SOLUTIONS OF A CAUCHY-GOURSAT PROBLEM

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In a former paper [R] we studied a Goursat abstract problem and now we shall apply these results to a Cauchy-Gou rsat problem. The purpose is to verify how one can solve a concrete problem using the Goursat problem on Banach Spaces Scales and also, with convenient hypotheses, to privent qua-si-linearization processes.

§1. The Goursat non linear abstract problem [R].

Definition

A Banach Spaces Scales is a vector space over C, which is the union of a family $(X_s)_{0<s\leq 1}$ of complex Banach spaces such that if $0<s'<s\leq 1$ then $X_{s'}\supset X_s$ and $\| \ \|_{s'}\leq \| \ \|_s$.

Theorem - Consider the system,

$$(1.1) \qquad \begin{cases} D_t^{\alpha}u(t) = F(u,t) \\ u-u^o = 0(t^{\alpha}) \end{cases}$$

where $\alpha=(\alpha_1,\ldots,\alpha_N)$, $\alpha_i\in IN^*$, $D^{\alpha}=D_1^{\alpha_1}\ldots D_N^{\alpha_N}$ and $V=0(t^{\alpha}) \Longleftrightarrow$

$D_j^r V=0$ when $t_j=0$, $0\leq r<\alpha_j$, $j=1,\dots,N$. Let F be the map such that,

(1.2) $$F:V(0,R)\times D_\eta \longrightarrow X=\bigcup_{0<s\leq 1} X_s$$

where $V(0,R)=\bigcup_{0<s\leq 1} B_s(0,R)$, $B_s(0,R)$ is the ball with center 0 and radius R of X_s, $D_\eta=\{t\in C^N;\ |t|<\eta\}$.

(1.3) The restriction of F on $B_s(0,R)\times D_\eta$ is G-analytic with values in $X_{s'}$, $0<s'<s\leq 1$ and $\| F(u,t)\|_{s'}\leq \frac{C}{(s-s')^{m(\alpha)}}$, $m(\alpha)=\{\alpha_i,\ i=1,\dots,N\}$.

(1.4) The map u^o is holomorphic in D_η with values in X_1, where $D_t^\beta u(o)=0$, $\beta\geq\alpha$ and $\sup_{D_\eta}\| u^o(t)\|_1<\rho<R$. Then there exists a unique map u and a number Δ, $0<\Delta<\eta$, such that for each s, $0<s\leq 1$, u(t) is the unique solution of (1.1), holomorphic in $\{t\in C^N; |t|<\Delta(1-s)\}$ with the values in X_s. Further

$$\Delta = \min\{\frac{\eta}{2}, \sqrt[|\alpha|]{\frac{R}{(32e)^N eC\alpha^2}}\}$$

with $|\alpha|=\alpha_1+\dots+\alpha_N$ and $\alpha^2=\alpha_1^2\dots\alpha_N^2$.

Remark

If $\alpha_j=0$ the variable t_j can be considered as a parameter.

§2. Solution of a Cauchy-Goursat problem

Consider the system,

$$(2.1) \qquad \begin{cases} D_t^{\alpha}u(t) = f(t,z,D_z^{\beta_1}u,\ldots,D_z^{\beta_p}t) \\ u(z,t)-u^o(z,t) = O(t^{\alpha}) \end{cases}$$

where $t\in C^N$, $z\in C^n$, $\beta^i=(\beta_1^i,\ldots,\beta_N^i)$, $1\leqslant i\leqslant p$, with $0\leqslant|\beta^i|\leqslant m(\alpha)$ and f is a complex analytic function of variables $(t,z,w_1,\ldots,w_p)\in C^{N+n+p}$ when $|t|<\eta$, $|z|<1$, $|w_i|<R$, $i=1,\ldots,p$, which is bounded by $M>0$. The function u^o is analytic with $D_t^{\theta}u^o(0,z)=0$, $\theta\geqslant\alpha$, $\sup_{D_\eta}\|u^o(t,z)\|<R$ if $|z|<1$.

Then there exists a unique analytic solution u of (2.1).

Proof

We call $\overset{\circ}{X}_s=H(B_s,C)$, the complex Banach space of functions of n complex variables $z\in C^n$, with values in C, holomorphic and bounded, with partial derivatives bounded in the ball

$$B_s = \{z\in C^n, |z|<s\}, \quad |z| = \max_{1\leqslant j\leqslant n}|z_j|,$$

with the norm

$$\|u\|_s = \sup_{B_s}\max_{0\leqslant|\beta|\leqslant m(\alpha)}(|D_z^{\beta}u(z)|), \quad 0<s<1.$$

Let $F(u,t)$ be the function defined by

$$F(u,t)(z) = f(t,z,D_z^{\beta_1}u,\ldots,D_z^{\beta_p}u)$$

with $\|u\|_s<R$, $|z|<s$, if $0<s<1$.

We have

$$|F(u,t)(z)| \leqslant M \leqslant \frac{M}{(s-s')^{m(\alpha)}}$$

and by Cauchy's inequality

$$|D_z^{\beta_i} F(u,t)(z)| \leqslant \frac{M}{(s-s')^{(\beta_i)}} \leqslant \frac{M}{(s-s')^{m(\alpha)}}$$

if $\|u\|_s < R$, $0<|z|<s'<s\leqslant 1$, then

$$F(u,t) \in X_{s'}, \quad \| F(u,t)\|_{s'} \quad \frac{M}{(s-s')^{m(\alpha)}}, \quad 0<s'<1.$$

By §1 there exists a unique map u and a number Δ such that u(t) is the unique analytic solution in $\{t \in C^N, |t|<\Delta(1-s)\}$ with values in X_s. Then u(t,z) is the unique analytic solution in $\{(t,z);\ \frac{|t|}{\Delta}+|z|<1\}$.

This theorem generalizes a similar result in |D|, for a non-linear Cauchy-Kovalevska system.

|R| - SALVITTI, Reinaldo - Solution d'un système non lineaire abstrait de Gousart et dépendence des conditions initiales - C.R. Acad. Sc. Paris.

|D| - PISANELLI, Domingos - Solutions of a non-linear abstract Cauchy-Kovalevska system as a local Banach analytic manifold - Proceedings of the Brazilian Math. Soc. Symp. of Funcional Analysis - 1974 -ed. D.G. Figueiredo M. Dekker, Inc. New York - 1976.

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