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A criterion to characterize interacting theories in the Wightman framework

Christian D. Jäkel · **Walter F. Wreszinski**

Abstract We propose that a criterion to characterize interacting theories in a suitable Wightman framework of relativistic quantum field theories which incorporates a "singularity hypothesis", which has been conjectured for a long time, is supported by renormalization group theory, but has never been formulated mathematically. The (nonperturbative) wave function renormalization Z occurring in these theories is shown not to be necessarily equal to zero, except if the equal time commutation relations (ETCR) are assumed. Since the ETCR are not justified in general (because the interacting fields cannot in general be restricted to sharp times, as is known from model studies), the condition $Z = 0$ is not of general validity in interacting theories. We conjecture that it characterizes either unstable (composite) particles or the charge-carrying particles, which become infraparticles in the presence of massless particles. In the case of QED, such "dressed" electrons are not expected to be confined, but in QCD, we propose a quark confinement criterion, which follows naturally from lines suggested by the works of Casher, Kogut and Susskind, and Lowenstein and Swieca.

1 Introduction

In his recent recollections, 'tHooft ([30], Sect. 5) emphasizes that an asymptotic (divergent) series, such as the power series for the scattering (S) matrix in the coupling constant $\alpha = \frac{1}{137}$ in quantum electrodynamics (qed_{1+3}) does not define a theory rigorously (or mathematically). In the same token, Feynman was worried about whether the qed S matrix would be "unitary at order 137" ([69], discussion on p. 126), and in Section 5 of [30], 'tHooft remarks that the uncertainties in the S matrix amplitudes at order 137 for qed are comparable to those associated with the "Landau ghost" (or pole) [38].

C. D. Jäkel
Department de Matemática Aplicada, Universidade de São Paulo (USP), São Paulo, Brazil
e-mail: jaekel@ime.usp.br

W. F. Wreszinski (✉)
Instituto de Fisica, Universidade de São Paulo (USP), São Paulo, Brazil
e-mail: wreszins@gmail.com

The main reason to believe in quantum field theory is, therefore, in spite of the spectacular success of perturbation theory ([67], Chap. 11 and [66] Chap. 15), strongly tied to non-perturbative approaches.

In his famous Erice lectures of 1979 [70], "Should we believe in quantum field theory?", Wightman remarks (p. 1011) that he once expressed to Landau his lack of confidence in the arguments which he and his co-workers had put forward for the inconsistency of field theory. "He then offered me the following: You agree that the essential problem of quantum field theory is its high-energy behavior? Yes. You agree that up to now no-one has suggested a consistent high-energy behavior for quantum field theory? Yes. Then you have to believe in the inconsistency of quantum field theory, because physicists are smart and if there was a consistent high-energy behavior, they would have found it!"

Landau's remarks concern, in particular, quantum electrodynamics in the Coulomb gauge in three space dimensions—qed_{1+3}. He refers, as we do throughout the paper, to "bona-fide" field theories in which cutoffs have been eliminated, and are thus invariant under certain symmetry groups: "effective" field theories are thereby excluded. Our recent result [35], however, establishes positivity of the (renormalized) energy, uniformly in the volume (V) and ultraviolet (Λ) cutoffs. This stability result may be an indication of the absence of Landau poles or ghosts in qed_{1+3}. To achieve this, the theory in Fock space (for fixed values of the cutoffs) is exchanged for a formulation in which (in the words of Lieb and Loss, who were the first to exhibit this phenomenon in a relativistic model [40]), "the electron Hilbert space is linked to the photon Hilbert space in an inextricable way". Thereby, "dressed photons" and "dressed electrons" arise as new entities.

The above picture, required by stability, provides a physical characterization of the otherwise only mathematically motivated non-Fock representations which arise in an interacting theory [70,71]. The non-unitary character of the transformations to the physical Hilbert space, when the space and ultraviolet cutoffs are removed, are intrinsic to the singular nature of field theory, and never really belonged to the usual lore of (even a smart!) theoretical physicist (for an exception, see the book by Barton [5]), which might explain Landau's remarks, as well as the fact that the answer to the question posed in the title involves considerations both of foundational and mathematical nature.

In this paper, we attempt to incorporate the findings of [35] in a proper general framework. For this purpose, we suggest a criterion characterizing interacting theories, incorporating a "singularity hypothesis" which has been conjectured for a long time, is supported by renormalization group theory, but has never been formulated mathematically. We do so in Definition 3.3. With this hypothesis, it is possible to prove Theorem 3.4, which is our main result. It permits to characterize interacting quantum field theories, in a suitable Wightman framework, by the property that the total spectral measure is infinite. The (non-perturbative) wave-function renormalization occurring in these theories, defined in Proposition 3.2, is shown not to be universally equal to zero in interacting theories satisfying the singularity hypothesis, except if equal time commutation relations (ETCR) are assumed (Corollaries 3.6 and 3.7). Since the ETCR are known, from model studies, not to be universally valid, the physical interpretation of the condition $Z = 0$ is open to question. We suggest that it may characterize either the existence of "dressed" particles in charged sectors, which may occur for specific theories such as qed_{1+3}) due to the presence of massless particles (the photon), by a theorem of Buchholz [13], or unstable (composite) particles. The latter conjecture, due to Weinberg ([67], p. 460), is, however, still unsupported by any rigorous result.

In the case of qed_{1+3} , such "dressed" electrons are not expected to be confined, but in qcd, we propose a quark confinement criterion, which follows naturally from lines suggested by the works of Casher, Kogut and Susskind [17], and Lowenstein and Swieca [41] in massless qed_{1+1} (the Schwinger model), in which the "dressing" of the electrons by photons is rather drastic, the electron field being expressed entirely as a functional of the photon field, which becomes massive.

2 The field algebra

To formulate the above-mentioned phenomena in qed_{1+3}, we consider a *field algebra* $\mathcal{F}^0 \equiv \{A_\mu^0, \psi^0, \bar{\psi}^0\}$ (containing the identity) generated by the free vector potential A_μ^0, $\mu = 0, 1, 2, 3$, and the electron–positron fields $\psi^0, \bar{\psi}^0$. We may assume that the field algebra is initially defined on the Fock–Krein (in general, indefinite-metric, see [8])

tensor product of photon and fermion Fock spaces. However, of primary concern for us will be an *inequivalent* representation of the field algebra $\mathcal{F}^0$ on a (physical) Hilbert space $\mathcal{H}$, with generator of time-translations—the physical Hamiltonian H—satisfying *positivity*, that is:

$$H \geq 0, \tag{1}$$

and such that:

$$H\Omega = 0, \tag{2}$$

where $\Omega \in \mathcal{H}$ is the vacuum vector, and:

$$A_\mu \equiv A_\mu(A_\mu^0, \psi^0, \bar{\psi}^0), \tag{3}$$
$$\Psi \equiv \Psi(A_\mu^0, \psi^0, \bar{\psi}^0), \tag{4}$$
$$\bar{\Psi} \equiv \bar{\Psi}(A_\mu^0, \psi^0, \bar{\psi}^0) . \tag{5}$$

We may define, as usual, the (restricted) class of c-number $U(1)$ local gauge transformations, acting on $\mathcal{F}^0$, by the maps:

$$A_\mu^0(f) \to A_\mu^0(f) + c \langle f, \partial^\mu u \rangle, \tag{6}$$
$$\psi^0(f) \to \psi^0(e^{iu} f), \tag{7}$$
$$\bar{\psi}^0(f) \to \bar{\psi}^0(e^{-iu} f), \qquad f \in \mathcal{S}(\mathbb{R}^{1+s}) . \tag{8}$$

In qed, $c = \frac{1}{e}$ and u satisfies certain regularity conditions, which guarantee that:

$$e^{\pm iu} f \in \mathcal{S}(\mathbb{R}^{1+s}) \quad \text{if } f \in \mathcal{S}(\mathbb{R}^{1+s}), \quad \text{and} \quad |\langle f, \partial^\mu u \rangle| < \infty .$$

In the following, $\mathcal{S}$ denotes Schwartz space (see, *e.g.*, [6]), $\langle \,.\,,\,.\, \rangle$ denotes the $L^2(\mathbb{R}^{1+s})$ scalar product, and s is the space dimension. Such transformations have been considered in a quantum context, that of (massless) relativistic qed in two space–time dimensions (the Schwinger model) by Raina and Wanders, but their unitary implementability is a delicate matter [47]. We shall use (9)–(11) merely as a guiding principle to construct the observable algebra, to which we now turn.

The *observable algebra* is assumed to consist of gauge-invariant objects, namely the tensor fields:

$$F_{\mu,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{9}$$

with A_μ given by (3), describing the dressed photons, and the quantities (14) below. To define them, we assume the existence of gauge-invariant quantities $\Psi, \bar{\Psi}$ in (4), (5), which create–destroy electrons–positrons "with their photon clouds".

While we hope that the results in [35] will eventually lead to an (implicit or explicit) expression for $\Psi, \bar{\Psi}$, it should be emphasized that this is a very difficult, open problem; see the important work of Steinmann in perturbation theory [54]. We also note that when we consider the *vacuum sector*, the fermion part of the observable algebra will be assumed to consist of the combinations:

$$A(f, g) \equiv \bar{\Psi}(f)\Psi(g) \text{ with } f, g \in \mathcal{S}(\mathbb{R}^{1+s}),$$
$$B(f, g) \equiv \Psi(f)\bar{\Psi}(g) \text{ with } f, g \in \mathcal{S}(\mathbb{R}^{1+s}) . \tag{10}$$

The quantities A and B above stand for rigorous versions of quantities of the form (12), smeared with functions f, g, which have never been constructed except in the Schwinger model. The existence of *charged sectors* is a related problem, which will concern us in Sect. 3.1.

3 A framework for relativistic quantum gauge theories

In this paper, we propose a framework which is not new, having been used by Lowenstein and Swieca [41] and Raina and Wanders [47] to construct a theory of qed_{1+1}, the Schwinger model.

The theory will be defined by its *n-point Wightman functions* [56] of observable fields. Alternatively, a Haag–Kastler theory [29] may be envisaged. It has been shown in the seminal work of the latter authors that the whole content of a theory can be expressed in terms of its observable algebra. In the case of gauge theories, the latter corresponds to the algebra generated by gauge invariant quantities [58]. As remarked by Lowenstein and Swieca [41], the observable algebra, being gauge-invariant, should have the same representations, independently of the gauge of the field algebra which it is constructed from. Thus, n-point functions constructed over the observable algebra should be the same in all gauges. These remarks fully justify the usage of non-covariant gauges, which, as we shall see, are of particular importance in a non-perturbative framework. For scattering theory and particle concepts within a theory of local observables, see [2,9,11], for a lucid review.

There exist various arguments supporting the use of non-covariant gauges in relativistic quantum field theory: they are of both physical and mathematical nature. In part one of his treatise, Weinberg notes ([67], p. 375, Ref. 2): "the use of Coulomb gauge in electrodynamics was strongly advocated by Schwinger on pretty much the same grounds as here: that we ought not to introduce photons with helicities other than ± 1". Indeed, as shown by Strocchi [55], a framework excluding "ghosts" necessarily requires the use of non-manifestly covariant gauges, such as the Coulomb gauge in qed_{1+3}, the Weinberg or unitary gauge in the Abelian Higgs model [65], and the Dirac [19] or light-cone gauge in quantum chromodynamics [18,48]. Another instance of the physical–mathematical advantage of a non-covariant gauge is the "$\alpha = \sqrt{\pi}$" gauge in massless qed_{1+1}—the Schwinger model [49]—see [41,47]. As in the Coulomb gauge in qed_{1+3}, there is no need for indefinite metric in this gauge, *i.e.*, the zero-mass longitudinal part of the current is gauged away, and one has a solution of Maxwell's equations (as an operator-valued, distributional identity):

$$\partial_\nu F^{\mu,\nu}(x) = -ej^\mu(x), \tag{11}$$

on the whole Hilbert space. This is an important ingredient in Buchholz's theorem [13], to which we come back in the sequel.

The structure of the observable algebra is quite simple in the Coulomb gauge: the field (9) is just the electric field, which is defined in terms of a massive scalar field, and the quantities (10) are, in this gauge, rigorous versions of the (path-dependent) quantities:

$$\psi(x)e^{ie\int_x^y dt_\mu\ A^\mu(t)}\psi^*(y), \tag{12}$$

and their adjoints (in the distributional sense), see [41,47]. In the case of qed_{1+3}, such quantities are plagued by infrared divergences, see the discussion in [54]. As a consequence of the simple structure of the observable algebra, one arrives at a correct physical–mathematical picture of spontaneous symmetry breakdown [41,47]; in covariant gauges, this picture is masked by the presence of spurious gauge excitations.

In [56], pp. 107–110, it is shown that if the (n-point) *Wightman functions* satisfy:

a. the relativistic transformation law;
b. the spectral condition;
c. hermiticity;

d. local commutativity;

e. positive-definiteness,

then they are the *vacuum expectation values of a field theory* satisfying the Wightman axioms, except, eventually, the uniqueness of the vacuum state. We refer to [46,56] for an account of Wightman theory, and for the description of these properties. It has been shown in [41,47] that qed_{1+1} in the "$\alpha = \sqrt{\pi}$" gauge satisfies a.–e. The crucial positive-definiteness condition e. has been shown in [41] to be a consequence of the positive-definiteness of a subclass of the n-point functions of the Thirring model [61] in the formulation of Klaiber [37]. Positive definiteness of the Klaiber n-point functions was rigorously proved by Carey, Ruijsenaars, and Wright [16]. Uniqueness of the vacuum holds in each irreducible subspace of the (physical) Hilbert space $\mathcal{H}$ [47], as a result of the cluster property; see also [41].

We shall *assume* a.–e. for the *n*-point functions of the observable fields, with, in addition, the following requirement:

f. interacting fields are assumed to satisfy the singularity hypothesis (the forthcoming Definition 3.3).

The crucial mathematical reason for choosing a non-covariant gauge is, as we shall see, the positive-definiteness condition e. Concerning the uniqueness of the vacuum, we shall assume that it is valid by restriction to an irreducible component of $\mathcal{H}$, as in qed_{1+1}.

3.1 The Källén–Lehmann representation

A major dynamical issue in quantum field theory is the (LSZ or Araki–Haag) asymptotic condition (see, *e.g.*, [11] and references given there), which relates the theory, whose objects are the fundamental observable fields, to *particles*, described by physical parameters (*mass* and *charge* in qed). This issue is equivalent to the renormalization (or normalization) of perturbative quantum field theory, which itself is related to the construction of continuous linear extensions of certain functionals, such as to yield well-defined Schwartz distributions (see [51] and references given there). On the other hand, in a non-perturbative framework, a theory of renormalization of masses and fields also exists, and "has nothing directly to do with the presence of infinities" ([67], p. 441, Sect. 10.3). We adopt a related proposal, which we formulate here, for simplicity, for a theory of a self-interacting scalar field A of mass m satisfying the Wightman axioms (modifications are mentioned in the sequel). We assume that A is an operator-valued tempered distribution on the Schwartz space $\mathcal{S}$ (see [46], Ch. IX).

We have the following result, concerning the spectral representation of the two-point function W_2 ([46], p. 70, Theorem IX-34):

Theorem 3.1 (The Källén–Lehmann representation)

$$W_2^m(x - y) = \langle \Omega, A(x)A(y)\Omega \rangle = \frac{1}{i} \int_0^\infty \mathrm{d}\rho(m_\circ^2) \, \Delta_+^{m_\circ}(x - y), \tag{13}$$

where Ω denotes the vacuum vector, $x = (x_0, \vec{x})$, and:

$$\Delta_+^{m_\circ}(x) = \frac{i}{2(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k} \, \frac{\mathrm{e}^{-ix_0\sqrt{m_\circ^2 + \vec{k}^2} + i\vec{x}\cdot\vec{k}}}{\sqrt{m_\circ^2 + \vec{k}^2}} \tag{14}$$

is the two-point function of the free scalar field of mass $m_\circ$, and ρ is a polynomially bounded measure on $[0, \infty)$, that is:

$$\int_0^L \mathrm{d}\rho(m_\circ^2) \leq C(1 + L^N) \tag{15}$$

for some constants C and N. It is further assumed that:

$$\langle \Omega, A(f)\Omega \rangle = 0 \qquad \forall f \in \mathcal{S} . \tag{16}$$

Note that (13) is symbolic; for its proper meaning, which relies on (15), see [46]. In the next proposition, since there is only a finite number of masses in nature, we assume a priori that the pure point part (which could eventually include a dense point spectrum, i.e., accumulation points) is, in fact, discrete, containing only a finite number of mass values. We further assume that this discrete part of the measure is associated with the (renormalized) physical masses existent in nature, *i.e.*, just one for a scalar field (and latter the electron and the photon mass in qed). This assumption is verified for the free fields (scalar, spinor, and vector), with $Z = 1$.

It should be remarked that a scalar field of mass m creates, with non-vanishing probability, the state of a particle of mass m *in the sense of Wigner* (see, e.g., [67]) from the vacuum. In particular, the electron in qed is not a particle, although the photon is (expected to be) a particle. It is, nevertheless, true that stability in [35,40] is achieved by the introduction of "dressed photons".

Proposition 3.2 *For a scalar field of mass $m \geq 0$, the measure $\mathrm{d}\rho(m_\circ^2)$ appearing in the Källén–Lehmann spectral representation allows a decomposition:*

$$\mathrm{d}\rho(m_\circ^2) = Z\delta(m_\circ^2 - m^2) + d\sigma(m_\circ^2), \tag{17}$$

where

$$0 < Z < \infty \tag{18}$$

and

$$\int_0^L \mathrm{d}\sigma(m_\circ^2) \leq C_1(1 + L^{N_1}) \tag{19}$$

for some constants C_1 and N_1.

Proof By the Lebesgue decomposition (see, *e.g.*, Theorems I.13, I.14, p. 22 of [45]):

$$\mathrm{d}\rho = \mathrm{d}\rho_{p.p.} + \mathrm{d}\rho_{s.c.} + \mathrm{d}\rho_{a.c.}, \tag{20}$$

where p.p. denotes the *pure point*, s.c. denotes the *singular continuous*, and a.c. denotes the *absolutely continuous* parts of $\mathrm{d}\rho$. By the assumptions, the pure point part of the measure is, in fact, discrete and, for a scalar field of mass m, we obtain:

$$\mathrm{d}\rho_{p.p.}(m_\circ^2) = Z\delta(m_\circ^2 - m^2), \tag{21}$$

where Z satisfies (18), and, by (15), (17) and (20), and $\mathrm{d}\sigma$ satisfies (19). □

Remark 3.1 Of course, $Z = 0$ in (21), if there is no discrete component of mass m in the total mass spectrum of the theory. In general, Z in each discrete component of $d\rho$ has only to satisfy:

$$0 \leq Z < \infty \tag{22}$$

because of the positive-definiteness conditions e. (or the positive-definite Hilbert space metric).

Remark 3.2 Expression (21) corresponds precisely to ([67], p. 461, Equ. (10.7.20)). Thus, Proposition 3.2 is just a mathematical statement of the nonperturbative renormalization theory, as formulated by Weinberg. Thus, the physical interpretation of Z is that $0 < Z < \infty$ is the wave function renormalization constant, due to the fact that the physical field A_{phys} is normalized (or renormalized) by the one-particle condition ([67], (10.3.6)) which stems from the LSZ (or Haag–Ruelle) asymptotic condition (see also Sections 2 and 5 of [11] and references given there for the appropriate assumptions). A general field, as considered in (13), does not have this normalization. By the same token, the quantity m^2 in (21) is interpreted as the physical (or renormalized) mass associated with the scalar field.

For $F_{\mu,\nu}$ and Ψ, we have the analogues of (13), namely:

$$\langle F_{\mu,\nu}(x)\Omega, F_{\mu,\nu}(y)\Omega \rangle = \int \mathrm{d}\rho_{ph}(m_{\circ}^2) \int \frac{\mathrm{d}^3 p}{2p_0} (-p_{\mu}^2 g_{\nu\nu} - p_{\nu}^2 g_{\mu\mu}) \mathrm{e}^{ip \cdot (x-y)}, \tag{23}$$

with $\mu \neq \nu$, no summations involved, and $p_0 = \sqrt{\vec{p}^{\,2} + m_{\circ}^2}$. Denoting spinor indices by α, β, we have:

$$S_{\alpha,\beta}^+(x - y) = \langle \Omega, \Psi_\alpha(x)\bar{\Psi}_\beta(y)\Omega \rangle$$
$$= \int_0^\infty \mathrm{d}\rho_1(m_{\circ}^2) S_{\alpha,\beta}^+(x - y; m_{\circ}^2) + \delta_{\alpha,\beta} \int_0^\infty \mathrm{d}\rho_2(m_{\circ}^2) \Delta^+(x - y; m_{\circ}^2), \tag{24}$$

with $\mathrm{d}\rho_{ph}, \mathrm{d}\rho_1, \mathrm{d}\rho_2$ positive, polynomially bounded measures, and ρ_1 satisfying certain bounds with respect to ρ_2 (see [39], p. 350 for the notation). Again, as in (21):

$$\mathrm{d}\rho_{ph}(m_{\circ}^2) = Z_3 \delta(m_{\circ}^2) + \mathrm{d}\sigma_{ph}(m_{\circ}^2) \tag{25}$$

and

$$\mathrm{d}\rho_1(m_{\circ}^2) = Z_2 \delta(m_{\circ}^2 - m_e^2) + \mathrm{d}\sigma_1(m_{\circ}^2), \tag{26}$$

with m_e the renormalized electron mass, according to the conventional notation.

We have, by the general condition (22):

$$0 \leq Z_3 < \infty, \tag{27}$$
$$0 \leq Z_2 < \infty . \tag{28}$$

When $Z_3 > 0$, the renormalized electron charge follows from ([67], (10.4.18)). Assumption (16), which is also expected to be generally true on physical grounds, becomes:

$$\langle \Omega, F_{\mu,\nu}(f)\Omega \rangle = 0 \quad \forall f \in \mathcal{S}, \tag{29}$$
$$\langle \Omega, \Psi_\alpha(f)\Omega \rangle = 0 \quad \forall f \in \mathcal{S} \text{ and } \alpha \text{ a spinor index}. \tag{30}$$

In summary, Proposition 3.2 provides a rigorous (non-perturbative) definition of the wave-function renormalization constant. In the proof of Theorem 3.1 ([46], p. 70), the positive-definiteness condition e. plays a major role. Thus, the definition of Z and its range (22) (which depends on the positivity of the measure $\mathrm{d}\rho$) strongly hinge on the fact that the underlying Hilbert space has a positive metric. Parenthetically, the positive-definiteness condition on the Wightman functions is "beyond the powers of perturbation theory", as Steinmann aptly observes [54].

3.2 The singularity hypothesis

As remarked in the introduction, one of the most important features of relativistic quantum field theory is the behaviour of the theory at large momenta (or large energies). Renormalization group theory [66] has contributed a significant lore to this issue (even if none of it has been made entirely rigorous): see, in particular, the paper of Symanzik on the small-distance behavior analysis of the two-point functions in relativistic quantum field theory [60]. It strongly suggests that the light-cone singularity of the two-point functions of interacting theories is stronger than that of a free theory: this is expected even in asymptotically free quantum chromodynamics, where the critical exponents are anomalous. We refer to this as the "singularity hypothesis", which will be precisely stated in the next section.

3.2.1 Steinmann scaling degree and a theorem

To formulate the singularity hypothesis in rigorous terms, we recall the Steinmann scaling degree sd of a distribution [53]; for a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, let u_λ denote the "scaled distribution", defined by:

$$u_\lambda(f) \equiv \lambda^{-n} u(f(\lambda^{-1} \cdot)) .$$

As $\lambda \to 0$, we expect that $u_\lambda \approx \lambda^{-\omega}$ for some ω, the "degree of singularity" of the distribution u. Hence, we set:

$$sd(u) \equiv \inf \left\{ \omega \in \mathbb{R} \mid \lim_{\lambda \to 0} \lambda^\omega u_\lambda = 0 \right\}, \tag{31}$$

with the proviso that if there is no ω satisfying the limiting condition above, we set $sd(u) = \infty$. For the free scalar field of mass $m \geq 0$ in four-dimensional space–time, it is straightforward to show from the explicit form of the two-point function in terms of modified Bessel functions that:

$$sd(\Delta_+) = 2 . \tag{32}$$

In (32), and the forthcoming equations, we omit the mass superscript. From Theorem 3.1, we have that for $f \in \mathcal{S}(\mathbb{R}^4)$, the interacting two-point function satisfies:

$$W_+(f) = \int_0^\infty \mathrm{d}\rho(m_\circ^2) \int_{\mathbb{R}^3} \frac{d\vec{p}}{\sqrt{\vec{p}^{\,2} + m_\circ^2}} \, \tilde{f} \left(\sqrt{\vec{p}^{\,2} + m_\circ^2}, \vec{p} \, \right) . \tag{33}$$

Here, $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$ denotes the Fourier transform of f.

Definition 3.3 We say that the *singularity hypothesis* holds for an interacting scalar field if:

$$sd(W_+) > 2 . \tag{34}$$

Further support for Definition 3.3 comes from the fact that the singularity hypothesis is indeed satisfied at finite orders of perturbation theory (if the interaction density has engineering dimension larger than 2).

Theorem 3.4 *If the total spectral mass is finite, that is:*

$$\int_0^\infty \mathrm{d}\rho(a^2) < \infty, \tag{35}$$

then

$$sd(W_+) \leq 2 ; \tag{36}$$

i.e., *the scaling degree of W_+ cannot be strictly greater than that of a free theory, and thus, by Definition 3.3, the singularity hypothesis (34) is* not *satisfied.*

Proof The scaled distribution corresponding to W_+ is given by:

$$W_{+,\lambda}(f) = \lambda^{-2} \int_0^\infty \mathrm{d}\rho(m_\circ^2) \int_{\mathbb{R}^3} \frac{d\vec{p}}{\sqrt{\vec{p}^{\,2} + \lambda^2 m_\circ^2}} \tilde{f}\left(\sqrt{\vec{p}^{\,2} + \lambda^2 m_\circ^2}, \vec{p}\right) . \tag{37}$$

Assume the contrary to (36), *i.e.*, that $sd(W_+) = \omega_0 > 2$. Then, by the definition of the sd, if $\omega < \omega_0$, one must have:

$$\lim_{\lambda \to 0} \lambda^\omega W_{+,\lambda}(f) \neq 0 . \tag{38}$$

Choosing:

$$\omega = \omega_0 - \delta > 2 \tag{39}$$

in (38), we obtain from (37) and (38) that:

$$\lim_{\lambda \to 0} \lambda^{\omega-2} \left(\int_0^\infty \mathrm{d}\rho(m_\circ^2) \int_{\mathbb{R}^3} \frac{d\vec{p}}{\sqrt{\vec{p}^{\,2} + \lambda^2 m_\circ^2}} \tilde{f}\left(\sqrt{\vec{p}^{\,2} + \lambda^2 m_\circ^2}, \vec{p}\right) \right) \neq 0 . \tag{40}$$

The limit, as $\lambda \to 0$, of the term inside the brackets in (40), is readily seen to be finite by the Lebesgue dominated convergence theorem due to the assumption (35) and the fact that $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$; but this contradicts (38) because of (39). $\qquad \square$

Corollary 3.5 *The singularity hypothesis holds for an interacting scalar field only if $\int_0^\infty d\sigma(m_\circ^2) = \infty$. This necessary condition is independent of the value of $0 \le Z < \infty$.*

One importance of the above theorem, which is our main result, and especially of its corollary is that it provides, as we shall see next, a mathematical foundation for the forthcoming interpretation of the condition:

$$Z = 0 . \tag{41}$$

3.2.2 The ETCR hypothesis and its consequences for the singularity hypothesis

For the purposes of identification with Lagrangian field theory, one may equate the $A(.)$ of (13) with the "bare" scalar field ϕ_B ([67], p. 439, see also [31] and [25]), whereby:

$$A = \sqrt{Z} A_{phys} \tag{42}$$

under the condition (18). Under the same condition (18), the assumption of equal time commutation relations (ETCR) for the physical fields may be written (in the distributional sense):

$$\left[\frac{\partial A_{phys}(x_0, \vec{x})}{\partial x_0}, A_{phys}(x_0, \vec{y}) \right] = -\frac{i}{Z} \delta(\vec{x} - \vec{y}) . \tag{43}$$

Together with (13) and (42), (43) yields ([67], (10.7.18)):

$$1 = \int_0^\infty d\rho(m_\circ^2) .$$ (44)

Since $d\sigma$ in (17) is a positive measure, we obtain from (44) the inequality:

$$Z \leq 1$$ (45)

([67], p. 361). We thus find, comparing (44) with corollary 3.5:

Corollary 3.6 *The singularity hypothesis is incompatible with the ETCR hypothesis. Thus,* (45) *is not generally valid.*

In perturbation theory, $Z_3(\Lambda)$ ([67], p. 462) satisfies (45) for all ultraviolet cutoffs Λ, but it is just this condition which relies on the ETCR assumption and is not expected to be generally valid. In the limit $\Lambda \to \infty$, however, $Z_3(\Lambda)$ tends to $-\infty$ and hence violates (18) maximally. In fact, (18) is violated even for finite, sufficiently large Λ.

Although Corollary 3.6 is strong, in that it excludes the ETCR entirely, together with its consequences, it trivially implies that *all* values of Z compatible with (22) are allowed, without singling out the condition $Z = 0$.

In this connection, we should remark that the "bare" (conventional) field (42) in (13) should not be identified with the Wightman field, because the latter is supposed to have a particle interpretation, through scattering theory (see Remark 3.2). This is due to the one-particle space normalization (see also Jost's monograph [33], (1), p.120 for a detailed discussion of this point in connection with the Haag–Ruelle theory). We are thus led to replace the A on the l.h.s. of (13) by the physical, renormalized field (43). When (18) holds, the resulting spectral measure is, of course, $dg(m_\circ^2) = \frac{1}{Z} d\rho(m_\circ^2)$ which satisfies, by (44):

$$\frac{1}{Z} = \int dg(m_\circ^2).$$ (46)

(46) is Wightman's (17) in his famous paper [68]. The same formula is found in Lehmann [39], Källén [36], and Barton [5]. Of course, Theorem 3.2 remains valid as long as (18) holds. As a consequence, we have the following result:

Corollary 3.7 *If, in* (13), *the field A is assumed to be the physical field* (42), *and the ETCR* (43) *is assumed as well, only the value $Z = 0$ remains in* (22) *as possibly compatible with the singularity hypothesis.*

It is (46) which seems to underlie the general belief that $Z = 0$ is generally true in an interacting theory, a conjecture which has been made even, for instance, by the great founders of axiomatic (or general) quantum field theory, Wightman and Haag. Indeed, in [70], p. 201, it is observed that "$\int_0^\infty d\rho(m_\circ^2) = \infty$ is what is usually meant by the statement that the field-strength renormalization is infinite". This follows from (46), with "field-strength renormalization" interpreted as $\frac{1}{Z}$. The connection with the singularity hypothesis comes next ([70], p. 201), with the observation that, by (13), W_2 will have the same singularity, as $(x - y)^2 = 0$, as does $\Delta_+(x - y; m^2)$. As for Haag, he remarks ([25], p. 55):"In the renormalized perturbation expansion one relates formally the true field A_{phys} to the canonical field A (our notation) which satisfies (43), where Z is a constant (in fact, zero). This means that the fields in an interacting theory are more singular objects than in the free theory, and we do not have the ETCR".

Although both assertions above seem to substantiate the conjecture that $Z = 0$ is expected to be a *general* condition for interacting fields, it happens, however, that the resulting "$\infty \times \delta$" behavior of (43), while suggestive, is entirely misleading. In other, equivalent, words, the relation (46) is rigorously justified if $\int dg(m_\circ^2) < \infty$ as shown by Wightman in [68], p. 863, but in the opposite case $\int dg(m_\circ^2) = \infty$, as required by Theorem 3.2, (46) is misleading, as the example in ([68], p. 863) demonstrates.

It follows from Corollary 3.6 and Corollary 3.7 that for both choices of fields in (13), $Z = 0$ is not generally true. The latter case hinges on the fact that the ETCR is not generally valid for interacting fields, as briefly reviewed in the forthcoming paragraph. We conclude that the singularity hypothesis opens the possibility of the non-universal validity of the equation $Z = 0$.

The hypothesis of ETCR has been in serious doubt for a long time, see, *e.g.*, the remarks in [56], p. 101. Its validity has been tested [73] in a large class of models in two-dimensional space–time; the Thirring model [61], the Schroer model [50], the Thirring–Wess model of vector mesons interacting with zero-mass fermions (see [20,62]), and the Schwinger model [49], using, throughout, the formulation of Klaiber [37] for the Thirring model, and its extension to the other models by Lowenstein and Swieca [41]—for the Schwinger model, the previously mentioned noncovariant gauge "$\alpha = \sqrt{\pi}$" was adopted. Except for the Schwinger model, whose special canonical structure is due to its equivalence (in an irreducible sector) to a theory of a free scalar field of positive mass, the quantity:

$$\{\psi(x), \psi(y)\} - \langle \Omega, \{\psi(x), \psi(y)\}\Omega \rangle \cdot \mathbf{1}, \tag{47}$$

where the ψ's are the interacting fermi fields in the models and $\{\,.\,,\,.\,\}$ denotes the anti-commutator and Ω denotes the vacuum, do not exist in the equal time limit as operator-valued distributions, for a certain range of coupling constants. Two different definitions of the equal time limit were used and compared, one of them due to Schroer and Stichel [52]. The models also provide the examples of the validity of the singularity hypothesis (for the currents, analogous assertions hold if the commutator is used in place of the anti-commutator). Thus, the ETCR is definitely not true in general.

Although (see Remark 3.2), when $0 < Z < \infty$, Z is interpreted as the non-perturbative field strength renormalization, relating "bare" fields to physical fields, as in (43), the remaining case (41) remains to be understood. As stated in ([67], pg. 461), "the limit $Z = 0$ has an interesting interpretation as a condition for a particle to be composite rather than elementary". This brings us to our next topic.

4 A proposal for the meaning of the condition $Z = 0$: the presence of massless and unstable particles

Buchholz [13] used Gauss' law to show that the discrete spectrum of the mass operator:

$$P_\sigma P^\sigma = M^2 = P_0^2 - \vec{P}^2 \tag{48}$$

in a charged sector is empty. Above, P^0 is the generator of time translations in the physical representation, *i.e.*, the physical hamiltonian H, and $\vec{P}$ is the physical momentum. This fact is interpreted as a confirmation of the phenomenon that, in a charged sector, and given certain interactions (satisfying Gauss' law) between massless particles and others carrying an electric charge, the latter are converted to infraparticles and are accompanied by clouds of soft photons.

Buchholz formulates adequate assumptions which must be valid in order that one may determine the electric charge of a physical state Φ with the help of Gauss' law:

$$e\langle \Phi, j_\mu \Phi \rangle = \langle \Phi, \partial^\nu F_{\nu,\mu} \Phi \rangle . \tag{49}$$

(49) is assumed to hold in the sense of distributions on $\mathcal{S}(\mathbb{R}^n)$.

When endeavouring to apply Buchholz's theorem to concrete models such as qed_{1+3}, problems similar to those occurring in connection with the charge superselection rule [58] arise. The most obvious one is that Gauss' law (49) is only expected to be valid (as an operator equation in the distributional sense) in non-covariant gauges (see (11)). Recalling (28), we find that, in an interacting theory satisfying the axioms a. to f. for observable fields, in which massless particles (photons), as well as infraparticles ("electrons with their photon clouds") occur in a charged sector—that is, in quantum electrodynamics in Buchholz's characterization [13]—we have:

Corollary 4.1

$$Z_2 = 0 \ . \tag{50}$$

It is interesting to recall, in connection with Corollary 4.1, that in [35], *both* the photon field and the electron-positron field are "dressed". The photon is, however, believed to be a particle [44], and it might be expected that the "clouds" around the photon vanish asymptotically. Indeed, the condition $Z_3 = 0$ does not characterize massless particles (photons), see [12]; only (27) remains true.

We now return to the subject of *composite* or *unstable particles*. Classically speaking, these are particles whose field does not appear in the Lagrangian ([67], p. 461, [64]). We were, however, unable to render this notion precise even in the classical case (and we thank the referee for convincing us of this fact).

There are, as yet, no rigorous results on the quantum theory, except for cut-off theories, see [1] and references given there. Since the very notion of particle involves the Poincaré group, there are serious difficulties with the concept of unstable particle in a theory where the cutoffs are not removed. For a model of Galilei invariant molecular dynamics with particle production, see [27].

The condition $Z = 0$ appears, however, in various non-rigorous approaches to the concept of unstable particle [3,42,63]. Turning to scalar fields for simplicity, we consider the case of a scalar particle C, of mass m_C, which may decay into a set of two (for simplicity) stable particles, each of mass m. We have energy conservation in the rest frame of C, that is:

$$m_C = \sum_{i=1}^{2} \sqrt{\vec{q}_i\,^2 + m_i^2} \geq \sum_{i=1}^{2} m_i,$$

with $m_i = m, i = 1, 2$, and $\vec{q}_i$ the momenta of the two particles in the rest frame of C:

$$m_C > 2m \ . \tag{51}$$

To check that:

$$Z_C = 0 \tag{52}$$

when (51) holds, while $0 < Z_C < \infty$ is valid in the stable case $m_C < 2m$, in a model, we are beset with the difficulty to obtain information on the two-point function. An exception are those rare cases in which the (Fock) zero particle state is persistent ([26]),*i.e.*, Lee-type models.

The quantum model of Lee type of a composite (unstable) particle, satisfying (51), where (52) was indeed found, is that of Araki et al. [4]. Unfortunately, however, the (heuristic) results in [4] have one major defect: their model contains "ghosts". The paper [28], cited by Weinberg, assumes, however, the inequality opposite to (51), and so is concerned with stable particles. In fact, their reference to (51) treats it as the *general* conjecture based on [36,68] previously referred to.

There exists, however, a ghostless version of the model treated heuristically in [4], with the correct kinematics, due to Hepp (Theorem 3.4, p. 54, of [26]). In this version, the masses in (51), which are, of course, renormalized masses, may be determined rigorously from the selfadjointness of the renormalized Hamiltonian. It is an open problem of great importance to carry out this investigation in detail: it would be the first rigorous model of unstable (composite) particles in quantum field theory.

For atomic resonances, the model in [4] may be treated rigorously, see [72].

5 A proposal for a criterion for quark confinement in quantum chromodynamics (qcd)

The fundamental objects of quantum chromodynamics (qcd) ([66], Section 18.7) are the color gauge-covariant field strength tensor $F_a^{\mu\nu}$, where $a = 1, 2, 3$ is the colour index, and μ, ν are Minkowski space indexes and the quark field Ψ_a. The color gauge vector potential A_a^μ describes massless gluons. The unobservability of quarks has never been explained, but, in the 70s and 80s, several models, notably the Schwinger model [49], *i.e.*, the electrodynamics of massless electrons in two spacetime dimensions, have been suggested as models of the confinement of quarks. In this model, the rigorous version of the observable (12) ([41], Section IV) is a bilocal quantity which creates a charge dipole with an electric field in between, according to Gauss' law: this picture has been analysed in great detail by Casher, Kogut and Susskind [17]. The electron field becomes a functional of the photon field, which acquires a positive mass: let the corresponding field at time zero be denoted by ϕ. If Ω is the Fock vacuum, we define the dipole state (corresponding to placing the "string" at time $t = 0$; we omit the time variable for simplicity):

$$\Psi_{R,\epsilon} = T_{R,\epsilon}\Omega, \tag{53}$$

where

$$T_{R,\epsilon} = \exp[i\sqrt{\pi}\,\pi(g_{R,\epsilon})]\,. \tag{54}$$

Above π denotes the field momentum operator at zero time, conjugate to ϕ and $g_{R,\epsilon}$ is the function:

$$g_{R,\epsilon}(x) = \begin{cases} 1 & \text{if } 0 \le x \le R\,; \\ 0 & \text{if } x \le -\epsilon \text{ or } x \ge R + \epsilon\,. \end{cases} \tag{55}$$

The above definitions follow [41], pp. 183–185, to which we refer the reader for more details, except that we introduced $\epsilon > 0$ fixed and let $R \to \infty$, for reasons of rigor. The connection between ϕ and the electric field (electromagnetic field tensor F_{01}) at time zero is thereby given as:

$$\phi(x) = F_{01}(x)\,. \tag{56}$$

As in ([15], Appendix A, where in (A.1), the mass term should be replaced by $\frac{e^2}{\pi}\phi(x, 0)^2$ inside the Wick dots), we consider the family of states:

$$\omega_{R,\epsilon}(\,\cdot\,) = \langle \Psi_{R,\epsilon},\,\cdot\,\Psi_{R,\epsilon}\rangle \tag{57}$$

on the Weyl CCR algebra generated by $\exp(i\phi(f))$ and $\exp(i\pi(g))$, $f, g \in \mathcal{S}_{\mathbb{R}}^0(\mathbb{R})$, the Schwartz space of real-valued functions on the real line, whose Fourier transform vanishes at the origin. By compactness (Theorem 2.3.15 of [10]), there exists at least one limit in the weak* topology:

$$\omega_\epsilon = \lim_{R\to\infty} \omega_{R,\epsilon}\,. \tag{58}$$

The generator H of time translations in the Fock (vacuum) representation is given by:

$$H = \frac{1}{2}\int \mathrm{d}x\; :[\pi^2(x) + (\nabla\phi)^2(x) + m^2\phi^2(x)]:\,. \tag{59}$$

Above, the dots indicate Wick ordering and:

$$m = \frac{e}{\sqrt{\pi}} \tag{60}$$

is the dynamically generated photon mass, whose origin is topological (see the preface of [21,59], p. 317). Let $V(g) = \exp(i\pi(g))$. Using:

$$: \phi(x)^2 := \lim_{x_1,x_2 \to x} [\phi(x_1)\phi(x_2) - \langle \Omega, \phi(x_1)\phi)(x_2)\Omega\rangle \mathbf{1}]$$

and similarly for $: \nabla(\phi)^2(x) :$, we obtain:

$$V(h)^{-1}H(x)V(h) = H(x) - \nabla h(x) \cdot \nabla\phi(x) + \frac{1}{2}(\nabla h)^2(x) + \frac{e^2}{\pi}\left(h^2(x) - 2h(x)\phi(x)\right)$$

from which a. of the following proposition follows:

Proposition 5.1 *a.*

$$\langle \Psi_{R,\epsilon}, H\Psi_{R,\epsilon}\rangle = \frac{\pi}{2}\int_{-\infty}^{+\infty} dx \left((g'_{R,\epsilon})^2(x) + \frac{e^2}{\pi}g^2_{R,\epsilon}(x)\right),$$

where the prime above denotes the derivative;
b. Any state ω_ϵ, defined by (58), has unit charge.

Proof See Proposition A.1 of Appendix A of [15] for the definition of charge and the proof of item b. □

The above proposition suggests the following definition:

Definition 5.2 We say that ω_ϵ has *finite energy* if $\Psi_{R,\epsilon}$ lies in the quadratic form domain of H for any $R < \infty$ and $\epsilon > 0$, and:

$$\limsup_{R\to\infty} \langle \Psi_{R,\epsilon}, H\Psi_{R,\epsilon}\rangle < \infty. \tag{61}$$

If ω_ϵ does not have finite energy (with respect to the vacuum, which has, by definition, energy zero), *i.e.*, if:

$$\limsup_{R\to\infty} \langle \Psi_{R,\epsilon}, H\Psi_{R,\epsilon}\rangle = \infty, \tag{62}$$

we say that the associated fermions are *confined*.

By [15], Proposition A.2 of Appendix A, the representation defined by ω_ϵ is inequivalent to the Fock (vacuum) representation. This result follows from the fact that the state corresponds to a nonzero eigenvalue, namely one, of the charge operator, but is not due to the existence of a macroscopic energy barrier between ω_ϵ and the vacuum state. Indeed, when charged particles exist, a system containing a finite number of these charged particles should have finite energy relative to the vacuum, which is normalized to zero: this is the physical meaning of (61); this condition is satisfied in the Streater–Wilde model of soliton sectors, where the analogue of the l.h.s. of (61) is seen to be identical to the energy of classical soliton solutions of the two-dimensional wave equation [57]. Note, however, that in their model, the scalar field is massless. In our case, we have the following corollary of Proposition 5.1:

Corollary 5.3 *In the Schwinger model, Definition 5.2 implies that the fermions are confined.*

Proof The first term in the r.h.s. of a. of Proposition 5.1 is uniformly bounded in R, and the second is, by (55):

$$E_d \equiv \int_{-\infty}^{\infty} dx \, \langle \Psi_{R,\epsilon}, : F_{01}(x)^2 : \Psi_{R,\epsilon}\rangle \geq \text{const. } R. \tag{63}$$

(63) implies (62) and, thus, the fermions are confined. □

Corollary 5.3 provides the following physical interpretation for the mass term in the Schwinger model (see also [17]) as $R \to \infty$: it represents the energy of a dipole in the non-relativistic limit, since, in one dimension, the Coulomb potential is linear. In general, we have the following distributional formula ([23], p. 361):

$$\mathcal{F}(|\vec{p}|^{\lambda}) = \text{const.} \, |\vec{x}|^{-\lambda-s}, \tag{64}$$

where $\mathcal{F}$ denotes the distributional Fourier transform, and s denotes the space dimension.

For $Re\lambda \leq -s$, the function $|p|^{\lambda}$ is not locally integrable, but it defines a distribution in the sense of Guelfand and Chilov (see [23], p. 71).

If $\lambda = -2$ and $s = 1$, we have the present case. If $s = 3$, we find R^{-1} for large R as expected. In the non-abelian case, we would have in (63), $: F_{\mu\nu}^a(x) F_a^{\mu\nu}(x) :$ instead of $: F_{01}(x)^2 :$, where the summation over the color index a is understood. Due to the $A \wedge A$ term in the gluon field tensor, we have in (64) the leading infrared singularity with $\lambda = -4$ as $|\vec{p}| \to 0$, which implies the same linear behavior as (63)! Thus, if observable fields of type (10) may be constructed for qcd, whereby f and g would have compact support around points growing linearly with a parameter R along a radial direction, yielding a quark–antiquark pair, and if the present analogy is sound, one expects confinement. Definition 5.2 would have, however, to allow for a "path" dependence of the l.h.s. of (62) and require the validity of (62) independently of the "path".

It should be remarked that in qcd, it is not the fermion number which is expected to be confined, but color, which is a multiplicative charge. A very interesting soluble abelian model of triality ("charm", an abelian version of the three color states of a free quark) is found in Casher, Kogut, and Susskind [17]: all states of nonzero "charm" are confined.

It was Casher, Kogut and Susskind [17] who first proposed that the deep inelastic structure functions of the theory might have the scaling laws of the underlying Fermi fields in the Schwinger model, although only massive Bosons appear as asymptotic states. This was proved by Swieca: the short-distance limit of the n-point functions of the observables in the model is those of a free theory of charged massless fermions and massless photons ([59], p. 317).

In qcd, it was already suggested by Cornwall in 1982 [18] that a gluon mass is dynamically generated. He used the Dirac or light-cone gauge, in which ghosts are absent. Further work in covariant gauges suggests that the dynamically generated gluon mass is primarily due to the simplest severe infrared singularity $|\vec{p}|^{-4}$ mentioned above [24], see also [7]. If this was so, the analogy between qcd and the Schwinger model would be complete: the short-distance limit would imply the vanishing of the dynamically generated gluon mass, and, with it, the quarks would reappear as free particles, together with massless gluons. Note that due to (60), there is no interaction in the limit when the dynamically generated mass of the photon tends to zero: there is "asymptotic freedom". The scaling dimension of the gluon fields is, however, expected to be anomalous, as previously observed.

The gluon mass also fits well in the general conjecture of the mass gap in general Yang–Mills theories, see the problem posed by Jaffe and Witten in [34].

6 Conclusion

We have suggested a criterion which characterizes interacting theories in a proper Wightman framework, based on a "singularity hypothesis". The associated framework relies on the axioms for the Wightman n-point functions for observable fields, including positivity, and, thus, requires the use of non-covariant gauges. The singularity hypothesis is either incompatible with the ETCR (Corollary 3.6) or requires that the (nonperturbative) wave function renormalization constant Z equals zero, if the ETCR is assumed (Corollary 3.7), depending on which type of field—the "bare" (conventional), or the physical (renormalized) field in Lagrangian perturbation theory is identified with the Wightman field in the theory proposed in [68].

Since the ETCR is generally not valid for interacting field theories, this opens up the possibility that the condition $Z = 0$, assumed to be universal for interacting theories, is of specific nature. We propose that it describes "dressed"

infraparticles (in the presence of massless particles) in certain specific theories, such as quantum electrodynamics in Buchholz's characterization [13], or of composite (unstable) particles.

Since the gluons are massless, "dressed" quarks and gluons may occur in the theory of strong interactions. A proposed caricature of (some aspects of) qcd such as quark confinement is the Schwinger model revisited in Section 5. There, the "dressing" of the electrons assumes a drastic form: by Bosonization, the electron field is a functional of the photon field, which acquires a mass. We propose a criterion of confinement which is valid in this model, and whose extrapolation to qcd, if valid, predicts a surprisingly realistic picture. It is important that this conjectured extrapolation depends on a description in terms of *observables* (compare ref. [14]), whose role in the Schwinger model was emphasized by Lowenstein and Swieca [41]. Accordingly, in qcd, we also expect the analogue(s) of the condition $Z = 0$ due to the massless gluons; the Schwinger model is, however, canonical, as explicitly verified in [73]. This is due to the fact that the "dressing" of the electrons by the photons is such that, apart from the θ angle characterizing the irreducible representation, they entirely disappear from the picture, originating a free theory of (massive) photons in Fock space, and thereby accounting for the model's dynamical solubility. An example of a canonical interacting field theory, for entirely different reasons, is given in [32].

The proposed framework poses several difficult problems. The construction of the "dressed" observables remains open even in qed_{1+3}, in spite of the results in [35]. In qcd, the methods of [35] are not directly applicable, due to the gluon self-interaction.

In [59], there is a final remark: "After almost a century of existence the main question about quantum field theory seems still to be: what does it really describe? and not yet: does it provide a good description of nature?". The fact that all but the lightest particles are unstable and there is as yet no rigorous model in quantum field theory to describe them (see the end of Section 4) is a clear instance of the fact that quantum field theory has lost contact with its prime object of study, the elementary particles, and, therefore, with nature itself. As an important subarea of mathematical physics, it seems to have moved in the direction contrary to Geoffrey Sewell's suggestion that "in the words 'mathematical physics', 'physics' is the noun and 'mathematical' is the adjective". If this tendency can be inverted, it may even be hoped that, in spite of all the difficulties—those mentioned above, concerning our approach, as well as others ([22,34,43])—the question whether we should believe in quantum field theory posed by Wightman in the title of [70] may be answered in the affirmative.

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