HOMOLOGY AND COHOMOLOGY VIA THE PARTIAL GROUP ALGEBRA

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ABSTRACT

We study partial homology and cohomology from the ring theoretic point of view via the partial group algebra $\mathbb{K}_{par}G$. In particular, we link the partial homology and cohomology of a group G with coefficients in an irreducible (resp. indecomposable) $\mathbb{K}_{par}G$-module M with the ordinary homology and cohomology groups of a subgroup H of G, where H depends on M, with coefficients in an appropriate irreducible (resp. indecomposable) $\mathbb{K}H$-module. Furthermore, we compare the standard cohomological dimension $cd_{\mathbb{K}}(G)$ (over a field $\mathbb{K}$) with the partial cohomological dimension $cd^{par}_{\mathbb{K}}(G)$ (over $\mathbb{K}$) and show that $cd^{par}_{\mathbb{K}}(G) \geq cd_{\mathbb{K}}(G)$ and that there is equality for $G = \mathbb{Z}$.

1. INTRODUCTION

Partial actions and partial representations enjoy diverse theoretic developments and keep finding notable applications. They have been brought into use in the theory of C^*-algebras, providing, in particular, an efficient approach to algebras generated by partial isometries, with the Cuntz–Krieger algebras and the more general Exel–Laca algebras being prominent early examples of the success of the new notions [26, 29]. In addition, they allowed for an isomorphic identification of any second countable C^*-algebraic bundle, satisfying a mild regularity condition, with the semidirect product bundle associated with a continuous twisted partial group action on the unit fibre algebra [25]. Apart from their relevance in the study of K-theory, ideal structure and representations of the C^*-algebras under consideration, the new tools also turned out to be productive in dealing with the Kubo–Martin–Schwinger(KMS) equilibrium states of C^*-dynamical systems [30].

The C^*-algebraic advances, Exel's definition of a partial action of a group on an abstract set [27], the concept of a partial representation introduced in [27] and [43] and the relation of these notions to the

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inverse semigroup $S(G)$ defined in [27] stimulated the algebraic investigations on the subject, initially made in [19, 36, 46, 47]. In algebra the new concepts are useful to graded algebras, Hecke algebras, Leavitt path algebras, inverse semigroups, restriction semigroups and automata (see the survey article [16] and the references therein). Moreover, the partial action definition of Thompson's groups V and $\mathcal{G}_{2,1}$ provides an appropriate framework to deal with algorithmic problems for them [9]. Some important decision problems, properly placed in the language of inverse semigroups [49], are related to extension properties of partial bijections, to some of which partial actions are also handy [14]. In particular, they participate in an interaction between model theory, profinite topology of groups and formal languages, being relevant this way to the Ribes–Zalesskii properties of groups and other topologically defined conditions on groups.

Among the recent applications, we mention the significance of partial actions and partial representations to C^*-algebras related to dynamical systems of type (m, n) [5], to dynamical systems associated with separated graphs and related C^*-algebras [4, 6], to paradoxical decompositions [4], to shifts [6], to full or reduced C^*-algebras of E-unitary or strongly E^*-unitary inverse semigroups [41], to topological higher-rank graphs [44], to Matsumoto and Carlsen–Matsumoto C^*-algebras of arbitrary subshifts [18] (see also [31, 32]), to ultragraph C^*-algebras [33] and to expansions of monoids in the class of two-sided restriction monoids [40], with the latter involving a previous construction from [39] based on partial actions (see also [13]).

Besides being related to inverse semigroups, partial actions and partial representations also have various connections to groupoids [16]; one of them is based on the partial action groupoid, another one involves the partial group algebra, which is crucial for the present article, and one more is the fact proved in [8] and [34], stating that Steinberg algebras ([48, 12]), associated with Hausdorff ample groupoids, can be seen as partial skew inverse semigroup rings (see also [15]). More information on partial actions, partial representations and their applications may be found in Exel's book [28] and in [16]. In particular, the former reference contains detailed partial crossed product descriptions of graph C^*-algebras and the Wiener–Hopf C^*-algebras associated with quasi-lattice ordered groups.

Among the theoretic developments, two partial group cohomology theories were introduced: one in [20] based on partial actions and the other one in [1] based on partial representations. Despite the fact that partial actions and partial representations are related notions, the two cohomology theories have little in common. In particular, the cohomology in [1] deals with an abelian category, whereas the category of partial modules from [20] is not even additive. The cohomology from [20] turned out to be useful to extensions of semilattices of groups by groups, to a Chase–Harrison–Rosenberg exact sequence for partial Galois extensions and to partial projective group representations and to reduced C^*-crossed products (see [16]). In particular, it was shown in [23] that each group component of the partial Schur multiplier is a partial cohomology group. Furthermore, partial 2-cocycles appeared in [38] as certain obstructions in the study of the ideal structure of the reduced crossed product of a C^*-algebra by a global action. In addition, it was extended from groups to groupoids in [42] and used to classify the equivalence classes of certain groupoid graded rings. Partial actions of Hopf algebras were introduced in [11] and were reformulated in [35] and [51] with the aim of obtaining a richer theory of partial actions of algebraic groups. The partial cohomology theory developed in [20] stimulated its Hopf theoretic treatment in [7], where a natural generalization of Sweedler's cohomology to the partial action setting was given.

The partial modules in [20] are unital partial actions of groups on commutative monoids, whereas the authors in [1] deal with modules over the partial group algebra $\mathbb{K}_{par}G$. The latter is an algebra which governs the partial $\mathbb{K}$-representations of a group G, and it has as a $\mathbb{K}$-basis, the above-mentioned semigroup $S(G)$, introduced by Exel in [27] to deal with partial actions and partial representations. Exel defined $S(G)$ by generators and relations and gave a canonical form of the elements of $S(G)$ which he used to show that $S(G)$ is an inverse monoid [27]. Later in [37] Kellendonk and Lawson proved that $S(G)$ is isomorphic to the Szendrei expansion of G, and, as a consequence of a result by Szendrei [50], $S(G)$ is isomorphic to the Birget–Rhodes expansion of G. The information on $S(G)$ given in [27] and [37] is an important technical tool when dealing with partial linear and partial projective group representations, as well as with the partial cohomologies.

The algebraic study of $\mathbb{K}_{par}G$ was initiated in [19]. In particular, using a groupoid $\Gamma(G)$ associated with G, a result on the structure of $\mathbb{K}_{par}G$ was obtained assuming that G is a finite group. The latter fact permits one to make conclusions about the partial representations of finite groups. The connection of partial representations of G with the groupoid $\Gamma(G)$ also inspired the development of a theory of partial representations of Hopf algebras in [2], [3], where it was shown that partial representations correspond to left modules over a Hopf algebroid. The groupoid technique was further used in [24] to give a structure result on the indecomposable/irreducible finite dimensional partial representations of arbitrary groups. Furthermore, it was shown in [17] that $\mathbb{K}_{par}G$ is isomorphic to the partial skew group ring $B \rtimes_\tau G$, where τ is a partial action of G on the subalgebra B of $\mathbb{K}_{par}G$ generated by the idempotents of $S(G)$.

It is very natural to consider a (co)homology theory for G using $\mathbb{K}_{par}G$-modules. A key point in [1] is to choose as the 'trivial' module the $\mathbb{K}_{par}G$-module structure on B given by the partial representation $G \to \text{End}_{\mathbb{K}}(B)$, which corresponds to the partial action τ (Subsection 2.1). Then the nth partial cohomology group $H^n_{par}(G, M)$ of G with values in a left $\mathbb{K}_{par}G$-module M is defined in [1] by means of the corresponding Ext functor (Subsection 2.2). The authors of [1] prove the existence of a spectral sequence relating the Hochschild cohomology of the partial skew group ring $\mathcal{A} \rtimes_\tau G$ with $H^n_{par}(G, -)$ and the Hochschild cohomology of $\mathcal{A}$. The globalization problem for $H^n_{par}(G, M)$ was investigated in [21] in the case when M is an algebra, whose the $\mathbb{K}_{par}G$-module structure comes from a unital partial action of G on M. For more information around $\mathbb{K}_{par}G$ the reader is referred to [16] and [28]. In particular, the difference between the cohomologies in [1] and [20] is discussed in [16] and [21].

In this paper we introduce homology groups for partial representations as the left derived functors of the tensor functors $- \otimes_{\mathbb{K}_{par}G} B$, $B \otimes_{\mathbb{K}_{par}G} -$, and we study homological and cohomological properties of the partial group algebra $\mathbb{K}_{par}G$ using ring theoretic homological methods.

Let G be an arbitrary group, $\mathbb{K}$ be a field and V be an irreducible (resp. indecomposable) left $\mathbb{K}_{par}G$-module, which is finite dimensional over $\mathbb{K}$. By [24, Theorem 2.3] there is a unique connected component Δ of $\Gamma(G)$ with a finite number of vertices, an elementary bimodule $_{\mathbb{K}_{par}G}W_{\mathbb{K}H}$ related to Δ and a left irreducible (resp. indecomposable) $\mathbb{K}H$-module U, finite dimensional over $\mathbb{K}$, such that $V \simeq W \otimes_{\mathbb{K}H} U$ as left $\mathbb{K}_{par}G$-modules. Here H is the stabilizer in G of a fixed vertex of the connected component Δ.

Our main goal is to establish the following result.

THEOREM 1.1 *Let G be an arbitrary group, Δ be a connected component of the groupoid $\Gamma(G)$ with a finite number of vertices and W be the elementary $\mathbb{K}_{par}G$- $\mathbb{K}H$-bimodule related to Δ. Let U be an arbitrary left $\mathbb{K}H$-module. Then*

$$H^i_{par}(G, W \otimes_{\mathbb{K}H} U) \simeq H^i(H, U) \text{ and } H_i^{par}(G, W \otimes_{\mathbb{K}H} U) \simeq H_i(H, U).$$

In particular, if V is an irreducible (resp. indecomposable) left $\mathbb{K}_{par}G$-module, which is finite dimensional over $\mathbb{K}$ and as above $V \simeq W \otimes_{\mathbb{K}H} U$ as left $\mathbb{K}_{par}G$-modules, then

$$H_i^{par}(G, V) \simeq H_i(H, U) \text{ and } H^i_{par}(G, V) \simeq H^i(H, U).$$

By definition for a left $\mathbb{K}_{par}G$-module V we have $H^n_{par}(G, V) = Ext^n_{\mathbb{K}_{par}G}(B, V)$ where B is considered as a left $\mathbb{K}_{par}G$-module and $H_n^{par}(G, V) = Tor_n^{\mathbb{K}_{par}G}(B, V)$, where B is considered as a right $\mathbb{K}_{par}G$-module (Subsection 2.2).

Theorem 1.1 follows from Theorem 4.11 and Theorem 4.13 Its proof needs homological and cohomological versions of the Shapiro Lemma that we give in Section 3. Their use in the proof of Theorem 1.1 requires $B \otimes_{\mathbb{K}_{par}G} W$ to be isomorphic to the trivial right $\mathbb{K}H$-module $\mathbb{K}$ and $\mathbb{K}\Delta$ to be

a flat left $\mathbb{K}_{par}G$-module (Proposition 4.6 and Theorem 6.6). Proposition 4.6 is a consequence of Theorem 4.5 whose proof is separated in Section 5. In the case when G is finite by [**19**] $\mathbb{K}_{par}G \simeq \mathbb{K}\Gamma \simeq \oplus \mathbb{K}\Delta$, hence $\mathbb{K}\Delta$ is a projective left $\mathbb{K}_{par}G$-module. Furthermore, we show in Section 6.1 that if A is a vertex of a component Δ with finitely many vertices and $G \backslash A$ is infinite, then $\mathbb{K}\Delta$ is not a projective left $\mathbb{K}_{par}G$-module. Moreover, if G is an infinite group, then for the connected component Δ with unique vertex G, $\mathbb{K}\Delta = \mathbb{K}G$ is not projective as a $\mathbb{K}_{par}G$-module.

In Section 4 we establish the following consequences of Theorem 1.1 for finite groups G.

COROLLARY 1.2 Suppose that G is a finite group. Then for $i \geq 0$

$$H_i^{par}(G,B) \simeq \oplus_\Delta H_i(H, \mathbb{K}) \text{ and } H_{par}^i(G,B) \simeq \oplus_\Delta H^i(H, \mathbb{K}),$$

where the sum is taken over the connected components Δ of the groupoid $\Gamma(G)$ and H is the stabilizer of a chosen vertex of Δ (thus it is defined up to conjugation).

We define the *partial cohomological dimension* $cd_{\mathbb{K}}^{par}(G)$ of G (over $\mathbb{K}$) as the projective dimension of B as a left $\mathbb{K}_{par}G$-module, that is,

$$cd_{\mathbb{K}}^{par}(G) = \max\{i \mid H_{par}^i(G,V) \neq 0 \text{ for some } \mathbb{K}_{par}G\text{-module } V\}.$$

Note that *the cohomological dimension $cd_{\mathbb{K}}(G)$ of G* (over $\mathbb{K}$) is the projective dimension of the trivial left $\mathbb{K}G$-module $\mathbb{K}$, that is,

$$cd_{\mathbb{K}}(G) = \max\{i \mid H^i(G,M) \neq 0 \text{ for some } \mathbb{K}G\text{-module } M\}.$$

As a corollary of Theorem 1.1 we prove the following result.

COROLLARY 1.3 For any group G we have $cd_{\mathbb{K}}^{par}(G) \geq cd_{\mathbb{K}}(G)$. In particular, if $char(\mathbb{K}) = p > 0$ and G has p-torsion then $cd_{\mathbb{K}}^{par}(G) = cd_{\mathbb{K}}(G) = \infty$.

We believe that the following stronger version of Corollary 1.3 holds.

COROLLARY 1.4 For any group G we have $cd_{\mathbb{K}}^{par}(G) = cd_{\mathbb{K}}(G)$.

In the particular case when G is a free group we have that $cd_{\mathbb{K}}(G) = 1$ and Conjecture D is equivalent to the following:

CONJECTURE 1.5 *Let G be a free group and $R = \mathbb{K}_{par}G$. Then the kernel IG of the augmentation map $\epsilon : R \to B$ is a projective left R-module.*

We prove in Section 8 that Conjecture 1.5 holds for $G = \mathbb{Z}$, so that $cd_{\mathbb{K}}^{par}(G) = cd_{\mathbb{K}}(G) = 1$. It is interesting to note that in this case IG is not a free $\mathbb{K}_{par}G$-module but only a projective one.

2. NOTATION AND PRELIMINARY RESULTS
2.1 The partial group algebra $\mathbb{K}_{par}G$

Let G be a group and $\mathbb{K}$ be a field. We recall some well-known definitions and facts.

DEFINITION 2.1. A *partial representation* of G into a unital (associative) $\mathbb{K}$-algebra $\mathcal{A}$ is a map $\pi : G \to \mathcal{A}$ such that for every $g, h \in G$ we have
  1) $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$;
  2) $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh)$;

3) $\pi(1) = 1_{\mathcal{A}}$, where 1 is the neutral element of G.

If $\mathcal{A} = End_{\mathbb{K}}(V)$, where V is a vector space over $\mathbb{K}$, then we say that π is a partial representation of G on V.

A concept closely related to partial representations is that of a partial action:

DEFINITION 2.2. A partial action α of G on a $\mathbb{K}$-algebra $\mathcal{A}$ is given by a collection $\{D_g\}_{g \in G}$ of ideals in $\mathcal{A}$ and a family $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g \in G}$ of non-necessarily unital algebra isomorphisms satisfying:
1) $D_e = \mathcal{A}$ and $\alpha_e = id_{\mathcal{A}}$;
2) $\alpha_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) = D_h \cap D_{g^{-1}}$;
3) if $y \in D_{h^{-1}} \cap D_{(gh)^{-1}}$ then $\alpha_g \alpha_h(y) = \alpha_{gh}(y)$.

A partial action is said to be *unital* if each ideal D_g is a unital algebra, that is, $D_g = 1_g \mathcal{A}$ for each $g \in G$, where 1_g is a central idempotent of $\mathcal{A}$. To a unital partial action α of G on $\mathcal{A}$ one may associate two partial representations. One of them is the map $G \to End_{\mathbb{K}}(\mathcal{A})$ given by

$$g \mapsto [a \mapsto \alpha_g(a1_{g^{-1}})], \quad g \in G, a \in \mathcal{A}. \tag{1}$$

The other one involves the partial skew group algebra. The latter is defined as follows.

DEFINITION 2.3. Suppose that there is a unital partial action α of G on a $\mathbb{K}$-algebra $\mathcal{A}$. Then there is an associative partial skew group ring (also called the partial smash product), denoted by $\mathcal{A} \rtimes_\alpha G$, where

$$\mathcal{A} \rtimes_\alpha G = \oplus_{g \in G} D_g \# g$$

and

$$(a1_g \# g)(b1_h \# h) = a\alpha_g(b1_h 1_{g^{-1}}) \# gh \text{ for } a, b \in \mathcal{A}.$$

The second partial representation associated with α is the map $G \to \mathcal{A} \rtimes_\alpha G$ given by $g \to 1_g \# g$.

There is a bijective correspondence between the partial $\mathbb{K}$-representations of G and the $\mathbb{K}$-representations of the partial group algebra $\mathbb{K}_{par} G$, whose definition is as follows.

DEFINITION 2.4. The *partial group algebra* $\mathbb{K}_{par} G$ is the $\mathbb{K}$-algebra with a generating set $\{[g] \mid g \in G\}$ subject to the relations
1) $[g][h][h^{-1}] = [gh][h^{-1}]$;
2) $[g^{-1}][g][h] = [g^{-1}][gh]$;
3) $[1] = 1_{\mathbb{K}_{par} G}$,
for all $g, h \in G$.

For simplicity, we will also denote the identity of $\mathbb{K}_{par} G$ by 1 instead of $1_{\mathbb{K}_{par} G}$. Evidently,

$$G \ni g \mapsto [g] \in \mathbb{K}_{par} G \tag{2}$$

is a partial representation. Each partial representation $\pi : G \to \mathcal{A}$ gives rise to a unital partial action on a commutative subalgebra of $\mathcal{A}$ [17, Lemma 6.5]. We specify this for the case of the partial representation (2). It is an easily seen and known fact that the elements $e_g = [g][g^{-1}]$ are pairwise commuting

idempotents of $\mathbb{K}_{par}G$ such that

$$[g]e_h = e_{gh}[g] \text{ and } e_1 = 1,$$

for all $g, h \in G$. Let

$$B =_{alg} \langle e_g \mid g \in G \rangle \qquad (3)$$

be the commutative subalgebra of $\mathbb{K}_{par}G$, generated by the elements $\{e_g\}_{g \in G}$, and let $D_g = e_g B$. Then the partial representation (2) induces a unital partial action τ of G on B given by the family of isomorphisms $\tau_g : D_{g^{-1}} \to D_g$ defined by

$$\tau_g(e_{g^{-1}}e_{h_1} \cdots e_{h_n}) = [g]e_{g^{-1}}e_{h_1} \cdots e_{h_n}[g^{-1}] = e_g e_{gh_1} \cdots e_{gh_n}. \qquad (4)$$

LEMMA 2.5. $[17, \text{Theorem 6.9}]$ *There is an isomorphism*

$$\varphi : \mathbb{K}_{par}G \to B \rtimes_\tau G,$$

given by

$$\varphi([g_1] \cdots [g_n]) = e_{g_1} e_{g_1 g_2} \cdots e_{g_1 \cdots g_n} \# g_1 \cdots g_n$$

and

$$\varphi^{-1}(e_g e_{h_1} \cdots e_{h_n} \# g) = e_g e_{h_1} \cdots e_{h_n}[g].$$

It is an immediate consequence of Lemma 2.5 that $\mathbb{K}_{par}G$ is a G-graded algebra:

$$\mathbb{K}_{par}G = \oplus_{g \in G} B_g \text{ and } B_g B_h \subseteq B_{gh},$$

where $B_g = D_g \# g$. It is readily seen that each D_g is the $\mathbb{K}$-vector subspace of B generated by $[h_1] \cdots [h_k]$ where $k \geq 1$ and $h_1 \cdots h_k = g$.

According to (1) the unital partial action (4) gives rise to the partial representation $\pi : G \to End_{\mathbb{K}}(B)$ given by $\pi(g)(x) = [g]x[g^{-1}]$ for $g \in G, x \in B$. This endows B with a structure, a left $\mathbb{K}_{par}G$-module:

$$^{[g]}b = [g]b[g^{-1}].$$

Symmetrically, we shall consider B as a right $\mathbb{K}_{par}G$-module defined by

$$b^{[g]} = [g^{-1}]b[g].$$

2.2. Introduction to homological algebra

We refer the reader to Rotman's book $[45]$ for the definitions of the derived functors *Tor* and *Ext*. For a fixed associative ring R we consider the derived functors $Tor_i^R(-,-)$ of the functor tensor product $- \otimes_R -$ and the derived functors $Ext_R^i(-,-)$ of the functor $\text{Hom}_R(-,-)$. Then for a left $\mathbb{K}_{par}G$-module V and a right $\mathbb{K}_{par}G$-module M the n-th partial homology groups of G with coefficients in M and V

are

$$H_n^{par}(G,M) = Tor_n^{\mathbb{K}_{par}G}(M,B) \text{ and } H_n^{par}(G,V) = Tor_n^{\mathbb{K}_{par}G}(B,V).$$

Note that B in $Tor_n^{\mathbb{K}_{par}G}(M,B)$ is a left $\mathbb{K}_{par}G$-module and that B in $Tor_n^{\mathbb{K}_{par}G}(B,V)$ is a right $\mathbb{K}_{par}G$-module. Similarly the n-th partial cohomology groups of G with coefficients in M and V are

$$H_{par}^n(G,M) = Ext_{\mathbb{K}_{par}G}^n(B,M) \text{ and } H_{par}^n(G,V) = Extn_{\mathbb{K}_{par}G}(B,V),$$

where B in $Ext_{\mathbb{K}_{par}G}^n(B,M)$ is a right $\mathbb{K}_{par}G$-module and B in $Extn_{\mathbb{K}_{par}G}(B,V)$ is a left $\mathbb{K}_{par}G$-module.

A fundamental result that we will need later on is the existence of the long exact sequence in homology associated with a short exact sequence. Suppose

$$0 \to V_1 \to V \to V_2 \to 0$$

is a short exact sequence of left S-modules for some associative ring S. Then there is a long exact sequence [45, Cor. 6.30]

$$... \to Tor_n^S(M,V_1) \to Tor_n^S(M,V) \to Tor_n^S(M,V_2) \to Tor_{n-1}^S(M,V_1)$$

$$\to ... \to Tor_1^S(M,V_2) \to M \otimes_S V_1 \to M \otimes_S V \to M \otimes_S V_2 \to 0.$$

Similarly, for a short exact sequence of right S-modules

$$0 \to M_1 \to M \to M_2 \to 0,$$

there is a long exact sequence

$$... \to Tor_n^S(M_1,V) \to Tor_n^S(M,V) \to Tor_n^S(M_2,V) \to Tor_{n-1}^S(M_1,V)$$

$$\to ... \to Tor_1^S(M_2,V) \to M_1 \otimes_S V \to M \otimes_S V \to M_2 \otimes_S V \to 0.$$

We observe that if

$$... \to W_1 \to W_2 \to W_3 \to W_4 \to ...$$

is a long exact sequence of S-modules with $W_1 = 0 = W_4$, then

$$\text{the middle map } W_2 \to W_3 \text{ is an isomorphism}.$$

We will apply this argument several times later on together with the fact that for a right S-module A and a left S-module B the functors $Tor_i^S(A,-)$ and $Tor_i^S(-,B)$ vanish on projective S-modules; in particular

$$Tor_i^S(A,S) = 0 = Tor_i^S(S,B).$$

2.3. On idempotents and projective modules

In this subsection R is a unital associative ring. The following lemma is well known, but for completeness, we give a proof.

Lemma 2.6. *Let $e_1, \ldots, e_n \in R$ be idempotents that commute with each other. Then*

$$Re_1 + Re_2 + \ldots + Re_n = Re,$$

where

$$e = e_1 + (1 - e_1)e_2 + \ldots + (1 - e_1) \ldots (1 - e_{n-1})e_n$$

and $R/(Re_1 + \ldots + Re_n)$ is a projective R-module.

Proof. Note that $e_1, (1 - e_1)e_2, (1 - e_1)(1 - e_2)e_3, \ldots, (1 - e_1) \ldots (1 - e_{n-1})e_n$ are pairwise orthogonal idempotents (that is, with mutual product 0). Then e is an idempotent and $Re_1 + Re_2 + \ldots + Re_n = Re$. Since $R = Re \oplus R(1 - e)$ we conclude that

$$R/(Re_1 + \ldots + Re_n) = R/Re \simeq R(1 - e)$$

is a projective R-module. $\square$

3. PARTIAL GROUP HOMOLOGY VERSUS ORDINARY HOMOLOGY

Theorem 3.1. *(Homological Shapiro lemma for partial actions) Let W be a $\mathbb{K}_{par}G$- $\mathbb{K}H$-bimodule for some groups G and H such that W is flat as a right $\mathbb{K}H$-module, W is flat as a left $\mathbb{K}_{par}G$-module and $B \otimes_{\mathbb{K}_{par}G} W$ is isomorphic to the trivial right $\mathbb{K}H$-module $\mathbb{K}$. Then, for any left $\mathbb{K}H$-module U, considering $W \otimes_{\mathbb{K}H} U$ as a left $\mathbb{K}_{par}G$-module, we have*

$$H_i^{par}(G, W \otimes_{\mathbb{K}H} U) \simeq H_i(H, U), \quad i \geq 0.$$

Proof. Let

$$\mathcal{P} : \ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to U \to 0$$

be a free resolution of U as a left $\mathbb{K}H$-module. Since W is a flat right $\mathbb{K}H$-module, the functor $W \otimes_{\mathbb{K}H} -$ is exact and hence we obtain an exact complex

$$\mathcal{S} = W \otimes_{\mathbb{K}H} \mathcal{P} : \ldots \to S_i \to S_{i-1} \to \ldots \to S_0 \to W \otimes_{\mathbb{K}H} U \to 0,$$

where $S_i = W \otimes_{\mathbb{K}H} P_i$. We view each S_i as a left $\mathbb{K}_{par}G$-module via the left action of $\mathbb{K}_{par}G$ on W. Since each P_i is a free $\mathbb{K}H$-module, we have that

$$S_i = W \otimes_{\mathbb{K}H} P_i = W \otimes_{\mathbb{K}H} (\oplus \mathbb{K}H) = \oplus W, \tag{5}$$

hence

$$S_i \text{ is a flat left } \mathbb{K}_{par}G - \text{module},$$

so $\mathcal{S}$ is a flat resolution of the left $\mathbb{K}_{par}G$-module $W \otimes_{\mathbb{K}H} U$. By [**45**, Theorem 7.5] not only projective but flat resolutions can be used to calculate $\mathrm{Tor}_i^{\mathbb{K}_{par}G}(B, -)$. Then

$$H_i^{par}(G, W \otimes_{\mathbb{K}H} U) = \mathrm{Tor}_i^{\mathbb{K}_{par}G}(B, W \otimes_{\mathbb{K}H} U) = H_i(B \otimes_{\mathbb{K}_{par}G} \mathcal{S}^{del}),$$

where the upper index *del* means the deleted resolution, that is, the complex in dimension -1 is substituted with 0. Note that

$$B \otimes_{\mathbb{K}_{par}G} S_i = B \otimes_{\mathbb{K}_{par}G} (W \otimes_{\mathbb{K}H} P_i) \simeq (B \otimes_{\mathbb{K}_{par}G} W) \otimes_{\mathbb{K}H} P_i,$$

hence

$$H_i(B \otimes_{\mathbb{K}_{par}G} \mathcal{S}^{del}) \simeq H_i((B \otimes_{\mathbb{K}_{par}G} W) \otimes_{\mathbb{K}H} \mathcal{P}^{del}).$$

Since $B \otimes_{\mathbb{K}_{par}G} W$ is the trivial right $\mathbb{K}H$-module $\mathbb{K}$,

$$H_i((B \otimes_{\mathbb{K}_{par}G} W) \otimes_{\mathbb{K}H} \mathcal{P}) \simeq H_i(\mathbb{K} \otimes_{\mathbb{K}H} \mathcal{P}^{del}) = H_i(H,U).$$

$\square$

For a $\mathbb{K}_{par}G$-$\mathbb{K}H$-bimodule W and a right $\mathbb{K}H$-module U consider $Hom_{\mathbb{K}H}(W,U)$ as a right $\mathbb{K}_{par}G$-module in the following way; for $f \in Hom_{\mathbb{K}H}(W,U)$ and $r \in \mathbb{K}_{par}G$ we have $fr \in Hom_{\mathbb{K}H}(W,U)$ defined by $fr(w) = f(rw)$.

THEOREM 3.2. *(Cohomological Shapiro lemma for partial actions) Let W be a $\mathbb{K}_{par}G$-$\mathbb{K}H$-bimodule for some groups G and H such that W is projective as a right $\mathbb{K}H$-module, W is flat as a left $\mathbb{K}_{par}G$-module and $B \otimes_{\mathbb{K}_{par}G} W$ is isomorphic to the trivial right $\mathbb{K}H$-module $\mathbb{K}$. Then for any right $\mathbb{K}H$-module U, considering $Hom_{\mathbb{K}H}(W,U)$ as a right $\mathbb{K}_{par}G$-module, we have*

$$H^i_{par}(G, Hom_{\mathbb{K}H}(W,U)) \simeq H^i(H,U), \quad i \geq 0.$$

Proof. Let

$$\mathcal{E} : 0 \to U \to E^0 \to E^1 \to E^2 \to \dots$$

be an injective resolution of U as a right $\mathbb{K}H$-module. Since $B \otimes_{\mathbb{K}_{par}G} W$ is isomorphic to the trivial right $\mathbb{K}H$-module $\mathbb{K}$ we have

$$H^i(H,U) = Ext^i_{\mathbb{K}H}(\mathbb{K},U) = Ext^i_{\mathbb{K}H}(B \otimes_{\mathbb{K}_{par}G} W, U)$$
$$= H^i(Hom_{\mathbb{K}H}(B \otimes_{\mathbb{K}_{par}G} W, \mathcal{E}^{del})).$$

By the Adjoint Isomorphism Theorem [45, Theorem 2.75] we obtain

$$Hom_{\mathbb{K}H}(B \otimes_{\mathbb{K}_{par}G} W, E^i) \simeq Hom_{\mathbb{K}_{par}G}(B, Hom_{\mathbb{K}H}(W, E^i)),$$

hence

$$H^i(Hom_{\mathbb{K}H}(B \otimes_{\mathbb{K}_{par}G} W, \mathcal{E}^{del})) \simeq H^i(Hom_{\mathbb{K}_{par}G}(B, Hom_{\mathbb{K}H}(W, \mathcal{E}^{del}))).$$

We claim that $Hom_{\mathbb{K}H}(W, E^i)$ is injective as a right $\mathbb{K}_{par}G$-module. This is equivalent to the exactness of the functor $Hom_{\mathbb{K}_{par}G}(-, Hom_{\mathbb{K}H}(W, E^i))$. Consider again the adjoint isomorphism

$$Hom_{\mathbb{K}_{par}G}(-, Hom_{\mathbb{K}H}(W, E^i)) \simeq Hom_{\mathbb{K}H}(- \otimes_{\mathbb{K}_{par}G} W, E^i).$$

Note that the right-hand side is the composition of two exact functors $- \otimes_{\mathbb{K}_{par}G} W$ and $Hom_{\mathbb{K}H}(-, E^i)$, because W is flat as a left $\mathbb{K}_{par}G$-module and E^i is injective as a right $\mathbb{K}H$-module. As a composition of two exact functors is exact, we obtain that $Hom_{\mathbb{K}H}(W, E^i)$ is injective as a right $\mathbb{K}_{par}G$-module. Consequently, since W is a projective right

$\mathbb{K}H$-module, $Hom_{\mathbb{K}H}(W,-)$ is an exact functor and $Hom_{\mathbb{K}H}(W,\mathcal{E})$ is an injective resolution of the right $\mathbb{K}_{par}G$-module $Hom_{\mathbb{K}H}(W,U)$. Then

$$H^i(H,U) \simeq H^i(Hom_{\mathbb{K}_{par}G}(B,Hom_{\mathbb{K}H}(W,\mathcal{E}^{del})))$$

$$\simeq Ext^i_{\mathbb{K}_{par}G}(B,Hom_{\mathbb{K}H}(W,U)))$$

$$= H^i_{par}(G,Hom_{\mathbb{K}H}(W,U)).$$

$\square$

4. APPLICATIONS: THE BIMODULE *W*

We are going to show the existence of a bimodule W as in Theorem 3.1 and Theorem 3.2. First we recall some notions from [19] and [24]. Given an arbitrary groupoid Γ and a field $\mathbb{K}$, the groupoid algebra $\mathbb{K}\Gamma$ is defined as the vector space with basis formed by the morphisms of Γ, which are multiplied by the rule

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1 \circ \gamma_2, & \text{if } \gamma_1 \circ \gamma_2 \text{ exists in } \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

As it is usual for functions, we apply morphisms from right to left, so that in the composition $\gamma_1 \circ \gamma_2$ first γ_2 acts and then γ_1.

For a group G consider the groupoid $\Gamma(G)$ whose objects (seen as vertices) are subsets of G which contain 1 and whose morphisms are pairs (A,g) where A is a subset of G which contains 1 and g^{-1}. Then both A and gA are objects in $\Gamma(G)$ and the morphism (A,g) can be seen as an arrow from A to gA, which is interpreted as multiplication by g from the left. The product $(B,g') \circ (A,g)$ is defined in $\Gamma(G)$ if and only if $B = gA$, that is, when the range gA of (A,g) coincides with the domain B of (B,g'). In this case $(B,g') \circ (A,g) = (gA,g') \circ (A,g) = (A,g'g)$. We have the decomposition of the groupoid algebra

$$\mathbb{K}\Gamma(G) = \oplus \, \mathbb{K}\Delta$$

into a direct sum of two-sided ideals, where Δ runs over the connected components of $\Gamma(G)$.

Let now Δ be a connected component of $\Gamma(G)$ whose set of vertices $\mathcal{V}_\Delta$ is finite. By [24, Theorem 2.2] the map

$$\lambda_\Delta(g) = \sum_{A \in \mathcal{V}_\Delta, g^{-1} \in A} (A,g) \tag{6}$$

is a partial representation $G \to \mathbb{K}\Delta$, where the sum is taken over all vertices A of Δ which contain g^{-1}. By the universal property of $\mathbb{K}_{par}G$, the map λ_Δ extends to a homomorphism of $\mathbb{K}$-algebras

$$\mathbb{K}_{par}G \to \mathbb{K}\Delta, \ [g] \mapsto \lambda_\Delta(g),$$

which, with a slight abuse of notation, will be denoted by the same symbol λ_Δ. We consider $\mathbb{K}\Delta$ as a left $\mathbb{K}_{par}G$-module by means of λ_Δ, that is,

$$[g] \cdot a = \lambda_\Delta(g)a \tag{7}$$

for $a \in \mathbb{K}\Delta$ and $g \in G$.

Let n be the number of vertices in Δ and fix a vertex $A \in \mathcal{V}_\Delta$. Then there exist $g_1,g_2,\dots,g_n \in G$, with $g_i^{-1} \in A$, such that $\mathcal{V}_\Delta = \{g_1A,g_2A,\dots,g_nA\}$. It is convenient to assume that $g_1 = 1$. Let H be the stabilizer of A, that is,

$$H = \{h \in G : hA = A\}.$$

Then H is a subgroup of G and the groupoid algebra $\mathbb{K}\Delta$ is isomorphic to the algebra $M_n(\mathbb{K}H)$ of all $n \times n$-matrices with entries in the group algebra $\mathbb{K}H$. The isomorphism is given as follows. Let

(g_iA, g) be a morphism in Δ whose range is $gg_iA = g_jA$ for some $j \in \{1, \ldots, n\}$. Notice that $g_j^{-1}gg_i = h_i \in H$ and write $g = g_jh_ig_i^{-1}$. Then the desired isomorphism is given by

$$\eta : \mathbb{K}\Delta \to M_n(\mathbb{K}H), \; (g_iA, g) \mapsto E_{ji}(h_i), \tag{8}$$

where $E_{ji}(h_i)$ stands for the $n \times n$-matrix whose unique non-zero entry h_i is placed in the position (j, i).

For $g \in G$ denote by M_g the matrix $\lambda_\Delta([g])$ after identifying $\mathbb{K}\Delta$ with $M_n(\mathbb{K}H)$. Then

$$M_g = \sum_{g_iA \ni g^{-1}} E_{ji}(h_i), \tag{9}$$

in which the sum is taken over all vertices g_iA of Δ which contain g^{-1}. Thus M_g is a monomial matrix over H, in the sense that each row and each column of M_g contain at most one non-zero entry, which is an element of H. Denote by M_g^* the matrix obtained from M_g by transposition and replacement of each entry $h \in H$ by its inverse.

LEMMA 4.1 *Let G be an arbitrary group and let M_g be as defined above. For every $g \in G$ we have that $M_{g^{-1}} = M_g^*$.*

Proof. While in (9) g_iA runs over all vertices of Δ which contain g^{-1}, the target vertices $g_jA = gg_iA$ of the arrows (g_iA, g) are all those vertices of Δ which contain g. Evidently, (g_jA, g^{-1}) is the inverse of (g_iA, g) in the groupoid Δ, and, consequently, it follows from (9) that

$$M_{g^{-1}} = \sum_{g_jA \ni g} E_{ij}(h_i^{-1}) = M_g^*,$$

as desired. □

LEMMA 4.2 *Let Δ, λ_Δ and B be as above, with G being an arbitrary group. Then $\lambda_\Delta(B)$ is the subalgebra of $\mathbb{K}\Delta$ whose $\mathbb{K}$-basis is formed by the elements $(C, 1)$, where C runs over $\mathcal{V}_\Delta$.*

Proof. We need to show that $\{(g_iA, 1); i = 1, 2, \ldots, n\}$ is a basis for $\lambda_\Delta(B)$. First of all, we have the formula

$$\lambda_\Delta(e_g) = \sum_{g \in C, C \in \mathcal{V}_\Delta} (C, 1) = \sum_{g_iA \ni g} (g_iA, 1) \tag{10}$$

since, by the definition of λ_Δ and the expression of the product in $\mathbb{K}\Gamma(G)$,

$$\lambda_\Delta(e_g) = \lambda_\Delta([g])\lambda_\Delta([g^{-1}]) = \sum_{g_iA \ni g^{-1}, g_jA \ni g} (g_iA, g)(g_jA, g^{-1}) = \sum_{g_jA \ni g} (g_jA, 1).$$

Consequently, for every finite $X \subseteq G$,

$$\lambda_\Delta\left(\prod_{t \in X} e_t\right) = \prod_{t \in X} \lambda_\Delta(e_t) = \sum_{C \in \mathcal{V}_\Delta, \; C \supseteq X} (C, 1), \tag{11}$$

assuming that this sum is zero if no such vertex C exists.

Fixing now an arbitrary $C_0 \in \mathcal{V}_\Delta$ we shall show that $(C_0, 1) \in \lambda_\Delta(B)$.

We shall proceed using induction on $d(C_0)$, where the function $d : \mathcal{V}_\Delta \to \mathbb{N}$ was defined in [**22**, p. 2571] as follows. If $C \in \mathcal{V}_\Delta$ is maximal, that is, if

$$C \subseteq \tilde{C} \text{ for some } \tilde{C} \in \mathcal{V}_\Delta \text{ implies } C = \tilde{C},$$

we set $d(C) = 0$. Otherwise, we define

$$d(C) = \max\{m \in \mathbb{N} \mid \exists D_1, \dots, D_m \in \mathcal{V}_\Delta, \text{ with } D_m \supsetneq \dots \supsetneq D_1 \supsetneq C\}.$$

Since $\mathcal{V}_\Delta$ is finite, there is a finite subset $X_0 \subseteq C_0$ such that if $X_0 \subseteq C$ for some $C \in \mathcal{V}_\Delta$ then $C_0 \subseteq C$. Indeed, let $\mathcal{J} \subseteq \{1, \dots, n\}$ be the set of all indexes j such that $C_0 \nsubseteq g_j A$. For each $j \in \mathcal{J}$ choose an element $t_j \in C_0 \backslash g_j A$. Then $X_0 = \{t_j \mid j \in \mathcal{J}\}$ is the desired finite set.

If $d(C_0) = 0$, then (11) for $X = X_0$ readily yields $(C_0, 1) \in \lambda_\Delta(B)$, so suppose that $d(C_0) > 0$ and $(C, 1) \in \lambda_\Delta(B)$ for all $C \in \mathcal{V}_\Delta$ with $d(C) < d(C_0)$. Then we conclude from (11) applied for $X = X_0$ that

$$(C_0, 1) = \lambda_\Delta \left(\prod_{t \in X_0} e_t \right) - \sum_{C \in \mathcal{V}_\Delta, \, C \supsetneq C_0} (C, 1) \in \lambda_\Delta(B),$$

completing our proof. □

COROLLARY 4.3 With the notation of Lemma 4.2, we have that $\dim_{\mathbb{K}}(\lambda_\Delta(B)) = n$.

Proof. It is clear that the linearly independent elements $(g_i A, 1)$ generate the image of B by λ_Δ, and hence $\dim_{\mathbb{K}}(\lambda_\Delta(B)) = n$. □

The free right $\mathbb{K}H$-module

$$W = (\mathbb{K}H)^n = M_{n,1}(\mathbb{K}H),$$

viewed as column matrices, possesses a left $M_n(\mathbb{K}H)$-action given by matrix multiplication, transforming W into an $M_n(\mathbb{K}H)$-$\mathbb{K}H$-bimodule. Then the $\mathbb{K}$-algebra homomorphism

$$\lambda_\Delta : \mathbb{K}_{par}G \to \mathbb{K}\Delta,$$

followed by the isomorphism

$$\eta : \mathbb{K}\Delta \to M_n(\mathbb{K}H),$$

endows W with a structure of a $\mathbb{K}_{par}G$-$\mathbb{K}H$-bimodule, called *elementary*. The latter gives rise to an elementary partial matrix representation considered in [**24**].

We recall the next fact, whose statement is a slight modification of Theorem 2.3 from [**24**].

THEOREM 4.4. *Let G be an arbitrary group and V be an irreducible (resp. indecomposable) left $\mathbb{K}_{par}G$-module, which is finite dimensional over $\mathbb{K}$. Then there is a unique connected component Δ of $\Gamma(G)$ with a finite number of vertices, an elementary bimodule $_{\mathbb{K}_{par}G}W_{\mathbb{K}H}$ related to Δ and a left irreducible (resp. indecomposable) $\mathbb{K}H$-module U, finite dimensional over $\mathbb{K}$, such that $V \simeq W \otimes_{\mathbb{K}H} U$ as left $\mathbb{K}_{par}G$-modules.*

It is shown in [**24**, Theorem 2.2] that for each irreducible (indecomposable) $\mathbb{K}_{par}G$-module V, which has a finite dimension over $\mathbb{K}$, there exists a unique connected component Δ of $\Gamma(G)$, with a finite number of vertices, and a $\mathbb{K}\Delta$-module structure on V such that $_{\mathbb{K}_{par}G}V$ is obtained from $_{\mathbb{K}\Delta}W$

$\otimes_{\mathbb{K}H} U$ by means of λ_Δ, so that Theorem 4.4 is a direct consequence of [**24**, Theorem 2.3]. Notice that the map

$$M : G \to M_n(\mathbb{K}H), \quad g \mapsto M_g = \sum_{g_i A \ni g^{-1}} E_{ji}(h_i),$$

is the elementary matrix representation afforded by the bimodule $_{\mathbb{K}_{par}G}W_{\mathbb{K}H}$. Given $d \in \lambda_\Delta(B)$ and $\sum_{\gamma \in \Delta} \alpha_\gamma \gamma \in \mathbb{K}\Delta$, define the action

$$d \lhd \left(\sum_{\gamma \in \Delta} \alpha_\gamma \gamma\right) = \sum_{\gamma \in \Delta} \alpha_\gamma \gamma^{-1} d\gamma. \tag{12}$$

Using Lemma 4.2 it is easy to check that $\lambda_\Delta(B)$ is a right $\mathbb{K}\Delta$-module with respect to (12): it suffices to check that $(tD, t^{-1})(C, 1)(D, t) \in \lambda_\Delta(B)$, for any $(C, 1), (D, t) \in \Delta$, and this product is either zero or equal to $(D, 1) \in \lambda_\Delta(B)$.

The next fact is crucial for our construction of W as in Theorem 3.1 and Theorem 3.2, and since its proof is long it will be given in Section 5.

4.5. *Let Δ be a connected component of $\Gamma(G)$ with a finite number of vertices and λ_Δ be as in (6). Then there is an isomorphism of right $\mathbb{K}\Delta$-modules*

$$\lambda_\Delta(B) \simeq B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta,$$

where the left $\mathbb{K}_{par}G$-module structure on $\mathbb{K}\Delta$ is given by (7).

Let * be the involution of $\mathbb{K}_{par}G$ determined by $[g]^* = [g^{-1}]$. Thus $e_g^* = ([g][g^{-1}])^* = [g^{-1}]^*[g]^* = [g][g^{-1}] = e_g$ for every $g \in G$. Note that $\mathbb{K}\Delta$ also has a natural involution determined by

$$\left(\sum_{\gamma \in \Delta} \alpha_\gamma \gamma\right) \mapsto \sum_{\gamma \in \Delta} \alpha_\gamma \gamma^{-1},$$

which, by a slight abuse of notation, will also be denoted by *. It is clear from the equality (6) which defines λ_Δ that $\lambda_\Delta(a^*) = \lambda_\Delta(a)^*$ for every $a \in \mathbb{K}_{par}G$.

PROPOSITION 4.6 *Let G be an arbitrary group, Δ be a connected component of $\Gamma(G)$ with a finite number of vertices and W be the elementary $\mathbb{K}_{par}G$- $\mathbb{K}H$-bimodule associated with Δ by means of the homomorphism λ_Δ. Then $B \otimes_{\mathbb{K}_{par}G} W \simeq \mathbb{K}$ as right $\mathbb{K}H$-modules, where $\mathbb{K}$ is the trivial $\mathbb{K}H$-module and B is a right $\mathbb{K}_{par}G$-module via $b^{[g]} = [g^{-1}]b[g]$ for $b \in B, g \in G$.*

Proof. The algebra isomorphism $\mathbb{K}\Delta \simeq M_n(\mathbb{K}H)$ endows $\mathbb{K}\Delta$ with the structure of a right $\mathbb{K}H$-module. Consequently, as a right $\mathbb{K}H$-module, $\mathbb{K}\Delta$ is isomorphic to the direct sum of n copies of the free right $\mathbb{K}H$-module W, and we have that

$$B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta \simeq B \otimes_{\mathbb{K}_{par}G} W^n \simeq (B \otimes_{\mathbb{K}_{par}G} W)^n$$

as right $\mathbb{K}H$-modules. Then it follows from Corollary 4.3 and Theorem 4.5 that

$$n = \dim_{\mathbb{K}}(\lambda_\Delta(B)) = \dim_{\mathbb{K}}(B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta) = \dim_{\mathbb{K}}((B \otimes_{\mathbb{K}_{par}G} W)^n),$$

and therefore $\dim_{\mathbb{K}}(B \otimes_{\mathbb{K}_{par}G} W) = 1$. It remains to show that the right $\mathbb{K}H$ action on this module is trivial.

Let $\{e_i; i = 1, 2, \ldots, n\}$ be the canonical $\mathbb{K}H$-basis of $W \simeq M_{n,1}(\mathbb{K}H)$.

CLAIM 4.7. $1 \otimes e_i h = 1 \otimes e_i$ in $B \otimes_{\mathbb{K}_{par}G} W$ for any $h \in H$ and any $i = 1, 2, \ldots, n$.

In order to see this, fix $h \in H$ and recall that $\mathcal{V}_\Delta = \{g_1 A, g_2 A, \ldots, g_n A\}$ where $g_1 = 1$. Notice that according to (6) the element $(g_i A, g_i h g_i^{-1}) \in \Delta$ is a summand of $\lambda_\Delta(g_i h g_i^{-1})$ for each $i = 1, 2, \ldots, n$, as $A \ni g_i^{-1}$ implies $g_i A \ni g_i h^{-1} g_i^{-1}$. By (8), $\eta(g_i A, g_i h g_i^{-1}) = E_{ii}(h)$ and h is the (i, i)-entry of the matrix $\eta(\lambda_\Delta(g_i h g_i^{-1}))$. Recalling that $\eta(\lambda_\Delta(g_i h g_i^{-1}))$ is monomial over H and that the left $\mathbb{K}_{par}G$-module structure on W is given via the composition $\eta \circ \lambda_\Delta$, we obtain that

$$1 \otimes e_i h = 1 \otimes \eta(\lambda_\Delta(g_i h g_i^{-1})) e_i = 1^{[g_i h g_i^{-1}]} \otimes e_i$$

$$= [g_i h^{-1} g_i^{-1}][g_i h g_i^{-1}] \otimes e_i = e_{g_i h^{-1} g_i^{-1}} \otimes e_i,$$

recalling that if $b \in B$ and $r \in G$ then $b^{[r]} = [r^{-1}] b [r] \in B$ (Proposition 4.6). On the other hand

$$1 \otimes \eta(\lambda_\Delta(e_{g_i h^{-1} g_i^{-1}})) e_i = 1 \otimes \eta(\lambda_\Delta([g_i h^{-1} g_i^{-1}][g_i h g_i^{-1}])) e_i = 1^{[g_i h^{-1} g_i^{-1}][g_i h g_i^{-1}]} \otimes e_i$$

$$= [g_i h^{-1} g_i^{-1}][g_i h g_i^{-1}][g_i h^{-1} g_i^{-1}][g_i h g_i^{-1}] \otimes e_i = e_{g_i h^{-1} g_i^{-1}} \otimes e_i.$$

Since by equality (10) the element $(g_i A, 1) \in \Delta$ is a summand of $\lambda_\Delta(e_{g_i h^{-1} g_i^{-1}})$, the (i, i)-entry of the diagonal matrix $\eta(\lambda_\Delta(e_{g_i h^{-1} g_i^{-1}}))$ is 1, and thus

$$1 \otimes e_i h = 1 \otimes \eta(\lambda_\Delta(e_{g_i h^{-1} g_i^{-1}})) e_i = 1 \otimes e_i,$$

as claimed.

CLAIM 4.8. $\{1 \otimes e_i\}$ is a $\mathbb{K}$-basis of $B \otimes_{\mathbb{K}_{par}G} W$ for any fixed $i = 1, 2, \ldots, n$.

Indeed, note that the elements $1 \otimes e_1, \ldots, 1 \otimes e_n$ generate $B \otimes_{\mathbb{K}_{par}G} W$ as a right $\mathbb{K}H$-module, since $b \otimes w = 1 \otimes \eta(\lambda_\Delta(b)) w$ for any $b \otimes w \in B \otimes_{\mathbb{K}_{par}G} W$. It follows from Claim 4.7 that $1 \otimes e_1, \ldots, 1 \otimes e_n$ generate $B \otimes_{\mathbb{K}_{par}G} W$ as a $\mathbb{K}$-vector space.

Proposition 2.2 of [22] implies that $\lambda_\Delta : \mathbb{K}_{par}G \to \mathbb{K}\Delta$ is an epimorphism. Hence, given $1 \leq i, j \leq n$, we may write $E_{j,i}(1) = \eta(\lambda_\Delta(a))$ for some $a \in \mathbb{K}_{par}G$, and it follows that

$$1 \otimes e_j = 1 \otimes E_{j,i}(1) e_i = 1 \otimes \eta(\lambda_\Delta(a)) e_i = 1^a \otimes e_i.$$

For any $b \in B$, the element $\lambda_\Delta(b) \in \mathbb{K}\Delta$ is a $\mathbb{K}$-linear combination of idempotents of the form $(C, 1)$, $C \in \mathcal{V}_\Delta$, and therefore $\eta(\lambda_\Delta(b))$ is a diagonal matrix with entries in $\mathbb{K}$ thanks to equality (8). In particular $\eta(\lambda_\Delta(1^a))$ is such a diagonal matrix. Therefore

$$1 \otimes e_j = 1^a \otimes e_i = 1 \otimes \eta(\lambda_\Delta(1^a)) e_i = \alpha_{j,i}(1 \otimes e_i)$$

for some $\alpha_{j,i} \in \mathbb{K}$. Hence all elements $1 \otimes e_i$ are scalar multiples of each other and, given that those elements generate $B \otimes_{\mathbb{K}_{par}G} W$ as a $\mathbb{K}$-vector space, at least one of the scalars $\alpha_{j,i}$ is nonzero; but this implies that all are nonzero. Due to the fact that $\dim_{\mathbb{K}} B \otimes_{\mathbb{K}_{par}G} W = 1$, any set $\{1 \otimes e_i\}$ is a $\mathbb{K}$-basis, proving the claim.

The fact that the right $\mathbb{K}H$-action on $B \otimes_{\mathbb{K}_{par}G} W$ is trivial follows from Claim 4.7. $\square$

The following remark will be justified in Section 6.1.

REMARK 4.9. *When G is a finite group and Δ is a connected component of $\Gamma(G)$, the left $\mathbb{K}_{par}G$-module $\mathbb{K}\Delta$ (via λ_Δ) is a projective module. However, in the case of an infinite group G, $\mathbb{K}\Delta$ may not be a projective left $\mathbb{K}_{par}G$-module even when Δ has a finite number of vertices.*

Although projectivity may fail, in Section 6.2 we will prove a strong result regarding the modules $\mathbb{K}\Delta$.

THEOREM 4.10. *Let G be an arbitrary group and Δ be a connected component of $\Gamma(G)$ with a finite number of vertices. Then $\mathbb{K}\Delta$ is flat as a left $\mathbb{K}_{par}G$-module (via λ_Δ).*

THEOREM 4.11. *Let G be an arbitrary group, Δ be a connected component of $\Gamma(G)$ with a finite number of vertices and W be the elementary $\mathbb{K}_{par}G$- $\mathbb{K}H$-bimodule related to Δ. Let U be an arbitrary left $\mathbb{K}H$-module. Then*

$$H_i^{par}(G, W \otimes_{\mathbb{K}H} U) \simeq H_i(H, U), \quad i \geq 0.$$

Proof. By the isomorphism $\mathbb{K}\Delta \simeq M_n(\mathbb{K}H)$ given in (8), W is a direct summand of $\mathbb{K}\Delta$ as a $\mathbb{K}\Delta$-module. Since $\mathbb{K}\Delta$ is flat as a left $\mathbb{K}_{par}G$-module, we deduce that

$$W \text{ is flat as a left } \mathbb{K}_{par}G\text{-module.}$$

Thanks to Proposition 4.6, $B \otimes_{\mathbb{K}_{par}G} W$ is the trivial right $\mathbb{K}H$-module $\mathbb{K}$, and, because W is also flat as a right $\mathbb{K}H$-module (being free), our statement follows by Theorem 3.1. $\square$

COROLLARY 4.12 Let Δ be a connected component of the groupoid $\Gamma(G)$ such that Δ has finitely many vertices and $V = W \otimes_{\mathbb{K}H} U$ be an irreducible (resp. indecomposable) left $\mathbb{K}_{par}G$-module with decomposition given by Theorem 4.4. Then

$$H_i^{par}(G, V) \simeq H_i(H, U), \quad i \geq 0.$$

THEOREM 4.13. *Let G be an arbitrary group, Δ be a connected component of $\Gamma(G)$ with a finite number of vertices and W be the elementary $\mathbb{K}_{par}G$- $\mathbb{K}H$-bimodule related to Δ. Let U be an arbitrary left $\mathbb{K}H$-module. Then*

$$H_{par}^i(G, W \otimes_{\mathbb{K}H} U) \simeq H^i(H, U), \quad i \geq 0.$$

Proof. We write U_0 for U considered as a right $\mathbb{K}H$-module by $uh = h^{-1}u$. Using Theorem 3.2, Proposition 4.6 and Theorem 4.10, we obtain $\mathbb{K}$-isomorphisms

$$H_{par}^i(G, Hom_{\mathbb{K}H}(W, U_0)) \simeq H^i(H, U_0) \simeq H^i(H, U),$$

in which the latter isomorphism comes from the fact that $H^i(H, U_0) = Ext_{\mathbb{K}H}^i(\mathbb{K}, U_0)$, where $\mathbb{K}$ is considered as a trivial right $\mathbb{K}H$-module, and $H^i(H, U) = Ext_{\mathbb{K}H}^i(\mathbb{K}, U)$, with $\mathbb{K}$ considered as a trivial left $\mathbb{K}H$-module. Note that by definition

$$H_{par}^i(G, Hom_{\mathbb{K}H}(W, U_0)) = Ext_{\mathbb{K}_{par}G}^i(B, Hom_{\mathbb{K}H}(W, U_0)),$$

where both B and $Hom_{\mathbb{K}H}(W, U_0)$ are right $\mathbb{K}_{par}G$-modules.

On the other hand $H_{par}^i(G, V) = Ext_{\mathbb{K}_{par}G}^i(B, V)$, where both B and $V = W \otimes_{\mathbb{K}H} U$ are left $\mathbb{K}_{par}G$-modules. Write V_0 for V considered as right $\mathbb{K}_{par}G$-module by $v[g] = [g^{-1}]v$. Note that $Ext_{\mathbb{K}_{par}G}^i(B, V) \simeq Ext_{\mathbb{K}_{par}G}^i(B, V_0)$, where B in $Ext_{\mathbb{K}_{par}G}^i(B, V_0)$ is considered as a right $\mathbb{K}_{par}G$-module. Then it remains to show that

$$V_0 \simeq Hom_{\mathbb{K}H}(W, U_0) \text{ as right } \mathbb{K}_{par}G\text{-modules.} \tag{13}$$

Note that

$$V = W \otimes_{\mathbb{K}H} U \simeq (\mathbb{K}H)^n \otimes_{\mathbb{K}H} U \simeq (\mathbb{K}H \otimes_{\mathbb{K}H} U)^n \simeq U^n,$$

as $\mathbb{K}$-vector spaces, and composing these isomorphisms we obtain the $\mathbb{K}$-isomorphism

$$\varphi_1 : V_0 \to U^n$$

given by

$$\varphi_1((\lambda_1, \ldots, \lambda_n) \otimes u) = (\lambda_1 u, \ldots, \lambda_n u),$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{K}H$ and $u \in U$.

Also consider the standard isomorphism

$$\varphi_2 : Hom_{\mathbb{K}H}(W, U_0) \to U^n$$

defined by

$$\varphi_2(f) = (f(e_1), \ldots, f(e_n)),$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of the free right $\mathbb{K}H$-module $W \simeq (\mathbb{K}H)^n$.

Recall that the right action of $[g]$ on $f \in Hom_{\mathbb{K}H}(W, U_0)$ is determined by

$$(f \cdot [g])(w) = f([g] \cdot w).$$

Then via the isomorphism φ_2 we can consider U^n as a right $\mathbb{K}_{par}G$-module with $[g]$ acting in the following way : if $u = (u_1, \ldots, u_n) \in U^n$ then $u \cdot [g] = \tilde{u}$, where $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) \in U^n$,

$$M_g^* \cdot u^t = \tilde{u}^t$$

and $M_g, M_g^* \in M_n(\mathbb{K}H)$ are the matrices defined in (9). Indeed to see that the action is precisely this one, we consider

$$f(e_i) = u_i, (f \cdot [g])(e_i) = \tilde{u}_i \text{ and } M_g = (\lambda_{i,j}) \in M_n(\mathbb{K}H).$$

Then

$$\tilde{u}_i = f([g]e_i) = f\left(\sum_{1 \leq j \leq n} e_j \lambda_{j,i}\right) = \sum_{1 \leq j \leq n} f(e_j)\lambda_{j,i} = \sum_{1 \leq j \leq n} u_j \lambda_{j,i} = \sum_{1 \leq j \leq n} \lambda_{j,i}^* u_j.$$

On the other hand, for $\lambda \in (W \otimes_{\mathbb{K}H} U)_0 = V_0$ we have $\lambda \cdot [g] = [g^{-1}] \cdot \lambda$, so the action of $[g]$ on U^n induced by φ_1 is given as follows: if $u = (u_1, \ldots, u_n) \in U^n$, then $u \cdot [g] = \tilde{u}$, where

$$M_{g^{-1}} \cdot u^t = \tilde{u}^t,$$

that is, it is given by left multiplication by the matrix $M_{g^{-1}}$, where the n-tuples are considered as columns.

Finally, $M_g^* = M_{g^{-1}}$ by Lemma 4.1, thus proving (13). □

COROLLARY 4.14 Let Δ be a connected component of the groupoid $\Gamma(G)$ such that Δ has finitely many vertices and $V = W \otimes_{\mathbb{K}H} U$ be an irreducible (resp. indecomposable) left $\mathbb{K}_{par}G$-module with decomposition given by Theorem 4.4. Then

$$H^i_{par}(G,V) \simeq H^i(H,U), \quad i \geq 0.$$

COROLLARY 4.15 (**Corollary 1.2**) Suppose G is a finite group. Then

$$H^{par}_i(G,B) \simeq \oplus_\Delta H_i(H, \mathbb{K}) \text{ and } H^i_{par}(G,B) \simeq \oplus_\Delta H^i(H, \mathbb{K}), \quad i \geq 0,$$

where the sum is over the connected components Δ of the groupoid $\Gamma(G)$ and H is the stabilizer of a chosen vertex of Δ (thus is defined up to conjugation).

Proof. We will split B as a direct sum of indecomposable left $\mathbb{K}_{par}G$-modules. Write

$$e_A = \prod_{g \in A} e_g \prod_{s \in G \backslash A} (1 - e_s) \in B,$$

where A is a subset of G. The action of $[g]$ on e_A gives $[g]e_A[g^{-1}] = e_{gA}$ if $1 \in A$. Note that $e_A = 0$ if $1 \notin A$, and thus $e_{gA} = 0$ if $g^{-1} \notin A$. Observe that there is a $\mathbb{K}$-isomorphism

$$B \simeq \oplus_{1 \in A} \mathbb{K}e_A.$$

Indeed, let $G \backslash \{1\} = \{t_1, \ldots, t_k\}$. Then

$$B = Be_{t_1} \oplus B(1 - e_{t_1}) = (Be_{t_2} \oplus B(1 - e_{t_2}))e_{t_1} \oplus (Be_{t_2} \oplus B(1 - e_{t_2}))(1 - e_{t_1}).$$

Continuing the decomposition of B as $Be_{t_3} \oplus B(1 - e_{t_3})$ and so on, we obtain

$$B = \oplus_{1 \in A} Be_A.$$

Note that $B = \mathbb{K}[e_{t_1}, \ldots, e_{t_k}]$, and for $g \in A$ we have $e_g e_A = e_A$ and for $g \notin A$ we have $e_g e_A = 0$. Thus $Be_A = \mathbb{K}e_A$.

Furthermore, for each subset A with $1 \in A$ the sum

$$\oplus_{g \in G, g^{-1} \in A} \mathbb{K}e_{gA}$$

is a left $\mathbb{K}_{par}G$-module. Notice that in the latter sum, gA are the vertices of a connected component Δ of the groupoid $\Gamma(G)$.

Thus B splits as a direct sum of left $\mathbb{K}_{par}G$-modules over the connected components Δ of $\Gamma(G)$, where each summand is isomorphic to $\mathbb{K}\mathcal{V}_\Delta$. Then consider the epimorphisms

$$\lambda_\Delta : \mathbb{K}_{par}G \to \mathbb{K}\Delta \simeq M_n(\mathbb{K}H)$$

and

$$\theta : M_n(\mathbb{K}H) \to M_n(\mathbb{K}),$$

where θ is induced by the epimorphism $\mathbb{K}H \to \mathbb{K}$ that sends each element of H to 1. Then $\mathbb{K}\mathcal{V}_\Delta \simeq \mathbb{K}^n = M_{n,1}(\mathbb{K})$, where $M_{n,1}(\mathbb{K})$ are all $n \times 1$-matrices over $\mathbb{K}$, via the isomorphism

that sends e_{g_iA} to e_i. Thus $M_{n,1}(\mathbb{K})$ is a left $M_n(\mathbb{K})$-module via matrix multiplication, and via the composition map $\theta \circ \lambda_\Delta$ it is a left $\mathbb{K}_{par}G$-module. Thus

$$\mathbb{K}\mathcal{V}_\Delta \simeq W \otimes_{\mathbb{K}H} \mathbb{K},$$

where W is the elementary bimodule related to the connected component Δ. Note that the above isomorphism is of left $\mathbb{K}_{par}G$-modules because $\theta(M_g)e_i = e_j$ for $g_j^{-1}gg_i = h_i \in H$ since $M_g = \sum_{g_iA \ni g^{-1}} E_{ji}(h_i)$. Then by Theorem 4.11 and Theorem 4.13

$$H^i_{par}(G, B) = \oplus_\Delta H^i_{par}(G, W \otimes_{\mathbb{K}H} \mathbb{K}) \simeq \oplus_\Delta H^i(H, \mathbb{K})$$

and

$$H^{par}_i(G, B) = \oplus_\Delta H^{par}_i(G, W \otimes_{\mathbb{K}H} \mathbb{K}) \simeq \oplus_\Delta H_i(H, \mathbb{K}). \qquad \square$$

THEOREM 4.16. *Let G be a finite group. Then for $i \geq 1$*

$$H^i_{par}(G, \mathbb{K}_{par}G) = 0.$$

Proof. Since G is finite we have

$$\mathbb{K}_{par}G \simeq \mathbb{K}\Gamma(G) = \oplus_\Delta \mathbb{K}\Delta,$$

where the direct sum is over the connected components Δ of the groupoid $\Gamma(G)$. Thus $\mathbb{K}\Delta$ is a projective left $\mathbb{K}_{par}G$-module and we can decompose $\mathbb{K}\Delta \simeq W_1 \oplus \ldots \oplus W_n$, where $W_1 \simeq \ldots \simeq W_n \simeq W$ as $\mathbb{K}_{par}G$-$\mathbb{K}H$-bimodules. Then

$$\mathbb{K}\Delta \simeq \oplus_{1 \leq j \leq n} W_j \simeq \oplus_{1 \leq j \leq n} W_j \otimes_{\mathbb{K}H} U_j,$$

where each $U_j \simeq \mathbb{K}H$. Thus

$$H^i_{par}(G, \mathbb{K}_{par}G) \simeq \oplus_\Delta H^i_{par}(G, \mathbb{K}\Delta) \simeq \oplus_\Delta \oplus_{1 \leq j \leq n = n(\Delta)} H^i_{par}(G, W_j \otimes_{\mathbb{K}H} \mathbb{K}H),$$

and by Theorem 4.13 for $i \geq 1$

$$H^i_{par}(G, W_j \otimes_{\mathbb{K}H} \mathbb{K}H) \simeq H^i(H, \mathbb{K}H) = 0,$$

where the last equality holds for any finite group H. Indeed by [10, Prop. III.6.2] for any group T with a subgroup of finite index T_0 there is an isomorphism $H^i(T, \mathbb{K}T) \simeq H^i(T_0, \mathbb{K}T_0)$, observing that for subgroups of finite index, induced and co-induced modules are isomorphic (see [10, Prop. III.5.9]). Thus we obtain our result by applying this fact for $T = H$ and the trivial subgroup T_0. $\qquad \square$

5. PROOF OF THEOREM 4.5

In this section whenever we make calculations in $\mathbb{K}\Delta$ all sums are over subsets of G that are vertices in the connected component Δ, that is, belong to $\mathcal{V}_\Delta$.

5.1. The morphisms φ and ψ connecting $B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta$ and $\lambda_\Delta(B)$.

We will introduce now an equivalence relation on G which will be essential in what follows. Given $g \in G$, let

$$\varepsilon_g = \lambda_\Delta(g)\lambda_\Delta(g^{-1}) = \lambda_\Delta(e_g) \in \mathbb{K}\Delta.$$

We define an equivalence relation in G by

$$g \sim t \iff \varepsilon_g = \varepsilon_t. \tag{14}$$

Fix a complete set of representatives $X \subset G$ for the equivalence relation defined above. We claim that X is finite. In fact, Δ has a finite number of vertices and, by equality (10), for every $t \in G$ we have the formula

$$\varepsilon_t = \lambda_\Delta(e_t) = \sum_{D \ni t}(D, 1), \tag{15}$$

which shows that the number of distinct ε_ts is finite.

REMARK 5.1 Note that $\varepsilon_t = \varepsilon_g$ if and only if for all $C \in \mathcal{V}_\Delta$,

$$C \ni g \iff C \ni t.$$

In fact, by the definition of the equivalence relation, if $\varepsilon_t = \varepsilon_g$ then $\sum_{D \ni t}(D, 1) = \sum_{C \ni g}(C, 1)$ and vice versa. Clearly the second equality holds if and only, for every vertex C of Δ, $C \ni g$ implies that $C \ni t$ and conversely.

Let C be a vertex of Δ. We define a partition $X = C' \cup C''$ of the system of representatives in the following manner:

$$C' = X \cap C \tag{16}$$

$$C'' = X \backslash C'. \tag{17}$$

Note that if $x \in X$ is equivalent to some $t \in C$ then, by Remark 5.1, $x \in C$. Thus we obtain the following description for C':

$$C' = \{x \in X \mid \exists t \in C \text{such that } x \sim t\}.$$

REMARK 5.2 C' determines C, that is, if $D, C \in \mathcal{V}_\Delta$ then D' and C' coincide if and only if $C = D$. In fact, suppose that $C' = D'$ and assume that there exists $d \in D \backslash C$. If $d \sim x \in X$ then $x \in D' = C'$ and hence $x \in C$. But $d \sim x$ and $x \in C$ then, by the previous remark, d also lies in C, a contradiction.

Consider the $\mathbb{K}$-linear map

$$\varphi : B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta \to \lambda_\Delta(B), \; b \otimes a \mapsto \lambda_\Delta(b) \lhd a,$$

where the right $\mathbb{K}\Delta$-action $\lhd$ was defined in (12). This map is well-defined, since for $b \in B$, $a \in \mathbb{K}\Delta$ and $[g] \in \mathbb{K}_{par}G$,

$$\varphi((b^{[g]}) \otimes a) = \varphi([g^{-1}]b[g] \otimes a)$$
$$= \lambda_\Delta(g^{-1})\lambda_\Delta(b)\lambda_\Delta(g) \lhd a$$

$$= \lambda_\Delta(g)^* \lambda_\Delta(b) \lambda_\Delta(g) \triangleleft a$$
$$= (\lambda_\Delta(b) \triangleleft \lambda_\Delta(g)) \triangleleft a$$
$$= \lambda_\Delta(b) \triangleleft (\lambda_\Delta(g)a)$$
$$= \varphi(b \otimes \lambda_\Delta(g)a)$$
$$= \varphi(b \otimes [g] \cdot a).$$

Note that φ is a map of right $\mathbb{K}\Delta$-modules: for $b \in B$ and $a_1, a_2 \in \mathbb{K}\Delta$,

$$\varphi((b \otimes a_1)a_2) = \varphi(b \otimes a_1 a_2) = \lambda_\Delta(b) \triangleleft (a_1 a_2) = (\lambda_\Delta(b) \triangleleft a_1) \triangleleft a_2 = \varphi(b \otimes a_1) \triangleleft a_2.$$

We are going to prove that φ is an isomorphism of right $\mathbb{K}\Delta$-modules. We begin by defining a right inverse to φ, which will later be shown to be a left inverse as well. Given $(C, 1) \in \lambda_\Delta(B)$, let

$$\bar{\pi}(C, 1) = \prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f) \tag{18}$$

and extend $\bar{\pi}$ linearly to $\lambda_\Delta(B)$. We now define

$$\psi : \lambda_\Delta(B) \to B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta, \ d \mapsto \bar{\pi}(d) \otimes 1_{\mathbb{K}\Delta}. \tag{19}$$

We will show that

$$\varphi \circ \psi = Id_{\lambda_\Delta(B)}. \tag{20}$$

In fact, we first remark that for any finite subset Y of G we have the equality

$$\prod_{t \in Y} \sum_{\substack{D \in \mathcal{V}_\Delta \\ D \ni t}} (D, 1) = \sum_{D \supseteq Y} (D, 1), \tag{21}$$

since the elements $(D, 1)$ are orthogonal idempotents in $\mathbb{K}\Delta$. Another simple fact needed in the computation below is that if C and F are subsets of G then

$$F \cap C'' = \emptyset \text{ if and only if } F' \subseteq C'.$$

LEMMA 5.3 $\lambda_\Delta \circ \bar{\pi} = Id_{\lambda_\Delta(B)}.$

Proof.

$$(\lambda_\Delta \circ \bar{\pi})(C, 1) = \lambda_\Delta\left(\prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f)\right)$$

$$= \prod_{t \in C'} \lambda_\Delta(e_t) \prod_{f \in C''} (1 - \lambda_\Delta(e_f))$$

$$= \prod_{t \in C'} \sum_{D \ni t} (D, 1) \prod_{f \in C''} \left(1 - \sum_{E \ni f} (E, 1)\right)$$

$$= \prod_{t \in C'} \sum_{D \ni t} (D, 1) \prod_{f \in C''} \sum_{F \not\ni f} (F, 1)$$

$$= \left(\sum_{D \supseteq C'} (D, 1)\right) \left(\sum_{F \cap C'' = \emptyset} (F, 1)\right)$$

$$= \sum_{\substack{D \supseteq C' \\ D \cap C'' = \emptyset}} (D,1).$$

Now consider a subset D such that $D \supseteq C'$ and $D \cap C'' = \emptyset$. The inclusion $D \supseteq C'$ implies $D' \supseteq C'$, and $D \cap C'' = \emptyset$ yields $D' \subseteq C'$. Hence $D' = C'$ and consequently $D = C$, thanks to Remark 5.2. Therefore

$$(\lambda_\Delta \circ \bar\pi)(C,1) = \sum_{\substack{D \supseteq C' \\ D \cap C'' = \emptyset}} (D,1) = (C,1).$$

$\square$

The proof of the above lemma immediately implies the following result, which is precisely (20).

COROLLARY 5.4 Given $(C,1)$ in $\mathbb{K}\Delta$, we have that

$$\varphi \circ \psi(C,1) = (C,1).$$

5.2. Linearity of ψ

We begin with two technical results which will be useful in the proof that ψ is a $\mathbb{K}\Delta$-linear map, that is, ψ is a homomorphism of right $\mathbb{K}\Delta$-modules.

REMARK 5.5 For all e, f in B, $f^e = fe$.

In fact, letting $e = e_g$ and using that B is a commutative ring, we have

$$f^{e_g} = f^{[g][g^{-1}]} = [g]([g^{-1}]f[g])[g^{-1}] = e_g f e_g = f e_g.$$

LEMMA 5.6 For all t_1, t_2 in G, if $t_1 \sim t_2$ then

(i) $e_{t_1} \otimes 1_{\mathbb{K}\Delta} = e_{t_2} \otimes 1_{\mathbb{K}\Delta} = e_{t_1} e_{t_2} \otimes 1_{\mathbb{K}\Delta} \in B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta$.

(ii) $[g^{-1}]e_{t_1}[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{t_2}[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{t_1} e_{t_2}[g] \otimes 1_{\mathbb{K}\Delta}$, for all $g \in G$.

Proof. (i)

$$e_{t_1} \otimes 1_{\mathbb{K}\Delta} = e_{t_1}^2 \otimes 1_{\mathbb{K}\Delta} = e_{t_1}^{e_{t_1}} \otimes 1_{\mathbb{K}\Delta} = e_{t_1} \otimes \lambda_\Delta(e_{t_1}) = e_{t_1} \otimes \lambda_\Delta(e_{t_2})$$
$$= e_{t_1}^{e_{t_2}} \otimes 1_{\mathbb{K}\Delta} = e_{t_1} e_{t_2} \otimes 1_{\mathbb{K}\Delta}.$$

Switching t_1 and t_2 we get $e_{t_2} \otimes 1_{\mathbb{K}\Delta} = e_{t_2} e_{t_1} \otimes 1_{\mathbb{K}\Delta} = e_{t_1} e_{t_2} \otimes 1_{\mathbb{K}\Delta}$.

(ii) Given $g \in G$,

$$[g^{-1}]e_{t_1}[g] \otimes 1_{\mathbb{K}\Delta} = (e_{t_1}^{[g]}) \otimes 1_{\mathbb{K}\Delta} = (e_{t_1}^{e_{t_1}})^{[g]} \otimes 1_{\mathbb{K}\Delta} = e_{t_1}^{e_{t_1}[g]} \otimes 1_{\mathbb{K}\Delta}$$
$$= e_{t_1} \otimes \lambda_\Delta(e_{t_1}[g]) = e_{t_1} \otimes \lambda_\Delta(e_{t_1})\lambda_\Delta([g]) = e_{t_1} \otimes \lambda_\Delta(e_{t_2})\lambda_\Delta([g])$$
$$= e_{t_1} \otimes \lambda_\Delta(e_{t_2}[g]) = (e_{t_1}^{e_{t_2}[g]}) \otimes 1_{\mathbb{K}\Delta} = (e_{t_1}^{e_{t_2}})^{[g]} \otimes 1_{\mathbb{K}\Delta}$$
$$= (e_{t_1} e_{t_2})^{[g]} \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{t_1} e_{t_2}[g] \otimes 1_{\mathbb{K}\Delta}.$$

Clearly we also have

$$[g^{-1}]e_{t_2}[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{t_2} e_{t_1}[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{t_1} e_{t_2}[g] \otimes 1_{\mathbb{K}\Delta},$$

concluding the proof. $\square$

In what follows, given a property P the symbol $[P]$ denotes the following Boolean operator:

$$[P] = \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

We are going to prove the following fact.

PROPOSITION 5.7. *The map* $\psi : \lambda_\Delta(B) \to B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta$ *defined in* (19) *is a homomorphism of right* $\mathbb{K}\Delta$-*modules.*

Proof. We shall show that

$$\psi((C,1) \triangleleft \lambda_\Delta(g)) = \psi(C,1)\lambda_\Delta(g)$$

for all $(C,1) \in \lambda_\Delta(B)$ and $g \in G$. On one hand, we have

$$\psi((C,1) \triangleleft \lambda_\Delta(g)) = \bar{\pi}((C,1) \triangleleft \lambda_\Delta(g)) \otimes 1_{\mathbb{K}\Delta}$$

$$= \bar{\pi}\left((C,1) \triangleleft \sum_{D \ni g^{-1}}(D,g)\right) \otimes 1_{\mathbb{K}\Delta}$$

$$= \sum_{D \ni g^{-1}} \bar{\pi}((gD,g^{-1})(C,1)(D,g)) \otimes 1_{\mathbb{K}\Delta}$$

$$= [g \in C]\bar{\pi}(g^{-1}C,1) \otimes 1_{\mathbb{K}\Delta}$$

$$= [g \in C] \prod_{t \in (g^{-1}C)'} e_t \prod_{f \in (g^{-1}C)''}(1 - e_f) \otimes 1_{\mathbb{K}\Delta}.$$

Since $C \ni 1$ we have that $g^{-1} \in g^{-1}C$ and as a consequence $g^{-1} \sim x_{g^{-1}} \in (g^{-1}C)'$. It follows from Lemma 5.6 that

$$e_{x_{g^{-1}}} \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta}$$

and hence

$$\psi((C,1) \triangleleft \lambda_\Delta(g)) = [g \in C] \prod_{t \in (g^{-1}C)'} e_t \prod_{f \in (g^{-1}C)''}(1 - e_f) \otimes 1_{\mathbb{K}\Delta}$$

$$= [g \in C] e_{x_{g^{-1}}} \prod_{t \in (g^{-1}C)'} e_t \prod_{f \in (g^{-1}C)''}(1 - e_f) \otimes 1_{\mathbb{K}\Delta}$$

$$= [g \in C] e_{g^{-1}} \prod_{t \in (g^{-1}C)'} e_t \prod_{f \in (g^{-1}C)''}(1 - e_f) \otimes 1_{\mathbb{K}\Delta}$$

$$= (*).$$

On the other hand

$$\psi(C,1)\lambda_\Delta(g) = (\bar{\pi}(C,1) \otimes 1_{\mathbb{K}\Delta})\lambda_\Delta(g) = \bar{\pi}(C,1) \otimes \lambda_\Delta(g)$$

$$= (\bar{\pi}(C,1)^{[g]}) \otimes 1_{\mathbb{K}\Delta}$$

$$= \left(\prod_{t \in C'} e_t \prod_{t \in C''}(1 - e_t)\right)^{[g]} \otimes 1_{\mathbb{K}\Delta}$$

$$= [g^{-1}]\left(\prod_{t \in C'} e_t \prod_{f \in C''}(1 - e_f)\right)[g] \otimes 1_{\mathbb{K}\Delta}$$

$$= \prod_{t \in C'} e_{g^{-1}t} \prod_{f \in C''} (1 - e_{g^{-1}f}) e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta}$$

$$= e_{g^{-1}} \prod_{t \in C'} e_{g^{-1}t} \prod_{f \in C''} (1 - e_{g^{-1}f}) \otimes 1_{\mathbb{K}\Delta}$$

$$= (**).$$

We shall check that $(*) = (**)$.

CLAIM 5.8. *If $g \in C$, then*

$$e_{g^{-1}} \prod_{t \in (g^{-1}C)'} e_t \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}} \prod_{t \in C'} e_{g^{-1}t} \otimes 1_{\mathbb{K}\Delta}. \tag{22}$$

Indeed, if $t \in C'$ then $g^{-1}t \in g^{-1}C' \subseteq g^{-1}C$ and hence $g^{-1}t \sim x_{g^{-1}t} \in (g^{-1}C)'$. Using once more Lemma 5.6, we obtain the equality

$$e_{g^{-1}t} \otimes 1_{\mathbb{K}\Delta} = e_{x_{g^{-1}t}} \otimes 1_{\mathbb{K}\Delta},$$

which ensures that every element $e_{g^{-1}t}$, with $t \in C'$, occurring in the right-hand side (RHS) of (22) may be substituted by some $e_{t'}, t' \in (g^{-1}C)'$, which occurs in the left-hand side (LHS) of (22).

Conversely, consider an element $e_{t'}$ with $t' \in (g^{-1}C)'$ as in the LHS of (22). There is $\beta \in C$ such that $t' = g^{-1}\beta$; let $x_\beta \in C'$ be its representative, that is, $\beta \sim x_\beta$. Then

$$e_{g^{-1}}e_{t'} = e_{g^{-1}}e_{g^{-1}\beta} = e_{g^{-1}\beta}e_{g^{-1}} = [g^{-1}]e_\beta[g],$$

hence, by Lemma 5.6,

$$e_{g^{-1}}e_{t'} \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_\beta[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{x_\beta}[g] \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}}e_{g^{-1}x_\beta} \otimes 1_{\mathbb{K}\Delta}.$$

It follows that every $e_{t'}$, where $t' \in (g^{-1}C)'$, occurring in the LHS of (22) may be replaced by an $e_{g^{-1}t}, t \in C'$, as in the RHS of (22), which completes the proof of our claim.

This takes care of the first products appearing in $(*)$ and $(**)$. Now we will consider the second products.

CLAIM 5.9. *If $g \in C$, then*

$$e_{g^{-1}} \prod_{f \in (g^{-1}C)''} (1 - e_f) \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}} \prod_{f \in C''} (1 - e_{g^{-1}f}) \otimes 1_{\mathbb{K}\Delta}. \tag{23}$$

If $f \in C''$ then $f \notin C'$ and therefore $g^{-1}f \notin g^{-1}C$. Let $x_{g^{-1}f} \in X$ be the representative of $g^{-1}f$. Since $g^{-1}f \notin g^{-1}C$, by Remark 5.1 we also have that $x_{g^{-1}f} \notin g^{-1}C$, which implies that $x_{g^{-1}f} \in (g^{-1}C)''$. As a consequence of Lemma 5.6,

$$(1 - e_{g^{-1}f})e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta} = (1 - e_{x_{g^{-1}f}})e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta}.$$

Hence each factor $1 - e_{g^{-1}f}$, where $f \in C''$, occurring in the RHS of (23) may be replaced by a factor $1 - e_t$, with $t \in (g^{-1}C)''$, in the LHS of (23).

Pick a factor $1 - e_f$ of the LHS of (23), where $f \in (g^{-1}C)''$,

$$f \in (g^{-1}C)'' \implies f \in X \backslash g^{-1}C \implies f \notin g^{-1}C \implies gf \notin C.$$

Let $x_{gf} \in X$ be the representative of gf. Since $gf \notin C$, once more Remark 5.1 implies that $x_{gf} \notin C$ and therefore $x_{gf} \in C''$. Using again Lemma 5.6, we have that

$$(1 - e_{g^{-1}x_{gf}})e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta} = (1 - e_{g^{-1}x_{gf}})[g^{-1}][g] \otimes 1_{\mathbb{K}\Delta} =$$
$$= [g^{-1}](1 - e_{x_{gf}})[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}](1 - e_{gf})[g] \otimes 1_{\mathbb{K}\Delta}$$
$$= (1 - e_f)e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta}.$$

Therefore each term $(1 - e_f)$, where $f \in (g^{-1}C)''$, occurring in the LHS of (23) may be replaced by a corresponding term $(1 - e_{g^{-1}x})$, where $x \in C''$, in the RHS of (23), and the claim follows.

Claims (22) and (23) above imply that if $g \in C$, then

$$e_{g^{-1}} \prod_{t \in (g^{-1}C)'} e_t \prod_{f \in (g^{-1}C)''} (1 - e_f) \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}} \prod_{t \in C'} e_{g^{-1}t} \prod_{f \in C''} (1 - e_{g^{-1}f}) \otimes 1_{\mathbb{K}\Delta}$$

and therefore the equality (*) = (**) holds when $g \in C$.

When $g \notin C$ we have (*) = 0; we shall see that in this case (**) = 0 as well. Assuming $g \notin C$, choose $x_g \in X$ such that $x_g \sim g$. In this case $x_g \in C''$ and, by Lemma 5.6,

$$e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_g[g] \otimes 1_{\mathbb{K}\Delta} = [g^{-1}]e_{x_g}[g] \otimes 1_{\mathbb{K}\Delta} = e_{g^{-1}x_g}e_{g^{-1}} \otimes 1_{\mathbb{K}\Delta}$$

and since the term $1 - e_{g^{-1}x_g}$ appears in (**) it follows that (**)=0.

This proves the equality $\psi((C, 1) \lhd \lambda_\Delta(g)) = \psi(C, 1)\lambda_\Delta(g)$ for all $(C, 1) \in \lambda_\Delta(B)$ and $g \in G$. Since by Proposition 2.2. of [22] the algebra $\mathbb{K}\Delta$ is generated by the elements $\lambda_\Delta(g)$, with $g \in G$, we conclude that ψ is a map of right $\mathbb{K}\Delta$-modules. □

5.3. The equality $\psi \circ \varphi = Id$. Conclusion of the proof of Theorem 4.5.

PROPOSITION 5.10 $\bar{\pi} : \lambda_\Delta(B) \to B$ *is a multiplicative map.*

Proof.

$$\bar{\pi}((C, 1)(D, 1)) = [C = D]\bar{\pi}(C, 1) = [C = D] \prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f),$$

and

$$\bar{\pi}(C, 1)\bar{\pi}(D, 1) = \prod_{t_1 \in C'} e_{t_1} \prod_{f_1 \in C''} (1 - e_{f_1}) \prod_{t_2 \in D'} e_{t_2} \prod_{f_2 \in D''} (1 - e_{f_2}).$$

Clearly, if $C = D$ then $\bar{\pi}(C, 1)^2 = \bar{\pi}((C, 1)^2)$. Assume that $C \neq D$. Then either $C \backslash D \neq \emptyset$ or $D \backslash C \neq \emptyset$. If there is $\alpha \in C \backslash D$ then $\alpha \sim x_\alpha \in D''$; but if $\alpha \in C$ then $x_\alpha \in C'$ and hence $\bar{\pi}(C, 1)\bar{\pi}(D, 1) = 0 = \bar{\pi}((C, 1)(D, 1))$. The other case is analogous, and hence $\bar{\pi}$ preserves products. □

LEMMA 5.11 $\bar{\pi}(1_{\mathbb{K}\Delta}) \otimes 1_{\mathbb{K}\Delta} = 1 \otimes 1_{\mathbb{K}\Delta}$.

Proof. For each $S \subseteq X$, let $P_S \in B$ be the element

$$P_S = \prod_{t \in S} e_t \prod_{f \in X \setminus S} (1 - e_f).$$

CLAIM 5.12 *Let $C \in \mathcal{V}_\Delta$. If $S \neq C'$ then $P_S \otimes (C, 1) = 0$ in $B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta$.*

Proof. Case 1. Assume that there exists $t_0 \in S \setminus C'$ (which is the same as requiring that $t_0 \in S \setminus C$).

$$
\begin{aligned}
P_S \otimes (C, 1) &= \prod_{t \in S} e_t \prod_{f \in X \setminus S} (1 - e_f) \otimes (C, 1) \\
&= e_{t_0} \prod_{t \in S} e_t \prod_{f \in X \setminus S} (1 - e_f) \otimes (C, 1) \\
&= (P_S^{e_{t_0}}) \otimes (C, 1) \\
&= P_S \otimes \lambda_\Delta(e_{t_0})(C, 1) \\
&= P_S \otimes \sum_{D \ni t_0} (D, 1)(C, 1) = 0,
\end{aligned}
$$

since $D \neq C$ for all such D.

Case 2. Suppose that there exists $t_0 \in C' \setminus S$. Then $t_0 \in X \setminus S$, and

$$
\begin{aligned}
P_S \otimes (C, 1) &= P_S (1 - e_{t_0}) \otimes (C, 1) \\
&= (P_S^{(1-e_{t_0})}) \otimes (C, 1) \\
&= P_S \otimes \lambda_\Delta(1 - e_{t_0})(C, 1) \\
&= P_S \otimes \left(1 - \sum_{D \ni t_0} (D, 1) \right) (C, 1) \\
&= P_S \otimes \sum_{E \not\ni t_0} (E, 1)(C, 1) = 0,
\end{aligned}
$$

where the last equality holds because $t_0 \in C' \subseteq C$. □

Note that

$$1 = \sum_{S \subseteq X} P_S \tag{24}$$

since

$$1 = \prod_{t \in X} 1 = \prod_{t \in X} ((1 - e_t) + e_t) = \sum_{S \subseteq X} P_S.$$

It follows that

$$1 \otimes 1_{\mathbb{K}\Delta} = \sum_{D \in \mathcal{V}_\Delta} P_{D'} \otimes 1_{\mathbb{K}\Delta}. \tag{25}$$

Indeed, by Claim 5.12,

$$1 \otimes 1_{\mathbb{K}\Delta} = \sum_{S \subseteq X} P_S \otimes 1_{\mathbb{K}\Delta} = \sum_{S \subseteq X} P_S \otimes \sum_{C \in \mathcal{V}_\Delta} (C, 1) = \sum_{C \in \mathcal{V}_\Delta} P_{C'} \otimes (C, 1)$$

$$= \sum_{D \in \mathcal{V}_\Delta} P_{D'} \otimes \sum_{C \in \mathcal{V}_\Delta} (C, 1) = \sum_{D \in \mathcal{V}_\Delta} P_{D'} \otimes 1_{\mathbb{K}\Delta}.$$

Finally

$$\tilde{\pi}(1_{\mathbb{K}\Delta}) \otimes 1_{\mathbb{K}\Delta} = \tilde{\pi}\left(\sum_{D \in \mathcal{V}_\Delta} (D, 1) \right) \otimes 1_{\mathbb{K}\Delta} = \sum_{D \in \mathcal{V}_\Delta} P_{D'} \otimes 1_{\mathbb{K}\Delta} = 1 \otimes 1_{\mathbb{K}\Delta}. \qquad \square$$

LEMMA 5.13 $\psi \circ \varphi \big|_{B \otimes 1_{\mathbb{K}\Delta}} = Id_{B \otimes 1_{\mathbb{K}\Delta}}.$

Proof. Let $g \in G$.

$$\psi \circ \varphi(e_g \otimes 1_{\mathbb{K}\Delta}) = \psi(\lambda_\Delta(e_g) \lhd 1_{\mathbb{K}\Delta}) = \psi(\lambda_\Delta(e_g)) = \tilde{\pi}\left(\lambda_\Delta(e_g) \right) \otimes 1_{\mathbb{K}\Delta}$$

$$= \tilde{\pi}\left(\sum_{g \in C} (C, 1) \right) \otimes 1_{\mathbb{K}\Delta} = \sum_{g \in C} \prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f) \otimes 1_{\mathbb{K}\Delta}.$$

Let C be any vertex of Δ containing g and let $x_g \in C'$ be such that $x_g \sim g$. From Lemma 5.6 it follows that $e_{x_g} \otimes 1_{\mathbb{K}\Delta} = e_{x_g} e_g \otimes 1_{\mathbb{K}\Delta}$ and therefore

$$\psi \circ \varphi(e_g \otimes 1_{\mathbb{K}\Delta}) = e_g \sum_{g \in C} \prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f) \otimes 1_{\mathbb{K}\Delta} = e_g (\psi \circ \varphi)(e_g \otimes 1_{\mathbb{K}\Delta}).$$

Let $D \in \mathcal{V}_\Delta$ be such that $g \notin D$. Then $g \sim x_g \in D''$ and it follows from Lemma 5.6 that $(1 - e_{x_g}) \otimes 1_{\mathbb{K}\Delta} = (1 - e_g) \otimes 1_{\mathbb{K}\Delta}$.

Hence,

$$(\psi \circ \varphi)(e_g \otimes 1_{\mathbb{K}\Delta}) = e_g \sum_{C \ni g} \prod_{t \in C'} e_t \prod_{f \in C''} (1 - e_f) \otimes 1_{\mathbb{K}\Delta} =$$

$$= e_g \left(\sum_{C \ni g} \prod_{t_1 \in C'} e_{t_1} \prod_{f_1 \in C''} (1 - e_{f_1}) + (1 - e_g) \sum_{D \not\ni g} \prod_{t_2 \in D'} e_{t_2} \prod_{f_2 \in D''} (1 - e_{f_2}) \right) \otimes 1_{\mathbb{K}\Delta}$$

$$= e_g \left(\sum_{C \ni g} \prod_{t_1 \in C'} e_{t_1} \prod_{f_1 \in C''} (1 - e_{f_1}) + \sum_{D \not\ni g} \prod_{t_2 \in D'} e_{t_2} \prod_{f_2 \in D''} (1 - e_{x_g})(1 - e_{f_2}) \right) \otimes 1_{\mathbb{K}\Delta}$$

$$= e_g \left(\sum_{C \ni g} \prod_{t_1 \in C'} e_{t_1} \prod_{f_1 \in C''} (1 - e_{f_1}) + \sum_{D \not\ni g} \prod_{t_2 \in D'} e_{t_2} \prod_{f_2 \in D''} (1 - e_{f_2}) \right) \otimes 1_{\mathbb{K}\Delta}$$

$$= e_g \left(\sum_{C \ni g} \tilde{\pi}(C, 1) + \sum_{D \not\ni g} \tilde{\pi}(D, 1) \right) \otimes 1_{\mathbb{K}\Delta} = e_g \left(\sum_{C \in \mathcal{V}_\Delta} \tilde{\pi}(C, 1) \right) \otimes 1_{\mathbb{K}\Delta}$$

$$= e_g \tilde{\pi}(1_{\mathbb{K}\Delta}) \otimes 1_{\mathbb{K}\Delta} \overset{Lemma\ 5.11}{=} e_g 1 \otimes 1_{\mathbb{K}\Delta} = e_g \otimes 1_{\mathbb{K}\Delta}. \qquad \square$$

To complete the proof of Theorem 4.5, we show that $\varphi : B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta \to \lambda_\Delta(B)$ is an isomorphism of right $\mathbb{K}\Delta$-modules with inverse $\varphi^{-1} = \psi$.

Indeed, given $b \otimes v \in B \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta$,

$$(\psi \circ \varphi)(b \otimes v) = \psi(\varphi(b \otimes 1_{\mathbb{K}\Delta})v) = ((\psi \circ \varphi)(b \otimes 1_{\mathbb{K}\Delta}))v$$

by Proposition 5.7, and Lemma 5.13 yields

$$((\psi \circ \varphi)(b \otimes 1_{\mathbb{K}\Delta}))v = (b \otimes 1_{\mathbb{K}\Delta})v = b \otimes v.$$

On the other hand, given $(D, 1) \in \lambda_\Delta(B)$, by Corollary we have that

$$(\varphi \circ \psi)(D, 1) = (D, 1),$$

and Theorem 4.5 is proved.

6. PROJECTIVE AND FLAT $\mathbb{K}_{par}G$-MODULES

In this section, whenever we make calculations in $\mathbb{K}\Delta$, all sums are over subsets of G that are vertices in the connected component Δ, that is, subsets that belong to $\mathcal{V}_\Delta$.

When G is a finite group, the partial group algebra $\mathbb{K}_{par}G$ is isomorphic to the algebra of the groupoid $\Gamma(G)$. This finite groupoid is the disjoint union of its connected components Δ, and therefore $\mathbb{K}_{par}G \simeq \mathbb{K}\Gamma(G) = \oplus_\Delta \mathbb{K}\Delta$ as an algebra, where the sum runs over the components Δ of $\Gamma(G)$, and each $\mathbb{K}\Delta$ is a two-sided ideal of $\mathbb{K}_{par}G$ which is generated by a central idempotent. It then follows that each $\mathbb{K}\Delta$ is a projective $\mathbb{K}_{par}G$-module when G is a finite group.

The isomorphism $\mathbb{K}_{par}G \simeq \mathbb{K}\Gamma(G)$ does not hold anymore for an infinite group, and we cannot apply the same reasoning for the $\mathbb{K}_{par}G$-modules $\mathbb{K}\Delta$. In fact, as we show in Section 6.1, when G is infinite there are certain components Δ such that $\mathbb{K}\Delta$ is not a projective $\mathbb{K}_{par}G$-module. However, we also show in Section 6.2 that each algebra $\mathbb{K}\Delta$ is still a *flat* $\mathbb{K}_{par}G$-module.

6.1. Non-projective $\mathbb{K}_{par}G$-modules

PROPOSITION 6.1 *Suppose that Δ is a connected component of the groupoid $\Gamma(G)$ with finitely many vertices, A is a vertex of Δ and $G \backslash A$ is infinite. Then $\mathbb{K}\Delta$ is not a projective left $\mathbb{K}_{par}G$-module.*

Proof. Recall that we view $\mathbb{K}\Delta$ as $\mathbb{K}_{par}G$-module via the map $\lambda_\Delta : \mathbb{K}_{par}G \to \mathbb{K}\Delta$. Note that by [22, Prop. 2.2], $\mathbb{K}\Delta$ is generated as a $\mathbb{K}$-algebra by $\{\lambda_\Delta([g]) \mid g \in G\}$ and since λ_Δ is a homomorphism of $\mathbb{K}$-algebras we deduce that λ_Δ is surjective.

Suppose that $\mathbb{K}\Delta$ is projective as a $\mathbb{K}_{par}G$-module. Then there is a homomorphism of $\mathbb{K}_{par}G$-modules $f : \mathbb{K}\Delta \to \mathbb{K}_{par}G$ such that $\lambda_\Delta \circ f = id_{\mathbb{K}\Delta}$. Then for $g^{-1} \in A$ we have

$$[g]f((A, 1)) = f(\lambda_\Delta([g])(A, 1)) = f(\sum_{g^{-1} \in C, C \in \mathcal{V}_\Delta} (C, g)(A, 1)) = f((A, g)),$$

and for $h \in G, g^{-1}h^{-1} \notin A, g^{-1} \in A$ we have

$$[h][g]f((A, 1)) = [h]f((A, g)) = f(\lambda_\Delta([h])(A, g)) = f(\sum_{h^{-1} \in C, C \in \mathcal{V}_\Delta} (C, h)(A, g)) = f(0) = 0.$$

Suppose $f((A, 1)) = \sum_{s \in S}[s]\mu_s$, where $\mu_s \in B$. Then for $g = 1_G, h^{-1} \in G \backslash A$ we have

$$0 = [h]f((A, 1)) = \sum_{s \in S}[h][s]\mu_s = \sum_{s \in S}[hs]e_{s^{-1}}\mu_s,$$

hence $[hs]e_{s^{-1}}\mu_s = 0$ for every $s \in S$. Thus $e_{s^{-1}}\mu_s \in \{b \in B \mid [hs]b = 0\} = (1 - e_{(hs)^{-1}})B$, hence

$$e_{s^{-1}}\mu_s \in \bigcap_{h^{-1} \in G \backslash A} (1 - e_{(hs)^{-1}})B = 0.$$

The last equality comes from the fact that $G \backslash A$ is infinite. Then $[s]\mu_s = [s]e_{s^{-1}}\mu_s = [s] \cdot 0 = 0$ and for $g_0^{-1} \in A$,

$$f((A, g_0)) = [g_0]f((A, 1)) = [g_0]\sum_{s \in S}[s]\mu_s = 0.$$

The same argument shows that for $C \in \mathcal{V}_\Delta, g^{-1} \in C$ we have $f((C, g)) = 0$, as $G \backslash C$ is also infinite. Therefore, f is the zero map. Finally $id_{\mathbb{K}\Delta} = \lambda_\Delta \circ f = 0$, a contradiction. $\qquad\square$

PROPOSITION 6.2. *Suppose that G is an infinite group and consider the connected component Δ with unique vertex G. Then $\mathbb{K}\Delta$ is not projective as a left $\mathbb{K}_{par}G$-module.*

Proof. First note that $\mathbb{K}\Delta \simeq M_1(\mathbb{K}G) \simeq \mathbb{K}G$; henceforth we will identify the algebras $\mathbb{K}\Delta$ and $\mathbb{K}G$. Suppose that $\mathbb{K}G$ is projective as a left $\mathbb{K}_{par}G$-module. Then there is a homomorphism of left $\mathbb{K}_{par}G$-modules

$$\theta : \mathbb{K}G \to \mathbb{K}_{par}G \text{ such that } \lambda_\Delta \circ \theta = id_{\mathbb{K}G}.$$

Note that

$$\lambda_\Delta([g]) = g \text{ and } \lambda_\Delta(e_g) = 1.$$

Then

$$\theta(g) = (1 + w_{g,g})[g] + \sum_{g_1 \in G\backslash\{g\}} w_{g,g_1}[g_1],$$

where each $w_{g,g_1} \in \Omega = \mathrm{Ker}(\lambda_\Delta) \cap B$. Then since θ is a homomorphism of left $\mathbb{K}_{par}G$-modules

$$[t]\theta(g) = \theta(tg) \text{ for } t, g \in G,$$

hence

$$[t](1 + w_{g,g})[g] + \sum_{g_1 \in G\backslash\{g\}}[t]w_{g,g_1}[g_1] = (1 + w_{tg,tg})[tg] + \sum_{g_2 \in G\backslash\{tg\}} w_{tg,g_2}[g_2]. \qquad (26)$$

This together with the rules of multiplication in $\mathbb{K}_{par}G = \oplus_{g \in G}B[g]$:

$$[g]e_h = e_{gh}[g] \text{ and } [g][h] = e_g[gh] \text{ for } g, h \in G,$$

imply that

$$[t](1 + w_{g,g})[g] = (1 + w_{tg,tg})[tg] \text{ and } [t]w_{g,g_1}[g_1] = w_{tg,tg_1}[tg_1] \text{ for } g, t \in G, g_1 \in G\backslash\{g\}.$$

Note that $[t](1 + w_{g,g}) \in [t]B = B[t]$, hence $(1 + w_{tg,tg})[tg] = [t](1 + w_{g,g})[g] \in B[t][g] = Be_t[tg]$. Therefore,

$$1 + w_{tg,tg} \in Be_t + B(1 - e_{tg}) \text{ for every } t, g \in G.$$

Fix $tg = g_0$, thus

$$1 + w_{g_0,g_0} \in \cap_{h \in G}(Be_h + B(1 - e_{g_0})). \qquad (27)$$

CLAIM 6.3. *If G is infinite we have $\cap_{h \in G}(Be_h + B(1 - e_{g_0})) = B(1 - e_{g_0})$.*

Note that the Claim together with (27) implies that $1 + w_{g_0,g_0} \in B(1 - e_{g_0}) \subseteq \Omega$, that together with $w_{g_0,g_0} \in \Omega$ gives $1 \in \Omega$, a contradiction.

Proof of Claim 6.3 Note that $\mathbb{K}_{par} G$ is isomorphic to $\mathbb{K} S(G)$, where $S(G)$ is Exel's inverse semigroup. Every element of $S(G)$ has a canonical form $e_{g_1} \ldots e_{g_s}[g]$, that is unique up to permutation of the factors $e_{g_1}, \ldots, e_{g_s}$. Thus the elements of B are $\mathbb{K}$-linear combinations of $e_{g_1} \ldots e_{g_s}$ and $e_1 = 1$, where $g_1, \ldots, g_s \in G \backslash \{1\}$ are pairwise distinct, and this decomposition is unique (up to permutation of the factors in each product). Thus one such $\mathbb{K}$-linear combination is in $Be_h e_{g_0}$ for $h \neq g_0$ precisely when two of $e_{g_1}, \ldots, e_{g_s}$ are e_{g_0} and e_h for every summand of the $\mathbb{K}$-linear combination. This together with the fact that G is infinite implies that $\cap_{h \in G} Be_h e_{g_0} = 0$.

Suppose that $\lambda \in \cap_{h \in G}(Be_h + B(1 - e_{g_0}))$. Then $\lambda e_{g_0} \in \cap_{h \in G} Be_h e_{g_0} = 0$, hence $\lambda \in B(1 - e_{g_0})$. □

6.2. Flat $\mathbb{K}_{par} G$-modules

LEMMA 6.4. *Let T be a subset of $G \backslash \{1\}$, and let B_0 be the unital $\mathbb{K}$-subalgebra of B generated by $\{e_g \mid g \in T\}$. a) Consider $\mathbb{K}$ as B_0-module, where each e_g acts on $\mathbb{K}$ as 1. Then $\mathbb{K}$ is flat as B_0-module. b) Consider $\mathbb{K}$ as B_0-module, where each e_g acts on $\mathbb{K}$ as 0. Then $\mathbb{K}$ is flat as B_0-module.*

Proof. Note that there is a $\mathbb{K}$-automorphism of B_0 that sends e_g to $1 - e_g$ for each $g \in T$. Note that in case b) the idempotent $1 - e_g$ acts as 1, thus we can apply case a).

Now consider case a). We will show that

$$Tor_1^{B_0}(-, \mathbb{K}) = 0,$$

which is equivalent to the fact that $\mathbb{K}$ is flat as a left B_0-module.

Let $g_1, \ldots, g_k$ be pair-wise different elements of T. Then by induction on k we have

$$B_0 = (\oplus_{0 \leq i \leq k-1} e_{g_1} \ldots e_{g_i}(1 - e_{g_{i+1}})B_0) \oplus e_{g_1} \ldots e_{g_k} B_0, \tag{28}$$

where for $i = 0$ the element $e_{g_1} \ldots e_{g_i}$ is 1. Set

$$M_{\{g_1, \ldots, g_k\}} = \sum_{1 \leq i \leq k}(1 - e_{g_i})B_0 \text{ and } W_{\{g_1, \ldots, g_k\}} = \oplus_{0 \leq i \leq k-1} e_{g_1} \ldots e_{g_i}(1 - e_{g_{i+1}})B_0.$$

We claim that $M_{\{g_1, \ldots, g_k\}} = W_{\{g_1, \ldots, g_k\}}$, that is,

$$M_{\{g_1, \ldots, g_k\}} = \oplus_{0 \leq i \leq k-1} e_{g_1} \ldots e_{g_i}(1 - e_{g_{i+1}})B_0. \tag{29}$$

Indeed, $W_{\{g_1, \ldots, g_k\}} \subseteq M_{\{g_1, \ldots, g_k\}}$, so by (28),

$$M_{\{g_1, \ldots, g_k\}} = W_{\{g_1, \ldots, g_k\}} \oplus (M_{\{g_1, \ldots, g_k\}} \cap e_{g_1} \ldots e_{g_k} B_0).$$

Note that multiplication with $e_{g_1} \ldots e_{g_k}$ acts on $e_{g_1} \ldots e_{g_k} B_0$ as the identity map and acts on $M_{\{g_1, \ldots, g_k\}}$ as the zero map, hence $M_{\{g_1, \ldots, g_k\}} \cap e_{g_1} \ldots e_{g_k} B_0 = 0$ and (29) holds. By (28) and (29) $M_{\{g_1, \ldots, g_k\}}$ is a B_0-module direct summand of B_0, hence $M_{\{g_1, \ldots, g_k\}}$ is a projective B_0-module, and thus it is a flat B_0-module.

Let Ω be the ideal of B_0 generated by $\{e_g - 1 \mid g \in T\}$, that is, the B_0-submodule generated by $\{e_g - 1 \mid g \in T\}$. Note that Ω is the union of the submodules $M_{\{g_1, \ldots, g_k\}}$, where $\{g_1, \ldots, g_k\} \subseteq T$. Thus Ω is the direct limit of $M_{\{g_1, \ldots, g_k\}}$, where the direct system is defined by

the inclusion maps $M_{N_1} \to M_{N_2}$ for finite subsets $N_1 \subseteq N_2$ of T. The direct limit of flat B_0-modules over a directed index set is a flat B-module [**45**, Prop. 5.34], hence

$$\Omega \text{ is a flat } B_0\text{-module.}$$

In order to prove that $\mathbb{K}$ is a flat left B_0-module we need to show that

$$Tor_1^{B_0}(N, \mathbb{K}) = 0 \qquad (30)$$

for any right B_0-module N. Note that N is the direct limit (in this case the union) of its finitely generated B_0-submodules and the functor $Tor_i^{B_0}(\ , \mathbb{K})$ commutes with direct limits for every $i \geq 1$ since tensor product commutes with direct limits (see [**45**, Prop. 7.8]). Hence it suffices to prove (30) with N being a finitely generated B_0-module.

Consider the short exact sequence

$$0 \to \Omega \to B_0 \to \mathbb{K} \to 0$$

of B_0-modules, where the map $B_0 \to \mathbb{K}$ is given by $e_g \mapsto 1, g \in G$. The corresponding long exact sequence in homology for N is

$$\ldots \to Tor_1^{B_0}(N, \Omega) \to Tor_1^{B_0}(N, B_0) \to Tor_1^{B_0}(N, \mathbb{K}) \to N \otimes_{B_0} \Omega \to N \otimes_{B_0} B_0 \to N \otimes_{B_0} \mathbb{K} \to 0.$$

Note that B_0 is a free B_0-module, hence it is a projective B_0-module and $Tor_1^{B_0}(N, B_0) = 0$. Consequently, we obtain the exact sequence

$$0 \to Tor_1^{B_0}(N, \mathbb{K}) \to N \otimes_{B_0} \Omega \to N \otimes_{B_0} B_0 \to N \otimes_{B_0} \mathbb{K} \to 0,$$

and thus (30) is equivalent to

$$N \otimes_{B_0} \Omega \to N \otimes_{B_0} B_0 \simeq N \text{ is an injective map} \qquad (31)$$

for any finitely generated B_0-module N.

CLAIM 6.5 *If (31) holds for B_0-modules N_1 and N_2 and $0 \to N_1 \to N \to N_2 \to 0$ is a short exact sequence of B_0-modules, then (31) holds for N.*

Indeed, the proof is a diagram chasing similar to the proof of the 3x3-Lemma (see [**45**, Exer. 2.32]). Consider the commutative diagram

$$
\begin{array}{ccccccc}
N_1 \otimes_{B_0} \Omega & \xrightarrow{\alpha_1} & N_1 \otimes_{B_0} B_0 & \longrightarrow & N_1 \otimes_{B_0} \mathbb{K} & \longrightarrow & 0 \\
\downarrow \beta & & \downarrow \delta & & \downarrow & & \\
N \otimes_{B_0} \Omega & \xrightarrow{\alpha} & N \otimes_{B_0} B_0 & \longrightarrow & N \otimes_{B_0} \mathbb{K} & \longrightarrow & 0 \\
\downarrow \gamma & & \downarrow & & \downarrow & & \\
N_2 \otimes_{B_0} \Omega & \xrightarrow{\alpha_2} & N_2 \otimes_{B_0} B_0 & \longrightarrow & N_2 \otimes_{B_0} \mathbb{K} & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Observe that $Im(\beta) = \text{Ker}(\gamma)$, as the B_0-module Ω is flat. Then since α_1, α_2 and δ are injective, a simple diagram chasing implies that α is injective. This completes the proof of the Claim.

Using induction on the number of generators, Claim 6.5 implies that (31) holds for every finitely generated B_0-module precisely when (31) holds for every cyclic B_0-module $N = B_0/I$. Consider the short exact sequence $0 \to I \to B_0 \to B_0/I \to 0$ of B_0-modules and the corresponding long exact sequence in homology

$$\ldots \to 0 = Tor_1^{B_0}(B_0,\, \mathbb{K}) \to Tor_1^{B_0}(B_0/I,\, \mathbb{K}) \to I \otimes_{B_0} \mathbb{K} \to$$

$$B_0 \otimes_{B_0} \mathbb{K} \xrightarrow{\partial} (B_0/I) \otimes_{B_0} \mathbb{K} \to 0, \tag{32}$$

where we have used that B_0 is a free B_0-module, hence it is a projective B_0-module and thus $Tor_1^{B_0}(B_0,\, \mathbb{K}) = 0$. Since ∂ is an epimorphism, and $B_0 \otimes_{B_0} \mathbb{K}$ is a $\mathbb{K}$-vector space of dimension 1, we have the following two cases. a) If $(B_0/I) \otimes_{B_0} \mathbb{K} \simeq \mathbb{K}$ then since $B_0 \otimes_{B_0} \mathbb{K} \simeq \mathbb{K}$ and ∂ is an epimorphism we can deduce that ∂ is an isomorphism. Furthermore, for every right B_0-module M we have $M \otimes_{B_0} \mathbb{K} \simeq M \otimes_{B_0} (B_0/\Omega) \simeq M/M\Omega$, hence

$$\mathbb{K} \simeq (B_0/I) \otimes_{B_0} \mathbb{K} \simeq (B_0/I) \otimes_{B_0} (B_0/\Omega) \simeq B_0/(I+\Omega),$$

which implies $B_0 \neq I + \Omega$ that, together with the fact that Ω is a maximal ideal (that is, B_0-submodule of B_0 with $B_0/\Omega \simeq \mathbb{K}$), implies $I \subseteq \Omega$. Note that the long exact sequence (32) together with the fact that ∂ is injective implies

$$Tor_1^{B_0}(B_0/I,\, \mathbb{K}) \simeq I \otimes_{B_0} \mathbb{K} \simeq I \otimes_{B_0} B_0/\Omega \simeq I/I\Omega.$$

This last quotient is the zero module; in fact, if $b \in I \subseteq \Omega$, then $b \in \sum_{1 \leq i \leq k} (1 - e_{g_i})B_0$ for some $g_1, \ldots, g_k \in T$. Hence $be_{g_1} \ldots e_{g_k} = 0$ and $b = b(1 - e_{g_1} \ldots e_{g_k}) \in b\Omega \subseteq I\Omega$, that is, $I \subseteq I\Omega$ and so

$$Tor_1^{B_0}(B_0/I,\, \mathbb{K}) \simeq I/I\Omega = 0.$$

b) If $(B_0/I) \otimes_{B_0} \mathbb{K} = 0$ then, since, as above, $(B_0/I) \otimes_{B_0} \mathbb{K} \simeq B_0/(I+\Omega)$, we deduce that $I \not\subseteq \Omega$ and by the long exact sequence (32) there is a short exact sequence

$$0 \to Tor_1^{B_0}(B_0/I,\, \mathbb{K}) \to I \otimes_{B_0} \mathbb{K} \to \mathbb{K} \to 0.$$

Since $I \otimes_{B_0} \mathbb{K} \simeq I/I\Omega$ we obtain

$$dim_{\mathbb{K}}(I/I\Omega) = dim_{\mathbb{K}}(I \otimes_{B_0} \mathbb{K}) = 1 + dim_{\mathbb{K}} Tor_1^{B_0}(B_0/I,\, \mathbb{K}). \tag{33}$$

On the other hand, $I/(I \cap \Omega) \simeq (I+\Omega)/\Omega = B_0/\Omega \simeq \mathbb{K}$, hence

$$dim_{\mathbb{K}} I/(I \cap \Omega) = 1.$$

Notice that $I \cap \Omega = I\Omega$. Indeed, as in a) an element $b \in I \cap \Omega$ can be written as $b = b(1 - e_{g_1} \ldots e_{g_k}) \in I\Omega$, so that $I \cap \Omega \subseteq I\Omega$, and thus $I \cap \Omega = I\Omega$. It follows that $dim_{\mathbb{K}} I/I\Omega = 1$ and by (33)

$$Tor_1^{B_0}(B_0/I,\, \mathbb{K}) = 0.$$

$\square$

THEOREM 6.6. *Suppose that Δ is a connected component of the groupoid $\Gamma(G)$ with finitely many vertices. Then $\mathbb{K}\Delta$ is flat as a left $\mathbb{K}_{par}G$-module.*

Proof. As the proof is long we split it in several steps.

    Step 1. In this step we prove that

$$\mathrm{Ker}(\lambda_\Delta) = \mathbb{K}_{par}G(\mathrm{Ker}(\lambda_\Delta) \cap B) = (\mathrm{Ker}(\lambda_\Delta) \cap B)\mathbb{K}_{par}G. \qquad (34)$$

Recall that

$$\lambda_\Delta([g]) = \sum_{g^{-1} \in C, C \in \mathcal{V}_\Delta} (C, g)$$

and that $\{(C, 1)\}_{C \in \mathcal{V}_\Delta}$ is a basis of $\lambda_\Delta(B)$ as $\mathbb{K}$-vector space (by Lemma 4.2). Thus

$$\lambda_\Delta(B[g]) = V_g,$$

where V_g is the $\mathbb{K}$-subspace of $\mathbb{K}\Delta$ spanned by $\{(C, g)\}_{C \ni g^{-1}, C \in \mathcal{V}_\Delta}$. Note that

$$V_g \cap \sum_{t \in G, t \neq g} V_t = 0.$$

Suppose $m = \sum_{g \in G} b_g[g] \in \mathrm{Ker}(\lambda_\Delta)$, where $b_g \in B$. Then

$$0 = \sum_{g \in G} \lambda_\Delta(b_g[g]) \in \oplus_{g \in G} V_g,$$

hence

$$\lambda_\Delta(b_g)\lambda_\Delta([g]) = \lambda_\Delta(b_g[g]) = 0 \text{ for every } g \in G.$$

Then if

$$\lambda_\Delta(b_g) = \sum_i k_i(C_i, 1)$$

for some $k_i \in \mathbb{K} \backslash \{0\}$, we have $(C_i, 1)(C, g) = 0$ for every $C \in \mathcal{V}_\Delta$ such that $g^{-1} \in C$, hence $C_i \neq gC$, that is, $g \notin C_i$. Recall that

$$\lambda_\Delta(e_g) = \sum_{g \in C, C \in \mathcal{V}_\Delta} (C, 1) \text{ and } \lambda_\Delta(1 - e_g) = \sum_{g \notin C, C \in \mathcal{V}_\Delta} (C, 1).$$

Thus $\lambda_\Delta(b_g) \in \lambda_\Delta(B(1 - e_g))$, hence

$$b_g \in B(1 - e_g) + (B \cap \mathrm{Ker}(\lambda_\Delta))$$

and

$$b_g[g] \in B(1 - e_g)[g] + (B \cap \mathrm{Ker}(\lambda_\Delta))[g] = (B \cap \mathrm{Ker}(\lambda_\Delta))[g],$$

where we used that $(1 - e_g)[g] = 0$. Note that

$$(B \cap \mathrm{Ker}(\lambda_\Delta))[g] = [g](B \cap \mathrm{Ker}(\lambda_\Delta)) \text{ for } g \in G,$$

hence (34) holds.

Step 2. As a consequence of Step 1, we shall show that there is a natural isomorphism

$$- \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \simeq - \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B).$$

For an arbitrary right $\mathbb{K}_{par}G$-module M we have the well-defined map

$$M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B) \to M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta), \quad m \otimes_B b \mapsto m \otimes_{\mathbb{K}_{par}G} b,$$

which is clearly natural in M. For the inverse map note that $\mathbb{K}_{par}G = \oplus_{g \in G}[g]D_{g^{-1}}$, where $D_{g^{-1}}$ is the ideal of B generated by the element $e_{g^{-1}} = [g^{-1}][g]$. It follows by Step 1 that an arbitrary element $x \in \mathrm{Ker}(\lambda_\Delta)$ can be uniquely written in the form

$$x = \sum_{g \in G}[g]d_{g^{-1}} \in \mathbb{K}_{par}G, \quad (d_{g^{-1}} \in D_{g^{-1}} \cap \mathrm{Ker}(\lambda_\Delta)).$$

Then taking any $m \in M$, set

$$m \otimes_{\mathbb{K}_{par}G} x \mapsto \sum_{g \in G} m[g] \otimes_B d_{g^{-1}}.$$

This is a well-defined map $M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \to M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B)$, because for any $h \in G$, on the one hand we have

$$(m[h], [g]d_{g^{-1}}) \mapsto m[h][g] \otimes_B d_{g^{-1}} = m[h][g]e_{g^{-1}} \otimes_B d_{g^{-1}} =$$

$$m[hg]e_{g^{-1}} \otimes_B d_{g^{-1}} = m[hg] \otimes_B e_{g^{-1}}d_{g^{-1}} = m[hg] \otimes_B d_{g^{-1}},$$

and on the other, $[h][g]d_{g^{-1}} = [h][g]e_{g^{-1}}d_{g^{-1}} = [hg]e_{g^{-1}}d_{g^{-1}} = [hg]d_{g^{-1}} = [hg]e_{(hg)^{-1}}d_{g^{-1}}$, so that

$$(m, [h][g]d_{g^{-1}}) \mapsto m[hg] \otimes_B e_{(hg)^{-1}}d_{g^{-1}} = m[hg]e_{(hg)^{-1}} \otimes_B d_{g^{-1}} = m[hg] \otimes_B d_{g^{-1}}.$$

Clearly, the two maps are inverses to each other.

Step 3. We aim to show that $\lambda_\Delta(B)$ is flat as B-module, where we consider $\lambda_\Delta(B)$ as a left B-module via the restriction map $\lambda_\Delta|_B$.

Let A be a vertex of Δ, let $\mathcal{V}_\Delta = \{g_1 A, \dots, g_n A\}$ (with $g_1 = 1$) and let H be the stabilizer of A. Recall from (8) that there is an isomorphism of algebras

$$\eta: \mathbb{K}\Delta \to M_n(\mathbb{K}H), \ (g_i A, g) \mapsto E_{ji}(g_j^{-1}gg_i),$$

where $gg_i A = g_j A$ (and hence $g_j^{-1}gg_i \in H$). Via this isomorphism we identify $\lambda_\Delta(B)$ with the algebra of diagonal matrices with entries in $\mathbb{K}$. Thus

$$\lambda_\Delta(B) \simeq K_1 \oplus \dots \oplus K_n, \tag{35}$$

where each $K_i \simeq \mathbb{K}$ as a vector space, but we need to specify the B-action on K_i. For $C = g_i A$ the element $(C, 1) \in \mathbb{K}\Delta$ is identified with $E_{ii}(1)$. Then $K_i = E_{ii}(\mathbb{K})$ and $e_g \in B$ acts on K_i via λ_Δ, that is, $\lambda_\Delta(e_g)$ as an element of $M_n(\mathbb{K}H)$ is a diagonal matrix with diagonal $(k_1, \dots, k_n) \in \mathbb{K}^n$ and e_g acts on K_i as multiplication by k_i. Note that $k_i = 1$ if $g \in g_i A$ and $k_i = 0$ if $g \notin g_i A$.

Let B_i be the $\mathbb{K}$-subalgebra of B generated by $\{e_g \mid g \in g_i A\}$ and consider the epimorphism of $\mathbb{K}$-algebras

$$\lambda_i: B \to B_i$$

that sends e_g to e_g for $g \in g_i A$ and sends e_g to 0 if $g \notin g_i A$. Thus B_i acts trivially on K_i (that is, each e_g for $g \in g_i A$ acts as 1) and the B-action on K_i is via the epimorphism λ_i.

By (35) $\lambda_{\Delta}(B)$ is flat as B-module if and only if K_i is flat as B-module for all $1 \leq i \leq n$. Thus it remains to show that $- \otimes_B K_i$ is an exact functor for $1 \leq i \leq n$. Note that

$$- \otimes_B K_i \simeq (- \otimes_B B_i) \otimes_{B_i} \mathbb{K}.$$

It suffices to show that both functors $- \otimes_B B_i$ and $- \otimes_{B_i} \mathbb{K}$ are exact.

Note that for the inclusion $j_i : B_i \to B$ we have that $\lambda_i j_i = id_{B_i}$ but j_i is not a homomorphism of B-modules, hence we cannot claim that B_i is projective as B-module, where the B-action is via λ_i.

Let $\hat{B}_i$ be the unital $\mathbb{K}$-subalgebra of B generated by $\{e_g \mid g \in G \backslash g_i A\}$. Then

$$\xi : B_i \otimes_{\mathbb{K}} \hat{B}_i \to B, \ x \otimes y \mapsto xy$$

is an isomorphism of algebras and of right $\hat{B}_i$-modules. Considering any right B-module also as a right $\hat{B}_i$-module (by restricting the action of B to the subalgebra $\hat{B}_i$) we can show that

$$- \otimes_B B_i \simeq - \otimes_{\hat{B}_i} \mathbb{K}, \tag{36}$$

where we view $\mathbb{K}$ as $\hat{B}_i$-module with e_g acting as 0 for $g \in G \backslash g_i A$. Indeed, $B \otimes_B B_i \simeq B_i$ and

$$B \otimes_{\hat{B}_i} \mathbb{K} \simeq B_i \otimes_{\mathbb{K}} \hat{B}_i \otimes_{\hat{B}_i} \mathbb{K} \simeq B_i \otimes_{\mathbb{K}} \mathbb{K} \simeq B_i \simeq B \otimes_B B_i.$$

Keeping in mind the involved actions on modules, one can directly check that the resulting isomorphism $B \otimes_{\hat{B}_i} \mathbb{K} \to B \otimes_B B_i$ maps $b \otimes_{\hat{B}_i} \alpha$ to $b \otimes_B \alpha$, where $\alpha \in \mathbb{K}$.

Since both tensor functors are right exact and commute with direct sums, using a free presentation of M as a B-module, we conclude that

$$M \otimes_{\hat{B}_i} \mathbb{K} \simeq M \otimes_B B_i, \ \ m \otimes_{\hat{B}_i} \alpha \mapsto m \otimes_B \alpha,$$

for every right B-module M. Clearly this respects homomorphisms of right B-modules (that is, it is natural in M), so that (36) follows.

By Lemma 6.4 both functors $- \otimes_{B_i} \mathbb{K}$ and $- \otimes_{\hat{B}_i} \mathbb{K}$ are exact, which completes the proof of Step 3.

Step 4. In this step we complete the proof by showing that $Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}\Delta) = 0$ for an arbitrary right $\mathbb{K}_{par}G$-module M.

Consider the short exact sequence of left $\mathbb{K}_{par}G$-modules

$$0 \to \text{Ker}(\lambda_{\Delta}) \to \mathbb{K}_{par}G \xrightarrow{\lambda_{\Delta}} \mathbb{K}\Delta \to 0$$

and the associated long exact sequence in homology

$$... \to Tor_1^{\mathbb{K}_{par}G}(M, \text{Ker}(\lambda_{\Delta})) \to Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}_{par}G) \to Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}\Delta) \to$$

$$M \otimes_{\mathbb{K}_{par}G} \text{Ker}(\lambda_{\Delta}) \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}_{par}G \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta \to 0.$$

Since $\mathbb{K}_{par}G$ is a free $\mathbb{K}_{par}G$-module, it is projective and, consequently, $Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}_{par}G) = 0$. Thus we obtain the exact sequence

$$0 \to Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}\Delta) \to M \otimes_{\mathbb{K}_{par}G} \text{Ker}(\lambda_{\Delta}) \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}_{par}G \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}\Delta \to 0.$$

Hence the equality $Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}\Delta) = 0$ is equivalent to the fact of

$$M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}_{par}G \simeq M$$

being an injective map. By Step 2 we have

$$M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \simeq M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B). \tag{37}$$

The short exact sequence of B-modules

$$0 \to \mathrm{Ker}(\lambda_\Delta) \cap B \to B \to \lambda_\Delta(B) \to 0$$

gives rise to the associated long exact sequence in homology

$$\ldots \to 0 = Tor_1^B(M,B) \to Tor_1^B(M, \lambda_\Delta(B)) \to$$

$$M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B) \to M \otimes_B B \to M \otimes_B \lambda_\Delta(B) \to 0.$$

Keeping in mind the proof of Step 2, it is easy to see that the isomorphism (37) maps the kernel of $M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}_{par}G$ onto the kernel of $M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B) \to M \otimes_B B$. Then

$$Tor_1^{\mathbb{K}_{par}G}(M, \mathbb{K}\Delta) \simeq \mathrm{Ker}\left(M \otimes_{\mathbb{K}_{par}G} \mathrm{Ker}(\lambda_\Delta) \to M \otimes_{\mathbb{K}_{par}G} \mathbb{K}_{par}G\right) \simeq$$

$$\mathrm{Ker}\left(M \otimes_B (\mathrm{Ker}(\lambda_\Delta) \cap B) \to M \otimes_B B\right) \simeq Tor_1^B(M, \lambda_\Delta(B)) = 0,$$

where the last equality comes from the fact that by Step 3 $\lambda_\Delta(B)$ is a flat B-module. $\square$

COROLLARY 6.7 **(Corollary 1.3)** For any group G we have $cd_{\mathbb{K}}^{par}(G) \geq cd_{\mathbb{K}}(G)$. In particular, if $\mathrm{char}(\mathbb{K}) = p > 0$ and G has p-torsion then $cd_{\mathbb{K}}^{par}(G) = cd_{\mathbb{K}}(G) = \infty$.

Proof. We give two proofs.

1. Suppose $cd_{\mathbb{K}}^{par}(G) = m < \infty$. Let

$$\mathcal{P}: 0 \to P_m \to \ldots \to P_1 \to P_0 \to B \to 0$$

be a projective resolution of the right $\mathbb{K}_{par}G$-module B. By Theorem 6.6 applied for the connected component Δ of the groupoid $\Gamma(G)$ with unique vertex G, that is, $\mathbb{K}\Delta = \mathbb{K}G$, we have that $- \otimes_{\mathbb{K}_{par}G} \mathbb{K}G$ is an exact functor. Note that $B \otimes_{\mathbb{K}_{par}G} \mathbb{K}G \simeq B \otimes_{\mathbb{K}_{par}G} W \simeq \mathbb{K}$, where W is defined before Theorem 4.4 where G acts trivially on $\mathbb{K}$. The latter isomorphism is a particular case of Proposition 4.6. Then $\mathcal{Q} = \mathcal{P} \otimes_{\mathbb{K}_{par}G} \mathbb{K}G$ is an exact complex and since each P_i is projective as a $\mathbb{K}_{par}G$-module, we have that $P_i \otimes_{\mathbb{K}_{par}G} \mathbb{K}G$ is a projective $\mathbb{K}G$-module. Thus $\mathcal{Q}$ is a projective resolution of length m of the trivial right $\mathbb{K}G$-module $B \otimes_{\mathbb{K}_{par}G} \mathbb{K}G \simeq \mathbb{K}$, that is, $cd_{\mathbb{K}}(G) \leq m$.

2. Suppose

$$n = cd_{\mathbb{K}}(G)$$
$$= max\{n_0 \mid \text{there is a } \mathbb{K}G-\text{module } U \text{ such that } H^{n_0}(G,U) \neq 0\} \leq \infty.$$

Consider the connected component Δ of the groupoid $\Gamma(G)$ with unique vertex G. Then by Theorem 4.13 applied to $W = \mathbb{K}H = \mathbb{K}G$,

$$H_{par}^{n_0}(G, U) \simeq H^{n_0}(G, U) \neq 0,$$

hence $cd_{\mathbb{K}}^{par}(G) \geq n_0$ and so $cd_{\mathbb{K}}^{par}(G) \geq n$.

Finally, if $\mathrm{char}(\mathbb{K}) = p > 0$ and G has p-torsion, let P be a cyclic subgroup of order p. Then $cd_{\mathbb{K}}(G) \geq cd_{\mathbb{K}}(P)$. Note that there is a free resolution $\ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to \mathbb{K} \to 0$ such that each $F_i = \mathbb{K}P$ and the differentials $\partial_i : F_i \to F_{i-1}$ for $i \geq 1$ alternate between the two maps D and N, defined by $D(\lambda) = \lambda(x - 1)$ and $N(\lambda) = \lambda \sum_{0 \leq i \leq p-1} x^i$, where x is a fixed generator of P. Then using this resolution to compute the (usual) cohomology of P with coefficients in the trivial $\mathbb{K}G$-module $\mathbb{K}$, we readily obtain that $H^i(P, \mathbb{K}) = \mathbb{K}$ for all $i \geq 0$, and hence $cd_{\mathbb{K}}(P) = \infty$. □

COROLLARY 6.8 Let G be an infinite group. Then $\mathbb{K}G$ is finitely generated (actually cyclic) but not finitely presented as a $\mathbb{K}_{par}G$-module via λ_Δ, where Δ is the connected component of the groupoid $\Gamma(G)$ with unique vertex G.

Proof. We give two proofs.

1. The first proof is homological. By Proposition 6.2, $\mathbb{K}\Delta = \mathbb{K}G$ is not projective as a $\mathbb{K}_{par}G$-module (via λ_Δ) and by Theorem 6.6, $\mathbb{K}\Delta$ is flat as a $\mathbb{K}_{par}G$-module. By [45, Thm. 3.56] a finitely presented flat module is projective, and hence $\mathbb{K}G$ is not finitely presented as $\mathbb{K}_{par}G$-module.

2. The second proof is ring theoretic and uses the proof of Theorem 6.6. By Step 1 of the proof of Theorem 6.6., $\mathrm{Ker}(\lambda_\Delta)$ is finitely generated as $\mathbb{K}_{par}G$-module if and only if $\mathrm{Ker}(\lambda_\Delta) \cap B$ is finitely generated as B-module. This is equivalent to $\lambda_\Delta(B)$ being finitely presented as B-module (via λ_Δ). If this is the case then by (35) from the proof of Step 3 of Theorem 6.6, K_1 is finitely presented as B-module. Note that in our case, using the notation of Step 3 of Theorem 6.6, $n = 1$, $A = G$ and $B = B_1$, hence $K_1 = \mathbb{K}$ is the trivial B-module, that is, each e_g acts as 1. Then Ω the ideal of B generated by by all elements $e_g - 1$, $g \in G$, is finitely generated as a B-module, which contradicts the fact of G being infinite. □

7. AUXILIARY FACTS ON IDEMPOTENTS AND CANCELLATION

In this section R is a unital associative ring.

REMARK 7.1 Suppose that $r_1 e_1 + \ldots + r_k e_k = 0$, where $e_1, \ldots, e_k$ are idempotents that commute with each other. Then for each $i = 1, 2, \ldots, k$ we have that

$$r_i \in Re_1 + \cdots Re_{i-1} + Re_{i+1} + \cdots + Re_k + R(1 - e_i).$$

In fact,

$$r_i = r_i e_i + r_i(1 - e_i) = \sum_{j \neq i} -r_j e_j + r_i(1 - e_i), \ \ 1 \leq i \leq k.$$

These equalities may be arranged in a matrix form: if v_1 is the column vector $(e_1, \ldots, e_k)^t$ and b is the column vector $(r_1(1 - e_1), \ldots, r_k(1 - e_k))^t$, then

$$(r_1, \ldots, r_k)^t = N v_1 + b, \tag{38}$$

where $N = (n_{i,j})$ is the $k \times k$ matrix which has entries $n_{i,j} = -r_j$ if $i \neq j$ and $n_{i,i} = 0$.

The next result shows that there is an equality analogous to (38) involving a skew-symmetric matrix M. It will be needed in the next section for studying the left augmentation ideal IG when $G = \mathbb{Z}$.

LEMMA 7.2 *Suppose that $r_1 e_1 + \ldots + r_k e_k = 0$, where $e_1, \ldots, e_k$ are idempotents that commute with each other. Let v_1 be the column vector $(e_1, \ldots, e_k)^t$. Then there are elements $b_1, \ldots, b_k \in R$ and a skew-symmetric matrix $M \in M_k(R)$ (that is, $M^t = -M$ and all diagonal entries of M are 0) such that*

$$(r_1, \ldots, r_k)^t = M v_1 + b,$$

where b is the matrix column $(b_1(1 - e_1), \ldots, b_k(1 - e_k))^t$ for some $b_1, \ldots, b_k \in R$.

Proof. 1. Suppose first that $k = 2$, i.e., $r_1 e_1 = -r_2 e_2$. Then
$r_1 = r_1 e_1 + r_1(1 - e_1) = -r_2 e_2 + r_1(1 - e_1)$ and we have

$$r_1 = -r e_2 + b_1(1 - e_1),$$

with $r = r_2$ and $b_1 = r_1$. Since $(r_2 - r e_1)e_2 = r_2 e_2 - r_2 e_2 e_1 = 0$ there is $b_2 \in R$ such that

$$r_2 = r e_1 + b_2(1 - e_2).$$

Then we can set

$$M = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix}.$$

2. The general case is obtained by induction on k. Suppose the result holds for $k - 1$. Note that $r_1 e_1(1 - e_k) + \ldots + r_{k-1} e_{k-1}(1 - e_k) = (r_1 e_1 + \ldots + r_{k-1} e_{k-1})(1 - e_k) = -r_k e_k(1 - e_k) = 0$ and that $e_1(1 - e_k), \ldots, e_{k-1}(1 - e_k)$ are idempotents that commute with each other. Then by the inductive hypothesis there is a skew-symmetric matrix $M_0 \in M_{k-1}(R)$ and $\tilde{b}_1, \ldots, \tilde{b}_{k-1} \in R$ such that

$$(r_1, \ldots, r_{k-1})^t =$$

$$M_0(e_1(1 - e_k), \ldots, e_{k-1}(1 - e_k))^t + (\tilde{b}_1(1 - e_1(1 - e_k)), \ldots, \tilde{b}_{k-1}(1 - e_{k-1}(1 - e_k)))^t.$$

Then

$$r_k e_k = -(r_1, \ldots, r_{k-1})(e_1, \ldots, e_{k-1})^t =$$

$$-(M_0(e_1(1 - e_k), \ldots, e_{k-1}(1 - e_k))^t)^t(e_1, \ldots, e_{k-1})^t$$

$$-(\tilde{b}_1(1 - e_1(1 - e_k)), \ldots, \tilde{b}_{k-1}(1 - e_{k-1}(1 - e_k)))(e_1, \ldots, e_{k-1})^t.$$

Then by multiplying the above equality on the right with e_k we deduce that

$$r_k e_k = r_k e_k^2 = -(M_0(e_1(1 - e_k), \ldots, e_{k-1}(1 - e_k))^t)^t(e_1, \ldots, e_{k-1})^t e_k$$

$$-(\tilde{b}_1(1 - e_1(1 - e_k)), \ldots, \tilde{b}_{k-1}(1 - e_{k-1}(1 - e_k)))(e_1, \ldots, e_{k-1})^t e_k =$$

$$-(\tilde{b}_1(1 - e_1(1 - e_k)), \ldots, \tilde{b}_{k-1}(1 - e_{k-1}(1 - e_k)))(e_1, \ldots, e_{k-1})^t e_k =$$

$$-(\tilde{b}_1 e_k, \ldots, \tilde{b}_{k-1} e_k)(e_1, \ldots, e_{k-1})^t = -\tilde{b}_1 e_1 e_k - \ldots - \tilde{b}_{k-1} e_{k-1} e_k,$$

hence $(r_k + \tilde{b}_1 e_1 + \ldots + \tilde{b}_{k-1} e_{k-1}) e_k = 0$ and so $r_k + \tilde{b}_1 e_1 + \ldots + \tilde{b}_{k-1} e_{k-1} \in R(1 - e_k)$. Then there is $b \in R$ such that

$$r_k = -\tilde{b}_1 e_1 - \ldots - \tilde{b}_{k-1} e_{k-1} + b(1 - e_k).$$

Finally we can define $M \in M_k(R)$ in the following way:
the last row is $(-\tilde{b}_1 e_1, \ldots, -\tilde{b}_{k-1} e_{k-1}, 0)$, the last column is $(\tilde{b}_1 e_1, \ldots, \tilde{b}_{k-1} e_{k-1}, 0)^t$ and if we delete from M the last row and the last column we are left with $M_0(1 - e_k)$. Then

$$(r_1, \ldots, r_k)^t = M(e_1, \ldots, e_k)^t + (\tilde{b}_1(1 - e_1), \ldots, \tilde{b}_{k-1}(1 - e_{k-1}), b(1 - e_k))^t$$

and it suffices to set $b_i = \tilde{b}_i$ for $1 \le i \le k - 1$ and $b_k = b$. □

8. THE STRUCTURE OF THE LEFT AUGMENTATION IDEAL *IG* FOR $G = \mathbb{Z}$

8.1. Some remarks on the general case

From now on $R = \mathbb{K}_{par} G$. Recall that R is a $\mathbb{K}$-vector space with a generating set $e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_n} \# g_1 \ldots g_n$, $(n \ge 1)$ and that IG is the kernel of the augmentation map

$$\varepsilon : R \to B,$$

which is given by

$$\varepsilon(e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_n} \# g_1 \ldots g_n) = e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_n}.$$

Thus IG is generated as a $\mathbb{K}$-vector space by $\{f_{g_1, \ldots, g_n}\}_{n \ge 1, g_1, \ldots, g_n \in G}$, where

$$f_{g_1, \ldots, g_n} = e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_n} \# g_1 \ldots g_n - e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_n} \# 1.$$

We have the following product formula:

$$(b_1 \# g_1)(b_2 \# g_2) = b_1 [g_1] b_2 [g_1^{-1}] \# g_1 g_2,$$

where $b_1 \in B e_{g_1}, b_2 \in B e_{g_2}$.

LEMMA 8.1. *IG is a left R-module generated by* $\{f_g\}_{g \in G \setminus 1}$.

Proof. It suffices to apply the relation below many times and use induction on n

$$f_{g_1, \ldots, g_n} = (e_{g_1} \# g_1) f_{g_2, \ldots, g_n} + (e_{g_1 g_2} e_{g_1 g_2 g_3} \ldots e_{g_1 \ldots g_n} \# 1) f_{g_1}.$$

□

Observe that

$$(e_{g_1} \# g_1) f_{g_2} + (e_{g_1 g_2} \# 1) f_{g_1} - (e_{g_1} \# 1) f_{g_1 g_2} = 0 \text{ for all } g_1, g_2 \in G \tag{39}$$

and

$$((e_g - 1) \# 1) f_g = 0 \text{ for all } g \in G. \tag{40}$$

LEMMA 8.2. *a) There is an isomorphism of left B-modules*

$$IG = \oplus_{g \in G \setminus \{1\}} Bf_g \simeq \oplus_{g \in G \setminus \{1\}} Be_g.$$

In particular, IG is a projective left B-module; b) IG is a left R-module via the multiplication in R, moreover,

$$[g_1]f_{g_2} = e_{g_1} f_{g_1 g_2} - e_{g_1 g_2} f_{g_1};$$

c) for every $g \in G$ the set

$$\widetilde{X}_g = \{ e_h f_{hg} - e_{hg} f_h \}_{h \in G}$$

generates Rf_g as a left B-module and $Rf_g = Rf_{g^{-1}}$.

Proof. a), b) Identifying $[g_1]$ with $e_{g_1} \# g_1$ and e_g with $e_g \# 1$, we have by (39) that

$$[g_1]f_{g_2} = e_{g_1} f_{g_1 g_2} - e_{g_1 g_2} f_{g_1}. \tag{41}$$

Hence IG is a left B-module generated by $\{ f_g \}_{g \in G \setminus \{1\}}$ and

$$IG = \sum_{g \in G \setminus \{1\}} Bf_g = \oplus_{g \in G \setminus \{1\}} Bf_g,$$

and we have an isomorphism of left B-modules $Bf_g \simeq B/B(1 - e_g)$, observing that $B(1 - e_g)$ is the annihilator of f_g in B. Thus IG as a B-module is isomorphic to

$$\oplus_{g \in G \setminus \{1\}} B/B(1 - e_g) \simeq \oplus_{g \in G \setminus \{1\}} Be_g,$$

which is a projective B-module, since $B = B(1 - e_g) \oplus Be_g$. c) By (41)

$$Rf_g = Bf_g + \sum_{h \in G \setminus \{1\}} B(e_h f_{hg} - e_{hg} f_h) =$$

$$Bf_g + Bf_{g^{-1}} + \sum_{g_1, g_2 \in G \setminus \{1\}, g_2^{-1} g_1 = g} B(e_{g_2} f_{g_1} - e_{g_1} f_{g_2}) =$$

$$Bf_{g^{-1}} + \sum_{h \in G \setminus \{1\}} B(e_{hg^{-1}} f_h - e_h f_{hg^{-1}}) = Rf_{g^{-1}},$$

where $f_1 = 0$, $e_1 = 1$ and the latter equality is obtained from the first one taking g^{-1} instead of g. Note that $f_g = e_1 f_g - e_g f_1$ and so

$$\widetilde{X}_g = \{ e_h f_{hg} - e_{hg} f_h \}_{h \in G} = \{ f_g \} \cup \{ e_h f_{hg} - e_{hg} f_h \}_{h \in G \setminus \{1\}}$$

generates Rf_g as a left B-module. $\square$

8.2. The case $G = \mathbb{Z}$

Let $G = \mathbb{Z} = \langle g \rangle$. Write f_i for $f_{g^i} = e_{g^i} \# g^i - e_{g^i} \# 1$ and e_i for e_{g^i}. Thus $e_0 = 1$, $f_0 = 0$ and the basic relations (39) and (40) are

$$e_i f_{i+j} = e_{i+j} f_i + (e_i \# g^i) f_j, \text{ and } (e_i - 1) f_i = 0 \text{ for } i, j \in \mathbb{Z}. \tag{42}$$

LEMMA 8.3. *For each $i \in \mathbb{Z}$, $i \neq 0$, the annihilator of f_i in R is $R(1 - e_i)$.*

Proof. The inclusion $ann_R(f_i) \supseteq R(1-e_i)$ obviously follows from (42). For the opposite inclusion take an arbitrary $\lambda \in ann_R(f_i)$ and write

$$\lambda = \sum_{j=-l}^{m} b_j[g^j],$$

with $b_j \in B$ and $m, l \geq 0$. We apply induction on $l+m$. For $l = 0 = m$ we have $\lambda = b \in B$ and $0 = \lambda f_i = b([g^i] - e_i)$ implies $b[g^i] = 0 = be_i$, hence $\lambda = b \in B(1-e_i) \subseteq R(1-e_i)$, as required.

Assume now that $l + m > 0$. Then

$$0 = \sum_{j=-l}^{m} b_j[g^j]f_i = \sum_{j=-l}^{m} b_j[g^j](e_i[g^i] - e_i) = \sum_{j=-l}^{m} b_j[g^j]([g^i] - e_i) = \sum_{j=-l}^{m} b_j e_i[g^{i+j}] - \sum_{j=-l}^{m} b_j e_{i+j}[g^j].$$

Without loss of generality we can assume that $i > 0$. Since $R = \oplus_{t \in G} B[t]$, the above equality implies that $b_m e_m[g^{i+m}] = 0$. Hence $b_m e_m \in \{b \in B \mid b[g^{i+m}] = 0\} = B(1-e_{i+m})$, thus $b_m \in B(1-e_{i+m})$. Then

$$b_m[g^m] \in B(1-e_{i+m})[g^m] = B[g^m](1-e_i) \subseteq R(1-e_i)$$

and

$$\lambda_0 = \lambda - b_m[g^m] = \sum_{j=-l}^{m-1} b_j[g^j] \in ann_R(f_i) = \{r \in R \mid rf_i = 0\}.$$

By induction $\lambda_0 \in R(1-e_i)$, hence $\lambda = \lambda_0 + b_m[g^m] \subseteq R(1-e_i)$. □

Note that

$$IG = \oplus_{i \in \mathbb{Z}\setminus\{0\}} Bf_i \subseteq R = \oplus_{i \in \mathbb{Z}} B(e_i \# g^i),$$

where $Bf_i \simeq B/B(e_i - 1)$ and $Rf_i = Rf_{-i}$. Define for $k \geq 1$

$$V_k = Rf_1 + \dots + Rf_k = Rf_{-1} + \dots + Rf_{-k} \subset IG$$

and hence, using (42), we have

$$Rf_j = \sum_{i \in \mathbb{Z}} B(e_i \# g^i)f_j = Bf_j + \sum_{i \in \mathbb{Z}\setminus\{0\}} B(e_i f_{i+j} - e_{i+j} f_i).$$

Note that V_k has a generating set X_k as a B-module, where

$$X_k = \{\pm(e_i f_{i+j} - e_{i+j} f_i)\}_{1 \leq j \leq k, i \in \mathbb{Z}} \supset \{\pm f_1, \dots, \pm f_k, \pm f_{-1}, \dots, \pm f_{-k}\}.$$

Obviously the above definition will work if we remove the formally unnecessary sign $\pm$ but we prefer to keep it for symmetry.

We have a natural order in $\{f_i\}_{i \in \mathbb{Z}} : f_{i_1} < f_{i_2}$ if $|i_1| < |i_2|$ and $f_{-i} < f_i$ for $i > 0$. For a non-zero element $v = \sum_i b_i f_i$, where $b_i \in B$, we define the *leading term* as $b_{i_0} f_{i_0}$ where f_{i_0} is maximal with $b_{i_0} f_{i_0} \neq 0$ and we call b_{i_0} a *leading coefficient*. Note that any element of $b_{i_0} + B(1 - e_{i_0})$ is a leading coefficient of v since $B(1 - e_{i_0})$ is the annihilator of f_{i_0} in B. We call f_{i_0} the *degree* of v with respect to $\{f_i\}$ and write $f_{i_0} = deg(v)$. The *support* $supp(v)$ of the non-zero element v is the set of all f_i for which $b_i f_i \neq 0$.

LEMMA 8.4. *If $v \in V_k\setminus\{0\}$ has degree f_s with $|s| \geq k+1$ then any leading coefficient of v with respect to $\{f_i\}$ belongs to I, where a) $I = Be_{s-1} + \dots + Be_{s-k} + B(1-e_s)$ if $s > 0$; b) $I = Be_{s+1} + \dots + Be_{s+k} + B(1-e_s)$ if $s < 0$.*

Proof.

CLAIM 8.5. *If $v = \sum_i b_i x_i$, where each $b_i \in B$, $x_i \in X_k$,*

$$f_{s_0} = max\{f_j \mid f_j \in \cup_i supp(x_i)\}$$

and $f_{s_0} \notin supp(v)$, that is, $f_{s_0} > f_s$, then there is a decomposition $v = \sum_i \tilde{b}_i \tilde{x}_i$ with $\tilde{b}_i \in B$, $\tilde{x}_i \in X_k$ and

$$f_{s_1} = max\{f_j \mid f_j \in \cup_i supp(\tilde{x}_i)\} < f_{s_0}.$$

Applying Claim 8.5 several times we obtain

CLAIM 8.6. *If $v \in V_k \backslash \{0\}$ then we can write $v = \sum_i b_i x_i$, where each $b_i \in B$, $x_i \in X_k$ and $max\{f_j \mid f_j \in \cup_i supp(x_i)\} = f_s = deg(v)$.*

Note that Lemma 8.4 easily follows from Claim 8.6. Indeed, if $s > 0$, then the leading term of v is of the form $be_i f_{i+j}$, $b \in B$, $i + j = s$, $1 \leq j \leq k$, hence $e_i \in \{e_{s-k}, \ldots, e_{s-1}\}$. Recalling that the leading coefficient is given modulo $B(1 - e_s)$, we obtain a). Case b) is symmetric.

Proof of Claim 8.5. We arrange the indexes of x_i so that $f_{s_0} \in supp(x_i)$ for $1 \leq i \leq d$ and $f_{s_0} \notin supp(x_i)$ for $i > d$. Thus $x_j = e_{i_j} f_{s_0} - e_{s_0} f_{i_j}$ for $1 \leq j \leq d$ and f_{s_0} is greater than the degree of $\tilde{v}$, where $\tilde{v} = \sum_{d+1 \leq j} b_j x_j$, and

$$v = b_1(e_{i_1} f_{s_0} - e_{s_0} f_{i_1}) + \ldots + b_d(e_{i_d} f_{s_0} - e_{s_0} f_{i_d}) + \tilde{v}.$$

Since $f_{s_0} \notin supp(v)$ we have that $\sum_{1 \leq j \leq d} b_j e_{i_j} f_{s_0} = 0$, that is, $\sum_{1 \leq j \leq d} b_j e_{i_j} \in ann_B(f_{s_0}) = B(1 - e_{s_0})$, which means that for some $\hat{b} \in B$ we have

$$b_1 e_{i_1} + \ldots + b_d e_{i_d} + \hat{b}(1 - e_{s_0}) = 0.$$

Then by Lemma 7.2 there exists some skew-symmetric $(d + 1) \times (d + 1)$-matrix $M = (m_{j,t})$ with coefficients in B and $\bar{b}_1, \ldots, \bar{b}_{d+1} \in B$ such that

$$(b_1, \ldots, b_d, \hat{b})^t = M(e_{i_1}, \ldots, e_{i_d}, 1 - e_{s_0})^t + (\bar{b}_1(1 - e_{i_1}), \ldots, \bar{b}_d(1 - e_{i_k}), \bar{b}_{d+1} e_{s_0})^t. \tag{43}$$

Note that

$$v = \Big(\sum_{1 \leq j \leq d} b_j e_{i_j} \Big) f_{s_0} - e_{s_0} \Big(\sum_{1 \leq j \leq d} b_j f_{i_j} \Big) + \tilde{v} = -e_{s_0} \Big(\sum_{1 \leq j \leq d} b_j f_{i_j} \Big) + \tilde{v}.$$

Then it follows by (43), using $(1 - e_{s_0}) e_{s_0} = 0$, $(1 - e_{i_j}) f_{i_j} = 0$, that

$$e_{s_0} \sum_{1 \leq j \leq d} b_j f_{i_j} = e_{s_0} \sum_{1 \leq j \leq d} \sum_{1 \leq t \leq d} m_{j,t} e_{i_t} f_{i_j}.$$

Since $m_{j,j} = 0$ and $m_{j,t} = -m_{t,j}$ we deduce that

$$e_{s_0} \sum_{1 \leq j \leq d} b_j f_{i_j} = e_{s_0} \sum_{1 \leq j < t \leq d} m_{j,t} (e_{i_t} f_{i_j} - e_{i_j} f_{i_t}),$$

and hence we have a new decomposition

$$v = -e_{s_0} \Big(\sum_{1 \leq j \leq d} b_j f_{i_j} \Big) + \tilde{v} = -e_{s_0} \sum_{1 \leq j < t \leq d} m_{j,t} (e_{i_t} f_{i_j} - e_{i_j} f_{i_t}) + \tilde{v}$$

$$\in \sum_{1 \leq j < t \leq d} B(e_{i_j} f_{i_j} - e_{i_t} f_{i_t}) + \tilde{v}. \tag{44}$$

Note that since $T_k = \{e_{i_r} f_{s_0} - e_{s_0} f_{i_1}, \ldots, e_{i_r} f_{s_0} - e_{s_0} f_{i_d}\} \subseteq X_k$ with f_{s_0} maximal element among the supports of the elements of T_k then either

    1. $s_0 \geq k+1, \{i_1, \ldots, i_d\} \subseteq \{s_0 - k, \ldots, s_0 - 1\}$; or
    2. $s_0 \leq -k-1, \{i_1, \ldots, i_d\} \subseteq \{s_0 + 1, \ldots, s_0 + k\}$.

    In both cases $\{e_{i_j} f_{i_j} - e_{i_t} f_{i_t}\}_{1 \leq j < t \leq d} \subseteq X_k$. This together with the new decomposition (44) completes the proof of Claim 8.5.   □

PROPOSITION 8.7

$$V_{k+1}/V_k \simeq R/(R(1 - e_{k+1}) + \sum_{1 \leq i \leq k} Re_i).$$

Proof. Note that V_{k+1}/V_k is a cyclic R-module, and thus

$$V_{k+1}/V_k \simeq R/J_k,$$

where J_k is the left ideal of R defined by

$$J_k := \{r \in R \mid rf_{k+1} \in V_k\}.$$

To prove the proposition we will show that

$$J_k = R(1 - e_{k+1}) + \sum_{1 \leq i \leq k} Re_i. \tag{45}$$

The easy part is the inclusion $R(1 - e_{k+1}) + \sum_{1 \leq i \leq k} Re_i \subseteq J_k$ that is equivalent to $1 - e_{k+1}, e_1, \ldots, e_k \in J_k$. To see this, note by (40) that $(1 - e_{k+1})f_{k+1} = 0$ and using (41), for $1 \leq i \leq k$,

$$e_i f_{k+1} = [g^i][g^{-i}] f_{k+1} = \tag{46}$$

$$[g^i](e_{-i} f_{k+1-i} - e_{k+1-i} f_{-i}) \in R(e_{-i} f_{k+1-i} - e_{k+1-i} f_{-i}) \subseteq \sum_{-k \leq j \leq k} Rf_j = V_k.$$

Thus to complete the proof of the proposition it remains to be shown that $J_k \subseteq R(1 - e_{k+1}) + \sum_{1 \leq i \leq k} Re_i$.

    Let

$$\lambda \in J_k.$$

By the structure of $R = \mathbb{K}_{par} G$ we have a decomposition

$$\lambda = \sum_i b_i (e_i \# g^i) \in R, \tag{47}$$

where $b_i \in B$. Hence, using again (41),

$$v = \lambda f_{k+1} = \sum_i b_i((e_i \# g^i)f_{k+1}) = \sum_i b_i(e_i f_{k+1+i} - e_{k+1+i} f_i). \tag{48}$$

If $v = 0$ then $\lambda \in ann_R(f_{k+1}) = R(1 - e_{k+1})$ in view of Lemma 8.3.

Suppose now that $v \neq 0$, and let f_s be the degree of v with respect to $\{f_i\}$.
Case 1. Suppose $|s| \leq k$. Then

$$v = \lambda f_{k+1} \in \sum_{-k \leq i \leq k} Bf_i \subseteq V_k. \tag{49}$$

We decompose

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3,$$

where

$$\lambda_1 = \sum_{i < -k} b_i(e_i \# g^i), \quad \lambda_2 = \sum_{-k \leq i \leq -1} b_i(e_i \# g^i) \text{ and } \quad \lambda_3 = \sum_{i \geq 0} b_i(e_i \# g^i).$$

Note that by (48) and $f_0 = 0$ we have

$$v_1 := \lambda_1 f_{k+1} \in \sum_{j < 0} Bf_j, v_2 := \lambda_2 f_{k+1} \in \sum_{-k \leq j \leq k} Bf_j$$

and

$$v_3 := \lambda_3 f_{k+1} \in \sum_{j > 0} Bf_j.$$

Then by (49) and the fact that $v = v_1 + v_2 + v_3$ we deduce that

$$v_1 \in \sum_{-k \leq j < 0} Bf_j \text{ and } v_3 \in \sum_{0 < j \leq k} Bf_j,$$

that is,

$$v_1 = \lambda_1 f_{k+1} = \sum_{i < -k} b_i(e_i f_{k+1+i} - e_{k+1+i} f_i) \in \sum_{-k \leq j < 0} Bf_j$$

and so

$$v_1 = \sum_{t < 0} a_t f_t$$

where $a_t = -b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1}$ for $t < -k$ and $a_t = b_{t-k-1} e_{t-k-1}$ for $-k \leq t < 0$. Then

$$(-b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1})f_t = 0 \text{ for } t < -k. \tag{50}$$

Hence

$$-b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1} \in B(1 - e_t),$$

and by multiplying with e_t we get

$$b_t e_t e_{k+1+t} = b_{t-k-1} e_{t-k-1} e_t \text{ for } t < -k. \tag{51}$$

Since only finitely many b_t can be non-zero we deduce from (51) that

$$b_t e_t e_{k+1+t} = 0 \text{ for } t < -k. \tag{52}$$

Note that (52) is equivalent to

$$b_t e_t \in B(1 - e_{k+1+t}) \text{ for } t < -k. \tag{53}$$

Therefore for $t < -k$

$$b_t e_t \# g^t = b_t e_t e_t \# g^t \in B(1 - e_{k+1+t}) e_t \# g^t = B(1 - e_{k+1+t})[g^t] = B[g^t](1 - e_{k+1}),$$

hence

$$\lambda_1 \in R(1 - e_{k+1}). \tag{54}$$

Similar calculations can be done for λ_3 to show that

$$\lambda_3 \in R(1 - e_{k+1}). \tag{55}$$

For completeness we give the details. Recall that

$$v_3 = \lambda_3 f_{k+1} = \sum_{i \geq 0} b_i(e_i f_{k+1+i} - e_{k+1+i} f_i) \in \sum_{0 < j \leq k} B f_j \tag{56}$$

and so

$$v_3 = \sum_{t > 0} c_t f_t,$$

where $c_t = -b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1}$ for $t > k$ and $c_t = -b_t e_{k+1+t}$ for $0 < t \leq k$. Then by (56)

$$(-b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1}) f_t = 0 \text{ for } t > k. \tag{57}$$

Hence

$$-b_t e_{k+1+t} + b_{t-k-1} e_{t-k-1} \in B(1 - e_t),$$

and by multiplying with e_t we get

$$b_t e_t e_{k+1+t} = b_{t-k-1} e_{t-k-1} e_t \text{for } t > k. \tag{58}$$

Since only finitely many b_t can be non-zero we deduce from (58) that

$$b_t e_t e_{k+1+t} = 0 \text{ for } t \geq 0. \tag{59}$$

Note that (59) is equivalent to

$$b_t e_t \in B(1 - e_{k+1+t}) \text{for } t \geq 0. \tag{60}$$

Then for $t \geq 0$

$$b_t e_t \# g^t = b_t e_t e_t \# g^t \in B(1 - e_{k+1+t}) e_t \# g^t = B(1 - e_{k+1+t})[g^t] = B[g^t](1 - e_{k+1}).$$

This implies (55). Finally since $[g^i] e_{-i} = [g^i]$ we have that

$$\lambda_2 = \sum_{-k \leq i \leq -1} b_i(e_i \# g^i) \in \sum_{-k \leq i \leq -1} B[g^i] = \sum_{-k \leq i \leq -1} B[g^i] e_{-i} \subseteq \sum_{1 \leq j \leq k} R e_j. \tag{61}$$

Then by (54), (55) and (61) we obtain that

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 \in R(1 - e_{k+1}) + \sum_{1 \leq j \leq k} R e_j.$$

Case 2. Suppose $|s| \geq k + 1$, recalling that f_s is the degree of v. We can assume that $s \geq k + 1$, the case $s \leq -k - 1$ is similar.

Recall that

$$\lambda = \sum_i b_i e_i \# g^i = \sum_i b_i [g^i].$$

Denote by c the highest i for which $b_i[g^i] \neq 0$. Then by (48) there are two possibilities:

1) $b_c e_c f_{k+1+c} \neq 0$ and f_{k+1+c} is the degree of $v = \lambda f_{k+1}$, that is, $k+1+c = s$;

2) $b_c e_c f_{k+1+c} = 0$.

If 1) holds the leading term of v is $b_{s-k-1} e_{s-k-1} f_s$ and $b_{s-k-1} e_{s-k-1}$ is a leading coefficient of v, so $b_{s-k-1} e_{s-k-1} \neq 0$. By Lemma 8.4

$$e_{s-k-1} b_{s-k-1} \in Be_{s-k} + \ldots + Be_{s-1} + B(1 - e_s).$$

Then by Remark 7.1

$$b_{s-k-1} \in Be_{s-k} + \ldots + Be_{s-1} + B(1 - e_s) + B(1 - e_{s-k-1})$$

and

$$b_{s-k-1}(e_{s-k-1} \# g^{s-k-1}) \in$$

$$\left(Be_{s-k} + \ldots + Be_{s-1} + B(1 - e_s) + B(1 - e_{s-k-1})\right)(e_{s-k-1} \# g^{s-k-1}) =$$

$$\left(Be_{s-k} + \ldots + Be_{s-1} + B(1 - e_s)\right)(e_{s-k-1} \# g^{s-k-1}) =$$

$$(e_{s-k-1} \# g^{s-k-1})(Be_1 + \ldots + Be_k + B(1 - e_{k+1})). \tag{62}$$

Then since $(1 - e_{k+1})f_{k+1} = 0$ we have

$$(e_{s-k-1} \# g^{s-k-1})(Be_1 + \ldots + Be_k + B(1 - e_{k+1}))f_{k+1} =$$

$$(e_{s-k-1} \# g^{s-k-1})(Be_1 + \ldots + Be_k)f_{k+1} \subseteq V_k, \tag{63}$$

since by (46)

$$(Be_1 + \ldots + Be_k)f_{k+1} \subseteq V_k. \tag{64}$$

Then by (62) and (63) we have

$$b_{s-k-1}(e_{s-k-1} \# g^{s-k-1})f_{k+1} \in V_k. \tag{65}$$

Finally for $\tilde{\lambda} = \lambda - b_{s-k-1} e_{s-k-1} \# g^{s-k-1} = \lambda - b_c e_c \# g^c \in R$ we have by (65)

$$\tilde{\lambda} \cdot f_{k+1} = v - b_{s-k-1}(e_{s-k-1} \# g^{s-k-1}) \cdot f_{k+1} \in V_k$$

and by (47) $supp_G(\tilde{\lambda}) = supp_G(\lambda) \backslash \{b_{s-k-1} e_{s-k-1} \# g^{s-k-1}\}$ has strictly fewer elements than $supp_G(\lambda)$, where we define the support with respect to G as

$$supp_G\left(\sum_i b_i e_i \# g^i\right) = \{b_i e_i \# g^i \mid b_i e_i \# g^i \neq 0\}.$$

Note that $supp_G$ should not be confused with $supp$ used in the proof of Lemma 8.4. Then by induction on the number of elements in $supp_G(\lambda)$, where $supp_G(0)$ has 0 elements, we have

that $supp_G(\tilde{\lambda}) \subseteq Re_1 + \ldots + Re_k + R(1 - e_{k+1})$, hence, in view of (62),

$$supp_G(\lambda) = supp_G(\tilde{\lambda}) \cup \{b_{s-k-1}e_{s-k-1}\#g^{s-k-1}\} \subseteq Re_1 + \ldots + Re_k + R(1 - e_{k+1})$$

and so $\lambda \in Re_1 + \ldots + Re_k + R(1 - e_{k+1})$.

If 2) holds then $b_c e_c \in ann_B(f_{k+1+c}) = B(1 - e_{k+1+c})$ and hence

$$b_c[g^c] = b_c e_c[g^c] \in B(1 - e_{k+1+c})[g^c] = B[g^c](1 - e_{k+1}) \subseteq R(1 - e_{k+1}).$$

Thus for $\tilde{\lambda} = \lambda - b_c e_c \# g^c$ we have

$$\tilde{\lambda}f_{k+1} = \lambda f_{k+1} - b_c e_c \# g^c f_{k+1} = v - b_c[g^c]f_{k+1} \in v - R(1 - e_{k+1})f_{k+1} = v - 0 = v \in V_k,$$

and we can continue as in the last paragraph of 1).

It follows from Case 1 and Case 2 that

$$J_k \subseteq R(1 - e_{k+1}) + \sum_{1 \le j \le k} Re_j,$$

completing the proof. □

LEMMA 8.8. *Let $G \ne 1$ be a group and $R = \mathbb{K}_{par}G$. Then*
a) IG is not a free left R-module,
b) $\mathbb{K}_{par}G$ has the invariant basis number property, that is, $(\mathbb{K}_{par}G)^n \simeq (\mathbb{K}_{par}G)^m$ as left $\mathbb{K}_{par}G$-modules implies that $n = m$.

Proof. a) Consider $\mathbb{K}$ as a right B-module via the epimorphism of $\mathbb{K}$-algebras $B \to \mathbb{K}$ that sends e_g to 0 for every $g \in G\backslash\{1\}$ and sends $e_1 = 1$ to 1. We view R as a left B-module via the multiplication in R. Note that

$$\mathbb{K} \otimes_B \mathbb{K}_{par}G = \mathbb{K} \otimes_B (\oplus_{g \in G} Be_g[g]) \simeq \oplus_{g \in G}(\mathbb{K} \otimes_B Be_g[g]) =$$

$$\oplus_{g \in G}(\mathbb{K}e_g \otimes_B B[g]) = \mathbb{K}e_1 \otimes_B B[1] = \mathbb{K} \otimes_B B = \mathbb{K}.$$

Thus for every non-zero left free R-module F we have $\mathbb{K} \otimes_B F \ne 0$.

We will show that

$$\mathbb{K} \otimes_B IG = 0,$$

which implies that IG is not a free R-module. Recall that by Lemma 8.2 $IG = \oplus_{g \in G\backslash\{1\}} Bf_g$. Then using that $(1 - e_g)f_g = 0$ we have

$$\mathbb{K} \otimes_B IG = \mathbb{K} \otimes_B (\oplus_{g \in G\backslash\{1\}} Bf_g) \simeq \oplus_{g \in G\backslash\{1\}} \mathbb{K} \otimes_B Bf_g =$$

$$\oplus_{g \in G\backslash\{1\}} \mathbb{K} \otimes_B Be_g f_g = \oplus_{g \in G\backslash\{1\}} \mathbb{K}e_g \otimes_B Bf_g = 0.$$

b) Suppose that $(\mathbb{K}_{par}G)^n \simeq (\mathbb{K}_{par}G)^m$ as left $\mathbb{K}_{par}G$-modules and recall that by part a) $\mathbb{K} \otimes_B \mathbb{K}_{par}G \simeq \mathbb{K}$. Then there is an isomorphism of $\mathbb{K}$-vector spaces

$$\mathbb{K}^n \simeq \mathbb{K} \otimes_B (\mathbb{K}_{par}G)^n \simeq \mathbb{K} \otimes_B (\mathbb{K}_{par}G)^m \simeq \mathbb{K}^m,$$

and hence $n = m$. □

THEOREM 8.9. *Let $G = \mathbb{Z}$ and $R = \mathbb{K}_{par}G$. Then $IG = \cup_{k \geq 1}V_k$ such that each V_k is a projective R-module and*

$$IG \simeq \oplus_{k \geq 1}P_k,$$

where $P_k = V_k/V_{k-1}$ is a projective left R-module and $V_0 = 0$. In particular IG is a projective R-module, hence $cd^{par}_{\mathbb{K}}(\mathbb{Z}) = 1$. Furthermore, IG is not a finitely generated R-module and it is not a free left R-module.

Proof. Note that $V_1 = Rf_1 \simeq R/R(1 - e_1)$ and by Proposition 8.7, $V_{k+1}/V_k \simeq R/(Re_1 + \ldots + Re_k + R(1 - e_{k+1}))$. Then by Lemma 2.6, V_1 and each $P_{k+1} = V_{k+1}/V_k$ are projective R-modules, and therefore the short exact sequence of R-modules

$$0 \to V_k \to V_{k+1} \to P_{k+1} \to 0,$$

where $V_0 = 0$, splits, that is, $V_{k+1} \simeq V_k \oplus P_{k+1}$. Thus

$$IG = \cup_{k \geq 1}V_k \simeq \oplus_{k \geq 1}P_k$$

is a projective R-module.

We note that if IG is a finitely generated R-module, say by a set T, then for some V_k we have that $T \subseteq V_k$. Thus $IG = V_k$ and $V_{k+1} = V_k$, a contradiction with $V_{k+1}/V_k \simeq R/(Re_1 + \ldots + Re_k + R(1 - e_{k+1})) \neq 0$. □

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