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SYMMETRIC ELEMENTS UNDER ORIENTED INVOLUTIONS IN GROUP RINGS

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ABSTRACT. Let RG denote the group ring of a group G over a commutative ring with unity R. Given a homomorphism $\sigma : G \to \{\pm 1\}$ and an involution of the group G, an *oriented invoultion* $^{\sigma\varphi}$ of RG is defined in a natural way. We characterize when the set of symmetric elements under this involution is a subring. This gives a unified setting for earlier work of several authors.

INTRODUCTION

Let RG denote the group ring of a group G over a commutative ring with unity R and let $\mathcal{U}(RG)$ be its group of units. Given an involution * in RG, the set of symmetric elements under this involution is a subring if and only if it is commutative.

The commutativity of such a set, when * is the classical involution of RG, (obtained by extending linearly the map $g \mapsto g^{-1}$, for all $g \in G$), was studied in [4] and [5]. Also commutativity of symmetric units, in the case when G is a torsion group and R is of odd prime characteristic was studied.

Later, Jespers and Ruiz [7] considered involutions of RG obtained by extending R-linearly arbitrary involutions of G, gave a characterization for the commutativity of the set of symmetric elements and, when G is torsion and R a G-favourable integral domain, also of symmetric units. This restriction was needed by Bovdi [3] in earlier work for the classical involution of G.

Let $\sigma : G \to \{\pm 1\}$ be a homomorphism. Such a map is called an *orientation* of the group G. Novikov [10], introduced in the context of

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K-theory, an involution of RG, defined by

$$\left(\sum r_i g_i\right)^{\sigma} = \sum r_i \sigma(g_i) g_i^{-1},$$

which was subsequently studied by various authors (see [1], [2], [8] and [9]).

Given both an orientation $\sigma : G \to \{\pm 1\}$ and a group involution $\varphi : G \to G$, an *oriented involution* of RG is defined by

$$\left(\sum r_i g_i\right)^{\sigma\varphi} = \sum r_i \sigma(g_i)\varphi(g_i).$$

The set of symmetric elements of RG under $\sigma\varphi$ will be denoted by $(RG)^{\sigma\varphi}$ and $\mathcal{U}^{\sigma\varphi}(RG)$ will denote the set the symmetric units.

We characterize the commutativity of $(RG)^{\sigma\varphi}$ for arbitrary non-trivial orientations σ and involutions φ of G. Also, we show that if G is a torsion group and R is a G-favourable domain, then the commutativity of $\mathcal{U}^{\sigma\varphi}(RG)$ is equivalent to that of $(RG)^{\sigma\varphi}$.

1. Preliminary lemmas.

Let R be a commutative ring with unity and let G be a group with a non-trivial orientation homomorphism σ and an involution φ. Notice that, as σ is non-trivial, $char(R)$ must be different from 2.

If N denotes the kernel of σ, then N is a subgroup of index 2 in G. It is obvious that the involution $\sigma\varphi$ coincides on the subring RN with the ring involution φ. Also, we have that the symmetric elements in G, under $\sigma\varphi$, are the symmetric elements in N under φ. If we denote the sets of symmetric elements in G, under the involutions $\sigma\varphi$ and φ, by $G_{\sigma\varphi}$ and G_{φ}, respectively, then we can to write $G_{\sigma\varphi} = N \cap G_{\varphi} = N_{\varphi}$. Thus, with this notations, we have that, as an R-module, $(RG)^{\sigma\varphi}$ is generated by the set

$$\begin{aligned} S &= G_{\sigma\varphi} \cup \{g + g^{\sigma\varphi} \mid g \in G \setminus G_{\sigma\varphi}\} \\ &= N_{\sigma} \cup \{g + g^{\varphi} \mid g \in N \setminus N_{\varphi}\} \cup \{g - g^{\varphi} \mid g \in G \setminus N\}. \end{aligned}$$

Therefore $(RG)^{\sigma\varphi}$ is a commutative ring if and only if the elements in S commute.

Now, suppose that $(RG)^{\sigma\varphi}$ is commutative. Then $(RN)^{\varphi}$ is commutative, and thus, by [7], we know the struture of N and the action of φ on N. In order to determine the structure of G and the action of $\sigma\varphi$ we shall begin with two technical lemma. To simplify the notation, we shall use the Lie bracket of elements in RG, defined by $[\alpha, \beta] = \alpha\beta - \beta\alpha$

Lemma 1.1. *Suppose that* $(RG)^{\sigma\varphi}$ *is commutative and let* $g \in (G \setminus N) \setminus G_\varphi$, $h \in G$. *Then one of the following holds:*

(*i*) $gh = hg$;

(*ii*) $char(R) = 4$ *and* $gh = g^\varphi h^\varphi = hg^\varphi = h^\varphi g$.

Furthermore, $gg^\varphi = g^\varphi g$.

Proof: We divide the proof into 4 different cases.

(*a*) Suppose first that $h \in N \cap G_\varphi$. Then $0 = [g + g^{\sigma\varphi}, h] = [g - g^\varphi, h] = gh - g^\varphi h - hg + hg^\varphi$. Since $char(R) \neq 2$ and $g \notin G_\varphi$ we have that $gh = hg$, and (*i*) follows. In particular, as $gg^\varphi \in N \cap G_\varphi$, we obtain that $gg^\varphi = g^\varphi g$, the last statement of the lemma.

(*b*) In case $h \in (G \setminus N) \setminus G_\varphi$, then $0 = [g+g^{\sigma\varphi}, h+h^{\sigma\varphi}] = [g-g^\varphi, h-h^\varphi]$ $= gh - gh^\varphi - g^\varphi h + g^\varphi h^\varphi - hg + hg^\varphi + h^\varphi g - h^\varphi g^\varphi$. Thus, as $g, h \notin G_\varphi$ and $char(R) \neq 2$, we have four possibilities to consider either:

- $gh = hg$ or
- $gh = h^\varphi g^\varphi$ or
- $char(R) = 3$ and gh is equal to two elements among $g^\varphi h^\varphi, hg^\varphi, h^\varphi g$ or
- $char(R) = 4$ and $gh = g^\varphi h^\varphi = hg^\varphi = h^\varphi g$.

If $gh = h^\varphi g^\varphi = (gh)^\varphi$ then, as $gh \in N \cap G_\varphi$, by case (*a*), it follows that gh and g commute, that is, $gh = hg$. On the other hand, if $char(R) = 3$ and $gh = g^\varphi h^\varphi = hg^\varphi$ then $h^\varphi g^\varphi = hg = gh^\varphi$, and thus, $0 = [g + g^{\sigma\varphi}, h + h^{\sigma\varphi}] = h^\varphi g - g^\varphi h$. Therefore, $g^\varphi h = h^\varphi$, so $g^\varphi h \in N \cap G_\varphi$. Hence, by case (*a*), it follows that $g^\varphi hg = gg^\varphi h = g^\varphi gh$, and so $gh = hg$. Similarly, if either $gh = g^\varphi h^\varphi = h^\varphi g$ or $gh = hg^\varphi = h^\varphi g$ we have that $gh = hg$ and the result follows.

(*c*) Suppose now that $h \in N \setminus G_\varphi$. Then $0 = [g + g^{\sigma\varphi}, h + h^{\sigma\varphi}] = [g - g^\varphi, h + h^\varphi] = gh + gh^\varphi - g^\varphi h - g^\varphi h^\varphi - hg + hg^\varphi - h^\varphi g + h^\varphi g^\varphi$. Thus, since $char(R) \neq 2$, it follows that gh must be either equal to at least two terms with positive coefficient or to some term with a negative coefficient. If the first possibility occurs we have that either $gh = gh^\varphi$ or $hg^\varphi = h^\varphi g^\varphi$, and so $h = h^\varphi$, a contradiction. Therefore, as $g \notin G_\varphi$, it follows that gh is equal to either $g^\varphi h^\varphi$ or hg or $h^\varphi g$. If $gh = h^\varphi g$ then $(gh)^\varphi = g^\varphi h$. Since $g \notin G_\varphi$, it follows that $gh \notin G_\varphi$. On the other hand, if $gh = g^\varphi h^\varphi = (hg)^\varphi$ then $gh \in G_\varphi$ if and only if $gh = hg$. Suppose now that $gh \notin G_\varphi$. As $gh \notin N$ it follows, by case (*b*), that either gh and g commute or $char(R) = 4$ and $g^2h = g^\varphi h^\varphi g^\varphi = ghg^\varphi = h^\varphi g^\varphi g$ ($= h^\varphi gg^\varphi$, as in (*a*)). Therefore, either $gh = hg$ or $char(R) = 4$ and $gh = g^\varphi h^\varphi = hg^\varphi = h^\varphi g$, and the result follows.

(*d*) Finally, if $h \in (G \setminus N) \cap G_\varphi$, then $gh \in N$. If $gh \in G_\varphi$ then, by case (*a*), it follows that gh and g commute, and (*i*) follows. Now, if

$gh \notin G_\varphi$ then, by case (c), we have that either gh and g commute or $ghg^\varphi = (gh)^\varphi g$. Therefore, either g and h commute or $ghg^\varphi = h^\varphi g^\varphi g = hgg^\varphi$. Hence $gh = hg$ and again (i) follows.

□

The lemma above is enough to give a complete characterization of the commutativity of $(RG)^{\sigma\varphi}$ when $char(R) \neq 4$. Our next lemma is necessary to study the case when $char(R) = 4$.

Lemma 1.2. *Suppose that $(RG)^{\sigma\varphi}$ is commutative and let $g \in (G \setminus N) \cap G_\varphi$ and $h \in G$. Then one of the following holds:*

(i) $h \in N \cap G_\varphi$ and $gh = hg$;
(ii) $h \in N \setminus G_\varphi$ and, $gh = hg$ or $gh = h^\varphi g$;
(iii) $h \in (G \setminus N) \setminus G_\varphi$ and $gh = hg$;
(iv) $h \in (G \setminus N) \cap G_\varphi$ and, $gh = hg$ or $g^2h = hg^2$

Proof: Suppose that $h \in N \cap G_\varphi$. Then $gh \in G \setminus N$. If $gh \notin G_\varphi$ then, by part (a) of the proof of Lemma 1.1, we have that gh and h commute, and thus $gh = hg$. On the other hand, if $gh \in G_\varphi$ then it follows that $gh = (gh)^\varphi = h^\varphi g^\varphi = hg$, and thus (i) follows.

Now, if $h \in N \setminus G_\varphi$ then again $gh \in G \setminus N$. In case $gh \notin G_\varphi$, we have, by Lemma 1.1, that either gh and h commute or $(gh)^\varphi h^\varphi = h^\varphi gh$. So, since $(gh)^\varphi h^\varphi = h^\varphi g^\varphi h^\varphi = h^\varphi gh^\varphi$ and $h \neq h^\varphi$, it follows that $gh = hg$. In case $gh \in G_\varphi$, we have that $gh = (gh)^\varphi = h^\varphi g$ and so (ii) follows.

Now, if $h \in (G \setminus N) \setminus G_\varphi$ then, by part (d) of the proof of Lemma 1.1, we have that g and h commute, and (iii) follows.

Finally, assume that $h \in (G \setminus N) \cap G_\varphi$. Then $gh \in N$. If $gh \in G_\varphi$ then it follows, by part (i), that g and gh commute, so $gh = hg$. Now, if $gh \notin G_\varphi$ then we have, by part (ii), that either g and gh commute or $g^2h = (gh)^\varphi g = hg^2$ and the result follows. □

2. Commutativity of symmetric elements

Next, we study the commutativity of $(RG)^{\sigma\varphi}$. Denote the center of the group G by $Z(G)$ and recall that G satisfies the lack of commutativity property ("LC" for short) if, for any pair of elements $g, h \in G$, $gh = hg$ happens if and only if either $g \in Z(G)$, or $h \in Z(G)$, or $gh \in Z(G)$.

Assume that $(RG)^{\sigma\varphi}$ is commutative and let N be the kernel of σ. Then $(RN)^\varphi$ is a commutative ring. Hence, by [7, Theorem 2.4], we have two possibilities for N: either

(A) N is an abelian group, or

(*B*) N has the LC property, there exists a unique non-trivial commutator s and the involution φ, in N, is given by

$$g^{\varphi} = \begin{cases} g & \text{if } g \in Z(N). \\ sg & \text{if } g \in N \setminus Z(N). \end{cases} \tag{1}$$

Note that if the last possibility occurs, s is a central element in N of order 2.

The following technical lemma that will be needed in the sequel.

Lemma 2.1. *Let R be a commutative ring of characteristic 4, such that $(RG)^{\sigma\varphi}$ is commutative. If $G \neq N \cup G_{\varphi} \cup Z(G)$, then the set $(G \setminus N) \cap G_{\varphi}$ is either empty or central in G.*

Proof: From our assumption, there exists $g \notin N \cup G_{\varphi} \cup Z(G)$ and $h \in N$, such that $gh \neq hg$. Then part (*ii*) of Lemma 1.1 holds, that is, $gh = g^{\varphi}h^{\varphi} = hg^{\varphi} = h^{\varphi}g$. Therefore, it follows that $g^{\varphi} = h^{-1}gh$ and $h^{\varphi} = ghg^{-1}$. Furthermore, we have that $h, gh \notin G_{\varphi}$. In fact, if $h = h^{\varphi}$ then $gh = g^{\varphi}h^{\varphi} = g^{\varphi}h$ and thus $g \in G_{\varphi}$, a contradiction. On the other hand, if $gh \in G_{\varphi}$ then $h^{\varphi}g^{\varphi} = (gh)^{\varphi} = gh = h^{\varphi}g$ and so we get the same contradiction.

Assume that there exists an element $x \in (G \setminus N) \cap G_{\varphi}$. Then, by part (*iii*) of Lemma 1.2, it follows that x and g commute. Moreover, as $gh \in (G \setminus N) \setminus G_{\varphi}$, again applying part (*iii*) of Lemma 1.2 we have that x and gh commute. Therefore, x and h also commute.

Suppose that (*A*) holds, that is, N is abelian. Then, since h and xg are elements of N, they commute. Thus, as x and h commute, it follows that g and h also commute, a contradiction. Hence, if N is abelian then $(G \setminus N) \cap G_{\varphi} = \emptyset$.

Suppose that (*B*) holds. We claim that, in this case, x is central. First, since φ in N is given by (1), we note that $N_{\varphi} = Z(N)$. Furthermore, by part (*a*) of the proof of Lemma 1.1, it follows that g commutes with the elements of N_{φ}. Thus, as $G = N \cup Ng$, we have that $N_{\varphi} \subset Z(G)$. Therefore, since x and g commute, it is enough to show that x commutes with the elements of $N \setminus N_{\varphi}$. To prove this we remark first that $h^{\varphi} = sh$ and thus, since $gh = g^{\varphi}h^{\varphi} = hg^{\varphi} = h^{\varphi}g$, we also have that $gh = shg$, $g^{\varphi} = sg = gs$ and s is a central element in G.

Let y be an arbitary element of $N \setminus N_{\varphi}$. If $hy \neq yh$ then $hy \notin Z(N)$, and thus $s(hy) = (hy)^{\varphi} = y^{\varphi}h^{\varphi} = sysh = yh$. Therefore, either $hy = yh$ or $hy = syh$. On the other hand, by Lemma 1.1, we have that either $gy = yg$ or $gy = y^{\varphi}g = syg$. Furthermore, by part (*ii*) of Lemma 1.2, it follows that either $xy = yx$ or $xy = y^{\varphi}x = syx$.

Assume that $xy = syx$. If $gy \neq yg$ then we proceed as in the beginning of the proof, working with y instead of h, and conclude that $xy = yx$. Thus, we may also suppose that $gy = yg$.

If $hy = syh$ then $(hgy)^\varphi = y^\varphi g^\varphi h^\varphi = sysgsh = sygh = sgyh = sgshy = ghy = shgy$, and $hgy \in (G \setminus N) \setminus G_\varphi$. So, by part (iii) of Lemma 1.2, we have that x and hgy commute. Therefore, $xhgy = hgyx = hgsxy = shxgy = sxhgy$ and $s = 1$, a contradiction.

On the other hand, if $hy = yh$ then $(hxy)^\varphi = y^\varphi x^\varphi h^\varphi = syxsh = yxh = yhx = hyx = shxy$, and so $hxy \in (G \setminus N) \setminus G_\varphi$. Thus, by part (iii) of Lemma 1.2, it follows that x and hxy commute. Therefore, $xhxy = hxyx = xhsxy = sxhxy$ and $s = 1$, a contradiction. Hence, $xy = yx$ and x is central in G, as claimed.

□

Theorem 2.2. *Let R be a commutative ring with unity and let G be a non-abelian group with involution φ and non-trivial orientation homomorphism σ. Then, $(RG)^{\sigma\varphi}$ is a commutative ring if and only if one of the following conditions holds:*

(i) $N = Ker(\sigma)$ is an abelian group and $(G \setminus N) \subset G_\varphi$;

(ii) G and $N = Ker(\sigma)$ have the LC property, and there exists a unique non-trivial commutator s such that the involution φ is given by

$$\varphi(g) = \begin{cases} g & \text{if } g \in N \cap Z(G) \text{ or } g \in (G \setminus N) \setminus Z(G). \\ sg & \text{if otherwise.} \end{cases}$$

(iii) $char(R) = 4$, G has the LC property, and there exists a unique non-trivial commutator s such that the involution φ is given by

$$\varphi(g) = \begin{cases} g & \text{if } g \in Z(G). \\ sg & \text{if } g \notin Z(G). \end{cases}$$

Proof: Assume that $(RG)^{\sigma\varphi}$ is commutative and let N be the kernel of σ. We shall study separately two cases according to whether the elements in $G \setminus N$ are all symmetric under φ or not.

(1) $(G \setminus N) \subset G_\varphi$.

In this case (A) holds. In fact, assume that (B) holds. Let $x, y \in N$ be non-commuting elements. Then x, y and xy are not central in N. Thus, since φ is given by (1), we have that $x^\varphi = sx$, $y^\varphi = sy$ and $(xy)^\varphi = sxy$. Let $g \in G \setminus N$. As $xg \in G \setminus N$, we obtain $xg = (xg)^\varphi = g^\varphi x^\varphi = gsx$, that is, $x^g = sx$. Similarly, $y^g = sy$ and $(xy)^g = sxy$. But, $(xy)^g = x^g y^g = sxsy$ and $s = 1$, a contradiction. Hence (i) follows.

(2) $(G \setminus N) \not\subset G_\varphi$.

We shall consider two subcases:

(**2.1**) *If* $char(R) \neq 4$ *then* (ii) *holds.*

Let g be a fixed element in $G \setminus N$ such that $g \notin G_\varphi$. Then, by Lemma 1.1, we have that g is central in G. Moreover, since the index of N in G is equal to 2, it follows that $G = N \cup Ng$. Thus, clearly, if N is abelian then G is also abelian, a contradiction. Therefore, N has the LC property and a unique non-trivial commutator s. This implies that G has these properties too. In fact, as g is central, for all $x, y \in N$, we have that xg and yg commute if and only if x and y commute. As N has the LC property, this is equivalent to either $x \in Z(N)$, or $y \in Z(N)$ or $xy \in Z(N)$. But, $G = N \cup Ng$ and thus this is also equivalent to either $xg \in Z(G)$, or $yg \in Z(G)$ or $xgyg \in Z(G)$, for all $x, y \in N$. We can proceed in a similar way with elements of the form xg and y and conclude that G has the LC property. On the other hand, for all non-commutative $x, y \in N$ we have that $(xg, yg) = (x, y) = s$ and $(xg, y) = (x, y) = s$. Therefore, s is the unique non-trivial commutator of G and thus, central in G.

Finally, remembering that φ is given by (1) in N, we only need to prove that $Z(N) = N \cap Z(G)$ and to determine φ in $G \setminus N$. To see the later, let $h \in G \setminus N$. If h is not central then, by Lemma 1.1, it follows that $h \in G_\varphi$. If h is central, set $x \in N \setminus Z(N)$. Then $xh \in (G\setminus N)\setminus Z(G)$ and, as above, we obtain that $xh = (xh)^\varphi = h^\varphi x^\varphi = h^\varphi sx$. Therefore, we have that $h^\varphi = sh$.

Now, let $x \in Z(N) \setminus Z(G)$. Then $x^\varphi = x$ and there exists $y \in G \setminus N$ such that $xy \neq yx$ and $y^\varphi = y$. Thus $xy \in (G \setminus N) \setminus Z(G)$. Therefore $xy = (xy)^\varphi = y^\varphi x^\varphi = yx$, a contradiction. Hence $Z(N) = N \cap Z(G)$, and (ii) follows.

(**2.2**) *If* $char(R) = 4$ *then either (ii) or (iii) holds.*

If $G = N \cup G_\varphi \cup Z(G)$ then, because of condition (**2**), there exists a central element $g \in G \setminus N$ and, as in the previous argument, it is easy to see that (ii) follows.

Otherwise, it follows from Lemma 2.1 that if $g \in G \setminus N$ is not a central element, then $g \notin G_\varphi$. We claim that, for any non-commuting elements $g, h \in G$, we have that $g^\varphi = h^{-1}gh$. In fact, if $g \in G \setminus N$ then $g \notin G_\varphi$ and, by Lemma 1.1, it follows that $gh = hg^\varphi$, as claimed. On the other hand, if $g \in N$ and $h \in G \setminus N$ then $h \notin G_\varphi$ and, again by Lemma 1.1, we have that $h^\varphi g^\varphi = gh^\varphi$ or, equivalently, $gh = hg^\varphi$. Finally, if $g, h \in N$ then (B) holds. So, $g^\varphi = sg = (h, g^{-1})g = h^{-1}gh$, as claimed

Applying [6, Theorem III.3.3], we see that G has the LC property, a unique non-trivial commutator s and the involution φ is given by

$$g^{\varphi} = \begin{cases} g & \text{if } g \in Z(G) \\ sg & \text{if } g \notin Z(G) \end{cases}$$

Hence (*iii*) follows.

Proof of sufficiency.

Since $(RG)^{\sigma\varphi}$ is generated, as an R-module, by the set

$$\mathcal{S} = N_{\sigma} \cup \{g + g^{\varphi} \mid g \in N \setminus N_{\varphi}\} \cup \{g - g^{\varphi} \mid g \in G \setminus N\}$$

it is enough to show that the elements in $\mathcal{S}$ commute.

If (*i*) holds then, for all $g \in G \setminus N$, we have that $g + g^{\sigma\varphi} = g - g^{\varphi} = 0$. Thus, $(RG)^{\sigma\varphi}$ is commutative if and only if $(RN)^{\varphi}$ is commutative and the result follows.

Now, assume that (*ii*) holds. First, we prove that φ is an involution. Since s is a central element of order 2 in G it follows that $(g^{\varphi})^{\varphi} = g$. To show that φ is an antiautomorphism, given arbitrary elements $g, h \in G$ we shall consider two cases.

(1) $gh \neq hg$.

In this case $hg = sgh$. Also, neither g nor h nor gh are central. Thus, if either $g, h \in N$ or $g, h \notin N$ then $(gh)^{\varphi} = sgh = hg = h^{\varphi}g^{\varphi}$. Now, if $g \in N$ and $h \notin N$ then $(gh)^{\varphi} = gh = shg = h^{\varphi}g^{\varphi}$. Therefore, in this case, $(gh)^{\varphi} = h^{\varphi}g^{\varphi}$.

(2) $gh = hg$.

As G has the LC property, it follows that either g or h or gh is central. Suppose that either $g, h \in N$ or $g, h \notin N$. If either g and h are both central or both not central then $gh \in Z(G)$ and thus $(gh)^{\varphi} = gh = hg = h^{\varphi}g^{\varphi}$. If $g \in Z(G)$ and $h \notin Z(G)$ then $gh \notin Z(G)$ and thus $(gh)^{\varphi} = sgh = shg = h^{\varphi}g^{\varphi}$.

Now, suppose that $g \in N$ and $h \notin N$. Again, if either $g, h \in Z(G)$ or $g, h \notin Z(G)$ then $gh \in Z(G)$ and so $(gh)^{\varphi} = sgh = shg = h^{\varphi}g^{\varphi}$. If one of these elements is central and the other noncentral then $gh \notin Z(G)$, so $(gh)^{\varphi} = gh = hg = h^{\varphi}g^{\varphi}$.

Hence, if φ is given by (*ii*) then it is an involution. Furthermore,

$$\mathcal{S} = Z(N) \cup \{g + sg \mid g \in N \setminus Z(N)\} \cup \{g - sg \mid g \in (G \setminus N) \cap Z(G)\}$$

and, as s is central in G, it follows that the elements of

$$\{g - sg \mid g \in (G \setminus N) \cap Z(G)\}$$

commute with all elements of $\mathcal{S}$. On the other hand, since $G' = N' = \{1, s\}$, we have that the center of RN is generated, as an R-module, by the center of N and the elements of the form $g + sg$, with $g \in N \setminus Z(N)$.

Therefore, the commutativity of the elements of $\mathcal{S}$ follows, and hence also the commutativity of $(RG)^{\sigma\varphi}$.

Finally, suppose that (iii) holds. Then proceeding as above, we obtain that φ is an involution and

$$\mathcal{S} = (N \cap Z(G)) \cup \{g+sg \mid g \in N \setminus Z(G)\} \cup \{g-sg \mid g \in (G \setminus N) \setminus Z(G)\}.$$

Moreover, since $G' = \{1, s\}$, we have that the center of RG is generated, as an R-module, by the center of G and the elements of the form $g+sg$, with $g \in G \setminus Z(G)$. Hence $N \cap Z(G)$ and $\{g+sg \mid g \in N \setminus Z(G)\}$ are central subsets. Therefore, it remains to show that the elements of $\{g-sg \mid g \in (G \setminus N) \setminus Z(G)\}$ commute among themselves. To see this, let $g, h \in (G \setminus N) \setminus Z(G)$. Then, as s is a central element of order 2, it follows that

$$[g-sg, h-sh] = gh - sgh - sgh + gh - hg + shg + shg - hg.$$

Now, if $gh = hg$ then $[g-sg, h-sh] = 0$. Otherwise, $s = (g, h)$ and $gh = shg$. Therefore, $[g-sg, h-sh] = 4(gh-hg) = 0$. Hence, $(RG)^{\sigma\varphi}$ is commutative

□

In the rest of this section we shall assume that the involution φ is the classical involution of G, that is, $g^{\varphi} = g^{-1}$, for all $g \in G$. Moreover, we shall denote simply by $(RG)^{\sigma}$, the set of symmetric elements of RG, under corresponding involution. Notice that in the present case

$$G_{\varphi} = \{g \in G \mid g^2 = 1\}.$$

We are now ready to prove the following

Theorem 2.3. *Let R be a commutative ring with unity and let G be a non-abelian group with non-trivial orientation homomorphism σ. Then, $(RG)^{\sigma}$ is a commutative ring if and only if one of the following conditions holds:*

(i) $N = Ker(\sigma)$ is an abelian group and $(G \setminus N)^2 = 1$;

(ii) $N = Ker(\sigma) \cong \langle x, y \mid x^4 = 1,\ y^2 = x^2,\ y^{-1}xy = x^{-1}\rangle \times E$ and

$$G \cong \langle x, y, g \mid x^4 = 1,\ y^2 = x^2 = g^2,\ x^y = x^{-1},\ x^g = x,\ y^g = y\rangle \times E,$$

where E is an elementary abelian 2-group;

(iii) $char(R) = 4$ and G is a Hamiltonian 2-group.

Proof: We shall show that the conditions of Theorem 2.2 imply the conditions of the statement, in the present case. Notice first that (i) above is clearly equivalent to condition (i) of Theorem 2.2.

Assume that condition (iii) Theorem 2.2 holds. From the definition of φ, we get that if $g \in G$ is a central element then $g^2 = 1$, and if g is noncentral then $g^2 = s$. So, since the order of s is equal to

2, it follows that G is a 2-group with exponent less or equal to 4. Furthermore, every cyclic subgroup of G is normal. In fact, let $g, h \in G$ be non-commuting elements. Then, as remarked above, g^2, h^2 and $(gh)^2$ are equal to s and have order equal to 4. Thus $hgh = g$ so, $h^{-1}gh = h^{-1}(hgh)h = gh^2 = g^3 = g^{-1}$. Hence, G is a Hamiltonian 2-group and (iii) follows.

Finally, assume that condition (ii) of Theorem 2.2 holds. Then, $Z(N) = N \cap Z(G)$ as shown in the course of the proof of the theorem. By the above, we have that N is a Hamiltonian 2-group. Thus, $N = \langle x, y\rangle \times E$, where $\langle x, y\rangle$ is isomorphic to the Quaternion group of order 8 and E is an elementary abelian 2-group (see [11, Theorem 1.8.5]). On the other hand, by the definition of φ, we have that either $(G \setminus N)^2 = 1$ or $(G \setminus N) \cap Z(G) \neq \emptyset$.

Suppose that $(G \setminus N)^2 = 1$ and let $g \in G \setminus N$. Then $xg \in G \setminus N$ and thus $1 = (xg)^2$, that is, $gxg = x^{-1}$. Similarly, $gyg = y^{-1}$ and $g(xy)g = y^{-1}x^{-1} = yx$. But, $g(xy)g = gxggyg = x^{-1}y^{-1} = xy$, a contradiction. Thus, $(G \setminus N)^2 \neq 1$ and there exists an element $g \in (G \setminus N) \cap Z(G)$. Since $G = N \cup Ng$ we have that E is a central subgroup of G. Furthermore, $G = \langle x, y, g\rangle E$ and $\langle x, y, g\rangle$ is a normal subgroup of G. We claim that G is actually the direct product of the subgroups $\langle x, y, g\rangle$ and E. Note first that $g^2 = x^2$. In fact, as $g^{-1} = g^{\varphi} = sg$ and $x^{-1} = x^{\varphi} = sx$, it follows that $g^2 = s = x^2$. Thus, $\langle x, y, g\rangle = \{ag \mid a \in \langle x, y\rangle\} \cup \langle x, y\rangle$. But, $ag \notin N = \langle x, y\rangle \times E$, for all $a \in \langle x, y\rangle$. Therefore, $\langle x, y, g\rangle \cap E = \{1\}$ and $G = \langle x, y, g\rangle \times E$, as claimed.

Since $g^2 = x^2$ we have that

$$\langle x, y, g\rangle = \langle x, y, g \mid x^4, y^2 = x^2 = g^2, x^y = x^{-1}, x^g = x, y^g = y\rangle.$$

Conversely, suppose that (ii) holds and let

$$H = \langle x, y, g \mid x^4, y^2 = x^2 = g^2, x^y = x^{-1}, x^g = x, y^g = y\rangle.$$

Then, since E is a central subgroup of G, it is enough to prove that $Ker(\sigma_{|H}) = \langle x, y \mid x^4, y^2 = x^2, x^y = x^{-1}\rangle$ and $(RH)^{\sigma_{|H}}$ is commutative, which is straightforward.

If (iii) holds then $G \cong Q_8 \times E$, where Q_8 is the Quaternion group of order 8 and E is an elementary abelian 2-group. It is easy to see that, for any σ, $(RQ_8)^{\sigma}$ is commutative and the result follows. □

3. Symmetric units

We recall that in [3], a ring R is called *G-favourable* if for any $g \in G$, of finite order $|g|$, there is a nonzero $\alpha \in R$ such that $1 - \alpha^{|g|}$ is invertible in R. Notice that every infinite field is G-favourable.

Theorem 3.1. *Let R be a G-favourable integral domain and let G be a torsion group. Then, $\mathcal{U}^{\sigma\varphi}(RG)$ is commutative if and only if $(RG)^{\sigma\varphi}$ is commutative (or, equivalently, $\mathcal{U}^{\sigma\varphi}(RG)$ is a subgroup of $\mathcal{U}(RG)$ if and only if $(RG)^{\sigma\varphi}$ is a subring of RG).*

Proof: Clearly, if $(RG)^{\sigma\varphi}$ is a commutative ring then $\mathcal{U}^{\sigma\varphi}(RG)$ is an abelian group.

Assume that $\mathcal{U}^{\sigma\varphi}(RG)$ is commutative. First, we shall exibit some symmetric units in RG. To this end, let $g \in G$ be an arbitrary element. Since G is a torsion group and R is G-favourable, there is $\alpha \in R \setminus \{0\}$ such that $1-\alpha^{|g|}$ is an unit. Then

$$(g-\alpha)(\alpha^{|g|-1}+\alpha^{|g|-2}g+\cdots+\alpha g^{|g|-2}+g^{|g|-1})(1-\alpha^{|g|})^{-1}=1,$$

and thus $g-\alpha \in \mathcal{U}(RG)$. Moreover, if $g \in G \setminus N$ then, as $|g|$ is an even number, we have that

$$(g+\alpha)(\alpha^{|g|-1}-\alpha^{|g|-2}g+\cdots+\alpha g^{|g|-2}-g^{|g|-1})(\alpha^{|g|}-1)^{-1}=1,$$

and so $g+\alpha \in \mathcal{U}(RG)$. Define

$$u=\begin{cases}(g-\alpha)(g^{\varphi}-\alpha)=gg^{\varphi}+\alpha^2-\alpha(g+g^{\varphi}) & \text{if } g\in N\\ (g+\alpha)(\alpha-g^{\varphi})=\alpha^2-gg^{\varphi}+\alpha(g-g^{\varphi}) & \text{if } g\notin N\end{cases}$$

Then, as $|g|=|g^{\varphi}|$ it follows that $u \in \mathcal{U}^{\sigma\varphi}(RG)$. Also, for all $h \in G$, we have that $hh^{\varphi} \in \mathcal{U}^{\sigma\varphi}(RG)$. Thus, as the symmetric units commute, it follows that hh^{φ} commutes with gg^{φ} and with u. Therefore, we see that hh^{φ} commutes with $g+g^{\varphi}$, if $g \in N$, and with $g-g^{\varphi}$, if $g \notin N$.

Now, let $g,h \in G \setminus G_{\varphi}$, $f \in N_{\varphi}$ and let $\alpha,\beta \in R\setminus\{0\}$ be such that $1-\alpha^{|g|}$ and $1-\beta^{|h|}$ are units in R. If $g,h \in N$ then f, $(g-\alpha)(g^{\varphi}-\alpha)$ and $(h-\beta)(h^{\varphi}-\beta)$ are symmetric units. Thus again, since symmetric units commute and $[g+g^{\varphi},hh^{\varphi}]=0=[h+h^{\varphi},gg^{\varphi}]$, we obtain that $\alpha(g+g^{\varphi})$, $\beta(h+h^{\varphi})$ and f commute among themselves. Therefore, as R is a integral domain, $\alpha\beta=\beta\alpha\neq 0$ and so $g+g^{\varphi}$, $h+h^{\varphi}$ and f commute. Proceeding in a similar way, for $g,h \in G\setminus N$ we have that $g-g^{\varphi}$, $h-h^{\varphi}$ and f commute, and for $g \in N$ and $h \in G\setminus N$ we have that $g+g^{\varphi}$, $h-h^{\varphi}$ and f commute. Consequently, since N_{σ} is an abelian group, the elements of

$$N_{\sigma}\cup\{g+g^{\varphi} \mid g\in N\setminus N_{\varphi}\}\cup\{g-g^{\varphi} \mid g\in G\setminus N\}$$

commute. Thus, $(RG)^{\sigma\varphi}$ is commutative. □

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