomial algebra in k variables and set $\Lambda = \Lambda_1 \otimes \cdots \otimes \Lambda_N$. Let W_k be the Weyl group of type D_k and let $G = W_1 \times \cdots \times W_N$. As shown in [FH] there is an injective map $\varphi : U \to ((\operatorname{Frac}\Lambda) * \mathbb{Z}^{N(N-1)/2})^G$ such that φ restricts to an isomorphism $\varphi|_{\Gamma_q} : \Gamma_q \to \Lambda^G$ and $\varphi_m := \varphi|_{Z_m} : Z_m \to \Lambda_m^{W_m}$ for each $m \in [\![1,N]\!]$. Thus we have a commutative diagram

$$\begin{array}{ccc} \Gamma_q & \xrightarrow{\varphi|_{\Gamma_q}} & \Lambda^G \\ \uparrow f & & \uparrow g \\ Z_1 \otimes \cdots \otimes Z_N & \xrightarrow{\varphi_1 \otimes \cdots \otimes \varphi_N} & \Lambda_1^{W_1} \otimes \cdots \otimes \Lambda_N^{W_N} \end{array}$$

where the vertical arrows are given by multiplication. The horizontal maps and g are isomorphisms. Hence f is an isomorphism. Combining this fact with (2.15) we obtain the required isomorphism. □

3. Leading term of generators

In this section we prove the main theorem which determines the leading term of each of the generators d_{rs} of Γ_q with respect to the De Concini-Kac filtration.

Theorem 3.1. *The leading term of d_{rs} (see (2.13)), with respect to the De Concini-Kac filtration using (2.1) as decomposition of the longest Weyl group element, is obtained by taking*

$$\sigma = (1\ 2\ \cdots\ r)^s.$$

in the sum (2.13). That is,

$$\operatorname{lt}(d_{rs}) = \lambda \cdot t^{(0)}_{1+s,1} t^{(0)}_{2+s,2} \cdots t^{(0)}_{r,r-s} \cdot t^{(1)}_{1,r-s+1} t^{(1)}_{2,r-s+2} \cdots t^{(1)}_{s,r} \tag{3.1}$$

for some nonzero $\lambda \in \mathbb{C}$.

Example 3.2. As an example, we determine directly the leading term of d_{42}. By definition of the reverse lexicographical order, we must first compare the height of two elements to determine which is larger with respecto the total degree. Using (3.2)-(3.3), it is easy to see that there are four permutations in S_4 which gives the maximal possible height 8:

$$(13)(24), \quad (14)(23), \quad (1324), \quad (1423).$$

The monomial associated to such a permutation σ is

$$t^{(k_1)}_{\sigma(1)1} t^{(k_2)}_{\sigma(2)2} t^{(k_3)}_{\sigma(3)3} t^{(k_4)}_{\sigma(4)4}$$

where $k_i = 0$ if $\sigma(i) > i$ and $k_i = 1$ if $\sigma(i) < i$. After the height we need to compare the exponent of $F_{\beta_{34}}$ in the four different monomials, because β_{34} is the largest positive root in the ordering

$$\beta_{12} < \beta_{13} < \beta_{14} < \beta_{23} < \beta_{24} < \beta_{34}$$

(see (2.2)). This exponent is the same as the exponent (either 1 or 0) of $t^{(0)}_{43}$ due to the identifications (2.9). But this exponent is 0 in all four cases because none of the permutations map 3 to 4.

So we look at the second largest positive root, which is β_{24}. As in the previous case, we ask if $\sigma(2) = 4$ in any of the four permutations. There are two for which this holds, (13)(24) and (1324). The others do not map 2 to 4 which means their corresponding monomials are of lower total degree.

To compare the two candidates (13)(24) and (1324) we look at the third largest root, β_{23}. But $\sigma(2) \neq 3$ in both. Next is β_{14} but again $\sigma(1) \neq 4$ in both. Next is β_{13} and now $\sigma(1) = 3$ for both $\sigma = (13)(24)$ and $\sigma = (1324)$. Next is β_{12} and $\sigma(1) \neq 2$ in both. So we still don't know which monomial is largest. We have compared the $1 + 6$ biggest components of the total degree, namely the height and the 6 exponents of the negative root vectors F_β.

Thus we turn to comparing the remaining 6 exponents of the positive root vectors E_β. Now care must be taken since, by (2.4), these are ordered in reverse relative to the positive roots themselves. Therefore, the next component to compare is the exponent of $E_{\beta_{12}}$ because β_{12} is the smallest root. By (2.9), this is the same as the exponent of $t_{12}^{(1)}$ so we check if the permutations satisfy $\sigma(2) = 1$. None of them do, so we move on, checking $E_{\beta_{13}}$ which amounts to checking if $\sigma(3) = 1$. Here we finally get a discrepancy, (13)(24) satisfies this, but (1324) does not. Therefore (13)(24) is the permutation that gives the leading term in d_{42}.

Of course, $(13)(24) = (1234)^2$, so this proves Theorem 3.1 in the case $(r, s) = (4, 2)$.

We will need several lemmas. The following notation will be used for a permutation $\sigma \in S_r$:

$$\mathrm{EX}(\sigma) = \{i \in [\![1, r]\!] \mid \sigma(i) > i\}, \qquad \mathrm{AX}(\sigma) = \{i \in [\![1, r]\!] \mid \sigma(i) < i\}.$$

Elements of $\mathrm{EX}(\sigma)$ (respectively $\mathrm{AX}(\sigma)$ are called *exceedances* (respectively *anti-exceedances*) for σ.

The following lemma describes which nonzero terms appear in d_{rs}.

Lemma 3.3. *Let $s \in [\![1, r]\!]$ and let $\sigma \in S_r$. Then the following two statements are equivalent.*

(i) *There exists $k = (k_1, \ldots, k_r) \in \{0, 1\}^r$, depending on σ, such that $\sum_{i=1}^r k_i = s$ and $t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)} \neq 0$.*

(ii) *$\#\mathrm{AX}(\sigma) \leq s$ and $\#\mathrm{EX}(\sigma) \leq r - s$.*

Proof. This follows from the fact that $t_{ij}^{(1)} \neq 0$ iff $i \leq j$ and $t_{ij}^{(0)} \neq 0$ iff $i \geq j$. □

Define the *height* of a permutation $\sigma \in S_r$ by

$$\mathrm{ht}(\sigma) := \sum_{i=1}^{r} |\sigma(i) - i|. \tag{3.2}$$

The motivation for this terminology comes from the fact that

$$\mathrm{ht}(\sigma) = \mathrm{ht}(t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)}) \tag{3.3}$$

where the right hand side is given by (2.5) and the identification (2.9).

As the next step towards proving Theorem 3.1, we show that the permutation σ which gives the leading term of d_{rs} has to be a derangement (i.e. $\sigma(i) \neq i \ \forall i \in [\![1, r]\!]$).

Lemma 3.4. *Let $s \in [\![1, r]\!]$ and let $\sigma \in S_r$ be a permutation such that*

$$t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)} \neq 0$$

for some $k \in \{0, 1\}^r$ with $\sum_i k_i = s$. Then there exists a $\widetilde{\sigma} \in S_r$ such that

(i) $t^{(l_1)}_{\widetilde{\sigma}(1)1} \cdots t^{(l_r)}_{\widetilde{\sigma}(r)r} \neq 0$ *for some* $l \in \{0,1\}^r$ *with* $\sum_i l_i = s$;
(ii) $t^{(l_1)}_{\widetilde{\sigma}(1)1} \cdots t^{(l_r)}_{\widetilde{\sigma}(r)r} \geq t^{(k_1)}_{\sigma(1)1} \cdots t^{(k_r)}_{\sigma(r)r}$ *with respect to the De Concini-Kac filtration;*
(iii) $\widetilde{\sigma}$ *is a derangement.*

In particular, the permutation σ *such that* (3.1) *holds is a derangement.*

Proof. If σ already is a derangement, there is nothing to prove (take $\widetilde{\sigma} = \sigma$). So suppose $f := \#\{i \in S_r \mid \sigma(i) = i\} > 0$. It is enough to construct $\widetilde{\sigma}$ satisfying properties (i)-(ii) with $\#\{i \in S_r \mid \widetilde{\sigma}(i) = i\} = f - 1$ because then we can iterate this construction to arrive at a permutation satisfying all three conditions (i)-(iii).

For brevity, we call $(i, \sigma(i)) \in [\![1, r]\!]^2$ a *σ-jump* (respectively *σ-drop*) if i is an exceedance (respectively anti-exceedance) for σ. It will be useful to visualize a sequence $(i, \sigma(i), \ldots, \sigma^k(i))$ as a graph with vertex set $\{(x, \sigma^x(i)) \mid x \in [\![0, k]\!]\} \subset \mathbb{Z}^2$, connecting adjacent vertices (a, b) and $(a+1, \sigma(b))$, as in Figure 1. Then drops and jumps are simply as in Figure 2.

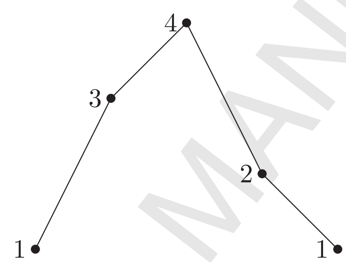
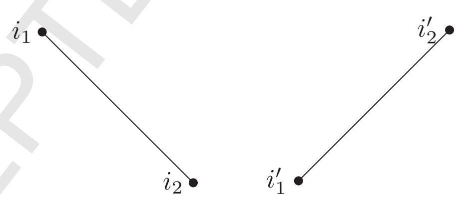


FIGURE 1. Pictorial representation of the cyclic permutation (1342).



(A) (B)

FIGURE 2. A σ-drop (A) and a σ-jump (B). The diagrams mean $i_2 = \sigma(i_1)$, $i_1 > i_2$ and $i'_2 = \sigma(i'_1)$, $i'_1 < i'_2$.

A σ-drop (i_1, i_2) will be called *drop-admissible* if we can "add another drop between i_1 and i_2", that is, if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $i_2 < j < i_1$. Then we can put $\widetilde{\sigma} = \sigma \circ (i_1\ j)$. With this $\widetilde{\sigma}$ we have

$$\#\mathrm{AX}(\widetilde{\sigma}) = 1 + \#\mathrm{AX}(\sigma), \qquad \#\mathrm{EX}(\widetilde{\sigma}) = \#\mathrm{EX}(\sigma).$$

Similarly, a σ-drop (i_1, i_2) is *jump-admissible* if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $j \notin [\![i_2, i_1]\!]$. Then $\widetilde{\sigma} = \sigma \circ (i_1\ j)$ satisfies

$$\#\mathrm{AX}(\widetilde{\sigma}) = \#\mathrm{AX}(\sigma), \qquad \#\mathrm{EX}(\widetilde{\sigma}) = 1 + \#\mathrm{EX}(\sigma).$$

See Figure 3 for an illustration of the possible scenarios in the case of a σ-drop.

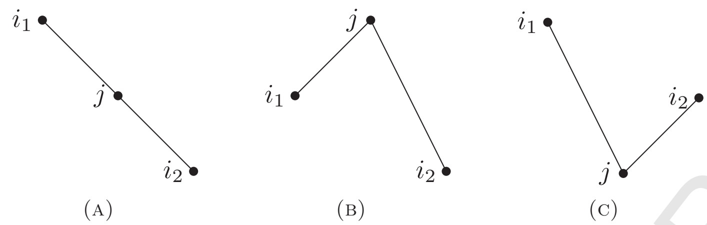


FIGURE 3. The three possible ways the i_1, j, i_2 piece of $\widetilde{\sigma} = \sigma \circ (i_1\ j)$ can look like, when (i_1, i_2) is a σ-drop: $i_1 < j < i_2$ (A), $j > i_1, i_2$ (B), and $j < i_1, i_2$ (C). The σ-drop (i_1, i_2) is drop-admissible in case (A), and jump-admissible in (B) and (C).

Analogously, a σ-jump (i_1, i_2) is *jump-admissible* if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $i_1 < j < i_2$. A σ-jump (i_1, i_2) is *drop-admissible* if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $j \notin [\![i_1, i_2]\!]$.

We will now show that, since σ is not a derangement, there exists a jump-admissible σ-drop or σ-jump.

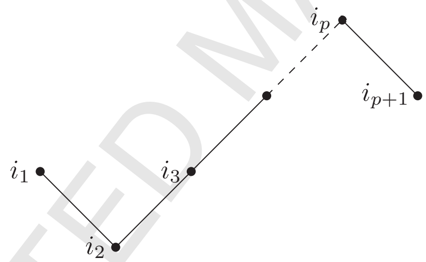


FIGURE 4. Illustration of a permutation σ satisfying conditions (a)-(d).

We know that σ is not the identity permutation since $\sum_i k_i = s \geq 1$. Thus there exists a tuple $(i_1, i_2, \ldots, i_p, i_{p+1}) \in [\![1, r]\!]^{p+1}$, where $p > 2$, such that (see Figure 4)

(a) $i_{j+1} = \sigma(i_j)$ for $j \in [\![1, p]\!]$;
(b) $i_1 > i_2$;
(c) $i_j < i_{j+1}$ for $j \in [\![2, p-1]\!]$;
(d) $i_p > i_{p+1}$.

Note that we do not exclude the possibility that $(i_p, i_{p+1}) = (i_1, i_2)$. Also, since σ is not a derangement, there is some $j \in [\![1, r]\!] \setminus \{i_1, \ldots, i_{p+1}\}$ fixed by σ.

If $j \notin [\![i_2, i_1]\!]$, then (i_1, i_2) is a jump-admissible σ-drop (as in case (B) or (C) in Figure 3). So suppose $i_1 > j > i_2$. If $j < i_p$ then (i_a, i_{a+1}) is a jump-admissible σ-jump for the $a \in [\![2, p-1]\!]$ with $i_a < p < i_{a+1}$. So suppose $j > i_p$. Then (i_p, i_{p+1}) is a jump-admissible σ-drop. This proves that, provided $\sigma(j) = j$ for some j, there always exists a jump-admissible σ-drop or σ-jump.

Similarly one proves there always exists a drop-admissible σ-drop or σ-jump.

If $\#\mathrm{AX}(\sigma) < s$ then we add a drop by putting $\widetilde{\sigma} = \sigma \circ (i\ j)$ where $(i, \sigma(i))$ is a drop-admissible σ-drop or σ-jump. Then $\widetilde{\sigma}$ will have one more drop than σ but the same number of jumps. That is, $\#\mathrm{AX}(\widetilde{\sigma}) = 1 + \#\mathrm{AX}(\sigma) + 1 \leq s$ and

$\#\mathrm{EX}(\widetilde{\sigma}) = \#\mathrm{EX}(\sigma) \leq r - s$ which by Lemma 3.3 ensures that property (i) is satisfied.

Analogously, if instead $\#\mathrm{EX}(\sigma) < r - s$ we add a jump by putting $\widetilde{\sigma} = \sigma \circ (i\ j)$ for appropriate i. Note that since σ is not a derangement, $\#\mathrm{AX}(\sigma) = s$ and $\#\mathrm{EX}(\sigma) = r - s$ cannot be true simultaneously.

Clearly $\widetilde{\sigma}$ has one less fixpoint than σ.

It remains to verify that property (ii) holds. The change from σ to $\widetilde{\sigma}$ has the following effect on monomials:

$$t_{jj}^{(k_j)} t_{\sigma(i)i}^{(k_i)} \longmapsto t_{\widetilde{\sigma}(j)j}^{(k_j)} t_{\widetilde{\sigma}(i)i}^{(k_i)} = t_{\sigma(i)j}^{(k_j)} t_{ji}^{(k_i)}$$

(unchanged factors omitted).

If j is not between i and $\sigma(i)$, then by definition of the height (3.2) one checks that $\mathrm{ht}(\widetilde{\sigma}) > \mathrm{ht}(\sigma)$ so (ii) holds by just looking at the height, which is the most significant part of the total degree (see (2.4)).

If j is between i and $\sigma(i)$, then $\mathrm{ht}(\widetilde{\sigma}) = \mathrm{ht}(\sigma)$ so we must compare roots in order to establish property (ii).

Suppose $i < j < \sigma(i)$. Then the change from σ to $\widetilde{\sigma}$ corresponds to

$$t_{\sigma(i)i}^{(0)} t_{jj}^{(k_j)} \longmapsto t_{\sigma(i)j}^{(0)} t_{ji}^{(0)}$$

The change in total degrees is

$$d(F_{\beta_{i,\sigma(i)}}) \longmapsto d(F_{\beta_{j,\sigma(i)}} F_{\beta_{ij}})$$

Since $\beta_{j,\sigma(i)} > \beta_{i,\sigma(i)}, \beta_{i,j}$ (recall the ordering (2.2)) it follows that property (ii) holds in this case. The case $i > j > \sigma(i)$ is analogous, keeping in mind that E_β are ordered in reverse. This finishes the proof of Lemma 3.4. □

The following result describes the height of the permutation giving rise to the leading term.

Lemma 3.5. *Fix $r \in \mathbb{Z}_{>0}$ and let $s \in [\![1, r]\!]$. Let $\sigma \in S_r$ be the permutation which gives rise to the leading term of d_{rs}. That is,*

$$\mathrm{lt}(d_{rs}) = \lambda t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)} \tag{3.4}$$

for some nonzero $\lambda \in \mathbb{C}$ and some $k \in \{0,1\}^r$ with $\sum_i k_i = s$. Then

$$\mathrm{ht}(\sigma) = 2s(r - s). \tag{3.5}$$

Proof. First we prove that $\mathrm{ht}(\sigma) \geq 2s(r-s)$. Let $\tau = (1\ 2\ \cdots\ r)^s$. We show that $\mathrm{ht}(\tau) = 2s(r-s)$. Since

$$\tau(i) = \begin{cases} i + s, & i + s \leq r \\ i + s - r, & i + s > r \end{cases}$$

we have by definition of $\mathrm{ht}(\tau)$

$$\mathrm{ht}(\tau) = \sum_{i=1}^{r-s} (i + s - i) + \sum_{i=r-s+1}^{r} (i - (i + s - r)) = 2s(r - s).$$

By Lemma 3.3 the monomial corresponding to τ appears in d_{rs}. Since (3.4) is the leading term of d_{rs}, we in particular have $\mathrm{ht}(\sigma) \geq \mathrm{ht}(\tau) = 2s(r-s)$ by definition of total degree of a monomial (2.4).

It remains to show that $\mathrm{ht}(\sigma) \leq 2s(r-s)$. By Lemma 3.4, σ is a derangement. Thus

$$\mathrm{ht}(\sigma) = \sum_{i=1}^{r} |\sigma(i) - i| = \sum_{i\colon \sigma(i)<i} (i - \sigma(i)) + \sum_{i\colon \sigma(i)>i} (\sigma(i) - i),$$

where the first sum has s terms and the second has $r-s$ terms. Clearly we have the estimate

$$\begin{aligned}\sum_{i\colon \sigma(i)<i} & (i - \sigma(i)) + \sum_{i\colon \sigma(i)>i} (\sigma(i) - i) \\ &\leq (r + (r-1) + \cdots + (r-s+1)) - (1 + 2 + \cdots + s) \\ &\quad + (r + (r-1) + \cdots (s+1)) - (1 + 2 + \cdots + (r-s)) = 2s(r-s).\end{aligned}$$

This proves the claim. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The case $r = s$ is trivial: By (2.13), $d_{rr} = \lambda \cdot t_{11}^{(1)} \cdots t_{rr}^{(1)}$, where $\lambda \in \mathbb{C}^{\times}$. Thus d_{rr} has only one term, corresponding to the identity permutation (1). Thus the conjecture holds in this case because $(1\ 2\ \cdots\ r)^r = (1)$. So we may assume $s < r$.

Let $\sigma \in S_r$ be the permutation which gives rise to the leading term of d_{rs}. That is,

$$\mathrm{lt}(d_{rs}) = \lambda t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)}$$

for some nonzero $\lambda \in \mathbb{C}$ and some $k \in \{0,1\}^r$ with $\sum_i k_i = s$. By Lemma 3.4, σ is a derangement. In particular, k is uniquely determined: $k_i = 0$ iff $\sigma(i) > i$ and $k_i = 1$ iff $\sigma(i) < i$. Moreover, since σ is a derangement, Lemma 3.3 implies that s equals the number of anti-exceedances for σ:

$$s = \#\{i \in [\![1, r]\!] \mid \sigma(i) < i\}. \tag{3.6}$$

We will now show that

$$\sigma^{-1}(r) = r - s. \tag{3.7}$$

This is equivalent to that $t_{r,r-s}^{(0)}$ occurs in $\mathrm{lt}(d_{rs})$. By (2.9) and that the K_i don't contribute to the total degree, we have $d(t_{r,r-s}^{(0)}) = d(F_{\beta_{r-s,r}})$. To show (3.7), note that $t_{r,r-s}^{(0)}$ occurs in the monomial corresponding to $\tau = (1\ 2\ \cdots\ r)^s$. Thus it is enough to prove that if $t_{ji}^{(0)}$ occurs in the leading monomial of d_{rs} then $\beta_{ij} \leq \beta_{r-s,r}$.

Suppose the opposite is true, i.e. that $\sigma^{-1}(j_0) = i_0 \in [\![r-s+1, j_0 - 1]\!]$ for some j_0 with $i_0 < j_0 \leq r$. We show that this leads to a contradiction in the height of σ. We have

$$\mathrm{ht}(\sigma) = \sum_{i=1}^{r} |\sigma(i) - i| = \sum_{i\colon \sigma(i)<i} (i - \sigma(i)) + \sum_{i\colon \sigma(i)>i} (\sigma(i) - i). \tag{3.8}$$

The first sum has s elements, by (3.6), and the second one has $r-s$ terms, since σ is a derangement. Since $\sigma(i_0) = j_0 > i_0$, we may estimate the first sum from above by assuming that i runs through the s largest elements of $[\![1,r]\!]\setminus\{i_0\}$, and $\sigma(i)$ just runs through the s smallest elements of $[\![1,r]\!]$. That is,

$$\sum_{i:\sigma(i)<i}(i-\sigma(i))\leq(r+(r-1)+\cdots+(r-s)-i_0)-(1+2+\cdots+s)$$
$$=r-i_0+s(r-s-1). \quad (3.9)$$

On the other hand, i_0 does belong to the summation range of the other sum and therefore

$$\sum_{i:\,\sigma(i)>i}(\sigma(i)-i)\leq(r+(r-1)+\cdots+(s+1))-(1+2+\cdots+(r-s-1)+i_0)$$
$$=(r-s-1)s+r-i_0, \quad (3.10)$$

i.e. the sum of the $r-s$ largest elements of $[\![1,r]\!]$ minus the smallest sum of $r-s$ elements of $[\![1,r]\!]$ requiring that one of them is i_0. Combining (3.8)-(3.10) we obtain

$$\mathrm{ht}(\sigma)\leq 2(r-s-i_0)+2s(r-s)<2s(r-s) \quad (3.11)$$

since $i_0>r-s$ by assumption. This contradicts Lemma 3.5 and finishes the proof of (3.7).

Then, since $\beta_{r-s-1,r-1}$ is the largest positive root of the form $\beta_{r-s-1,j}$ where $j<r$, $\beta_{r-s-2,r-2}$ is the largest positive root of the form $\beta_{r-s-2,j}$ with $j<r-1$, and so on, we conclude that the leading term of d_{rs} must have the form

$$\lambda\cdot t^{(0)}_{1+s,1}t^{(0)}_{2+s,2}\cdots t^{(0)}_{r,r-s}\cdot t^{(k_1)}_{\sigma(r-s+1),r-s+1}\cdots t^{(k_s)}_{\sigma(r)r}.$$

But $\sum k_i=s$ which forces $k_i=1$ for $i\in[\![1,s]\!]$. So $\sigma(i)<i$ for $i\in[\![r-s+1,r]\!]$. Since $d(t^{(1)}_{ij})=d(E_{\beta_{ij}})$ for $i<j$ and by definition (2.4) of the total degree, the E_β are ordered in *reverse* with respect to the order of the positive roots β, we are led to the question: What is the smallest possible root β_{ij} $(i<j)$ which may still occur in the monomial?

We know that $\{\sigma(r-s+1),\sigma(r-s+2),\dots,\sigma(r)\}=\{1,2,\dots,s\}$. Thus, the smallest root we can get is $\beta_{1,r-s+1}$, obtained iff $\sigma(r-s+1)=1$. But this happens for the permutation $\tau=(1\ 2\ \cdots\ r)^s$. Continuing, at each step we see that the smallest possible root is $\beta_{i,r-s+i}$ for $i=1,2,\dots,s$. This proves that $(1\ 2\ \cdots\ r)^s$ indeed is the permutation that gives the leading term of d_{rs}. □

For $a,b\in[\![1,N]\!]$, $a\neq b$, and $u\in U_q$, let $\deg_{ab}(u)\in\mathbb{Z}_{\geq 0}$ denote the component of the De Concini-Kac filtration degree $d(u)\in(\mathbb{Z}_{\geq 0})^{2M+1}$ corresponding to the root $\beta_{ab}=\varepsilon_a-\varepsilon_b$. The following result describes the positive roots that occur in the leading term of d_{rs}.

Corollary 3.6. *If $1\leq b<a\leq N$ and $1\leq s<r\leq N$, then*

$$\deg_{ba}(\mathrm{lt}(d_{rs}))=\begin{cases}1, & a-b=r-s \text{ and } a\leq r\\ 0, & \text{otherwise}\end{cases}$$

Proof. By Theorem 3.1, $\mathrm{lt}(d_{rs})$ is (up to multiplication by some K_i and a scalar) a product of distinct root vectors, and the positive root vector E_β, $\beta=\beta_{ba}$, occurs in $\mathrm{lt}(d_{rs})$ if and only if $(b,a)\in\{(1,r-s+1),(2,r-s+2),\dots,(s,r)\}$ which is equivalent to $a-b=r-s$ and $a\leq r$. □

Corollary 3.7. *If* $1 \leq b < a \leq N$, $1 \leq s < r \leq N$, *then*

$$\deg_{ba}\left(\mathrm{lt}(\prod_{1\leq s<r\leq N} d_{rs}^{k_{rs}})\right) = \sum_{r=a}^{N} k_{r,r-(a-b)}.$$

Remark 3.8. Define

$$X(r,s) = t_{sr}^{(1)} \tag{3.12}$$

for each $1 \leq s \leq r \leq N$. Then, by Theorem 3.1, $X(r,s)$ occurs in the leading term of d_{rs}, however it sometimes *does* occur in the leading term of some other d_{ab}, $(a,b) \neq (r,s)$. Thus one cannot use the technique from [FMO2] to prove that $U_q(\mathfrak{gl}_N)$ is a Galois order. In fact we were not able to generalize our approach in any way to make it work. That $U_q(\mathfrak{gl}_N)$ in fact is a Galois order over Γ_q was proved by the second author in [H] using different methods.

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