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TOPOLOGIES GENERATED BY WEAK SELECTION TOPOLOGIES

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ABSTRACT. A weak selection on an infinite set X is a function $f : [X]^2 \to X$ such that $f(\{x,y\}) \in \{x,y\}$ for each $\{x,y\} \in [X]^2$. A weak selection f on X defines a relation $x \prec_f y$ if $f(\{x,y\}) = x$ whenever $x, y \in X$ are distinct. The topology τ_f on X generated by the weak selection f is the one which has the family of all intervals $(\leftarrow, x)_f = \{y \in X : y \prec_f x\}$ and $(x, \rightarrow)_f = \{y \in X : x \prec_f y\}$ as a subbase. A weak selection on a space is said to be continuous if it is a continuous function with respect to the Vietoris topology on $[X]^2$. The paper deals with topological spaces (X, τ) for which there is a set W of continuous weak selections satisfying $\tau = \bigvee_{f \in W} \tau_f$ (we say that the topology of X is generated by continuous weak selections). We prove that for any infinite cardinal α, there exists a weakly orderable space whose topology cannot be generated by less than or equal to α-many continuous weak selections. We prove that any subspace of a space generated by continuous weak selections is also generated by continuous weak selections. Assuming that $\mathfrak{c}$ is regular, we construct a suborderable space whose topology is generated by $\mathfrak{c}$-many continuous weak selections but not by less than $\mathfrak{c}$. Also, under the assumption of GCH, for every infinite successor cardinal α^+ we construct a space X that is generated by α^+-many continuous weak selections but cannot be generated by α-many selections.

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1. Introduction and preliminaries

All spaces that we shall consider will be at least Hausdorff. For a set X, we let $[X]^2 = \{F \subseteq X : |F| = 2\}$. As usual, the symbols $\chi(X)$ and $w(X)$ denote the character and the weight of a space X, respectively. The symbols ω and ω_1 stand for the first infinite cardinal and the first uncountable cardinal, respectively. We use the symbols $\leq, \geq, <, >$ for the usual order of cardinals. If $\ll$ is a linear order on a set X, then the intervals are denoted by $(x,y)_{\ll}$, $[x,y]_{\ll}$, $(x,y]_{\ll}$, etc..

Given an infinite set X, we say that a function $f : [X]^2 \to X$ is a *weak selection* if $f(F) \in F$ for all $F \in [X]^2$. For a weak selection $f : [X]^2 \to X$, we define $x \prec_f y$ if $f(\{x,y\}) = x$, for each $\{x,y\} \in [X]^2$, and $x \preceq_f y$ if either $x \prec_f y$ or $x = y$. This relation $\preceq_f$ is reflexive and antisymmetric but could fail to be transitive. Following [6], if B and C are (not necessarily nonempty) subsets of X, we write $B \preceq_f C$ if $y \preceq_f z$ for every $y \in B$ and $z \in C$. For a weak selection $f : [X]^2 \to X$ and $x \in X$, we define

$$(\leftarrow, x)_f = \{y \in X : y \prec_f x\} \text{ and } (x, \rightarrow)_f = \{y \in X : x \prec_f y\}.$$

The topology on X having all open intervals $(\leftarrow, x)_f$ and $(x, \rightarrow)_f$ as a subbase will be denoted by τ_f. These kind of topologies are called *selection topologies.* Selection topologies were introduced in [9] where it was proved that they are Hausdorff; In [10] was proved that they are regular; in [14] was proved they are always completely regular.

Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X. The *Vietoris topology* τ_V on $\mathcal{F}(X)$ is the one which has a base consisting of all sets of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \cap V \neq \emptyset,\ V \in \mathcal{V} \text{ and } S \subseteq \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of X. If we consider $[X]^2$ as a subspace of the space $(\mathcal{F}(X), \tau_V)$, then we say that a weak selection is continuous if it is a continuous function with respect to the relative Vietoris topology. The following result is summarised in [12].

Theorem 1.1. *Let X be a space and let $f : [X]^2 \to X$ be a weak selection. Then, the following are equivalent.*

(1) *$\preceq_f$ is a closed subset of $X \times X$.*
(2) *$\prec_f$ is open in $X \times X$.*
(3) *If $x, y \in X$ and $x \prec_f y$, then there are open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \times V \subseteq \prec_f$.*

(4) *f is continuous.*

For a space X, we let $Sel_2(X)$ denote the set of all weak selections for X and the symbol $Sel_2^c(X)$ stands for the set of all continuous weak selections.

Recall that a space (X, τ) is *orderable* if there is a linear order $\ll$ such that $\tau = \tau_{\ll}$, where $\tau_{\ll}$ is the order topology induced by the order. A space is called *suborderable* if it is a subspace of an orderable space (suborderable spaces are also named generalized ordered space). These spaces were introduced by Herrlich's Ph. D. Dissertation in 1962. Čech [2, pp. 285-286] (see also [19, pp. 5-6]) proved that a topological space (X, τ) is suborderable iff there exits a linear order $\ll$ on X such that

(1) $\tau_{\ll} \subseteq \tau$, and
(2) τ has a base consisting of open convex subsets[1].

A suborderable space will be denoted by $(X, \tau, \ll)$, where τ is the topology of X and $\ll$ is the linear order on X witnessing the suborderability of (X, τ). It is not hard to see that $w(X) \leq |X|$ for every suborderable space X. A space (X, τ) is *weakly orderable* if there is a linear order $\ll$ on X such that $\tau_{\ll} \subset \tau$. Clearly every suborderable space is weakly orderable, and every weakly orderable space has a continuous weak selection. The converses are not true. Hrušák and Martińez Ruiz [13] construct a space with continuous weak selections but not weakly orderable. The class of weakly orderable spaces is much wider than that of suborderable spaces, for instance, there are examples of weakly orderable (Polish) spaces with arbitrary large covering dimension (hence, small inductive dimension) [11], while the small inductive dimension of suborderable spaces is always less than or equal to 1.

By using selection topologies explicitly, some preceding results concerning (sub -)orderability of spaces with continuous weak selections can be read as follows:

• (van Mill-Wattel [17]) For a compact space (X, τ) with $Sel_2^c(X) \neq \emptyset$, $\tau = \tau_f$ and τ_f is orderable for any $f \in Sel_2^c(X)$.

Combining theorems of Artico, Marconi, Pelant, Lotter, Tkachenko [1] or Miyazaki [18], and Garcia-Ferreira, Sanchis [4], we have

• For a completely regular, pseudo-compact space (X, τ) with $Sel_2^c(X) \neq \emptyset$, $\tau = \tau_f$ and τ_f is suborderable for any $f \in Sel_2^c(X)$.

[1]A subset C of a linearly ordered set $(X, \leq)$ is called *convex* if for each $x, y \in C$ with $x < y$ we have that $[x, y]_{\leq} \subseteq C$.

• (Gutev [7]) For a locally compact paracompact space (X,τ) with $Sel_2^c(X) \neq \emptyset$, $\tau = \tau_f$ and τ_f is semi-orderable for any $f \in Sel_2^c(X)$, where a space is *semi-orderable* if it is the topological sum of two orderable spaces (hence, suborderable).

The Sorgenfrey line is an example of a non-orderable, suborderable space. Here is a different type of a result concerning selection topologies:

• (Hrušák and Martínez-Ruiz [14]) For the Sorgenfrey line (S,τ), there is $f \in Sel_2^c(S)$ such that $\tau = \tau_f$

• For the subspace $(X,\tau) = (0,1)\bigcup\{2\}$ of the reals, there is no $f \in Sel_2^c(X)$ satisfying $\tau = \tau_f$ but there are two $f, g \in Sel_2^c(X)$ such that τ is generated by $\tau_f \cup \tau_g$.

Motivated by these results we introduce CWS-spaces and their cws-numbers as follows:

For a set X and a family $\{\tau_i : i \in I\}$ of topologies on X, we recall that the *supremum topology* of this family is the smallest topology, denoted by $\bigvee_{i\in I} \tau_i$, on X that contains τ_i for each $i \in I$. This topology has as a subbase the family $\{U_{i_0} \cap ..... \cap U_{i_n} : n < \omega \text{ and } \forall k \leq n(U_{i_k} \in \tau_{i_k})\}$.

Definition 1.1. A space (X,τ) is called a *continuous weak selection space* (CWS-space) if there is $W \subseteq Sel_2^c(X)$ such that $\tau = \bigvee_{f\in W} \tau_f$. For a CWS-space X, we define the *cws-number* of X, denoted by $cws(X)$, as the minimum cardinality of a subset $W \subseteq Sel_2^c(X)$ for which $\tau = \bigvee_{f\in W} \tau_f$. If $W \subseteq Sel_2(X)$, then we simply say that the space X is a WS-space and the *ws-number* of X, denoted by $ws(X)$, is defined as the minimum cardinality of a subset $W \subseteq Sel_2(X)$ for which $\tau = \bigvee_{f\in W} \tau_f$.

By definition, we have that every CWS-space is a WS-space and $ws(X) \leq cws(X)$, for every CWS-space X. We still do not have any example of a WS-space that is not a CWS-space. In this paper, we shall mainly address our attention to CWS-spaces and their cws-numbers. Automatically the WS-spaces are Tychonoff (see [14]).

Clearly, every orderable space X is a CWS-space with $cws(X) = 1$. We know that $cws(S) = 1$ for the Sorgenfrey line S [14].

In this paper we shall consider the following natural question.

Question 1.2. For each cardinal number $\alpha \geq 1$, is there a (suborderable) space (X,τ) such that $cws(X) = \alpha$?

The paper is organized as follows:

In the second section, we present some basic properties of CWS-spaces. The third section is devoted to the study of the cws-number of weakly orderable spaces by using spaces with only one non-isolated point. In particular, for every infinite cardinal α we give an example of weakly orderable space whose cws-number is bigger than α. The Michael line M is an example of a suborderable space that is not orderable. We will show that $cws(M) = 1$. By modifying the Michael line, in the last section, we construction, assuming that $\mathfrak{c}$ is regular, a suborderable space whose cws-number is $\mathfrak{c}$. By using the main idea of this construction, under the assumption of GCH, for every infinite successor cardinal α^+ we construct a space X such that $cws(X) = \alpha^+$.

2. Topologies generated by weak selections

For a given space (X, τ), it is pointed out in [9] that if the weak selection $f : [X]^2 \to X$ is continuous, then $\tau_f \subseteq \tau$. Also an example is given in [9, Ex. 3.6] showing that the condition $\tau_f \subseteq \tau$ does not imply the continuity of the weak selection f.

Yamauchi[20] shows that $cws(X) \leq 2$ for every subspace of the reals. In the next theorem, we shall show that every subspace of a CWS-space is a CWS-space.

Lemma 2.1. *Let Y be a subspace of a CWS-space X with $|Y| \geq 2$. Let A be a clopen subset of Y. Then A is $\bigvee\{\tau_f : f \in Sel_2^c(Y)\}$-open.*

Proof. If $A = Y$, obviously the topology of Y is generated by $\bigvee\{\tau_f : f \in Sel_2^c(Y)\}$-open. Let $p \in A$. We will show that there exists a $\bigvee\{\tau_f : f \in Sel_2^c(Y)\}$-open set W satisfying $p \in W \subset A$. Choose any $f \in Sel_2^c(X)$ and let $g = f|[Y]^2$. Then $g \in Sel_2^c(Y)$. First we show the case when $A = Y$. Choose $x \in A \backslash \{p\}$. Then $(\leftarrow, x)_g$ or $(x, \rightarrow)_g$ is a neighborhood of p in Y, depending on $p \preceq_g x$ or $x \preceq_g p$. The case $Y \setminus A \neq \emptyset$ follows Proposition 3.2 in [8]. Namely, whenever $g \in Sel_2^c(Y)$, define $g_A^+, g_A^- \in Sel_2^c(Y)$ such that $g_A^+(\{x, y\}) = x$ if and only if (i) $x \in A$ and $y \in Y \backslash A$,and (ii) $g_A^+(\{x, y\} = g(\{x, y\})$ for the other case; $g_A^-(\{x, y\}) = g(\{x, y\})$ if and only if (i) $y \in A$ and $x \in Y \setminus A$,and (ii) $g_A^-(\{x, y\}) = g(\{x, y\})$ for the other case. Choose any point $q \in Y \setminus A$. Then $(\leftarrow, q)_{g_A^+} \cap (q, \rightarrow)_{g_A^-} = A$ is $\bigvee\{\tau_f : f \in Sel_2^c(Y)\}$-open. □

Theorem 2.2. *Let (X, τ) be a CWS-space and Y be a subspace of X with $|Y| \geq 2$. Then Y is a CWS-space.*

Proof. For a point $x \in X$ and a weak selection f we denote by $O(x, f)$, either $(x \rightarrow)_f$ or $(\leftarrow, x)_f$. Let U be an open set of Y and V be a open subset of X

such that $V \cap Y = U$. Let $p \in U$. Then there exist a finite set $H \subset X$ and $\{f_x : x \in H\} \subset Sel_2^c(X)$ such that $p \in \bigcap_{x \in H} O(x, f_x) \subset V$.

Choose $q \in H$. We will show that there exist a $\bigvee\{\tau_f : f \in Sel_2^c(Y)\}$-open set W such that $p \in W \subset O(q, f_q)$.

For simplicity we set $f_q = f$ and $g = f|[Y]^2$. Note $g \in Sel_2^c(Y)$. Without loss of generality, we may assume $O(q, f_q) = (\leftarrow, q)_f$. In the case $q \in Y$, clearly $p \in (\leftarrow, q)_g \subset (\leftarrow, q)_f$, so, we need to show only the case $q \notin Y$. But in this case the set $Y \cap (\leftarrow, q)_f = Y \cap (\leftarrow, q]_f$ is clopen in Y and contains p. By Lemma 2.1, the proof is completed. □

In the next section, we shall prove that some weakly orderable CWS-spaces X with just one non-isolated point satisfy that $|X| < cws(X)$.

Example 2.3. There exists a weakly orderable space X (hence, $Sel_2^c(X) \neq \emptyset$) which is not a CWS-space. Indeed, consider the space $X = \{(0,0)\} \bigcup \{(x, \sin(1/x)) : x \in (0,1)\}$ as a subspace of the plane $\mathbb{R}^2$. It is known that X is connected and has exactly two continuous weak selections. But the space generated by the both selections is homeomorphic to $[0,1)$ which is not homeomorphic to X.

Let us give some topological properties of the weak selection topologies. First, notice that $w(X, \tau_f) \leq |X|$ for every $f \in Sel_2(X)$. To state a cardinal inequality involving the weight and the cws-number, we need the following lemma.

Lemma 2.4. *Let X be an infinite set and let $\{\tau_i : i \in I\}$ be a family of topologies on X. If $\tau = \bigvee_{i \in I} \tau_i$, then*

$$w(X, \tau) = \sum_{i \in I} w(X, \tau_i).$$

Moreover, there is $J \subseteq I$ such that $\tau = \bigvee_{j \in J} \tau_j$ and $|J| \leq w(X, \tau)$.

PROOF. The equality is easy to verify. Let $\mathcal{B}$ be a base for τ of size $w(X, \tau)$. Consider the set

$$\mathcal{A} = \{(A, B) \in \mathcal{B} \times \mathcal{B} : \exists I_{A,B} \in [I]^{<\omega} \forall i \in I_{A,B} \exists U_i \in \tau_i (A \subseteq \cap_{i \in I_{A,B}} U_i \subseteq B)\}.$$

For each pair $(A, B) \in \mathcal{A}$ fix $I_{A,B} \in [I]^{<\omega}$ and, for each $i \in I_{A,B}$, fix $U_i \in \tau_i$ so that $A \subseteq \cap_{i \in I_{A,B}} U_i \subseteq B$. Put $J = \bigcup_{(A,B) \in \mathcal{A}} I_{A,B}$. It is not difficult to show that $\tau = \bigvee_{j \in J} \tau_j$ and $|J| \leq |\mathcal{B} \times \mathcal{B}| \leq w(X, \tau)$. □

Observe that for every X and for every $f \in Sel_2(X)$, we have that $w(X, \tau_f) \leq |X|$. As a direct application of Lemma 2.4, we obtain the following theorem.

Theorem 2.5. *For a CWS-space X, the following inequality holds:*

$$cws(X) \leq w(X) \leq cws(X) \cdot |X|$$

Hence, if $|X| < w(X)$, then $w(X) = cws(X)$.

Since $w(X) \leq |X|$ for a suborderable space X, we have

Corollary 2.6. $cws(X) \leq |X|$ *for a suborderable space X.*

Recall that a space is *zero dimensional* if it has a clopen base.

Theorem 2.7. *Let X be a zero dimensional space. If $Sel_2^c(X) \neq \emptyset$, then X is a CWS-space.*

PROOF. Apply Lemma 2.1 for $Y = X$. Then each clopen subset of X is generated by continuous weak selections for X. □

Lemma 2.8. *A countable regular space (X, τ) is weakly orderable, in particular, $Sel_2^c(X) \neq \emptyset$.*

PROOF. Since a countable regular space is zero dimensional, X has a family $\mathcal{U} = \{U_n : n \in \omega\}$ of clopen sets such that for any $x \in X$, there exists a subfamily $\mathcal{V}_x \subset \mathcal{U}$ satisfying $\bigcap \mathcal{V}_x = \{x\}$. Now the topology τ' on X obtained by $\mathcal{U}$ as a subbase is metrizable, hence orderable ([15]). Since the topology τ' is weaker than τ, (X, τ) is weakly orderable. □

By the above Lemma 2.8 and Theorem 2.7, every countable regular space is a CWS-space.

Corollary 2.9. *If X is a countable regular space with $cws(X) \leq \omega$, then X is second countable.*

The next corollary is a direct application of Theorem 2.5 and the fact that a countable metrizable space is orderable ([15]).

Corollary 2.10. *For a countable regular X, $cws(X) \leq \omega$ if and only if $cws(X) = 1$.*

To finish this section we formulate the following question.

Question 2.11. Is there a CWS-space X with $cws(X) = n$ for any natural number $n > 2$ or $n = \omega$? Or does $cws(X) \leq \omega$ imply $cws(X) \leq 2$?

3. Selection numbers for weakly orderable spaces with only one non-isolated point

In this section, we shall prove that some spaces with only one non-isolated point are CWS-spaces. To work with this kind of spaces we shall introduce some notation. First of all, let us see how the discrete topology on an infinite set can be realized by a suitable weak selection:

Lemma 3.1. *Let α be a cardinal with the discrete topology and $|\alpha| \geq 2$. Then there is $f \in Sel_2^c(\alpha)$ such that τ_f is the discrete topology on α.*

PROOF. If α is finite, then the selection defined by any order on α gives the discrete topology. For an infinite α, fix a partition $\{\mathbb{Z}_\theta : \theta < \alpha\}$ of α in countable infinite subsets identifying each copy with the integers $\mathbb{Z}$. Then, we define $f : [\alpha]^2 \to \alpha$ as follows:

$$f(\{\mu,\nu\}) = \begin{cases} \mu & \text{if } \mu \in \mathbb{Z}_\gamma, \nu \in \mathbb{Z}_\theta \text{ and } \gamma < \theta < \alpha \\ \\ \min\{\mu,\nu\} & \text{if } \mu,\nu \in \mathbb{Z}_\theta, \theta < \alpha \end{cases}$$

for each $\{\mu,\nu\} \in [\alpha]^2$. □

Given an infinite cardinal α and a free filter $\mathcal{F}$ on α, $\xi(\mathcal{F}) = \alpha \cup \{\mathcal{F}\}$ will be the space where α is discrete and the neighborhoods of $\mathcal{F}$ are of the form $A \cup \{\mathcal{F}\}$ for $A \in \mathcal{F}$. From Theorem 1.1 we can see that if a weak selection $f : [\xi(\mathcal{F})]^2 \to \xi(\mathcal{F})$ is continuous then τ_f is contained in the topology of $\xi(\mathcal{F})$. It follows from Theorem 2.7 that the space $\xi(\mathcal{F})$ is a CWS-space iff $Sel_2^c(\xi(\mathcal{F})) \neq \emptyset$. But this theorem does not say anything about the cws-number of the space. For a filter $\mathcal{F}$ on α, we define the *norm* of $\mathcal{F}$ by $\|\mathcal{F}\| = \min\{|A| : A \in \mathcal{F}\}$.

Theorem 3.2. *Let α be an infinite cardinal and $\mathcal{F}$ be a filter on α with $Sel_2^c(\xi(\mathcal{F})) \neq \emptyset$. Then $\xi(\mathcal{F})$ is a CWS-space with $cws(\xi(\mathcal{F})) \leq \chi(\xi(\mathcal{F}))$. If $\chi(\xi(\mathcal{F})) > \|\mathcal{F}\|$, then $cws(\xi(\mathcal{F})) = \chi(\xi(\mathcal{F}))$.*

PROOF. By Theorem 2.7, $\xi(\mathcal{F})$ is a CWS-space. Let $B \in \mathcal{F}$ be a set with cardinality $|B| = \|\mathcal{F}\| = \beta$. We identify B with the cardinal β. Choose a neighborhood base $\mathcal{A}$ of $\mathcal{F}$ in $\xi(\mathcal{F})$ satisfying $A \subset \beta \cup \{\mathcal{F}\}$ for each $A \in \mathcal{A}$ and $|\mathcal{A}| = \chi(\xi(\mathcal{F}))$. For each $A \in \mathcal{A}$, define weak selections f_A^+ and f_A^- satisfying (i) $A \preceq_{f_A^+} \xi(\mathcal{F}) \setminus A \preceq_{f_A^-} A$, (ii) $f_A^+|[\xi(\mathcal{F}) \setminus A]^2 = f_A^-|[\xi(\mathcal{F}) \setminus A]^2$ and (iii) f_A^+ (hence f_A^-) generates the discrete topology on $\xi(\mathcal{F}) \setminus A$, (iv) $f_A^+|[A]^2 = f_A^-|[A]^2 = f|[A]^2$. Then clearly $f_A^+, f_A^- \in Sel_2^c(\xi(\mathcal{F}))$, and $(\lambda, \rightarrow)_{f_A^-} \cap (\leftarrow, \lambda)_{f_A^+} = A$ for any

$\lambda \in \xi(\mathcal{F}) \setminus A$ by the definition of f_A^+ and f_A^-. To get each $A \in \mathcal{A}$ we need two continuous weak selections f_A^+ and f_A^-, so $cws(\xi(\mathcal{F})) \le \chi(\xi(\mathcal{F}))$.

Put $\mathcal{G} = \{F \cap \beta : F \in \mathcal{F}\}$ and $\xi(\mathcal{G}) = \beta \cup \{\mathcal{G}\}$. The implication $cws(\xi(\mathcal{F})) \ge \chi(\xi(\mathcal{F}))$ when $\chi(\xi(\mathcal{F})) > \beta$ now follows from $\chi(\xi(\mathcal{F})) = \chi(\xi(\mathcal{G})) \le w(\xi(\mathcal{G})) \le cws(\xi(\mathcal{G})) \cdot |\beta| = cws(\xi(\mathcal{F})) \cdot |\beta| = cws(\xi(\mathcal{F}))$ in Theorem 2.5. □

A family $\mathcal{W}$ of subsets of a set X is *nested* if $U \subset V$ or $V \subset U$ for every $U, V \in \mathcal{W}$.

Lemma 3.3. *(Corollary 5.5 [5]) Let α be an infinite cardinal and $\mathcal{F}$ be a filter on α. Then $\xi(\mathcal{F})$ is weakly orderable so that $\mathcal{F}$ is the last or, respectively, the first point of $\xi(\mathcal{F})$ if and only if there exists a nested family $\mathcal{W}$ of neighborhoods of $\mathcal{F}$ such that $\{\mathcal{F}\} = \bigcap \mathcal{W}$.*

Corollary 3.4. *Let α be a cardinal and $\mathcal{F}$ be a free ultrafilter on α. Then $\xi(\mathcal{F})$ is a CWS-space and $cws(\xi(\mathcal{F})) = \chi(\xi(\mathcal{F}))$.*

Proof. All we need to show is that $Sel_2^c(\xi(\mathcal{F})) \neq \emptyset$. We show that $\xi(\mathcal{F})$ has a nested family of neighborhoods of $\mathcal{F}$ satisfying the condition of Lemma 3.3. Let $B = \{b_\mu : \mu \in \kappa\} \in \mathcal{F}$ such that $|B| = \|\mathcal{F}\|$. Then $W_\mu = \{b_\theta : \mu < \theta < \kappa\} \cup \{\mathcal{F}\}$, $\mu < \kappa$ is a nested family satisfying the condition of Lemma 3.3. □

A filter $\mathcal{F}$ on α is said to be *uniform* if $\|\mathcal{F}\| = \alpha$. For a uniform ultrafilter $\mathcal{F}$ on α, $\chi(\xi(\mathcal{F})) > \alpha$, so for any infinite cardinal α, we can get examples of weakly orderable spaces with arbitrarily large cws-numbers.

Let α be an infinite cardinal. Let ${}^\alpha\omega = \{f : \alpha \to \omega\}$ be the functions from α to ω. Let us recall that the *generalized sequential fan* S_α is the space $\xi(\mathcal{F}_\alpha) = (\alpha \times \omega) \cup \{\mathcal{F}_\alpha\}$ where $\mathcal{F}_\alpha$ is the filter on $\alpha \times \omega$ generated by the sets of the form $A_f = \{(\theta, n) \in \alpha \times \omega : f(\theta) < n\}$ for a function $f \in {}^\alpha\omega$. The points of $\alpha \times \omega$ are isolated, and the neighborhoods of $\mathcal{F}_\alpha$ are $F \cup \{\mathcal{F}_\alpha\}$, where $F \in \mathcal{F}_\alpha$. For $f, g \in {}^\alpha\omega$, we write that $f \le^* g$ if $|\{\theta < \alpha : g(\theta) < f(\theta)\}|$ is finite. A subset $D \subseteq {}^\alpha\omega$ is said to be *dominating* if for every $f \in {}^\alpha\omega$ there is $g \in D$ such that $f \le^* g$. Let $\mathfrak{d}_\alpha$ be the minimum cardinality of a dominating subset of ${}^\alpha\omega$.

We remark that $\chi(S_\alpha) = \mathfrak{d}_\alpha$, and $\mathfrak{d}_\omega$ is the *dominating number* $\mathfrak{d} \ge \omega_1$.

Corollary 3.5. *For every infinite cardinal α, S_α is a CWS-space with $cws(S_\alpha) = \mathfrak{d}_\alpha$.*

Proof. We need to check the condition of Lemma 3.3. Let $W_n = \{(\theta, m) \in \alpha \times \omega : m > n\} \cup \{\mathcal{F}_\alpha\}$ for $n \in \omega$, and $\mathcal{W} = \{W_n : n \in \omega\}$. Then $\mathcal{W}$ is nested and $\bigcap \mathcal{W} = \{\mathcal{F}_\alpha\}$. □

The space S_ω is countable and $cws(S_\omega) = \mathfrak{d} \geq \omega_1$, so Corollary 2.10 cannot be improved beyound ω.

We say that a space X is *p-orderable* [5], where $p \in X$, if X has an open nested base at p. It is proved in [5] that;

Theorem 3.6. *$\xi(\mathcal{F})$ is orderable so that $\mathcal{F}$ is the last or, respectively, the first point of $\xi(\mathcal{F})$ if and only if it is $\mathcal{F}$-orderable.*

Using this theorem we can get the following theorem.

Theorem 3.7. *Let $\mathcal{F}$ be a free filter on an infinite cardinal α. If $\xi(\mathcal{F})$ is suborderable, then it is orderable.*

Proof. Let $\ll$ be the linear order on $\xi(\mathcal{F})$ witnessing its suborderability. Let $L = \{\theta \in \xi(\mathcal{F}) : \theta \ll \mathcal{F}\}$ and $R = \{\theta \in \xi(\mathcal{F}) : \theta \gg \mathcal{F}\}$. Then $L \cap R = \emptyset$ and $\xi(\mathcal{F}) = L \cup R \cup \{\mathcal{F}\}$. If $\xi(\mathcal{F})$ is suborderable, then $L \cup \{\mathcal{F}\}$ and $R \cup \{\mathcal{F}\}$ are suborderable such that the point $\mathcal{F}$ is the last or the first point, respectively. Since $\{(\lambda, \mathcal{F}]_{\ll} : \lambda \in L\}$ is nested, $L \cup \mathcal{F}$ is $\mathcal{F}$-orderable with the last point $\mathcal{F}$. By the above Theorem, $L \cup \{\mathcal{F}\}$ is orderable. Similarly $R \cup \{\mathcal{F}\}$ is orderable with the first point $\mathcal{F}$. We can conclude that $\xi(\mathcal{F})$ is orderable. □

So, all the CWS-spaces given in this section are either orderable or non-suborderable. If a free filter $\mathcal{F}$ on $\alpha \geq \omega$ satisfies $\chi(\xi(\mathcal{F})) > \alpha$, then the space $\xi(\mathcal{F})$ cannot be suborderable.

4. Selection numbers for sub-orderable spaces-Variations of the Michael line

In the sequel, the usual order of the real line will be simply denoted by $\leq$ and the open intervals of $\mathbb{R}$ will be denoted by $(x, \rightarrow)$, $(\leftarrow, x)$ and (x, y), for each $x, y \in \mathbb{R}$. Similarly, for the closed intervals of $\mathbb{R}$. For $x, y \in \mathbb{R}$, $\min\{x, y\}$ and $\max\{x, y\}$ are taken in the usual order of $\mathbb{R}$. The set of irrational numbers will be denoted by $\mathbb{I}$.

The *Michael line* M is the space whose underlying set is $\mathbb{R}$ and its topology is the one generated by the Euclidean topology of $\mathbb{R}$ and the family of singletons $\{\{r\} : r \in \mathbb{I}\}$. This space was introduced by E. Michael [16]. To give more examples of CWS-spaces we shall use some variation of the Michael line:

Definition 4.1. Let B a subsets of the real line. We will denote by M_B the topological space whose underlying set is the real numbers and a base for its topology is given by $\{\{x\} : x \in \mathbb{R} \setminus B\} \cup \{(x - \frac{1}{n+1}, x + \frac{1}{n+1}) : x \in B \text{ and } n < \omega\}$.

When $B = \mathbb{Q}$ we obtain the Michael line. It is interesting to see that the space $M_{\mathbb{R}\setminus\{x\}}$ admits a continuous weak selection and $cws(M_{\mathbb{R}\setminus\{x\}}) = 1$ (indeed, this space is homeomorphic to $(0,1) \cup \{2\} \cup (3,4)$ and apply some characterizations from [20]), for every $x \in \mathbb{R}$, although $cws(X) = 2$ for $X = (0,1) \cup \{2\}$. For more than two points we have the following.

Lemma 4.1. *Let B be a proper dense subset of $\mathbb{R}$. For two distinct points $x, y \in M_B \setminus B$, there exists a continuous weak selection $f_{x,y} : [M_B]^2 \to M_B$ such that the topology τ_f is finer than the topology on M_B, and $\{x\}$ and $\{y\}$ are isolated points in the topology $\tau_{f_{x,y}}$.*

PROOF. Assume that $x < y$. We define the required weak selection $f_{x,y} : [M_B]^2 \to M_B$ as follows:

$$f_{x,y}(\{a,b\}) = \begin{cases} \min\{a,b\} & \text{if } \{a,b\} \neq \{x,y\} \\ y & \text{if } a = x \text{ and } b = y. \end{cases}$$

One can see without any difficulty that $f_{x,y}$ is continuous. Clearly,

$$(\leftarrow, x)_{f_{x,y}} = (\leftarrow, x) \cup \{y\} \text{ and } (y, \rightarrow)_{f_{x,y}} = (y, \rightarrow) \cup \{x\}.$$

Hence, the points x and y are isolated in the topology $\tau_{f_{x,y}}$. □

The next lemma follows directly from the previous lemma and the proof that the Michael line is suborderable (see [19]).

Lemma 4.2. *For every subset B of $\mathbb{R}$, the space M_B is a suborderable space.*

Theorem 4.3. *For the Michael line M, $cws(M) = 1$.*

PROOF. For each integer $n \in \mathbb{Z}$, fix a Cantor set $C_n \subsetneq [n, n+1] \setminus \mathbb{Q}$ and let $D_n = [n, n+1] \setminus (C_n \cup \mathbb{Q})$. Let $\{c_n^\xi : \xi < \mathfrak{c}\}$ and $\{d_n^\xi : \xi < \mathfrak{c}\}$ be a faithful indexation of C_n and D_n, respectively, for each $n \in \mathbb{Z}$. Our selection $f : [M]^2 \to M$ is defined by

$$f(\{x,y\}) = \begin{cases} c_{n+1}^\xi & \text{if } \{x,y\} = \{d_n^\xi, c_{n+1}^\xi\} \\ d_{n+1}^\xi & \text{if } \{x,y\} = \{c_n^\xi, d_{n+1}^\xi\} \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$. To prove the continuity we apply Theorem 1.1. To do that fix $\{x,y\} \in [\mathbb{R}]^2$ with $x < y$, and assume that $x \in [n, n+1)$ and $y \in [m, m+1)$, for some $n, m \in \mathbb{Z}$.

Case I. $x, y \in \mathbb{Q}$. Chose open intervals $I \subsetneq (n-1, n+1) \setminus (C_{n-1} \cup C_n)$ and $J \subsetneq (m-1, m+1) \setminus (C_{m-1} \cup C_m)$ so that $(x,y) \in I \times J$ and $i < j$ for every $(i,j) \in I \times J$. Then, $i \prec_f j$ for each $(i,j) \in I \times J$

Case II. $x \in \mathbb{Q}$ and $y \in \mathbb{I}$. As in the previous case, choose an open interval $I \subseteq (n-1, n+1) \setminus (C_{n-1} \cup C_n)$ containing x and satisfying $i < y$ for all $i \in I$. It is evident that $i \prec_f y$ for all $i \in I$.

Case III. $x \in \mathbb{I}$ and $y \in \mathbb{Q}$. This case is completely similar to the previous one.

Case IV. $x, y \in \mathbb{I}$. This case is trivial.

Thus, we have that the weak selection is continuous and so τ_f is contained in the original topology of M. Since

$$(d^{\xi}_{n+1}, \rightarrow)_f = (d^{\xi}_{n+1}, \rightarrow) \cup \{c^{\xi}_n\} \text{ and } (c^{\xi}_{n+1}, \rightarrow)_f = (c^{\xi}_{n+1}, \rightarrow) \cup \{d^{\xi}_n\},$$

for every $n \in \mathbb{Z}$ and for every $\xi < \mathfrak{c}$, it then follows that τ_f coincides with the topology of M. □

To construct more examples we recall that a *Bernstein* set is a subset B of $\mathbb{R}$ such that every compact subset of $\mathbb{R}$ contained either in B or in $\mathbb{R} \setminus B$ is countable. For the existence of a Bernstein set of cardinality $\mathfrak{c}$ see [3, P. 5.5.4(a)]. We need a special Bernstein set which will be describe in the next lemma.

In what follows, let $D = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Lemma 4.4. *Assume that $\mathfrak{c}$ is regular. Then, there exists a Bernstein set B of the reals such that if $U \subseteq D$ is an open subset of $\mathbb{R}^2$ containing $(B \times B) \cap D$, then $D \setminus U$ is contained in the union of fewer than $\mathfrak{c}$ vertical and horizontal lines. That is, there exists A of cardinality smaller than $\mathfrak{c}$ such that $D \setminus U \subseteq (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$.*

PROOF. First let us see some properties of the closed subsets of D that cannot be covered by fewer than $\mathfrak{c}$ many vertical and horizontal lines. We shall prove that for such a closed subset F of D we can find

($*$) $\{x_\alpha : \alpha < \mathfrak{c}\} \cup \{y_\alpha : \alpha < \mathfrak{c}\}$ pairwise distinct real numbers such that $(x_\alpha, y_\alpha) \in F$ for each $\alpha < \mathfrak{c}$.

Indeed, let $G = \{x \in \mathbb{R} : |(\{x\} \times \mathbb{R}) \cap F| = \mathfrak{c}$. If G has cardinality $\mathfrak{c}$ then using a diagonal argument we can show that F satisfies ($*$). Now, assume that G has cardinality smaller than $\mathfrak{c}$. Let $H = \{x \in \mathbb{R} : 0 < |(\{x\} \times \mathbb{R}) \cap F| < \mathfrak{c}\}$. Note that in this case $(\{x\} \times \mathbb{R}) \cap F$ is countable, for each $x \in H$, and the set H has cardinality $\mathfrak{c}$. Let L be the projections of $(H \times \mathbb{R}) \cap F$ into the second coordinate. If $|L| < \mathfrak{c}$, then F would be covered by fewer than $\mathfrak{c}$ vertical and horizontal lines. Therefore, L must have cardinality $\mathfrak{c}$. Suppose that for some $\beta < \mathfrak{c}$ we have defined $(x_\alpha, y_\alpha) \in (H \times \mathbb{R}) \cap F$ pairwise distinct, for each $\alpha < \beta$. Notice that the set $(\{x_\alpha : \alpha < \beta\} \times \mathbb{R}) \cap F$ has cardinality smaller than $\mathfrak{c}$. Fix $y_\beta \in L \setminus \{y_\alpha : \alpha < \beta\}$ so that y_β is not in the projection of $\{x_\alpha : \alpha < \beta\} \times \mathbb{R}) \cap F$ into the second coordinate. Now choose $x_\beta \in H$ such that $(x_\beta, y_\beta) \in F$. We can check that $\{x_\beta, y_\beta\} \cap \{x_\alpha, y_\alpha : \alpha < \beta\} = \emptyset$. Thus, F satisfies ($*$).

We know that there are $\mathfrak{c}$-many closed subsets of $\mathbb{R} \times \mathbb{R}$ of cardinality $\mathfrak{c}$. Hence, there are $\mathfrak{c}$-many closed subsets F of $\mathbb{R} \times \mathbb{R}$ satisfying $(*)$. Using a standard diagonal argument, we can find B such that $\mathbb{R} \setminus B$ is dense in $\mathbb{R}$, has cardinality $\mathfrak{c}$, B intersects every closed subset of $\mathbb{R}$ of cardinality $\mathfrak{c}$ and B intersects every closed set with property $(*)$. $\square$

Next, we shall require the following notation:
Given $f \in Sel_2(X)$ and two finite subsets L and R of X, we let

$$I_f(L, R) = \{x \in X : \forall y \in L \forall z \in R (y \prec_f x \wedge x \prec_f z)\}.$$

Lemma 4.5. *Assume that $\mathfrak{c}$ is regular. Let B be the Bernstein set of the reals from Lemma 4.4. For every finite subset $\{f_i : i \leq l\}$ of $Sel_2^c(M_B)$, the topology $\bigvee_{i \leq l} \tau_{f_i}$ on $\mathbb{R}$ has strictly less than $\mathfrak{c}$ many isolated points.*

Proof. Let B be a Bernstein set as in Lemma 4.4. Let $\{f_i : i \leq l\}$ be a finite subset of $Sel_2^c(M_B)$ and set $\tau = \vee_{i \leq l} \tau_{f_i}$. We remark that τ_{f_i} is contained in the original topology of M_B, for every $i \leq l$. Since f_i is continuous, by Theorem 1.1, for each $(x, y) \in (B \times B) \cap D$ and for each $i \leq l$, choose disjoint open intervals $I_{(x,y)}$ and $J_{(x,y)}$ such that $(x, y) \in I_{(x,y)} \times J_{(x,y)} \subseteq D$ and, for each $i \leq l$, either $a \prec_{f_i} b$ for all $(a, b) \in I_{(x,y)} \times J_{(x,y)}$, or $b \prec_{f_i} a$ for all $(a, b) \in I_{(x,y)} \times J_{(x,y)}$. Put

$$U = \bigcup \{I_{(x,y)} \times J_{(x,y)} : (x, y) \in (B \times B) \cap D\}.$$

Evidently, the set U is open and hence $D \setminus U$ is a closed set that does not intersect $B \times B$. Thus, $D \setminus U$ is contained in fewer than $\mathfrak{c}$ horizontal and vertical lines of $\mathbb{R}^2$. Hence, there exists A of cardinality smaller than $\mathfrak{c}$ such that $D \setminus U \subseteq (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$.

Let I be the set of isolated points of $(\mathbb{R}, \tau)$. We must have that $I \subseteq \mathbb{R} \setminus B$. For each isolated point $x \in I$ and for each $i \leq l$, choose $L_{x,i}, R_{x,i} \in [\mathbb{R}]^{<\omega}$ so that $\bigcap_{i \in l} I_{f_i}(L_{x,i}, R_{x,i}) = \{x\}$. Let us suppose that $|I| = \mathfrak{c}$. Since A has cardinality smaller than $\mathfrak{c}$ and $\mathfrak{c}$ is regular, we may choose $C \subseteq I \setminus A$ of cardinality $\mathfrak{c}$ so that

$$(*) \qquad L_{x,i} \cap A = L_{y,i} \cap A \text{ and } R_{x,i} \cap A = R_{y,i} \cap A,$$

for every $i \leq l$ and for every $x, y \in C$. Let $z \in C$ be an accumulation point of C in the Euclidean topology. We can find $\epsilon > 0$ so that for each $y \in (L_{z,i} \setminus A) \cup (R_{z,i} \setminus A)$ we have that $|z - y| > \epsilon$ and, for every $i \leq l$, f_i either chooses always the first coordinate or always chooses the second coordinate on any point of the square of side 2ϵ and center either (z, y) if $z < y$ or (y, z) if $y < z$. Since z is an accumulation point of C, there exists $t \in C$ such that $|z - t| < \epsilon$. Let $i \leq l$ and $y \in (L_{z,i} \setminus A) \cup (R_{z,i} \setminus A)$. Suppose that $(y, z) \in D$, then $(y, t) \in D$ and (y, t) is in the square centered in (y, z) thus, either f_i chooses z and t or f_i chooses y

for the pair (y,z) and (y,t). Similarly, if $(z,y) \in D$ then $(t,y) \in D$ and either f_i chooses z and t or f_i chooses y for the pair (z,y) and (t,y). This and $(*)$ imply that $t \in \bigcap_{i \leq l} I_{f_i}(L_{z,i}, R_{z,i}) = \{z\}$, a contradiction. □

Theorem 4.6. *Assume that $\mathfrak{c}$ is regular. Then, there exists a Bernstein set B of the reals such that $cws(M_B) = \mathfrak{c}$.*

Proof. Let B be a Bernstein set as in Lemma 4.4. By Corollary 3.7 $cws(M_B) \leq |M_B| = \mathfrak{c}$. For each $x \in \mathbb{R} \setminus B$ choose $f_x \in Sel_2^c(M_B)$ as in Lemma 4.1 so that $\{x\} \in \tau_{f_x}$. It is evident that $\{f_x : x \in \mathbb{R} \setminus B\}$ generates the topology on M_B.

On the other hand, according to Lemma 4.5, every finite subset of $Sel_2^c(M_B)$ produces strictly less than $\mathfrak{c}$-many isolated points of M_B. Therefore, by the regularity of $\mathfrak{c}$, the topology of M_B cannot be generated by less than $\mathfrak{c}$-many continuous weak selections. □

Theorem 4.7. [GCH] *For every infinite cardinal α there exists a suborderable space X such that $cws(X) = \alpha^+$.*

Proof. Let X be the set ${}^{\alpha}\mathbb{Q}$ and we equip X with the linear order given by $f < g$ if and only if $f \neq g$ and $f(\theta) < g(\theta)$ where $\theta = \min\{\gamma < \alpha : f(\gamma) \neq g(\gamma)\}$. We consider X as a space with the topology induced by this linear order. Then, we have that the set $\bigcup_{\theta<\alpha} {}^{\theta}\mathbb{Q} \times {}^{\alpha\setminus\theta}\{0\}$ is a dense subset of X of size α, and X has weight α. Since it is an ordered space with density α and size $\alpha^+ = 2^{\alpha}$, we can repeat the previous construction to obtain a subset B of X for which the "Michael line" X_B B has the desired properties. □

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