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Accuracy of classical conductivity theory at atomic scales for free fermions in disordered media

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A B S T R A C T

The growing need for smaller electronic components has recently sparked the interest in the breakdown of the classical conductivity theory near the atomic scale, at which quantum effects should dominate. In 2012, experimental measurements of electric resistance of nanowires in Si doped with phosphorus atoms demonstrate that quantum effects on charge transport almost disappear for nanowires of lengths larger than a few nanometers, even at very low temperature (4.2 K). We mathematically prove, for non-interacting lattice fermions with disorder, that quantum uncertainty of microscopic electric current density around their (classical) macroscopic values is suppressed, exponentially fast with respect to the volume of the region of the lattice where an external electric field is applied. This is in accordance with the above experimental observation. Disorder is modeled by a random external potential along with random, complex-valued, hopping amplitudes. The celebrated tight-binding Anderson model is one particular example of the general case considered here. Our mathematical analysis is based on Combes–Thomas estimates, the Akcoglu–Krengel ergodic theorem, and the large deviation formalism, in particular the Gärtner–Ellis theorem.

R É S U M É

Le besoin croissant de composants électroniques de plus en plus miniatures a rendu incontournable la connaissance des limites de la théorie classique de la conductivité électrique, sachant que les effets quantiques devraient dominer à l'échelle atomique. En 2012, une mesure expérimentale de la résistance électrique de fils nanoscopiques composés de silicium dopé avec des atomes de phosphore démontre que les effets quantiques disparaissent pour des fils de seulement quelques nanomètres, et cela même à très basses températures (4.2 K). Nous montrons mathématiquement, pour des fermions libres dans un milieu désordonné sur réseaux,

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que l'incertitude quantique de la densité de courants microscopiques autour de sa valeur macroscopique classique décroit exponentiellement avec le volume de la région où le champ électrique est appliqué. Ceci corrobore l'expérience de 2012. Le désordre est modélisé par un potentiel extérieur aléatoire, mais aussi par des amplitudes aléatoires de saut entre les sites. Le célèbre modèle d'Anderson sur réseaux est juste un exemple particulier du cas général traité ici. Notre analyse mathématique est basée sur l'estimée de Combes–Thomas, le théorème ergodique d'Akcoglu–Krengel, et le formalisme des grandes déviations, en particulier le théorème de Gärtner–Ellis.

1. Introduction

The classical conductivity theory of materials, based on the existence of a well-defined bulk resistivity, was expected to break down as atomic scales and low temperatures are reached, because quantum effects would dominate. In particular, the linear dependence of the resistance as a function of the length of conducting nanowires should be violated at atomic lengths, as explained in [1].

The growing need for smaller electronic components has recently sparked the interest in such a question. For instance, in 2006, the validity of the classical theory was experimentally verified, at room temperature, for nanowires in InAs with lengths down to ~ 200 nm [2]. Indeed, the measured resistivity for the nanowires is 23 Ω/nm, which is very near to the resistivity deduced from bulk properties of the material (24 Ω/nm). See [2, discussions after Eq. (2)]. A few years later, in 2012, the same property was observed [3], even at very low temperature (4.2 K) and lengths down to 20 nm (atomic scale), in experiments on nanowires in Si doped with phosphorus atoms. The breakdown of the classical description of these nanowires is expected [1] to be around ~ 10 nm (at similar temperature) since other experimental studies [4,5] on similar doped Si wires show strong deviations from bulk values of the resistivity around this length scale.

These experimental results demonstrate that quantum effects on charge transport can very rapidly disappear with respect to (w.r.t.) growing space-scales. We mathematically prove this fact by studying the suppression rate of the probability of finding microscopic current densities that differ from the macroscopic one. Observe that [6,7] already proved the convergence of the expectation values of microscopic current densities, but no information about the suppression of quantum uncertainty was obtained in the macroscopic limit.

There is a large mathematical literature on charged transport properties of fermions in disordered media, for instance by Bellissard and Schulz-Baldes in the nineties [8,9] or, more recently, by Klein, Müller and coauthors [10–14]. See also [15,16] and references therein, etc. However, it is not the purpose of this introduction to go into the details of the history of this specific research field. For a (non-exhaustive) historical perspective on linear conductivity (Ohm's law), see, e.g., [17] or our previous papers [6,7,18–22].

In spite of that large mathematical literature on quantum charged transport, the study performed in the current paper covers a completely new theoretical aspect of this problem, not exploited in the available literature, yet. Observe that although we were able in [7] to deal with interacting fermions, in the present paper we restrict ourselves to the non-interacting case, similar to [6]. Within the class of non-interacting particles the considered Hamiltonians are however completely general, since disorder is defined via random potentials and random, complex valued, hopping amplitudes, which are only assumed to have ergodic distributions. The celebrated tight-binding Anderson model is one particular example of the general case analyzed here and models with random vector potentials are also included within the present study.

We prove that quantum uncertainty of microscopic electric current densities (around their classical, macroscopic values) is suppressed, *exponentially fast* w.r.t. the volume $|\Lambda_L| = \mathcal{O}(L^d)$ (in lattice units (l.u.), $d \in \mathbb{N}$ being the space dimension) of the region of the lattice where an external electric field is applied. In order to achieve this, we use the large deviation formalism [23,24], which has been adopted in quantum statistical mechanics since the eighties [25, Section 7]. Other mathematical results which are pivotal in our analysis are the Combes–Thomas estimates [26,27], the Akcoglu–Krengel ergodic theorem [28] and the (Arzelà-) Ascoli theorem [29, Theorem A5]. Indeed, combined with the celebrated Gärtner–Ellis theorem (Theorem B.1), they allow us to prove a large deviation principle (LDP) for the current density distributions, which quantify the probability of deviations, due to quantum uncertainty, from the expected value.

The interacting case, as studied in [7,21], is technically much more involved. The mathematical techniques allowing to tackle such questions for interacting fermions are partially developed in [25,30], and use Grassmann integrals and Brydges–Kennedy tree expansions to construct Gärtner–Ellis generating functions. For the non-interacting case, in order to study properties of Gärtner–Ellis generating functions, one can use the Bogoliubov-type inequality

$$\left|\ln \mathrm{tr}\left(C\mathrm{e}^{H_1}\right) - \ln \mathrm{tr}\left(C\mathrm{e}^{H_0}\right)\right| \leq \sup_{\alpha\in[0,1]} \sup_{u\in[-1/2,1/2]} \left\| \mathrm{e}^{u(\alpha H_1+(1-\alpha)H_0)}\left(H_1 - H_0\right)\mathrm{e}^{-u(\alpha H_1+(1-\alpha)H_0)}\right\|_{\mathcal{B}(\mathbb{C}^n)},$$

where H_0, H_1 are arbitrary self-adjoint matrices, C is any positive matrix and tr denotes the normalized trace. See [31, Lemma 3.6] or Lemma 4.2 below. The above bound turns out to be useful for fermionic systems that are quasi-free (i.e. H_0, H_1 are polynomials of degree two in the fermionic creation and annihilation operators). In this special case, the right-hand side of the inequality can be efficiently bounded by $\|H_1 - H_0\|_{\mathcal{B}(\mathbb{C}^n)}$, using Combes–Thomas estimates. In contrast, for interacting fermions, explicit examples for which the right-hand side is arbitrarily bigger than $\|H_1 - H_0\|_{\mathcal{B}(\mathbb{C}^n)}$ at large volumes are known [32].

Our main results are Theorems 3.1, 3.4 and Corollaries 3.2, 3.5. From the technical point of view, Theorem 3.1 is the pivotal statement of the paper, the other assertions, basically the LDP for currents with a good rate function (Theorem 3.4 and Corollaries 3.2, 3.5), being all deduced from Theorem 3.1 by relatively standard methods of large deviations. Theorem 3.1 refers to the existence, continuity and differentiability of the (infinite volume) deterministic generating function for currents, which appears in the Gärtner–Ellis theorem (Theorem B.1). Besides the Bogoliubov-type inequality, as discussed above, its proof requires the Akcoglu–Krengel ergodic theorem [28] as an important argument, for one has to control the thermodynamical limit of (finite volume) generating functions that are random. To make possible the use of this important result from ergodic theory, various technical preliminaries are needed and the proof of Theorem B.1 is highly non-trivial, as a whole: We perform a rather complicated box decomposition of these random functions, which can be justified with the help of the Bogoliubov-type inequality and the "locality" (or space decay) of both the quasi-free dynamics and space correlations of KMS states, as a consequence of Combes–Thomas estimates (Appendix A).

To conclude, this paper is organized as follows:

- In Section 2, the mathematical setting is described in detail. It refers to quasi-free fermions on the lattice in disordered media. We also discuss the physical motivations of the model, which are supplemented by Appendix C to reduce the length of this section.
- In Section 3, the main results are stated and the large deviation (LD) formalism is shortly defined, being supplemented by Appendix B. More precisely, we present the mathematical statements related to the existence of generating functions of the LD formalism, an LD principle (LDP) for currents, as well as the behavior of the corresponding rate function. We finally combine them to state and discuss the exponentially fast suppression of quantum uncertainty of currents around the classical value of the current.
- Section 4 gathers all technical proofs. In particular, Bogoliubov-type inequalities discussed above are stated and proven in Section 4.1. Section 4.2 collects some useful, albeit elementary, properties of bilinear elements, which are basically quadratic elements in the CAR algebra resulting from the second-quantization of one-particle operators. Then, in Section 4.3, we show that current observable are bilinear elements associated with explicit one-particle operators that satisfy several explicit estimates. These upper bounds are pivotal for the proof of our main theorem, i.e., Theorem 3.1, which, effectively, only starts

in Section 4.4 and is finished in Section 4.5 with the use of the Akcoglu–Krengel ergodic theorem [28] and the (Arzelà-) Ascoli theorem [29, Theorem A5].

- We finally include Appendices A, B and C, stating general results used throughout the current paper, in a way well-adapted to our proofs. Appendix A is about the Combes–Thomas estimates while Appendix B explains the large deviation formalism, in particular the Gärtner–Ellis theorem. Appendix C contains supplementary information on the mathematical framework and relevant physical concepts, in order to make unnecessary the use of further references for a clear understanding of the subject of the current paper. More precisely, Appendix C summaries some important results on linear response current of our papers [6,7,18–22]. Appendix C.2 explains the origin of current observables in relation with the discrete continuity equation within the CAR algebra. Finally, Appendix C.3 makes explicit the link between the algebraic formulation we use here and the (more popular) one-particle Hilbert space formulation of non-interacting fermion systems.

Notation 1.1. *A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The scalar product of any Hilbert space $\mathcal{X}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. Note that $\mathbb{R}^+ \doteq \{x \in \mathbb{R} : x > 0\}$ while $\mathbb{R}_0^+ \doteq \mathbb{R}^+ \cup \{0\}$.*

2. Setup of the problem

We use the mathematical framework of [7,22] to study fermions on the lattice. For simplicity we take a cubic lattice $\mathbb{Z}^d$, even if other types of lattices can certainly be considered with the same, albeit adapted, methods. Disorder within the conductive material, due to impurities, crystal lattice defects, etc., is modeled by (a) a random external potential, like in the celebrated Anderson model, and (b) a random Laplacian, i.e., a self-adjoint operator defined by a next-nearest neighbor hopping term with random complex-valued amplitudes. In particular, random vector potentials can also be implemented.

Altogether, this yields the random tight-binding model mathematically described in Section 2.1: The underlying probability space is defined in Part (ii) of that subsection, while the one-particle Hamiltonian driven the non-interacting (or quasi-free) lattice-fermion system is explained in Part (iii), see in particular Equation (4). Then, we apply on the quasi-free fermion system in disordered media some time-dependent electromagnetic fields and look at the linear response current density in the thermodynamic limit of macroscopic electromagnetic fields. This study is already done in great generality in [7,21,22] and we shortly explain it in Section 2.3, with complementary explanations postponed to Appendix C. Then, we will be in a position to state the main results of the paper about the exponential rate of convergence of current densities in the limit of macroscopic electromagnetic fields.

Observe that *no* interaction between fermions are considered in the sequel and one can do all our study on the one-particle Hilbert space, as illustrated in Appendix C.3. Despite this, our approach is based on the algebraic formulation of fermion systems on lattices explained in Section 2.2 because it makes the role played by many-fermion correlations due to the Pauli exclusion principle, i.e., the antisymmetry of the many-body wave function, more transparent. For instance, the conductivity is naturally defined from current-current correlations, that is, four-point correlation functions, in this framework. The algebraic formulation also allows a clear link between transport properties of fermion systems and the CCR algebra of current fluctuations [20]. The latter is related to non-commutative central limit theorems (see, e.g., [33]). On top of this, the approach ensures a continuity with our previous results while making much clearer its extension to a study of *interacting* fermions for which the algebraic formulation is very advantageous. This paper can thus be seen as a preparation to do a similar study for interacting fermions. Such an analysis has already started with [25,30] via (highly technical) constructive methods used in quantum field theory, which will allow us to obtain convergent expansion schemes around the quasi-free case for generic generating functions.

2.1. Random tight-binding model

(i): The host material for conducting fermions is assumed to be a cubic crystal represented by the d-dimensional cubic lattice $\mathbb{Z}^d$ ($d \in \mathbb{N}$). Below, $\mathcal{P}_\mathrm{f}(\mathbb{Z}^d) \subset 2^{\mathbb{Z}^d}$ is the set of all non-empty *finite* subsets of $\mathbb{Z}^d$. Further,

$$\mathbb{D} \doteq \{z \in \mathbb{C}\colon\ |z| \leq 1\} \quad \text{and} \quad \mathfrak{b} \doteq \big\{\{x, x'\} \subset \mathbb{Z}^d\colon |x - x'| = 1\big\}$$

is the set of (non-oriented) edges of the cubic lattice $\mathbb{Z}^d$.

(ii): Disorder in the crystal is modeled by a random variable taking values in the measurable space $(\Omega, \mathfrak{A}_\Omega)$, with distribution $\mathfrak{a}_\Omega$:

Ω: Elements of Ω are pairs $\omega = (\omega_1, \omega_2) \in \Omega$, where ω_1 is a function on lattice sites with values in the interval $[-1, 1]$ and ω_2 is a function on edges with values in the complex closed unit disc $\mathbb{D}$. I.e.,

$$\Omega \doteq [-1, 1]^{\mathbb{Z}^d} \times \mathbb{D}^{\mathfrak{b}}.$$

$\mathfrak{A}_\Omega$: Let $\Omega_x^{(1)}$, $x \in \mathbb{Z}^d$, be an arbitrary element of the Borel σ-algebra $\mathfrak{A}_x^{(1)}$ of the interval $[-1, 1]$ w.r.t. the usual metric topology. Define

$$\mathfrak{A}_{[-1,1]^{\mathbb{Z}^d}} \doteq \bigotimes_{x \in \mathbb{Z}^d} \mathfrak{A}_x^{(1)},$$

i.e., $\mathfrak{A}_{[-1,1]^{\mathbb{Z}^d}}$ is the σ-algebra generated by the cylinder sets $\prod_{x \in \mathbb{Z}^d} \Omega_x^{(1)}$, where $\Omega_x^{(1)} = [-1, 1]$ for all but finitely many $x \in \mathbb{Z}^d$. In the same way, let

$$\mathfrak{A}_{\mathbb{D}^{\mathfrak{b}}} \doteq \bigotimes_{\mathbf{x} \in \mathfrak{b}} \mathfrak{A}_{\mathbf{x}}^{(2)} ,$$

where $\mathfrak{A}_{\mathbf{x}}^{(2)}$, $\mathbf{x} \in \mathfrak{b}$, is the Borel σ-algebra of the complex closed unit disc $\mathbb{D}$ w.r.t. the usual metric topology. Then

$$\mathfrak{A}_\Omega \doteq \mathfrak{A}_{[-1,1]^{\mathbb{Z}^d}} \otimes \mathfrak{A}_{\mathbb{D}^{\mathfrak{b}}} .$$

$\mathfrak{a}_\Omega$: The distribution $\mathfrak{a}_\Omega$ is an arbitrary *ergodic* probability measure on the measurable space $(\Omega, \mathfrak{A}_\Omega)$. I.e., it is invariant under the action

$$(\omega_1, \omega_2) \longmapsto \chi_x^{(\Omega)}(\omega_1, \omega_2) \doteq \left(\chi_x^{(\mathbb{Z}^d)}(\omega_1), \chi_x^{(\mathfrak{b})}(\omega_2)\right) , \qquad x \in \mathbb{Z}^d , \tag{1}$$

of the group $(\mathbb{Z}^d, +)$ of translations on Ω and $\mathfrak{a}_\Omega(\mathcal{X}) \in \{0, 1\}$ whenever $\mathcal{X} \in \mathfrak{A}_\Omega$ satisfies $\chi_x^{(\Omega)}(\mathcal{X}) = \mathcal{X}$ for all $x \in \mathbb{Z}^d$. Here, for any $\omega = (\omega_1, \omega_2) \in \Omega$, $x \in \mathbb{Z}^d$ and $y, y' \in \mathbb{Z}^d$ with $|y - y'| = 1$,

$$\chi_x^{(\mathbb{Z}^d)}(\omega_1)(y) \doteq \omega_1(y + x) ,\ \chi_x^{(\mathfrak{b})}(\omega_2)(\{y, y'\}) \doteq \omega_2(\{y + x, y' + x\}) . \tag{2}$$

As is usual, $\mathbb{E}[\cdot]$ denotes the expectation value associated with $\mathfrak{a}_\Omega$.

(iii): The one-particle Hilbert space is $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d; \mathbb{C})$ with scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$. Its canonical orthonormal basis is denoted by $\{\mathfrak{e}_x\}_{x \in \mathbb{Z}^d}$, which is defined by $\mathfrak{e}_x(y) \doteq \delta_{x,y}$ for all $x, y \in \mathbb{Z}^d$. ($\delta_{x,y}$ is the Kronecker delta.)

To any $\omega \in \Omega$ and strength $\vartheta \in \mathbb{R}_0^+$ of hopping disorder, we associate a self-adjoint operator $\Delta_{\omega,\vartheta} \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ describing the hoppings of a single particle in the lattice:

$$[\Delta_{\omega,\vartheta}(\psi)](x) \doteq 2d\psi(x) - \sum_{j=1}^{d} \Big((1 + \vartheta\overline{\omega_2(\{x, x-e_j\})})\ \psi(x-e_j) + \psi(x+e_j)(1+\vartheta\omega_2(\{x, x+e_j\})) \Big) \tag{3}$$

for any $x \in \mathbb{Z}^d$ and $\psi \in \ell^2(\mathbb{Z}^d)$, with $\{e_k\}_{k=1}^d$ being the canonical orthonormal basis of the Euclidean space $\mathbb{R}^d$. In the case of vanishing hopping disorder $\vartheta = 0$, (up to a minus sign) $\Delta_{\omega,0}$ is the usual d-dimensional discrete Laplacian. Since the hopping amplitudes are complex-valued (ω_2 takes values in $\mathbb{D}$), note additionally that random vector potentials can be implemented in our model. Then, the random tight-binding model is the one-particle Hamiltonian defined by

$$h^{(\omega)} \doteq \Delta_{\omega,\vartheta} + \lambda\omega_1 \ , \qquad \omega = (\omega_1, \omega_2) \in \Omega,\ \lambda, \vartheta \in \mathbb{R}_0^+, \tag{4}$$

where the function $\omega_1 \colon \mathbb{Z}^d \to [-1, 1]$ is identified with the corresponding (self-adjoint) multiplication operator. We use this operator to define a (infinite volume) dynamics, by the unitary group $\{\mathrm{e}^{ith^{(\omega)}}\}_{t\in\mathbb{R}}$, in the one-particle Hilbert space $\mathfrak{h}$. Note that the tight-binding Anderson model corresponds to the special case $\vartheta = 0$.

(iv): Let

$$\mathfrak{Z} \doteq \Big\{ \mathcal{Z} \subset 2^{\mathbb{Z}^d} \colon (\forall Z_1, Z_2 \in \mathcal{Z})\ Z_1 \neq Z_2 \Longrightarrow Z_1 \cap Z_2 = \emptyset \Big\},$$
$$\mathfrak{Z}_{\mathrm{f}} \doteq \{ \mathcal{Z} \in \mathfrak{Z} \colon |\mathcal{Z}| < \infty \text{ and } (\forall Z \in \mathcal{Z})\ 0 < |Z| < \infty \}.$$

One can restrict the dynamics to collections $\mathcal{Z} \in \mathfrak{Z}$ of disjoint subsets of the lattice by using the orthogonal projections P_Λ, $\Lambda \subset \mathbb{Z}^d$, defined on $\mathfrak{h}$ by

$$[P_\Lambda(\varphi)](x) \doteq \begin{cases} \varphi(x), & \text{if } x \in \Lambda. \\ 0, & \text{else.} \end{cases} \tag{5}$$

Then, the one-particle Hamiltonian within $\mathcal{Z} \in \mathfrak{Z}$ is

$$h_{\mathcal{Z}}^{(\omega)} \doteq \sum_{Z \in \mathcal{Z}} P_Z h^{(\omega)} P_Z, \tag{6}$$

leading to the unitary group $\{\mathrm{e}^{ith_{\mathcal{Z}}^{(\omega)}}\}_{t\in\mathbb{R}}$. This kind of decomposition over collections of disjoint subsets of the lattice is important in the technical proofs.

(v): By the Combes–Thomas estimate (Appendix A),

$$\left| \left\langle \mathfrak{e}_x, \mathrm{e}^{ith_{\mathcal{Z}}^{(\omega)}} \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \leq 36\mathrm{e}^{|t\eta| - 2\mu_\eta |x-y|} \tag{7}$$

for any $\eta, \mu \in \mathbb{R}^+$, $x, y \in \mathbb{Z}^d$, $\mathcal{Z} \in \mathfrak{Z}$, $\omega \in \Omega$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, where

$$\mu_\eta \doteq \mu \min\left\{ \frac{1}{2}, \frac{\eta}{8d\,(1+\vartheta)\,\mathrm{e}^{\mu}} \right\}. \tag{8}$$

See Corollary A.3, by observing that the parameter $\mathbf{S}$ defined by (A.4) is bounded in this case by $\mathbf{S}(h_{\mathcal{Z}}^{(\omega)}, \mu) \leq 2d(1+\vartheta)\mathrm{e}^{\mu}$.

2.2. Algebraic setting

Although all the problem can be formulated, in a mathematically equivalent way, in the one-particle (or Hilbert space) setting (Appendix C.3), since the underlying physical system is a many-body one, it is conceptually more appropriate to state the large deviation principle (LDP) related to microscopic current densities within the algebraic formulation for lattice fermion systems:

(i): We denote by $\mathcal{U} \equiv \mathcal{U}_{\mathfrak{h}}$ the CAR C^*-algebra generated by the identity $\mathfrak{1}$ and elements $\{a(\psi)\}_{\psi\in\mathfrak{h}}$ satisfying the canonical anticommutation relations (CAR): For all $\psi, \varphi \in \mathfrak{h}$,

$$a(\psi)a(\varphi) = -a(\varphi)a(\psi), \quad a(\psi)a(\varphi)^* + a(\varphi)^*a(\psi) = \langle \psi, \varphi \rangle_{\mathfrak{h}} \mathfrak{1}. \tag{9}$$

Note that CAR imply that, for all $\psi \in \mathfrak{h}$,

$$\|a(\psi)\|_{\mathcal{U}} \leq \|\psi\|_{\mathfrak{h}} , \tag{10}$$

and the map $\psi \mapsto a(\psi)^*$ from $\mathfrak{h}$ to $\mathcal{U}$ is linear. As is usual, $a(\psi)$ and $a(\psi)^*$ are called, respectively, annihilation and creation operators.

(ii): For all $\omega \in \Omega$ and $\lambda, \vartheta \in \mathbb{R}_0^+$, the dynamics on the CAR C^*-algebra $\mathcal{U}$ is defined by a strongly continuous group $\tau^{(\omega)} \doteq \{\tau_t^{(\omega)}\}_{t\in\mathbb{R}}$ of (Bogoliubov) $*$-automorphisms of $\mathcal{U}$ satisfying

$$\tau_t^{(\omega)}(a(\psi)) = a(\mathrm{e}^{ith^{(\omega)}}\psi) , \qquad t \in \mathbb{R}, \ \psi \in \mathfrak{h}. \tag{11}$$

See (4) as well as [34, Theorem 5.2.5] for more details on Bogoliubov automorphisms. Similarly, for any $\mathcal{Z} \in \mathfrak{Z}$, we define the strongly continuous group $\tau^{(\omega,\mathcal{Z})}$ by replacing $h^{(\omega)}$ in (11) with $h_{\mathcal{Z}}^{(\omega)}$ (see (6)). In order to define the thermodynamic limit, we introduce the increasing family

$$\Lambda_\ell \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \colon |x_1|, \ldots, |x_d| \leq \ell\}, \qquad \ell \in \mathbb{R}_0^+, \tag{12}$$

in $\mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d)$. Observe that, for any $t \in \mathbb{R}$, $\tau_t^{(\omega,\{\Lambda_\ell\})}$ converges strongly to $\tau_t^{(\omega)} \equiv \tau_t^{(\omega,\{\mathbb{Z}^d\})}$, as $\ell \to \infty$.

(iii): For any realization $\omega \in \Omega$ and disorder strengths $\lambda, \vartheta \in \mathbb{R}_0^+$, the thermal equilibrium state of the system at inverse temperature $\beta \in \mathbb{R}^+$ (i.e., $\beta > 0$) is by definition the unique $(\tau^{(\omega)}, \beta)$-KMS state $\varrho^{(\omega)}$, see [34, Example 5.3.2.] or [35, Theorem 5.9]. It is well-known that such a state is stationary w.r.t. the dynamics $\tau^{(\omega)}$, that is,

$$\varrho^{(\omega)} \circ \tau_t^{(\omega)} = \varrho^{(\omega)} , \qquad \omega \in \Omega, \ t \in \mathbb{R}. \tag{13}$$

The state $\varrho^{(\omega)}$ is also *gauge-invariant* and *quasi-free*, and it satisfies

$$\varrho^{(\omega)}(a^*(\varphi)\, a(\psi)) = \left\langle \psi, \frac{1}{1 + \mathrm{e}^{\beta h^{(\omega)}}}\varphi \right\rangle_{\mathfrak{h}} , \qquad \varphi, \psi \in \mathfrak{h}. \tag{14}$$

For $\beta = 0$, one gets the tracial state (or chaotic state), denoted by $\mathrm{tr} \in \mathcal{U}^*$.

Recall that gauge-invariant quasi-free states are positive linear functionals $\rho \in \mathcal{U}^*$ such that $\rho(\mathfrak{1}) = 1$ and, for all $N_1, N_2 \in \mathbb{N}$ and $\psi_1, \ldots, \psi_{N_1+N_2} \in \mathfrak{h}$,

$$\rho\left(a^*(\psi_1) \cdots a^*(\psi_{N_1}) a(\psi_{N_1+N_2}) \cdots a(\psi_{N_1+1})\right) = 0 \tag{15}$$

if $N_1 \neq N_2$, while in the case $N_1 = N_2 \equiv N$,

$$\rho\left(a^*(\psi_1)\cdots a^*(\psi_N)a(\psi_{2N})\cdots a(\psi_{N+1})\right) = \det\left[\rho\left(a^+(\psi_k)a(\psi_{N+l})\right)\right]_{k,l=1}^{N}. \tag{16}$$

See, e.g., [36, Definition 3.1], which refers to a more general notion of quasi-free states. The gauge-invariant property corresponds to Equation (15) whereas [36, Definition 3.1, Condition (3.1)] only imposes the quasi–free state to be even, which is a strictly weaker property than being gauge-invariant.

Similarly, for any $\mathcal{Z} \in \mathfrak{Z}$, we define the quasi-free state $\varrho_{\mathcal{Z}}^{(\omega)}$ by replacing $h^{(\omega)}$ in (14) with $h_{\mathcal{Z}}^{(\omega)}$ (see (6)). In the thermodynamic limit $\ell \to \infty$, $\varrho_{\{\Lambda_\ell\}}^{(\omega)}$ converges in the weak* topology to $\varrho^{(\omega)} \equiv \varrho_{\{\mathbb{Z}^d\}}^{(\omega)}$.

2.3. Current densities

(i) Currents: Fix $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$. For any oriented edge $(x, y) \in \left(\mathbb{Z}^d\right)^2$, we define the paramagnetic current observable by

$$I_{(x,y)}^{(\omega)} \doteq -2\Im\mathrm{m}\left(\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle_{\mathfrak{h}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_y)\right). \tag{17}$$

It is seen as a current because it satisfies a discrete continuity equation, as explained in Appendix C.2. Here, the self-adjoint operators $\Im\mathrm{m}(A) \in \mathcal{U}$ and $\Re\mathrm{e}(A) \in \mathcal{U}$ are the *imaginary* and *real parts* of $A \in \mathcal{U}$, that are, respectively,

$$\Im\mathrm{m}\,(A) \doteq \frac{1}{2i}\left(A - A^*\right) \quad \text{and} \quad \Re\mathrm{e}\,(A) \doteq \frac{1}{2}\left(A + A^*\right). \tag{18}$$

This "second-quantized" definition of current observable and the usual one in the one-particle setting, like in [8,10,11], are perfectly equivalent, in the case of non-interacting fermions. See for instance Equation (C.19).

Note that electric fields accelerate charged particles and induce so-called diamagnetic currents, which correspond to the ballistic movement of particles. In fact, as explained in [19, Sections III and IV], this component of the total current creates a kind of "wave front" that destabilizes the whole system by changing its state. The presence of diamagnetic currents leads then to the progressive appearance of paramagnetic currents which are responsible for heat production and the in-phase AC-conductivity of the system. For more details, see [7,19,21] as well as Appendix C on linear response currents.

(ii) Conductivity: As is usual, $[A, B] \doteq AB - BA \in \mathcal{U}$ denotes the commutator between the elements $A \in \mathcal{U}$ and $B \in \mathcal{U}$. For any finite subset $\Lambda \in \mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d)$, we define the space-averaged transport coefficient observable $\mathcal{C}_\Lambda^{(\omega)} \in C^1(\mathbb{R}; \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d))$, w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of the Euclidean space $\mathbb{R}^d$, by the corresponding matrix entries

$$\begin{aligned}\left\{\mathcal{C}_\Lambda^{(\omega)}(t)\right\}_{k,q} \doteq\ & \frac{1}{|\Lambda|}\sum_{x,y,x+e_k,y+e_q\in\Lambda}\int_0^t i[\tau_{-\alpha}^{(\omega)}(I_{(y+e_q,y)}^{(\omega)}), I_{(x+e_k,x)}^{(\omega)}]\mathrm{d}\alpha \\ & + \frac{2\delta_{k,q}}{|\Lambda|}\sum_{x\in\Lambda}\Re\mathrm{e}\left(\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle a(\mathfrak{e}_{x+e_k})^* a(\mathfrak{e}_x)\right)\end{aligned} \tag{19}$$

for any $\omega \in \Omega$, $t \in \mathbb{R}$, $\lambda, \vartheta \in \mathbb{R}_0^+$ and $k, q \in \{1, \ldots, d\}$. This object is the conductivity observable matrix associated with the lattice region Λ and time t. See Appendix C, in particular Equations (C.8)–(C.9). In fact, the first term in the right-hand side of (19) corresponds to the paramagnetic coefficient, whereas the second one is the diamagnetic component. For more details, see [21, Theorem 3.7].

(iii) Linear response current density: Fix a direction $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and a (time-dependent) continuous, compactly supported, electric field $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, i.e., the external electric field is a continuous function $t \mapsto \mathcal{E}(t) \in \mathbb{R}^d$ of time $t \in \mathbb{R}$ with compact support. Then, as it is explained in Appendix C, [7,21][1]

[1] Strictly speaking, these papers use smooth electric fields, but the extension to the continuous case is straightforward.

shows that the space-averaged linear response current observable in the lattice region Λ and at time $t = 0$ in the direction $\vec{w}$ is equal to

$$\mathbb{I}_{\Lambda}^{(\omega,\mathcal{E})} \doteq \sum_{k,q=1}^{d} w_k \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \left\{\mathcal{C}_{\Lambda}^{(\omega)}(-\alpha)\right\}_{k,q} \mathrm{d}\alpha \ . \tag{20}$$

To obtain the current density at any time $t \in \mathbb{R}$, it suffices to replace $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ in this equation with

$$\mathcal{E}_t(\alpha) \doteq \mathcal{E}(\alpha + t), \qquad \alpha \in \mathbb{R}. \tag{21}$$

Compare with Equations (C.8)–(C.9).

3. Main results

We study large deviations (LD) for the microscopic current density produced by any fixed, time-dependent electric field $\mathcal{E}$. Via the Gärtner–Ellis theorem (see, e.g., [24, Corollary 4.5.27]), this is a consequence of the following result:

Theorem 3.1 *(Generating functions for currents). There is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, the limit*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)}\left(\mathrm{e}^{|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}\right)$$

exists and equals

$$\mathrm{J}^{(\mathcal{E})} \doteq \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\left[\ln \varrho^{(\cdot)}\left(\mathrm{e}^{|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\cdot,\mathcal{E})}}\right)\right].$$

Moreover, for any $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable and convex.

Proof. The assertions directly follow from Corollaries 4.19 and 4.20. Note that the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ is a limit of convex functions, and hence, it is also convex. □

In probability theory, the law of large numbers refers to the convergence (at least in probability), as $n \to \infty$, of the average or empirical mean of n independent identically distributed (i.i.d.) random variables towards their expected value (assuming it exists). The large deviation formalism quantitatively describes, for large $n \gg 1$, the probability of finding an empirical mean that differs from the expected value. These are *rare* events, by the law of large numbers, and an LD principle (LDP) gives their probability as exponentially small (w.r.t. some speed) in the limit $n \to \infty$.

In the context of the algebraic formulation of quantum mechanics, observables (i.e., self-adjoint elements of some C^*-algebra, here $\mathcal{U}$) generalize the notion of random variables of classical probability theory. The link between both notions is given via the Riesz–Markov theorem and functional calculus: The commutative C^*-subalgebra of $\mathcal{U}$ generated by any self-adjoint element $A^* = A \in \mathcal{U}$ is isomorphic to the algebra of continuous functions on the compact set $\mathrm{spec}(A) \subset \mathbb{R}$. Hence, by the Riesz–Markov theorem, for any state $\rho \in \mathcal{U}^*$, there is a unique probability measure $\mathfrak{m}_{\rho,A}$ on $\mathbb{R}$ such that

$$\mathfrak{m}_{\rho,A}(\mathrm{spec}(A)) = 1 \qquad \text{and} \qquad \rho(f(A)) = \int_{\mathbb{R}} f(x)\mathfrak{m}_{\rho,A}(\mathrm{d}x) \tag{22}$$

for all complex-valued continuous functions $f \in C(\mathbb{R};\mathbb{C})$. $\mathfrak{m}_{\rho,A}$ is called the *distribution* of the observable A in the state ρ. The LD formalism naturally arises also in this more general framework: A *rate* function is a lower semi-continuous function $\mathrm{I}:\mathbb{R}\to[0,\infty]$. If I is not the ∞ constant function and has compact level sets, i.e., if $\mathrm{I}^{-1}([0,m])=\{x\in\mathbb{R}\colon \mathrm{I}(x)\leq m\}$ is compact for any $m\geq 0$, then one says that I is a *good* rate function. A sequence $(A_L)_{L\in\mathbb{N}}\subset\mathcal{U}$ of observables satisfies an LDP, in a state $\rho\in\mathcal{U}^*$, with speed $(\mathfrak{n}_L)_{L\in\mathbb{N}}\subset\mathbb{R}^+$ (a positive, increasing and divergent sequence) and rate function I if, for any Borel subset $\mathcal{G}$ of $\mathbb{R}$,

$$-\inf_{x\in\mathcal{G}^\circ}\mathrm{I}(x)\leq\liminf_{L\to\infty}\frac{1}{\mathfrak{n}_L}\ln\mathfrak{m}_{\rho,A_L}(\mathcal{G})\leq\limsup_{L\to\infty}\frac{1}{\mathfrak{n}_L}\ln\mathfrak{m}_{\rho,A_L}(\mathcal{G})\leq-\inf_{x\in\bar{\mathcal{G}}}\mathrm{I}(x).$$

Here, $\mathcal{G}^\circ$ is the interior of $\mathcal{G}$, while $\bar{\mathcal{G}}$ is its closure. Compare with Equations (B.1)–(B.2) in Appendix B.

A sufficient condition to ensure that a sequence of observables satisfies an LDP is given by the Gärtner–Ellis theorem. In particular, Theorem 3.1 combined with Theorem B.1 yields the following corollary:

Corollary 3.2 *(Large deviation principle for currents). Let $\tilde{\Omega}\subset\Omega$ be the measurable subset of full measure of Theorem 3.1. Then, for all $\beta\in\mathbb{R}^+$, $\vartheta,\lambda\in\mathbb{R}_0^+$, $\omega\in\tilde{\Omega}$, $l\in\mathbb{N}$, $\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\vec{w}\in\mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d}=1$, the sequence $(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})_{L\in\mathbb{N}}$ of microscopic current densities satisfies an LDP, in the KMS state $\varrho^{(\omega)}$, with speed $|\Lambda_L|$ and good rate function $\mathrm{I}^{(\mathcal{E})}$ defined on $\mathbb{R}$ by*

$$\mathrm{I}^{(\mathcal{E})}(x)\doteq\sup_{s\in\mathbb{R}}\left\{sx-\mathrm{J}^{(s\mathcal{E})}\right\}\geq 0.$$

Remark 3.3. By direct estimates, one verifies that, for any fixed state ρ, $(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})_{L\in\mathbb{N}}$ yields an exponentially tight family of probability measures, defined by (22) for $A=\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}$. Therefore, by [24, Lemma 4.1.23], $(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})})_{L\in\mathbb{N}}$ satisfies, along some subsequence, an LDP, *in any state* ρ, with speed $|\Lambda_L|$ and a good rate function. However, it is not clear whether this rate function depends on the choice of subsequences and $\omega\in\Omega$. Moreover, no information on minimizers of the rate function, like in Theorem 3.4, can be deduced from [24, Lemma 4.1.23].

Observe that, if an LDP holds true, then the law of large numbers follows [37, Theorem II.6.4] from the Borel–Cantelli lemma [37, Lemma A.5.2]. Therefore, by [6,7] and Corollary 3.2, the distributions of the microscopic current density observables, in the state $\varrho^{(\omega)}$, weak* converges, for $\omega\in\Omega$ almost surely, to the delta distribution at the (classical value of the) macroscopic current density. Using Theorem 3.1, we sharpen this result by proving that the microscopic current density converges *exponentially fast* to the macroscopic one, w.r.t. the volume $|\Lambda_L|$ of the region of the lattice where an external electric field is applied.

To this end, we remark from Corollary 4.20 (see (56)) that, for any $\beta\in\mathbb{R}^+$, $\vartheta,\lambda\in\mathbb{R}_0^+$, $\vec{w}\in\mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d}=1$, the macroscopic current density is equal to

$$x^{(\mathcal{E})}\doteq\partial_s\mathrm{J}^{(s\mathcal{E})}|_{s=0},\qquad\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d).\tag{23}$$

See also (C.10). Define

$$x_-\doteq\inf\left\{x\leq x^{(\mathcal{E})}\colon\mathrm{I}^{(\mathcal{E})}(x)<\infty\right\},\ x_+\doteq\sup\left\{x\geq x^{(\mathcal{E})}\colon\mathrm{I}^{(\mathcal{E})}(x)<\infty\right\}.$$

Obviously, $\mathrm{I}^{(\mathcal{E})}(x)=\infty$ for $x\in\mathbb{R}\backslash[x_-,x_+]$. We start by giving important properties of the rate function $\mathrm{I}^{(\mathcal{E})}$:

Theorem 3.4 *(Properties of the rate function). Fix $\beta\in\mathbb{R}^+$, $\vartheta,\lambda\in\mathbb{R}_0^+$, $\vec{w}\in\mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d}=1$ and $\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d)$. The rate function $\mathrm{I}^{(\mathcal{E})}$ is a lower-semicontinuous convex function satisfying: (i) $\mathrm{I}^{(\mathcal{E})}(x^{(\mathcal{E})})=0$;*

(ii) $\mathrm{I}^{(\mathcal{E})}(x) > 0$ *if* $x \neq x^{(\mathcal{E})}$; *(iii)* $\mathrm{I}^{(\mathcal{E})}(x) < \infty$ *for* $x \in (x_-, x_+)$ *with* $\mathrm{I}^{(\mathcal{E})}(x) \leq \mathrm{I}^{(\mathcal{E})}(x_-)$ *for* $x \in (x_-, x^{(\mathcal{E})}]$ *and* $\mathrm{I}^{(\mathcal{E})}(x) \leq \mathrm{I}^{(\mathcal{E})}(x_+)$ *for* $x \in [x^{(\mathcal{E})}, x_+)$; *(iv)* $\mathrm{I}^{(\mathcal{E})}$ *restricted to the interior of its domain, i.e., the (possibly empty) open interval* (x_-, x_+)*, is continuous.*

Proof. Fix all parameters of the theorem. Note that $\mathrm{I}^{(\mathcal{E})}$ is clearly a lower-semicontinuous convex function, by construction. As the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ is differentiable and convex (Theorem 3.1), the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ is the Legendre–Fenchel transform of $\mathrm{I}^{(\mathcal{E})}$, i.e.,

$$\mathrm{J}^{(s\mathcal{E})} = \sup_{x\in\mathbb{R}} \left\{ sx - \mathrm{I}^{(\mathcal{E})}(x) \right\}, \qquad s \in \mathbb{R},$$

and s_0 is a solution of the variational problem

$$\mathrm{I}^{(\mathcal{E})}(x) \doteq \sup_{s\in\mathbb{R}} \left\{ sx - \mathrm{J}^{(s\mathcal{E})} \right\}$$

if and only if s_0 solves $x = \partial_s \mathrm{J}^{(s\mathcal{E})}|_{s=s_0}$. By (23), it follows that

$$0 = \mathrm{J}^{(0)} = \inf_{x\in\mathbb{R}} \mathrm{I}^{(\mathcal{E})}(x) = \mathrm{I}^{(\mathcal{E})}(x^{(\mathcal{E})}).$$

This proves Assertion (i). To prove (ii), it suffices to show that $x^{(\mathcal{E})}$ is the only minimizer of $\mathrm{I}^{(\mathcal{E})}$. Note that x_0 is a minimizer of $\mathrm{I}^{(\mathcal{E})}$ if and only if 0 is a subdifferential of $\mathrm{I}^{(\mathcal{E})}$ at x_0 (Fermat's principle). By [38, Corollary 5.3.3] and the differentiability of the Legendre transform of $\mathrm{I}^{(\mathcal{E})}$, which is the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$, it follows that the minimizer of $\mathrm{I}^{(\mathcal{E})}$ is unique and Assertion (ii) follows. Assertion (iii) is a direct consequence of the fact that $\mathrm{I}^{(\mathcal{E})}$ is a convex function with $x^{(\mathcal{E})}$ as unique minimizer. Assertion (iv) is deduced from [38, Corollary 2.1.3]. □

Corollary 3.5 *(Exponentially fast suppression of quantum uncertainty of currents). Let* $\tilde{\Omega} \subset \Omega$ *be the measurable subset of full measure of Theorem 3.1. Then, for all* $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\vec{w} \in \mathbb{R}^d$ *with* $\|\vec{w}\|_{\mathbb{R}^d} = 1$*, and any open subset* $\mathcal{O} \subset \mathbb{R}$ *with* $x^{(\mathcal{E})} \notin \bar{\mathcal{O}}$*,*

$$\limsup_{L\to\infty} \frac{1}{|\Lambda_L|} \ln \mathfrak{m}_{\varrho^{(\omega)},\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}(\mathcal{O}) < 0.$$

The above limit does not depend on the particular realization of $\omega \in \tilde{\Omega}$*. If, additionally,* $\mathcal{O} \cap (x_-, x_+) \neq \emptyset$*, then*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \ln \mathfrak{m}_{\varrho^{(\omega)},\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}(\mathcal{O}) = -\inf_{x\in\mathcal{O}} \mathrm{I}^{(\mathcal{E})}(x) < 0.$$

See (22) for the definition of the distribution of $\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}$*, in the KMS state* $\varrho^{(\omega)}$*.*

Proof. It is a direct consequence of Corollary 3.2 and Theorem 3.4. □

Corollary 3.5 shows that the microscopic current density converges *exponentially fast* to the macroscopic one, w.r.t. the volume $|\Lambda_L|$ (in lattice units (l.u.)) of the region of the lattice where the electric field is applied. As discussed in the introduction, this is in accordance with the low temperature (4.2 K) experiment [3] on the resistance of nanowires with lengths down to approximately 40 l.u. ($L \simeq 20$). The breakdown of the classical description of these nanowires is expected [1,4,5] to be around 20 l.u. ($L \simeq 10$).

To conclude, note that, in the experimental setting of [2,3], contacts are used to impose an electric potential difference to the nanowires. These contacts yield supplementary resistances to the systems that are well-described by Landauer's formalism [46] when a *ballistic* charge transport takes place in the nanowires. In our model, the purely ballistic charge transport is reached when $\vartheta = 0$ and $\lambda \to 0^+$, as proven in [20, Theorem 4.6]. When the nanowire resistance becomes relatively small as compared to the contact resistances, then the charge transport in the nanowire is well-described by a ballistic approximation and Landauer's formalism applies, as also experimentally verified in [2]. This is the reason why [3] reaches much smaller length scales than [2]: the material used in [3] has a much larger linear resistivity (between 112 Ω/nm and 855 Ω/nm, see [3, Table 1]) than the one of [2] (23 Ω/nm, see [2, discussions after Eq. (2)]).

4. Technical proofs

4.1. Preliminary estimates

We start by giving two general estimates which will be used many times afterwards. The first one is an elementary observation:

Lemma 4.1 *(Operator norm estimate). For any operator $C \in \mathcal{B}(\mathfrak{h})$,*

$$\|C\|_{\mathcal{B}(\mathfrak{h})} \leq \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right| .$$

Proof. By the Cauchy–Schwarz inequality, for all $\varphi, \psi \in \mathfrak{h}$,

$$\begin{aligned}
\left|\langle \varphi, C\psi\rangle_{\mathfrak{h}}\right| &\leq \sum_{x,y\in\mathbb{Z}^d} \left|\varphi(x)\psi(y)\left\langle \mathfrak{e}_x, C\mathfrak{e}_y\right\rangle_{\mathfrak{h}}\right| \\
&= \sum_{x,y\in\mathbb{Z}^d} \left(|\varphi(x)| \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right|^{1/2}\right)\left(|\psi(y)| \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right|^{1/2}\right) \\
&\leq \sqrt{\sum_{x,y\in\mathbb{Z}^d} \left(|\varphi(x)|^2 \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right|\right)}\sqrt{\sum_{x,y\in\mathbb{Z}^d} |\psi(y)|^2 \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right|} \\
&\leq \|\varphi\|_{\mathfrak{h}} \|\psi\|_{\mathfrak{h}} \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \left|\langle \mathfrak{e}_x, C\mathfrak{e}_y\rangle_{\mathfrak{h}}\right| . \quad \square
\end{aligned}$$

The second one is a version of the Bogoliubov inequality. Recall that the tracial state $\mathrm{tr} \in \mathcal{U}^*$ is the quasi-free state satisfying (14) at $\beta = 0$.

Lemma 4.2 *(Bogoliubov-type inequalities). Let $C \in \mathcal{U}$ be any strictly positive element.*

(i) *For any continuously differentiable family $\{H_\alpha\}_{\alpha\in\mathbb{R}} \subset \mathcal{U}$ of self-adjoint elements,*

$$\left|\partial_\alpha \ln \mathrm{tr}\left(C\mathrm{e}^{H_\alpha}\right)\right| \leq \sup_{u\in[-1/2,1/2]} \left\|\mathrm{e}^{uH_\alpha}\left\{\partial_\alpha H_\alpha\right\}\mathrm{e}^{-uH_\alpha}\right\|_{\mathcal{U}} .$$

(ii) *Similarly, for any self-adjoint $H_0, H_1 \in \mathcal{U}$,*

$$\left|\ln \mathrm{tr}\left(C\mathrm{e}^{H_1}\right) - \ln \mathrm{tr}\left(C\mathrm{e}^{H_0}\right)\right| \leq \sup_{\alpha\in[0,1]} \sup_{u\in[-1/2,1/2]} \left\|\mathrm{e}^{u(\alpha H_1+(1-\alpha)H_0)}\left(H_1 - H_0\right)\mathrm{e}^{-u(\alpha H_1+(1-\alpha)H_0)}\right\|_{\mathcal{U}} .$$

Proof. (i) By Duhamel's formula, for any continuously differentiable family $\{H_\alpha\}_{\alpha\in\mathbb{R}} \subset \mathcal{U}$ of self-adjoint elements,

$$\partial_\alpha \left\{ \mathrm{e}^{H_\alpha} \right\} = \int_0^1 \mathrm{e}^{uH_\alpha} \left\{ \partial_\alpha H_\alpha \right\} \mathrm{e}^{(1-u)H_\alpha} \mathrm{d}u,$$

which implies that

$$\partial_\alpha \ln \mathrm{tr} \left(C\mathrm{e}^{H_\alpha} \right) = \int_0^1 \frac{\mathrm{tr} \left(C\mathrm{e}^{uH_\alpha} \left\{ \partial_\alpha H_\alpha \right\} \mathrm{e}^{(1-u)H_\alpha} \right)}{\mathrm{tr} \left(C\mathrm{e}^{H_\alpha} \right)} \mathrm{d}u.$$

Using the cyclicity of the trace, we then get

$$\begin{aligned}\partial_\alpha \ln \mathrm{tr} \left(C\mathrm{e}^{H_\alpha} \right) &= \int_0^1 \frac{\mathrm{tr} \left(\mathrm{e}^{\frac{H_\alpha}{2}} C \mathrm{e}^{\frac{H_\alpha}{2}} \mathrm{e}^{(u-\frac{1}{2})H_\alpha} \left\{ \partial_\alpha H_\alpha \right\} \mathrm{e}^{(\frac{1}{2}-u)H_\alpha} \right)}{\mathrm{tr} \left(\mathrm{e}^{\frac{H_\alpha}{2}} C \mathrm{e}^{\frac{H_\alpha}{2}} \right)} \mathrm{d}u \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{tr} \left(\mathrm{e}^{\frac{H_\alpha}{2}} C \mathrm{e}^{\frac{H_\alpha}{2}} \mathrm{e}^{uH_\alpha} \left\{ \partial_\alpha H_\alpha \right\} \mathrm{e}^{-uH_\alpha} \right)}{\mathrm{tr} \left(\mathrm{e}^{\frac{H_\alpha}{2}} C \mathrm{e}^{\frac{H_\alpha}{2}} \right)} \mathrm{d}u,\end{aligned}$$

which yields (i).

(ii) To prove the second assertion, it suffices to apply Assertion (i) to the family defined by

$$H_\alpha = H_0 + \alpha \left(H_1 - H_0 \right), \qquad \alpha \in [0,1]. \quad \square$$

Observe that Lemma 4.2 (ii) is proven in [31, Lemma 3.6]. Here, we give a proof of this estimate for completeness. These Bogoliubov-type inequalities are useful because we deal with quasi-free dynamics. In this case, we have a very good control on the norm of

$$\mathrm{e}^{uH_\alpha} \left\{ \partial_\alpha H_\alpha \right\} \mathrm{e}^{-uH_\alpha},$$

because H_α is a bilinear element, as explained in the next subsection.

4.2. Bilinear elements of CAR algebra

Similar to [39], bilinear elements are defined as follows:

Definition 4.3 *(Bilinear elements).* Fix an operator $C \in \mathcal{B}(\mathfrak{h})$ whose range $\mathrm{ran}(C)$ is finite dimensional. Given any finite-dimensional subspace $\mathcal{H} \subset \mathfrak{h}$, with orthonormal basis $\{\psi_i\}_{i\in I}$, such that $\mathcal{H} \supseteq \mathrm{ran}(C)$ and $\mathcal{H} \supseteq \mathrm{ran}(C^*)$, we define the bilinear element associated with C to be

$$\langle \mathrm{A}, C\mathrm{A} \rangle \doteq \sum_{i,j\in I} \langle \psi_i, C\psi_j \rangle_{\mathfrak{h}} \, a(\psi_i)^* a(\psi_j).$$

Note that such a finite dimensional $\mathcal{H}$ in this definition always exists, because

$$\dim (\mathrm{ran}(C)) = \dim (\mathrm{ran}(C^*)) < \infty,$$

and is an invariant space of C containing $(\ker(C))^\perp$. Hence, $\langle \mathrm{A}, C\mathrm{A} \rangle$ does not depend on the particular choice of $\mathcal{H}$ and its orthonormal basis.

Bilinear elements of $\mathcal{U}$ have adjoints equal to

$$\langle \mathrm{A}, C\mathrm{A}\rangle^* = \langle \mathrm{A}, C^*\mathrm{A}\rangle, \tag{24}$$

for any $C \in \mathcal{B}(\mathfrak{h})$ whose range is finite dimensional. In particular,

$$\Im\mathrm{m}\left\{\langle \mathrm{A}, C\mathrm{A}\rangle\right\} = \langle \mathrm{A}, \Im\mathrm{m}\left\{C\right\}\mathrm{A}\rangle, \tag{25}$$

where we recall that $\Im\mathrm{m}(A) \in \mathcal{U}$ is the imaginary part of $A \in \mathcal{U}$, see (18). For any $C \in \mathcal{B}(\mathfrak{h})$ whose range is finite dimensional and any $\varphi \in \mathfrak{h}$, note that

$$[\langle \mathrm{A}, C\mathrm{A}\rangle, a(\varphi)] = -a(C^*\varphi) \qquad \text{and} \qquad \left[\langle \mathrm{A}, C\mathrm{A}\rangle, a(\varphi)^*\right] = a(C\varphi)^*.$$

In particular, for any $C_1, C_2 \in \mathcal{B}(\mathfrak{h})$ whose ranges are finite dimensional,

$$[\langle \mathrm{A}, C_1\mathrm{A}\rangle, \langle \mathrm{A}, C_2\mathrm{A}\rangle] = \langle \mathrm{A}, [C_1, C_2]\,\mathrm{A}\rangle. \tag{26}$$

Moreover, by (11), for any $\varphi \in \mathfrak{h}$ and $C \in \mathcal{B}(\mathfrak{h})$, whose range is finite dimensional,

$$\mathrm{e}^{\langle \mathrm{A}, C\mathrm{A}\rangle} a(\varphi)\, \mathrm{e}^{-\langle \mathrm{A}, C\mathrm{A}\rangle} = a\left(\mathrm{e}^{-C^*}\varphi\right) \qquad \text{and} \qquad \mathrm{e}^{\langle \mathrm{A}, C\mathrm{A}\rangle} a(\varphi)^*\, \mathrm{e}^{-\langle \mathrm{A}, C\mathrm{A}\rangle} = a\left(\mathrm{e}^{C}\varphi\right)^*. \tag{27}$$

Because of the identities (27), bilinear elements can be used to represent the dynamics $\{\tau_t^{(\omega,\mathcal{Z})}\}_{t\in\mathbb{R}}$ for any $\omega \in \Omega$ and $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$. See (11), replacing $h^{(\omega)}$ with $h_{\mathcal{Z}}^{(\omega)}$ (cf. (6)), and observe that the range of $h_{\mathcal{Z}}^{(\omega)} \in \mathcal{B}(\mathfrak{h})$ is finite dimensional whenever $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$. Additionally, by using the tracial state $\mathrm{tr} \in \mathcal{U}^*$, i.e., the quasi-free state satisfying (14) for $\beta = 0$, the corresponding KMS state defined by (14), by replacing $h^{(\omega)}$ in this equation with $h_{\mathcal{Z}}^{(\omega)}$ (see (6)), is explicitly given by

$$\varrho_{\mathcal{Z}}^{(\omega)}(B) = \frac{\mathrm{tr}\left(B\mathrm{e}^{-\beta\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle}\right)}{\mathrm{tr}\left(\mathrm{e}^{-\beta\langle \mathrm{A}, h_{\mathcal{Z}}^{(\omega)}\mathrm{A}\rangle}\right)}, \qquad B \in \mathcal{U}, \tag{28}$$

for any $\omega \in \Omega$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\beta \in \mathbb{R}^+$ and $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$.

We conclude now by an additional observation used later to control quantum fluctuations:

Lemma 4.4. *For any self-adjoint operators $C_1, C_2 \in \mathcal{B}(\mathfrak{h})$ whose ranges are finite dimensional, let $C \doteq \ln\left(\mathrm{e}^{C_2}\mathrm{e}^{C_1}\mathrm{e}^{C_2}\right)$. Then,*

$$\mathrm{ran}(C) \subset \mathrm{lin}\left\{\mathrm{ran}(C_1) \cup \mathrm{ran}(C_2)\right\}$$

and there is a constant $D \in \mathbb{R}$ such that

$$\mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_1\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle} = \mathrm{e}^{\langle \mathrm{A}, C\mathrm{A}\rangle + D\mathbf{1}}.$$

Proof. Fix all parameters of the lemma. We give the proof in two steps:

Step 1: Let

$$\mathfrak{h}_0 \doteq \mathrm{lin}\left\{\mathrm{ran}(C_1) \cup \mathrm{ran}(C_2)\right\}$$

and $\mathcal{U}_{\mathfrak{h}_0} \subset \mathcal{U} \equiv \mathcal{U}_{\mathfrak{h}}$ be the (finite dimensional) CAR C^*-subalgebra generated by the identity $\mathbf{1}$ and $\{a(\varphi)\}_{\varphi\in\mathfrak{h}_0}$. Take two strictly positive elements M_1, M_2 of $\mathcal{U}_{\mathfrak{h}_0}$ satisfying the conditions

$$M_1 a(\varphi) M_1^{-1} = M_2 a(\varphi) M_2^{-1} \qquad \text{and} \qquad M_1 a(\varphi)^* M_1^{-1} = M_2 a(\varphi)^* M_2^{-1}$$

for any $\varphi \in \mathfrak{h}_0$. From this we conclude that

$$M_1 A M_1^{-1} = M_2 A M_2^{-1}, \qquad A \in \mathcal{U}_{\mathfrak{h}_0},$$

because all elements of $\mathcal{U}_{\mathfrak{h}_0}$ are polynomials in $\{a(\varphi), a(\varphi)^*\}_{\varphi\in\mathfrak{h}_0}$, by definition of $\mathcal{U}_{\mathfrak{h}_0}$ and finite dimensionality of $\mathfrak{h}_0$. In particular, by choosing, respectively, $A = M_2^{-1}$ and $A = M_2^{-1} B M_2$ for $B \in \mathcal{U}_{\mathfrak{h}_0}$, it follows that

$$M_1 M_2^{-1} = M_2^{-1} M_1 \qquad \text{and} \qquad M_1 M_2^{-1} B = B M_1 M_2^{-1}.$$

Hence, since any element of $\mathcal{U}_{\mathfrak{h}_0}$ commuting with all elements of $\mathcal{U}_{\mathfrak{h}_0}$ is a multiple of the identity, there is $D \in \mathbb{C}$ such that

$$M_1 M_2^{-1} = M_2^{-1} M_1 = D\mathbf{1}.$$

The constant D is non-zero because M_1, M_2 are assumed to be invertible. In fact, $M_1 = DM_2$ with $D > 0$ because $M_1, M_2 > 0$.

Step 2: Observe that $\mathrm{e}^{C_2}\mathrm{e}^{C_1}\mathrm{e}^{C_2} > 0$ because C_1, C_2 are both self-adjoint operators. In particular, $C \doteq \ln\left(\mathrm{e}^{C_2}\mathrm{e}^{C_1}\mathrm{e}^{C_2}\right)$ is well-defined as a bounded self-adjoint operator acting on $\mathfrak{h}$ with $\mathrm{ran}(C) \subset \mathfrak{h}_0$. Using (27), we obtain that

$$\mathrm{e}^{\langle \mathrm{A}, C\mathrm{A}\rangle} a(\varphi) \mathrm{e}^{-\langle \mathrm{A}, C\mathrm{A}\rangle} = \mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_1\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle} a(\varphi) \mathrm{e}^{-\langle \mathrm{A}, C_2\mathrm{A}\rangle}\mathrm{e}^{-\langle \mathrm{A}, C_1\mathrm{A}\rangle}\mathrm{e}^{-\langle \mathrm{A}, C_2\mathrm{A}\rangle}$$

and

$$\mathrm{e}^{\langle \mathrm{A}, C\mathrm{A}\rangle} a(\varphi)^* \mathrm{e}^{-\langle \mathrm{A}, C\mathrm{A}\rangle} = \mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_1\mathrm{A}\rangle}\mathrm{e}^{\langle \mathrm{A}, C_2\mathrm{A}\rangle} a(\varphi)^* \mathrm{e}^{-\langle \mathrm{A}, C_2\mathrm{A}\rangle}\mathrm{e}^{-\langle \mathrm{A}, C_1\mathrm{A}\rangle}\mathrm{e}^{-\langle \mathrm{A}, C_2\mathrm{A}\rangle}.$$

By Step 1, the assertion follows. □

4.3. Bilinear elements associated with currents

For simplicity, we fix, once and for all, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and $\eta, \mu \in \mathbb{R}^+$. For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, any collection $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}$, $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, we define the observables

$$\begin{aligned}\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq & \sum_{k,q=1}^{d} w_k \sum_{Z\in\mathcal{Z}} \sum_{x,y,x+e_k,y+e_q\in Z} \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \,\mathrm{d}\alpha \int_0^{-\alpha} \mathrm{d}s\; i[\tau_{-s}^{(\omega,\mathcal{Z}^{(\tau)})}(I_{(y+e_q,y)}^{(\omega)}), I_{(x+e_k,x)}^{(\omega)}] \\ & + 2\sum_{k=1}^{d} w_k \sum_{Z\in\mathcal{Z}} \sum_{x,x+e_k\in Z} \left(\int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \,\mathrm{d}\alpha\right) \Re\mathrm{e}\left(\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle a(\mathfrak{e}_{x+e_k})^* a(\mathfrak{e}_x)\right), \end{aligned} \tag{29}$$

where we recall that $\Re\mathrm{e}(A) \in \mathcal{U}$ is the real part of $A \in \mathcal{U}$, see (18). Note that

$$\mathfrak{K}_{\{\Lambda\},\{\mathbb{Z}^d\}}^{(\omega,\mathcal{E})} = |\Lambda|\, \mathbb{I}_{\Lambda}^{(\omega,\mathcal{E})}, \qquad \Lambda \in \mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d),$$

is a current observable (cf. (20)). These observables are bilinear elements (Definition 4.3):

(i) Single-hopping operators: For any $x \in \mathbb{Z}^d$, the shift operator $s_x \in \mathcal{B}(\mathfrak{h})$ is defined by

$$(s_x \psi)(y) \doteq \psi(x+y), \qquad y \in \mathbb{Z}^d. \tag{30}$$

Note that $s_x^* = s_{-x} = s_x^{-1}$ for any $x \in \mathbb{Z}^d$. Then, for any $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$, the single-hopping operators are

$$S_{x,y}^{(\omega)} \doteq \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} P_{\{x\}} s_{x-y} P_{\{y\}}, \qquad x, y \in \mathbb{Z}^d, \tag{31}$$

where $P_{\{x\}}$ is the orthogonal projection defined by (5) for $\Lambda = \{x\}$. Observe that

$$\left\langle \mathrm{A}, S_{x,y}^{(\omega)} \mathrm{A} \right\rangle = \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_y), \qquad x, y \in \mathbb{Z}^d.$$

Similarly, the paramagnetic current observables defined by (17) equal

$$I_{(x,y)}^{(\omega)} = -2 \langle \mathrm{A}, \Im\mathrm{m}\{S_{x,y}^{(\omega)}\} \mathrm{A} \rangle, \qquad x, y \in \mathbb{Z}^d, \tag{32}$$

for any $\omega \in \Omega$ and $\vartheta \in \mathbb{R}_0^+$. Compare with (25).

(ii) Local current observables: By (26), for any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, any collection $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}$, $\mathcal{Z} \in \mathfrak{Z}_{\mathrm{f}}$, and $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$,

$$\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} = \left\langle \mathrm{A}, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathrm{A} \right\rangle, \tag{33}$$

where

$$\begin{aligned} K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq {} & 4 \sum_{k,q=1}^{d} w_k \sum_{Z \in \mathcal{Z}} \sum_{x,y,x+e_k,y+e_q \in Z} \int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \, \mathrm{d}\alpha \\ & \int_0^{-\alpha} \mathrm{d}s \; i \left[\mathrm{e}^{-ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} \Im\mathrm{m}\{S_{y+e_q,y}^{(\omega)}\} \mathrm{e}^{ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}}, \Im\mathrm{m}\{S_{x+e_k,x}^{(\omega)}\} \right] \\ & + 2 \sum_{k=1}^{d} w_k \sum_{Z \in \mathcal{Z}} \sum_{x,x+e_k \in Z} \left(\int_{-\infty}^{0} \{\mathcal{E}(\alpha)\}_q \, \mathrm{d}\alpha \right) \Re\mathrm{e}\{S_{x+e_k,x}^{(\omega)}\} \end{aligned} \tag{34}$$

is an operator acting on $\mathfrak{h}$ whose range is finite dimensional. This one-particle operator satisfies the following decay bounds:

Lemma 4.5 *(Decay of local currents). For any $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $x, y \in \mathbb{Z}^d$, and two collections $\mathcal{Z} \in \mathfrak{Z}_f$ and $\mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$,*

$$\left| \left\langle \mathfrak{e}_x, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \leq D_{4.5} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} \mathrm{e}^{2|\alpha\eta|} \mathrm{d}\alpha \right) \left(\mathrm{e}^{-\mu_\eta |x-y|} + \eta \delta_{1,|x-y|} \right),$$

$$\frac{1}{|\cup \mathcal{Z}|} \sum_{x,y \in \mathbb{Z}^d} \left| \left\langle \mathfrak{e}_x, K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \leq D_{4.5} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} \mathrm{e}^{2|\alpha\eta|} \mathrm{d}\alpha \right) \sum_{z \in \mathbb{Z}^d} \mathrm{e}^{-\mu_\eta |z|} (1+\eta),$$

where

$$D_{4.5} \doteq 4d\eta^{-1} \times 36^2 (1+\vartheta)^2 \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{2\mu_\eta(1-|z|)} < \infty.$$

Recall that μ_η is defined by (8).

Proof. Fix the parameters of the lemma. By (7), note that for any $z_1, z_2, x, y \in \mathbb{Z}^d$, $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$ and $s \in \mathbb{R}$,

$$\left| \left\langle \mathfrak{e}_x, \mathrm{e}^{-ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} S_{z_2+e_q,z_2}^{(\omega)} \mathrm{e}^{ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} S_{z_1+e_k,z_1}^{(\omega)} \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \leq 36^2 (1+\vartheta)^2 \mathrm{e}^{2|s\eta| - 2\mu_\eta(|x-z_2-e_q|+|y-z_2+e_k|)} \delta_{y,z_1}. \quad (35)$$

By the Cauchy–Schwarz and triangle inequalities, observe also that

$$\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\mu_\eta(|x-z|+|y-z|)} \leq \mathrm{e}^{-\mu_\eta|x-y|} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\mu_\eta(|x-z|+|y-z|)} \leq \mathrm{e}^{-\mu_\eta|x-y|} \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\mu_\eta|z|} \right). \quad (36)$$

From (35)–(36), we obtain the bound

$$\sum_{Z\in\mathcal{Z}} \sum_{z_1,z_2,z_1+e_k,z_2+e_q\in Z} \left| \left\langle \mathfrak{e}_x, \mathrm{e}^{-ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} S_{z_2+e_q,z_2}^{(\omega)} \mathrm{e}^{ish_{\mathcal{Z}^{(\tau)}}^{(\omega)}} S_{z_1+e_k,z_1}^{(\omega)} \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right|$$
$$\leq 36^2 (1+\vartheta)^2 \mathrm{e}^{2|s\eta| - \mu_\eta|x-y|} \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{2\mu_\eta(1-|z|)} \right), \quad (37)$$

using that $|z-e_k| \geq |z|-1$ for any $z \in \mathbb{Z}^d$ and $k \in \{1,\ldots,d\}$. The other terms computed from (34) are estimated in the same way. We omit the details. This yields the first bound of the lemma. The second estimate is also proven in the same way. □

It is convenient to introduce at this point the notation

$$\partial_\Lambda(\tilde{\Lambda}) \doteq \left\{ \{x,y\} \subset \Lambda \colon \ |x-y| = 1, \ \{x,y\} \cap \tilde{\Lambda} \neq \emptyset \text{ and } \{x,y\} \cap \tilde{\Lambda}^c \neq \emptyset \right\} \quad (38)$$

for any set $\tilde{\Lambda} \subset \Lambda \subset \mathbb{Z}^d$ with complement $\tilde{\Lambda}^c \doteq \mathbb{Z}^d \backslash \tilde{\Lambda}$, while, for any $\mathcal{Z} \in \mathfrak{Z}$ such that $\cup\mathcal{Z} \subset \Lambda$,

$$\partial_\Lambda(\mathcal{Z}) \doteq \{\partial_\Lambda(Z) : Z \in \mathcal{Z}\}.$$

Then, the one-particle operators (34) also satisfy the following bounds:

Lemma 4.6 *(Box decomposition of local currents – I). For any $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\Lambda, \tilde{\Lambda} \in \mathcal{P}_f(\mathbb{Z}^d)$, $\lambda, \vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, and $\mathcal{Z} \in \mathfrak{Z}_f$ with $\cup\mathcal{Z} \subset \tilde{\Lambda}$,*

$$\sum_{x,y\in\mathbb{Z}^d} \left| \left\langle \mathfrak{e}_x, \left(K_{\{\Lambda\},\{\tilde{\Lambda}\}}^{(\omega,\mathcal{E})} - K_{\{\Lambda\},\mathcal{Z}}^{(\omega,\mathcal{E})} \right) \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right|$$
$$\leq D_{4.6} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} \alpha^2 \mathrm{e}^{2|\alpha\eta|} \mathrm{d}\alpha \right) \left(\sum_{x\in\Lambda} \sum_{z\in\tilde{\Lambda}\backslash\cup\mathcal{Z}} \mathrm{e}^{-\mu_\eta|x-z|} + \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\mu_\eta|z|} \sum_{x\in\cup\partial_{\tilde{\Lambda}}(\mathcal{Z})} 1 \right),$$

where

$$D_{4.6} \doteq 8 \times 36^4 \left(1+\vartheta\right)^3 \left(4d+\lambda\right) \mathrm{e}^{3\mu_\eta} \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\mu_\eta |z|}\right)^3 < \infty .$$

Proof. Fix all parameters of the lemma. Let

$$C_{\mathcal{Z}}^{(\omega)}(z_1,z_2,k,q) = \int_0^{-\alpha} \mathrm{d}s\ i\left[\mathrm{e}^{-ish_{\mathcal{Z}}^{(\omega)}} \Im\mathrm{m}\{S^{(\omega)}_{z_2+e_q,z_2}\}\mathrm{e}^{ish_{\mathcal{Z}}^{(\omega)}}, \Im\mathrm{m}\{S^{(\omega)}_{z_1+e_k,z_1}\}\right]$$

for any $z_1,z_2\in\mathbb{Z}^d$ and $k,q\in\{1,\ldots,d\}$. By Duhamel's formula,

$$\begin{aligned}
&\mathrm{e}^{-ish_{\{\tilde\Lambda\}}^{(\omega)}} A \mathrm{e}^{ish_{\{\tilde\Lambda\}}^{(\omega)}} - \mathrm{e}^{-ish_{\mathcal{Z}}^{(\omega)}} A \mathrm{e}^{ish_{\mathcal{Z}}^{(\omega)}} \\
&\quad = -i\int_0^s \mathrm{e}^{-i(s-u)h_{\mathcal{Z}}^{(\omega)}} \left[h_{\{\tilde\Lambda\}}^{(\omega)} - h_{\mathcal{Z}}^{(\omega)}, \mathrm{e}^{-iuh_{\{\tilde\Lambda\}}^{(\omega)}} A \mathrm{e}^{iuh_{\{\tilde\Lambda\}}^{(\omega)}}\right] \mathrm{e}^{i(s-u)h_{\mathcal{Z}}^{(\omega)}} \mathrm{d}u
\end{aligned}$$

and hence, for any $z_1,z_2\in\mathbb{Z}^d$ and $k,q\in\{1,\ldots,d\}$,

$$\begin{aligned}
&C_{\{\tilde\Lambda\}}^{(\omega)}(z_1,z_2,k,q) - C_{\mathcal{Z}}^{(\omega)}(z_1,z_2,k,q) \\
&\quad = 4\int_0^\alpha \mathrm{d}s \int_0^s \mathrm{d}u \left[\mathrm{e}^{-i(s-u)h_{\mathcal{Z}}^{(\omega)}} \left[h_{\{\tilde\Lambda\}}^{(\omega)} - h_{\mathcal{Z}}^{(\omega)}, \mathrm{e}^{-iuh_{\{\tilde\Lambda\}}^{(\omega)}} \Im\mathrm{m}\{S^{(\omega)}_{z_2+e_q,z_2}\} \mathrm{e}^{iuh_{\{\tilde\Lambda\}}^{(\omega)}}\right] \mathrm{e}^{i(s-u)h_{\mathcal{Z}}^{(\omega)}}, \Im\mathrm{m}\{S^{(\omega)}_{z_1+e_k,z_1}\}\right].
\end{aligned}$$

By developing the commutators and $\Im\mathrm{m}\{\cdot\}$ we get sixteen terms:

$$C_{\{\tilde\Lambda\}}^{(\omega)}(z_1,z_2,k,q) - C_{\mathcal{Z}}^{(\omega)}(z_1,z_2,k,q) = \int_0^\alpha \mathrm{d}s \int_0^s \mathrm{d}u \sum_{j=1}^{16} \mathbf{X}_j\left(s,u,z_1,z_2\right), \tag{39}$$

where, for instance,

$$\mathbf{X}_1\left(s,u,z_1,z_2\right) \doteq \mathrm{e}^{-i(s-u)h_{\mathcal{Z}}^{(\omega)}} \left(h_{\{\tilde\Lambda\}}^{(\omega)} - h_{\mathcal{Z}}^{(\omega)}\right) \mathrm{e}^{-iuh_{\{\tilde\Lambda\}}^{(\omega)}} S_{z_2+e_q,z_2} \mathrm{e}^{iuh_{\{\tilde\Lambda\}}^{(\omega)}} \mathrm{e}^{i(s-u)h_{\mathcal{Z}}^{(\omega)}} S_{z_1+e_k,z_1}. \tag{40}$$

Since $\cup\mathcal{Z}\subset\tilde\Lambda$, note that

$$\begin{aligned}
h_{\{\tilde\Lambda\}}^{(\omega)} - h_{\mathcal{Z}}^{(\omega)} &= \sum_{z_3,z_4\in\tilde\Lambda\backslash\cup\mathcal{Z}:\ \ |z_3-z_4|=1} S^{(\omega)}_{z_3,z_4} + \sum_{Z\in\mathcal{Z}} \sum_{\{z_3,z_4\}\in\partial_{\tilde\Lambda}(Z)} \left(S^{(\omega)}_{z_3,z_4} + S^{(\omega)}_{z_4,z_3}\right) \\
&\quad + \sum_{z_3\in\tilde\Lambda\backslash\cup\mathcal{Z}} \lambda\omega_1\left(z_3\right) S^{(\omega)}_{z_3,z_3}.
\end{aligned} \tag{41}$$

Meanwhile, for any $z_1,z_2,z_3,z_4,x,y\in\mathbb{Z}^d$ with $|z_3-z_4|\leq 1$, and real numbers $s\geq u\geq 0$, we infer from (7) and (36) that

$$\begin{aligned}
&\left|\left\langle \mathfrak{e}_x, \mathrm{e}^{-i(s-u)h_{\mathcal{Z}}^{(\omega)}} S^{(\omega)}_{z_3,z_4} \mathrm{e}^{-iuh_{\{\tilde\Lambda\}}^{(\omega)}} S^{(\omega)}_{z_2+e_q,z_2} \mathrm{e}^{iuh_{\{\tilde\Lambda\}}^{(\omega)}} \mathrm{e}^{i(s-u)h_{\mathcal{Z}}^{(\omega)}} S^{(\omega)}_{z_1+e_k,z_1} \mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \\
&\quad \leq 36^4 \left(1+\vartheta\right)^3 \mathrm{e}^{2|s\eta|+3\mu_\eta} \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\mu_\eta|z|}\right) \delta_{z_1,y} \mathrm{e}^{-\mu_\eta(|z_2-y|+|x-z_3|+|z_3-z_2|)}.
\end{aligned}$$

By (40)–(41), for any $\alpha\geq 0$, it follows that

$$\sum_{x,y\in\mathbb{Z}^d}\ \sum_{z_1,z_2,z_1+e_k,z_2+e_q\in\Lambda}\int_0^\alpha \mathrm{d}s\int_0^s \mathrm{d}u\left|\langle\mathfrak{e}_x,\mathbf{X}_1(s,u,z_1,z_2)\,\mathfrak{e}_y\rangle_\mathfrak{h}\right|$$
$$\leq \frac{36^4}{2}(1+\vartheta)^3(4d+\lambda)\,\alpha^2\mathrm{e}^{2|\alpha\eta|+3\mu_\eta}\left(\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\mu_\eta|z|}\right)^3$$
$$\times\left(\sum_{x\in\Lambda}\ \sum_{z\in\tilde{\Lambda}\backslash\cup\mathcal{Z}}\mathrm{e}^{-\mu_\eta|x-z|}+\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\mu_\eta|z|}\sum_{x\in\cup\partial_{\tilde{\Lambda}}(\mathcal{Z})}1\right).$$

The fifteen other terms $\mathbf{X}_j$ in (39) satisfy the same bound. By (34), the assertion follows for any $\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d)$. □

Lemma 4.7 *(Box decomposition of local currents – II). For any $\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)$, $\lambda,\vartheta\in\mathbb{R}_0^+$, $\omega\in\Omega$, $\mathcal{Z}_\tau\in\mathfrak{Z}$, and $\mathcal{Z}\in\mathfrak{Z}_f$ with $\cup\mathcal{Z}\subset\Lambda$,*

$$\sum_{x,y\in\mathbb{Z}^d}\left|\left\langle\mathfrak{e}_x,\left(K^{(\omega,\mathcal{E})}_{\{\Lambda\},\mathcal{Z}_\tau}-K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}_\tau}\right)\mathfrak{e}_y\right\rangle_\mathfrak{h}\right|\leq D_{4.7}\left(\int_\mathbb{R}\|\mathcal{E}(\alpha)\|^2_{\mathbb{R}^d}|\alpha|\,\mathrm{e}^{2|\alpha\eta|}\mathrm{d}\alpha\right)\sum_{z\in(\Lambda\backslash\cup\mathcal{Z})\cup(\cup\partial_\Lambda(\mathcal{Z}))}1,$$

where

$$D_{4.7}\doteq 16\times 36^2(1+\vartheta)^2\,d\mathrm{e}^{4\mu_\eta}\left(\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-2\mu_\eta|z|}\right)^2+d(1+\vartheta)<\infty.$$

Proof. Fix all parameters of the lemma. By combining (35) with direct estimates we observe that

$$\sum_{x,y\in\mathbb{Z}^d}\ \sum_{z_1,z_2,z_1+e_k,z_2+e_q\in\Lambda}\left|\left\langle\mathfrak{e}_x,\mathrm{e}^{-ish^{(\omega)}_{\mathcal{Z}^{(\tau)}}}S^{(\omega)}_{z_2+e_q,z_2}\mathrm{e}^{ish^{(\omega)}_{\mathcal{Z}^{(\tau)}}}S^{(\omega)}_{z_1+e_k,z_1}\mathfrak{e}_y\right\rangle_\mathfrak{h}\right|$$
$$-\sum_{x,y\in\mathbb{Z}^d}\sum_{Z\in\mathcal{Z}}\ \sum_{z_1,z_2,z_1+e_k,z_2+e_q\in Z}\left|\left\langle\mathfrak{e}_x,\mathrm{e}^{-ish^{(\omega)}_{\mathcal{Z}^{(\tau)}}}S^{(\omega)}_{z_2+e_q,z_2}\mathrm{e}^{ish^{(\omega)}_{\mathcal{Z}^{(\tau)}}}S^{(\omega)}_{z_1+e_k,z_1}\mathfrak{e}_y\right\rangle_\mathfrak{h}\right|$$
$$\leq 2\times 36^2(1+\vartheta)^2\mathrm{e}^{2|s\eta|+4\mu_\eta}\left(\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-2\mu_\eta|x|}\right)^2\sum_{z\in(\Lambda\backslash\cup\mathcal{Z})\cup(\cup\partial_\Lambda(\mathcal{Z}))}1 \qquad (42)$$

for any $s\in\mathbb{R}$. Similar to (39), the quantity

$$\sum_{x,y\in\mathbb{Z}^d}\left|\left\langle\mathfrak{e}_x,\left(K^{(\omega,\mathcal{E})}_{\{\Lambda\},\mathcal{Z}_\tau}-K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}_\tau}\right)\mathfrak{e}_y\right\rangle_\mathfrak{h}\right|$$

is a sum of nine terms. The first one is (42), the last one is related to $\Re\mathrm{e}\{S^{(\omega)}_{x+e_k,x}\}$ and gives the constant $d(1+\vartheta)$ in $D_{4.7}$. The seven remaining ones satisfy the same bound as the first one. □

4.4. Finite-volume generating functions

Fix $\beta\in\mathbb{R}^+$ and $\lambda,\vartheta\in\mathbb{R}_0^+$. Given $\mathcal{E}\in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\omega\in\Omega$ and three finite collections $\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_\mathrm{f}$, we define the finite-volume generating function

$$\mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq g_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} - g_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,0)}, \tag{43}$$

where

$$g_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})} \doteq \frac{1}{|\cup \mathcal{Z}|} \ln \mathrm{tr}\left(\exp(-\beta\langle \mathrm{A}, h_{\mathcal{Z}^{(\varrho)}}^{(\omega)} \mathrm{A}\rangle)\exp(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})})\right). \tag{44}$$

Recall that the tracial state $\mathrm{tr} \in \mathcal{U}^*$ is the quasi-free state satisfying (14) at $\beta = 0$, and $h_{\mathcal{Z}^{(\varrho)}}^{(\omega)}$ is the one-particle Hamiltonian defined by (6). See also Definition 4.3 and (29). By construction, note that

$$\frac{1}{|\Lambda_L|} \ln \varrho^{(\omega)}\left(\mathrm{e}^{|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}\right) = \lim_{L_\varrho\to\infty}\lim_{L_\tau\to\infty} \mathrm{J}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}}^{(\omega,\mathcal{E})}. \tag{45}$$

The family of functions $\mathcal{E} \mapsto \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}$ is equicontinuous with uniformly bounded second derivative:

Proposition 4.8 *(Equicontinuity of generating functions). Fix $n \in \mathbb{N}$. The family of maps $\mathcal{E} \mapsto \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}$ from $C_0^0([-n,n];\mathbb{R}^d) \subset C_0^0(\mathbb{R};\mathbb{R}^d)$ to $\mathbb{R}$, for $\beta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)} \in \mathfrak{Z}_f$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, and ϑ in a compact set of $\mathbb{R}_0^+$, is equicontinuous w.r.t. the sup norm for $\mathcal{E}$ in any bounded set of $C_0^0([-n,n];\mathbb{R}^d)$.*

Proof. Fix $n \in \mathbb{N}$, $\beta \in \mathbb{R}^+$, $\lambda,\vartheta \in \mathbb{R}_0^+$, $\omega \in \Omega$, $\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)} \in \mathfrak{Z}_\mathrm{f}$. By using Lemma 4.2 (ii), for any $\mathcal{E}_0,\mathcal{E}_1 \in C_0^0([-n,n];\mathbb{R}^d)$,

$$\begin{aligned} &\left| g_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E}_1)} - g_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E}_0)} \right| \\ &\leq \frac{1}{|\cup \mathcal{Z}|} \sup_{\alpha\in[0,\,1]} \sup_{u\in[-1/2,\,1/2]} \left\| \mathrm{e}^{u\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\alpha\mathcal{E}_1+(1-\alpha)\mathcal{E}_0)}} \mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E}_1-\mathcal{E}_0)} \mathrm{e}^{-u\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\alpha\mathcal{E}_1+(1-\alpha)\mathcal{E}_0)}} \right\|_{\mathcal{U}}. \end{aligned} \tag{46}$$

Recall that, for any $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, $\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}$ is the bilinear element associated with the operator $K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}$. See (33) and (34). In particular, from (27), we deduce the inequality

$$\sup_{u\in[-1/2,\,1/2]} \sup_{x,y\in\mathbb{Z}^d} \left\| \mathrm{e}^{u\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_y)\, \mathrm{e}^{-u\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}} \right\|_{\mathcal{U}} \leq \mathrm{e}^{\|K_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\|_{\mathcal{B}(\mathfrak{h})}}. \tag{47}$$

The assertion then follows by combining (33), (46) and Definition 4.3 with (47) and Lemmata 4.1, 4.5. □

Proposition 4.9 *(Uniform boundedness of second derivatives). Fix $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\beta_1, s_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$. Then,*

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1] \\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_f}} \left\{ \left|\partial_s \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})}\right| + \left|\partial_s^2 \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})}\right| \right\} < \infty.$$

Proof. Fix the parameters of the proposition. Then, by cyclicity of the tracial state,

$$\partial_s \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})} = \frac{1}{|\cup \mathcal{Z}|} \varpi_s\left(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\right)$$

and

$$\partial_s^2 \mathrm{J}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}^{(\omega,s\mathcal{E})} = \frac{1}{|\cup \mathcal{Z}|}\left(\varpi_s\left(\left(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\right)^2\right) - \varpi_s\left(\mathfrak{K}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}^{(\omega,\mathcal{E})}\right)^2\right),$$

where ϖ_s is the state defined, for any $B \in \mathcal{U}$, by

$$\varpi_s(B) = \frac{\mathrm{tr}\left(B\mathrm{e}^{\frac{s}{2}\mathfrak{K}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}\mathrm{e}^{-\beta\langle \mathrm{A}, h^{(\omega)}_{\mathcal{Z}^{(\varrho)}}\mathrm{A}\rangle}\mathrm{e}^{\frac{s}{2}\mathfrak{K}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}\right)}{\mathrm{tr}\left(\mathrm{e}^{\frac{s}{2}\mathfrak{K}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}\mathrm{e}^{-\beta\langle \mathrm{A}, h^{(\omega)}_{\mathcal{Z}^{(\varrho)}}\mathrm{A}\rangle}\mathrm{e}^{\frac{s}{2}\mathfrak{K}^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}\right)}.$$

By Lemma 4.4 and (33), observe that ϖ_s is the quasi-free state satisfying

$$\varpi_s(a^*(\varphi)\, a(\psi)) = \left\langle \psi, \frac{1}{1+\mathrm{e}^{-\frac{s}{2}K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}\mathrm{e}^{\beta h^{(\omega)}_{\mathcal{Z}^{(\varrho)}}}\mathrm{e}^{-\frac{s}{2}K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}}}\varphi \right\rangle_{\mathfrak{h}}, \qquad \varphi,\psi \in \mathfrak{h}. \tag{48}$$

Therefore, by (33) and Definition 4.3, we directly compute that

$$\partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} = \frac{1}{|\cup \mathcal{Z}|}\sum_{x,y\in\mathbb{Z}^d}\left\langle \mathfrak{e}_x, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_y\right\rangle_{\mathfrak{h}} \varpi_s\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_y)\right)$$

and

$$\begin{aligned}\partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}} = \frac{1}{|\cup \mathcal{Z}|}&\sum_{x,y,u,v\in\mathbb{Z}^d}\left\langle \mathfrak{e}_x, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_y\right\rangle_{\mathfrak{h}}\left\langle \mathfrak{e}_u, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_v\right\rangle_{\mathfrak{h}} \\ &\times \varpi_s\left(a(\mathfrak{e}_y)\, a(\mathfrak{e}_u)^*\right)\varpi_s\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_v)\right),\end{aligned}$$

because of the identity

$$\varpi_s\left(a(\mathfrak{e}_x)^*a(\mathfrak{e}_y)a(\mathfrak{e}_u)^*a(\mathfrak{e}_v)\right) = \varpi_s\left(a(\mathfrak{e}_x)^*a(\mathfrak{e}_y)\right)\varpi_s\left(a(\mathfrak{e}_u)^*a(\mathfrak{e}_v)\right) + \varpi_s\left(a(\mathfrak{e}_y)a(\mathfrak{e}_u)^*\right)\varpi_s\left(a(\mathfrak{e}_x)^*a(\mathfrak{e}_v)\right),$$

for $x, y, u, v \in \mathbb{Z}^d$, by (16) for $\rho = \varpi_s$. As a consequence,

$$\left|\partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\right| \le \frac{1}{|\cup \mathcal{Z}|}\sum_{x,y\in\mathbb{Z}^d}\left|\left\langle \mathfrak{e}_x, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_y\right\rangle_{\mathfrak{h}}\right|$$

and

$$\begin{aligned}\left|\partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\right| \le &\sup_{u,v\in\mathbb{Z}^d}\left|\left\langle \mathfrak{e}_u, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_v\right\rangle_{\mathfrak{h}}\right|\left(\frac{1}{|\cup \mathcal{Z}|}\sum_{x,y\in\mathbb{Z}^d}\left|\left\langle \mathfrak{e}_x, K^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}}\mathfrak{e}_y\right\rangle_{\mathfrak{h}}\right|\right) \\ &\times \sup_{y\in\mathbb{Z}^d}\sum_{u\in\mathbb{Z}^d}\left|\varpi_s\left(a(\mathfrak{e}_y)\, a(\mathfrak{e}_u)^*\right)\right|\sup_{x\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\left|\varpi_s\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_v)\right)\right|,\end{aligned}$$

which, by Lemma 4.5, implies that

$$\left|\partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\right| \le D_{4.5}\left(\int_{\mathbb{R}}\|\mathcal{E}(\alpha)\|_{\mathbb{R}^d}\,\mathrm{e}^{2|\alpha\eta|}\mathrm{d}\alpha\right)\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\mu_\eta|z|}(1+\eta) \tag{49}$$

as well as

$$\left|\partial_s^2 \mathrm{J}^{(\omega,s\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}}\right| \leq D_{4.5}^2 \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d} \mathrm{e}^{2|\alpha\eta|}\mathrm{d}\alpha\right)^2 (1+\eta)^2 \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\mu_\eta |z|} \times \sup_{y\in\mathbb{Z}^d} \sum_{u\in\mathbb{Z}^d} \left|\varpi_s\left(a(\mathfrak{e}_y)\, a(\mathfrak{e}_u)^*\right)\right| \sup_{x\in\mathbb{Z}^d} \sum_{v\in\mathbb{Z}^d} \left|\varpi_s\left(a(\mathfrak{e}_x)^*\, a(\mathfrak{e}_v)\right)\right|. \qquad (50)$$

Again by Lemma 4.5 together with (7)–(8), for any $\mu > \mu_\eta$,

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1]\\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}} \left\{\mathbf{S}_0(sK^{(\omega,\mathcal{E})}_{\mathcal{Z},\mathcal{Z}^{(\tau)}},\mu) + \mathbf{S}_0(\beta h^{(\omega)}_{\mathcal{Z}^{(\varrho)}},\mu)\right\} < \infty.$$

See (A.1). We thus infer from (48) and Corollary A.4 that there is a constant $\mu_1 \in \mathbb{R}^+$ such that, for any $x, y \in \mathbb{Z}^d$,

$$\sup_{\substack{\beta\in(0,\beta_1],\ \vartheta\in[0,\vartheta_1],\ \lambda\in[0,\lambda_1]\\ \omega\in\Omega,\ s\in[-s_1,s_1],\ \mathcal{Z},\mathcal{Z}^{(\varrho)},\mathcal{Z}^{(\tau)}\in\mathfrak{Z}_{\mathrm{f}}}} \left|\varpi_s\left(a(\mathfrak{e}_x)^*\, a(\mathfrak{e}_y)\right)\right| \leq 2\mathrm{e}^{-\mu_1|x-y|}.$$

Combining this estimate with (49)–(50), one gets the assertion. □

The local generating functionals (43) can be approximately decomposed into boxes of fixed volume: By using the boxes (12), for any subset $\Lambda \subset \mathbb{Z}^d$ and $l \in \mathbb{N}$, we define the l-th box decomposition $\mathcal{Z}^{(\Lambda,l)}$ of Λ by

$$\mathcal{Z}^{(\Lambda,l)} \doteq \left\{\Lambda_l + (2l+1)\,x : x \in \mathbb{Z}^d \text{ with } (\Lambda_l + (2l+1)\,x) \subset \Lambda\right\} \in \mathfrak{Z}.$$

Then, we get the following assertion:

Proposition 4.10 *(Box decomposition of generating functions). Fix $n \in \mathbb{N}$ and $\beta_1, \lambda_1, \vartheta_1 \in \mathbb{R}^+$. Then,*

$$\lim_{l\to\infty} \limsup_{L_\tau\geq L_\varrho\geq L\to\infty} \left|\mathrm{J}^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} - \frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}^{(\omega,\mathcal{E})}_{\{Z\},\{Z\},\{Z\}}\right| = 0,$$

uniformly w.r.t. $\beta \in [0,\beta_1]$, $\vartheta \in [0,\vartheta_1]$, $\lambda \in [0,\lambda_1]$, $\omega \in \Omega$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n,n];\mathbb{R}^d)$.

The proof of this statement is divided in a series of Lemmata:

Lemma 4.11 *(Box decomposition of generating functions – I). Fix $\beta_1, \lambda_1, \vartheta_1 \in \mathbb{R}^+$. Then,*

$$\limsup_{L_\tau\geq L_\varrho\geq L\to\infty} \left|g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L,\Lambda_L\},\{\Lambda_{L_\tau}\}}\right| = 0,$$

uniformly w.r.t. $\beta \in [0,\beta_1]$, $\vartheta \in [0,\vartheta_1]$, $\lambda \in [0,\lambda_1]$, $\omega \in \Omega$ and $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$.

Proof. Fix all parameters of the lemma. By Lemma 4.2 (ii),

$$\left|g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L,\Lambda_L\},\{\Lambda_{L_\tau}\}}\right| \leq \frac{\beta}{|\Lambda_L|} \sup_{\alpha\in[0,1]} \sup_{u\in[-1/2,1/2]} \left\|\mathrm{e}^{u\beta\langle \mathrm{A},h_\alpha \mathrm{A}\rangle} \langle \mathrm{A}, (h_1-h_0)\,\mathrm{A}\rangle\, \mathrm{e}^{-u\beta\langle \mathrm{A},h_\alpha \mathrm{A}\rangle}\right\|_{\mathcal{U}},$$

where

$$h_\alpha \doteq \alpha h^{(\omega)}_{\{\Lambda_{L_\varrho}\}} + (1-\alpha)\, h^{(\omega)}_{\{\Lambda_{L_\varrho}\backslash \Lambda_L, \Lambda_L\}}, \qquad \alpha \in [0,1]\,.$$

By using estimates similar to (47), we get

$$\begin{aligned} \left| g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L,\Lambda_L\},\{\Lambda_{L_\tau}\}} \right| &\leq \frac{\beta \mathrm{e}^{\beta(\lambda+2d)(1+\vartheta)}}{|\Lambda_L|} \sum_{x,y\in\mathbb{Z}^d} \left| \langle \mathfrak{e}_x, (h_1-h_0)\, \mathfrak{e}_y\rangle_{\mathfrak{h}} \right| \\ &\leq 4d\,(1+\vartheta)\,\beta \mathrm{e}^{\beta(\lambda+2d)(1+\vartheta)} \frac{1}{|\Lambda_L|} \sum_{z\in\cup\partial_{\Lambda_{L_\varrho}}(\Lambda_L)} 1. \qquad (51) \end{aligned}$$

See (41). Since

$$\limsup_{L_\varrho \geq L\to\infty} \frac{1}{|\Lambda_L|} \sum_{z\in\cup\partial_{\Lambda_{L_\varrho}}(\Lambda_L)} 1 = 0,$$

the assertion follows. □

Lemma 4.12 *(Box decomposition of generating functions – II). Fix $n\in\mathbb{N}$ and $\vartheta_1,\lambda_1\in\mathbb{R}^+$. Then,*

$$\lim_{l\to\infty} \limsup_{L_\tau\geq L_\rho\geq L\to\infty} \left| g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L,\Lambda_L\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L,\Lambda_L\},\mathcal{Z}^{(\Lambda_L,l)}} \right| = 0,$$

uniformly w.r.t. $\vartheta\in[0,\vartheta_1]$, $\lambda\in[0,\lambda_1]$, $\omega\in\Omega$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n,n];\mathbb{R}^d)$.

Proof. Fix all parameters of the lemma, in particular $L_\tau \geq L_\rho \geq L \geq l$, $\omega\in\Omega$ and $\lambda\in[0,\lambda_1]$. By Lemma 4.2 (ii) and (33),

$$\begin{aligned} &\left| g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash,\Lambda_L\},\mathcal{Z}^{(\Lambda_L,l)}} \right| \\ &\quad\leq \frac{1}{|\Lambda_L|} \sup_{\alpha\in[0,1]} \sup_{u\in[-1/2,1/2]} \left\| \mathrm{e}^{u\langle \mathrm{A},K_\alpha \mathrm{A}\rangle} \langle \mathrm{A}, (K_1-K_0)\,\mathrm{A}\rangle\, \mathrm{e}^{-u\langle \mathrm{A},K_\alpha \mathrm{A}\rangle} \right\|_{\mathcal{U}}, \end{aligned}$$

where

$$K_\alpha \doteq \alpha K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\tau}\}} + (1-\alpha)\, K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\mathcal{Z}^{(\Lambda_L,l)}}, \qquad \alpha\in[0,1]\,.$$

Like in the proof of Lemma 4.11, by (33) and Lemma 4.6,

$$\begin{aligned} &\left| g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L\},\{\Lambda_{L_\tau}\}} - g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash,\Lambda_L\},\mathcal{Z}^{(\Lambda_L,l)}} \right| \qquad (52) \\ &\leq D_{4.6} \left(\int_{\mathbb{R}} \|\mathcal{E}(\alpha)\|_{\mathbb{R}^d}\, \alpha^2 \mathrm{e}^{2|\alpha\eta|} \mathrm{d}\alpha \right) \mathrm{e}^{\sup_{\alpha\in[0,1]}\|K_\alpha\|_{\mathcal{B}(\mathfrak{h})}} \\ &\quad \times \frac{1}{|\Lambda_L|} \left(\sum_{x\in\Lambda_L} \sum_{z\in\Lambda_{L_\tau}\backslash\cup\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{e}^{-\frac{\mu_\eta}{2}|x-z|} + \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\frac{\mu_\eta}{2}|z|} \right) \sum_{x\in\cup\partial_{\Lambda_{L_\tau}}(\mathcal{Z}^{(\Lambda_L,l)})} 1 \right). \end{aligned}$$

By Lemmata 4.1 and 4.5, for any $n\in\mathbb{N}$, observe that the operator norm of K_α is uniformly bounded for $\alpha\in[0,1]$, $\vartheta\in[0,\vartheta_1]$, $\lambda\in\mathbb{R}_0^+$, $\omega\in\Omega$, $L, L_\tau, l\in\mathbb{N}$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n,n];\mathbb{R}^d)$. Note additionally that

$$\limsup_{L_\tau \geq L \to \infty} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \sum_{z \in \Lambda_{L_\tau} \backslash \cup \mathcal{Z}^{(\Lambda_L, l)}} \mathrm{e}^{-\frac{\mu_\eta}{2}|x-z|} = 0,$$

whereas

$$\limsup_{L_\tau \geq L \to \infty} \frac{1}{|\Lambda_L|} \sum_{x \in \cup \partial_{\Lambda_{L_\tau}}(\mathcal{Z}^{(\Lambda_L, l)})} 1 = \mathcal{O}\left(l^{-1}\right).$$

From these last observations combined with (52) the assertion follows. □

Lemma 4.13 *(Box decomposition of generating functions – III). Fix $\beta_1, \vartheta_1, \lambda_1 \in \mathbb{R}^+$. Then,*

$$\lim_{l \to \infty} \limsup_{L_\tau \geq L_\varrho \geq L \to \infty} \left| g^{(\omega, \mathcal{E})}_{\{\Lambda_L\}, \{\Lambda_{L_\varrho} \backslash \Lambda_L, \Lambda_L\}, \mathcal{Z}^{(\Lambda_L, l)}} - g^{(\omega, \mathcal{E})}_{\{\Lambda_L\}, \{\Lambda_{L_\varrho} \backslash \Lambda_L\} \cup \mathcal{Z}^{(\Lambda_L, l)}, \mathcal{Z}^{(\Lambda_L, l)}} \right| = 0,$$

uniformly w.r.t. $\beta \in [0, \beta_1]$, $\vartheta \in [0, \vartheta_1]$, $\lambda \in [0, \lambda_1]$, $\omega \in \Omega$ and $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$.

Proof. This lemma is proven exactly in the same way as Lemmata 4.11 and 4.12: Fix all parameters of the lemma and observe that

$$\begin{aligned} &\left| \left\langle \mathfrak{e}_x, \left(h^{(\omega)}_{\{\Lambda_{L_\varrho} \backslash \Lambda_L, \Lambda_L\}} - h^{(\omega)}_{\{\Lambda_{L_\varrho} \backslash \Lambda_L\} \cup \mathcal{Z}^{(\Lambda_L, l)}} \right) \mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \\ &\leq (1+\vartheta) \sum_{z_3, z_4 \in \Lambda_L \backslash \cup \mathcal{Z}^{(\Lambda_L, l)} \ : \ |z_3 - z_4| = 1} \delta_{z_3, y} \delta_{z_4, x} + \lambda \sum_{z_3 \in \Lambda_L \backslash \cup \mathcal{Z}^{(\Lambda_L, l)}} \delta_{z_3, x} \delta_{z_3, y} \\ &\quad + (1+\vartheta) \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} \sum_{\{z_3, z_4\} \in \partial_{\Lambda_L}(Z)} \left(\delta_{z_3, y} \delta_{z_4, x} + \delta_{z_4, y} \delta_{z_3, x} \right). \end{aligned}$$

See (41). Then, similar to (51), we get the bound

$$\begin{aligned} &\left| g^{(\omega, \mathcal{E})}_{\{\Lambda_L\}, \{\Lambda_{L_\varrho}\}, \{\Lambda_{L_\tau}\}} - g^{(\omega, \mathcal{E})}_{\{\Lambda_L\}, \{\Lambda_{L_\varrho} \backslash \Lambda_L, \Lambda_L\}, \{\Lambda_{L_\tau}\}} \right| \\ &\leq (4d + \lambda)(1+\vartheta) \beta \mathrm{e}^{\beta(\lambda + 2d)(1+\vartheta)} \frac{1}{|\Lambda_L|} \left(\sum_{z \in \Lambda_L \backslash \cup \mathcal{Z}^{(\Lambda_L, l)}} 1 + \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} \sum_{z \in \cup \partial_{\Lambda_L}(Z)} 1 \right), \end{aligned}$$

where

$$\limsup_{L \to \infty} \frac{1}{|\Lambda_L|} \left(\sum_{z \in \Lambda_L \backslash \cup \mathcal{Z}^{(\Lambda_L, l)}} 1 + \sum_{Z \in \mathcal{Z}^{(\Lambda_L, l)}} \sum_{z \in \cup \partial_{\Lambda_L}(Z)} 1 \right) = \mathcal{O}\left(l^{-1}\right). \quad \square$$

Lemma 4.14 *(Box decomposition of generating functions – IV). Fix $n \in \mathbb{N}$ and $\vartheta_1 \in \mathbb{R}^+$. Then,*

$$\lim_{l \to \infty} \limsup_{L_\tau \geq L_\varrho \geq L \to \infty} \left| g^{(\omega, \mathcal{E})}_{\{\Lambda_L\}, \{\Lambda_{L_\varrho} \backslash \Lambda_L\} \cup \mathcal{Z}^{(\Lambda_L, l)}, \mathcal{Z}^{(\Lambda_L, l)}} - g^{(\omega, \mathcal{E})}_{\mathcal{Z}^{(\Lambda_L, l)}, \{\Lambda_{L_\varrho} \backslash \Lambda_L\} \cup \mathcal{Z}^{(\Lambda_L, l)}, \mathcal{Z}^{(\Lambda_L, l)}} \right| = 0,$$

uniformly w.r.t. $\beta \in \mathbb{R}^+$, $\vartheta \in [0, \vartheta_1]$, $\lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n]; \mathbb{R}^d)$.

Proof. Fix all parameters of the lemma. Then, like for previous lemmata, we use again Lemma 4.2 (ii) and (33) to obtain the bound

$$\left| g^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\backslash\Lambda_L\}\cup\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}} - g^{(\omega,\mathcal{E})}_{\mathcal{Z}^{(\Lambda_L,l)},\{\Lambda_{L_\varrho}\backslash\Lambda_L\}\cup\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}} \right| \leq \frac{1}{|\Lambda_L|} \sup_{\alpha\in[0,1]} \sup_{u\in[-1/2,1/2]} \left\| \mathrm{e}^{u\langle \mathrm{A},K_\alpha \mathrm{A}\rangle} \langle \mathrm{A}, (K_1 - K_0)\,\mathrm{A}\rangle\, \mathrm{e}^{-u\langle \mathrm{A},K_\alpha \mathrm{A}\rangle} \right\|_{\mathcal{U}},$$

where

$$K_\alpha \doteq \alpha K^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\mathcal{Z}^{(\Lambda_L,l)}} + (1-\alpha)\, K^{(\omega,\mathcal{E})}_{\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}}, \qquad \alpha\in[0,1].$$

Therefore, by Lemmata 4.1, 4.5 and 4.7, the assertion follows. □

We are now in a position to prove Proposition 4.10:

Proof. Fix all parameters of Proposition 4.10. By Lemmata 4.11–4.14,

$$\limsup_{L_\tau\geq L_\varrho\geq L\to\infty} \left| \mathrm{J}^{(\omega,\mathcal{E})}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}} - \mathrm{J}^{(\omega,\mathcal{E})}_{\mathcal{Z}^{(\Lambda_L,l)},\{\Lambda_{L_\varrho}\backslash\Lambda_L\}\cup\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}} \right| = 0, \tag{53}$$

uniformly w.r.t. $\beta\in[0,\beta_1]$, $\vartheta\in[0,\vartheta_1]$, $\lambda\in[0,\lambda_1]$, $\omega\in\Omega$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n,n];\mathbb{R}^d)$. To conclude the proof, observe that

$$\mathrm{J}^{(\omega,\mathcal{E})}_{\mathcal{Z}^{(\Lambda_L,l)},\{\Lambda_{L_\varrho}\backslash\Lambda_L\}\cup\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}} = \mathrm{J}^{(\omega,\mathcal{E})}_{\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)},\mathcal{Z}^{(\Lambda_L,l)}} = \frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}^{(\omega,\mathcal{E})}_{\{Z\},\{Z\},\{Z\}}. \tag{54}$$

This follows from the fact that the tracial state $\mathrm{tr}\in\mathcal{U}^*$ is a product of single-site states. See, e.g., [40]. □

4.5. Akcoglu–Krengel ergodic theorem and existence of generating functions

For convenience, we shortly recall the Akcoglu–Krengel ergodic theorem. We restrict ourselves to *additive* processes associated with the probability space $(\Omega,\mathfrak{A}_\Omega,\mathfrak{a}_\Omega)$ defined in Section 2.1, even if the Akcoglu–Krengel ergodic theorem holds for superadditive or subadditive ones (cf. [28, Definition VI.1.6]).

Definition 4.15 *(Additive processes associated with random variables).* $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda\in\mathcal{P}_\mathrm{f}(\mathbb{Z}^d)}$ is an additive process associated with the probability space $(\Omega,\mathfrak{A}_\Omega,\mathfrak{a}_\Omega)$ if:

(i) the map $\omega\mapsto\mathfrak{F}^{(\omega)}(\Lambda)$ is bounded and measurable w.r.t. the σ-algebra $\mathfrak{A}_\Omega$ for any $\Lambda\in\mathcal{P}_\mathrm{f}(\mathbb{Z}^d)$.

(ii) For all disjoint $\Lambda_1,\Lambda_2\in\mathcal{P}_\mathrm{f}(\mathbb{Z}^d)$,

$$\mathfrak{F}^{(\omega)}(\Lambda_1\cup\Lambda_2) = \mathfrak{F}^{(\omega)}(\Lambda_1) + \mathfrak{F}^{(\omega)}(\Lambda_2)\ , \qquad \omega\in\Omega\ .$$

(iii) For all $\Lambda\in\mathcal{P}_\mathrm{f}(\mathbb{Z}^d)$ and any space shift $x\in\mathbb{Z}^d$,

$$\mathbb{E}\left[\mathfrak{F}^{(\cdot)}(\Lambda)\right] = \mathbb{E}\left[\mathfrak{F}^{(\cdot)}(x+\Lambda)\right]\ . \tag{55}$$

Recall that $\mathbb{E}[\,\cdot\,]$ is the expectation value associated with the distribution $\mathfrak{a}_\Omega$.

We now define *regular* sequences (cf. [28, Remark VI.1.8]) as follows:

Definition 4.16 *(Regular sequences).* The non-decreasing sequence $(\Lambda^{(L)})_{L\in\mathbb{N}}\subset\mathcal{P}_\mathrm{f}(\mathbb{Z}^d)$ of (possibly non-cubic) boxes in $\mathbb{Z}^d$ is a regular sequence if there is a finite constant $D\in(0,1]$ and a diverging sequence $(\ell_L)_{L\in\mathbb{N}}\subset\mathbb{N}$ such that $\Lambda^{(L)}\subset\Lambda_{\ell_L}$ and $0<|\Lambda_{\ell_L}|\leq D|\Lambda^{(L)}|$ for all $L\in\mathbb{N}$. Here, Λ_ℓ, $\ell\in\mathbb{R}^+$, is the family of boxes defined by (12).

Then, the form of Akcoglu–Krengel ergodic theorem we use in the sequel is the lattice version of [28, Theorem VI.1.7, Remark VI.1.8] for additive processes associated with the probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$:

Theorem 4.17 *(Akcoglu–Krengel ergodic theorem). Let* $\{\mathfrak{F}^{(\omega)}(\Lambda)\}_{\Lambda\in\mathcal{P}_f(\mathbb{Z}^d)}$ *be an additive process. Then, for any regular sequence* $(\Lambda^{(L)})_{L\in\mathbb{N}} \subset \mathcal{P}_f(\mathbb{Z}^d)$*, there is a measurable subset* $\tilde{\Omega} \subset \Omega$ *of full measure such that, for all* $\omega \in \tilde{\Omega}$*,*

$$\lim_{L\to\infty} \left\{ \left|\Lambda^{(L)}\right|^{-1} \mathfrak{F}^{(\omega)}\left(\Lambda^{(L)}\right) \right\} = \mathbb{E}\left[\mathfrak{F}^{(\cdot)}(\{0\})\right] .$$

See also [41].

The Akcoglu–Krengel (superadditive) ergodic theorem, cornerstone of ergodic theory, generalizes the celebrated Birkhoff additive ergodic theorem. It is used to deduce, via Proposition 4.8, the following Corollary:

Corollary 4.18 *(Akcoglu–Krengel ergodic theorem for generating functions). There is a measurable subset* $\tilde{\Omega} \subset \Omega$ *of full measure such that, for all* $\beta \in \mathbb{R}^+$*,* $\vartheta, \lambda \in \mathbb{R}_0^+$*,* $\omega \in \tilde{\Omega}$*,* $l \in \mathbb{N}$*,* $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ *and* $\vec{w} \in \mathbb{R}^d$ *with* $\|\vec{w}\|_{\mathbb{R}^d} = 1$*,*

$$\lim_{L\to\infty} \frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}^{(\omega,\mathcal{E})}_{\{Z\},\{Z\},\{Z\}} = \mathbb{E}\left[\mathrm{J}^{(\cdot,\mathcal{E})}_{\{\Lambda_l\},\{\Lambda_l\},\{\Lambda_l\}}\right].$$

Proof. Fix $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \Omega$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. For any $\Gamma \in \mathcal{P}_{\mathrm{f}}(\mathbb{Z}^d)$, let

$$\mathfrak{F}_l^{(\omega,\mathcal{E})}(\Gamma) \doteq \sum_{x\in\Gamma} \mathrm{J}^{(\omega,\mathcal{E})}_{\{\Lambda_l+(2l+1)x\},\{\Lambda_l+(2l+1)x\},\{\Lambda_l+(2l+1)x\}}.$$

Then, if

$$\Lambda^{(L)} \equiv \Lambda^{(L,l)} \doteq \left\{x \in \mathbb{Z}^d \colon (\Lambda_l + (2l+1)\,x) \subset \Lambda_L\right\} \subset \Lambda_L,$$

observe that

$$\left|\Lambda^{(L)}\right|^{-1} \mathfrak{F}_l^{(\omega,\mathcal{E})}\left(\Lambda^{(L)}\right) = \frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}^{(\omega,\mathcal{E})}_{\{Z\},\{Z\},\{Z\}}.$$

Therefore, since $(\Lambda^{(L)})_{L\in\mathbb{N}}$ is clearly a regular sequence, by Theorem 4.17, for any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R}; \mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$, there is a measurable subset $\hat{\Omega} \equiv \hat{\Omega}^{(\beta,\vartheta,\lambda,l,\mathcal{E},\vec{w})} \subset \Omega$ of full measure such that, for all $\omega \in \hat{\Omega}$,

$$\lim_{L\to\infty} \frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}^{(\omega,\mathcal{E})}_{\{Z\},\{Z\},\{Z\}} = \mathbb{E}\left[\mathrm{J}^{(\cdot,\mathcal{E})}_{\{\Lambda_l\},\{\Lambda_l\},\{\Lambda_l\}}\right].$$

Observe that, for any $n \in \mathbb{N}$, there is a countable dense set $\mathcal{D}_n \subset C_0^0(\mathbb{R}; \mathbb{R}^d)$. Let $\mathbb{S}^{d-1}$ be a dense countable subset of the $(d-1)$-dimensional sphere. Hence, by Proposition 4.8, we arrive at the assertion for any realization $\omega \in \tilde{\Omega} \subset \Omega$, where

$$\tilde{\Omega} \doteq \bigcap_{\vartheta,\lambda\in\mathbb{Q}\cap\mathbb{R}_0^+} \bigcap_{\beta\in\mathbb{Q}\cap\mathbb{R}^+} \bigcap_{\vec{w}\in\mathbb{S}^{d-1}} \bigcap_{n\in\mathbb{N}} \bigcap_{\mathcal{E}\in\mathcal{D}_n} \bigcap_{l\in\mathbb{N}} \hat{\Omega}^{(\beta,\vartheta,\lambda,l,\mathcal{E},\vec{w})} .$$

[Recall that any countable intersection of measurable sets of full measure has full measure.] □

Corollary 4.19 *(Almost surely existence of generating functions). Let $\tilde{\Omega} \subset \Omega$ be the measurable subset of Corollary 4.18. Then, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \mathbb{E}\left[\ln \varrho^{(\cdot)}\left(\mathrm{e}^{|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\cdot,\mathcal{E})}}\right)\right] = \lim_{L_\tau \geq L_\varrho \geq L \to\infty} \mathrm{J}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}}^{(\omega,\mathcal{E})} \doteq \mathrm{J}^{(\mathcal{E})}.$$

For all $n \in \mathbb{N}$, the convergence is uniform w.r.t. $\beta, \vartheta, \lambda$ in compact sets, $\omega \in \tilde{\Omega}$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n];\mathbb{R}^d)$.

Proof. By translation invariance of the distribution $\mathfrak{a}_\Omega$,

$$\mathbb{E}\left[\mathrm{J}_{\{\Lambda_l\},\{\Lambda_l\},\{\Lambda_l\}}^{(\cdot,\mathcal{E})}\right] = \mathbb{E}\left[\frac{1}{\left|\mathcal{Z}^{(\Lambda_L,l)}\right|} \sum_{Z\in\mathcal{Z}^{(\Lambda_L,l)}} \mathrm{J}_{\{Z\},\{Z\},\{Z\}}^{(\cdot,\mathcal{E})}\right].$$

Hence,

$$\left\{\mathbb{E}\left[\mathrm{J}_{\{\Lambda_l\},\{\Lambda_l\},\{\Lambda_l\}}^{(\cdot,\mathcal{E})}\right]\right\}_{l\in\mathbb{N}}$$

is a Cauchy sequence, by (53) and (54). By Proposition 4.10 and Corollary 4.18, there is a measurable subset $\tilde{\Omega} \subset \Omega$ of full measure such that, for all $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\omega \in \tilde{\Omega}$, $l \in \mathbb{N}$, $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$,

$$\lim_{L_\tau \geq L_\varrho \geq L \to\infty} \mathrm{J}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}}^{(\omega,\mathcal{E})} = \lim_{l\to\infty} \mathbb{E}\left[\mathrm{J}_{\{\Lambda_l\},\{\Lambda_l\},\{\Lambda_l\}}^{(\cdot,\mathcal{E})}\right].$$

For all $n \in \mathbb{N}$, the convergence is uniform w.r.t. $\beta, \vartheta, \lambda$ in compact sets, $\omega \in \tilde{\Omega}$, $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$ and $\mathcal{E}$ in any bounded set of $C_0^0([-n, n];\mathbb{R}^d)$. By (45), the assertion then follows. □

Corollary 4.20 *(Differentiability of generating functions). Fix $\beta, \lambda, \vartheta \in \mathbb{R}^+$ and $\vec{w} \in \mathbb{R}^d$ with $\|\vec{w}\|_{\mathbb{R}^d} = 1$. For any $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$, the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable, so that*

$$\partial_s \mathrm{J}^{(s\mathcal{E})} = \lim_{L\to\infty} \frac{\varrho^{(\omega)}\left(\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})} \mathrm{e}^{s|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}\right)}{\varrho^{(\omega)}\left(\mathrm{e}^{s|\Lambda_L|\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E})}}\right)}. \tag{56}$$

Proof. Take any $\mathcal{E} \in C_0^0(\mathbb{R};\mathbb{R}^d)$ and $\omega \in \tilde{\Omega}$. See Corollary 4.19. Then, for any $s \in \mathbb{R}$,

$$\mathrm{J}^{(s\mathcal{E})} = \lim_{L_\tau \geq L_\varrho \geq L \to\infty} \mathrm{J}_{\{\Lambda_L\},\{\Lambda_{L_\varrho}\},\{\Lambda_{L_\tau}\}}^{(\omega,s\mathcal{E})}.$$

By Proposition 4.9 combined with the mean value theorem and the (Arzelà-) Ascoli theorem [29, Theorem A5], there are three sequences $(L_\tau^{(n)})_{n\in\mathbb{N}}$, $(L_\varrho^{(n)})_{n\in\mathbb{N}}$, $(L^{(n)})_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$, with $L_\tau^{(n)} \geq L_\varrho^{(n)} \geq L^{(n)}$, such that the maps

$$s \mapsto \mathrm{J}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}}^{(\omega,s\mathcal{E})} \qquad \text{and} \qquad s \mapsto \partial_s \mathrm{J}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}}^{(\omega,s\mathcal{E})}$$

converge uniformly for s in any compact set of $\mathbb{R}$. In particular, the map $s \mapsto \mathrm{J}^{(s\mathcal{E})}$ from $\mathbb{R}$ to itself is continuously differentiable with

$$\partial_s \mathrm{J}^{(s\mathcal{E})} = \lim_{L_\tau \geq L_\varrho \geq L \to \infty} \partial_s \mathrm{J}^{(\omega,s\mathcal{E})}_{\{\Lambda_{L^{(n)}}\},\{\Lambda_{L_\varrho^{(n)}}\},\{\Lambda_{L_\tau^{(n)}}\}} = \lim_{L\to\infty} \frac{\varrho^{(\omega)}\left(\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L} \mathrm{e}^{s|\Lambda_L|\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\right)}{\varrho^{(\omega)}\left(\mathrm{e}^{s|\Lambda_L|\mathbb{I}^{(\omega,\mathcal{E})}_{\Lambda_L}}\right)}. \quad \square$$

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Appendix A. Combes–Thomas estimates

For any operator $h \in \mathcal{B}(\mathfrak{h})$ and $\mu \in \mathbb{R}_0^+$, let

$$\mathbf{S}_0(h,\mu) \doteq \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{\mu|x-y|} \left| \langle \mathfrak{e}_x, h\mathfrak{e}_y \rangle_{\mathfrak{h}} \right| \in \mathbb{R}_0^+ \cup \{\infty\}\,. \tag{A.1}$$

Note that

$$\mathbf{S}_0(h_1h_2,\mu) \leq \mathbf{S}_0(h_1,\mu)\mathbf{S}_0(h_2,\mu), \tag{A.2}$$

for any $h_1, h_2 \in \mathcal{B}(\mathfrak{h})$ and $\mu \in \mathbb{R}_0^+$. In particular, for any $z \in \mathbb{C}$, $h \in \mathcal{B}(\mathfrak{h})$ and $\mu \in \mathbb{R}_0^+$,

$$\mathbf{S}_0(\mathrm{e}^{zh},\mu) \leq \mathrm{e}^{\mathbf{S}_0(zh,\mu)} = \mathrm{e}^{|z|\mathbf{S}_0(h,\mu)} \tag{A.3}$$

and hence,

$$\left| \left\langle \mathfrak{e}_x, \mathrm{e}^{zh}\mathfrak{e}_y \right\rangle_{\mathfrak{h}} \right| \leq \mathrm{e}^{|z|\mathbf{S}_0(h,\mu)} \mathrm{e}^{-\mu|x-y|}.$$

The above bound can be sharpened if $z = it$ is imaginary by using the Combes–Thomas estimate, first proven in [26]. We give a version of this estimate that is adapted to the present setting: Given a self-adjoint operator $h = h^* \in \mathcal{B}(\mathfrak{h})$ whose spectrum is denoted by $\mathrm{spec}(h)$, we define the constants

$$\mathbf{S}(h,\mu) \doteq \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \left(\mathrm{e}^{\mu|x-y|} - 1 \right) \left| \langle \mathfrak{e}_x, h\mathfrak{e}_y \rangle_{\mathfrak{h}} \right| \in \mathbb{R}_0^+ \cup \{\infty\}\,, \tag{A.4}$$

for $\mu \in \mathbb{R}_0^+$, and

$$\Delta(h,z) \doteq \inf\{|z-\lambda| : \lambda \in \mathrm{spec}(h)\}\,, \qquad z \in \mathbb{C},$$

as being the distance from the point z to the spectrum of h. Since the function $x \mapsto (e^{xr}-1)/x$ is increasing on $\mathbb{R}^+$ for any fixed $r \geq 0$, it follows that

$$\mathbf{S}(h,\mu_1) \leq \frac{\mu_1}{\mu_2}\mathbf{S}(h,\mu_2)\,, \qquad \mu_2 \geq \mu_1 \geq 0. \tag{A.5}$$

The version of the Combes–Thomas estimate that is most convenient for the current study is the following:

Theorem A.1 *(Combes–Thomas). Let $h = h^* \in \mathcal{B}(\mathfrak{h})$, $\mu \in \mathbb{R}_0^+$ and $z \in \mathbb{C}$. If $\Delta(h,z) > \mathbf{S}(h,\mu)$ then, for all $x, y \in \mathbb{Z}^d$,*

$$\left|\left\langle \mathfrak{e}_x, (z-h)^{-1}\mathfrak{e}_y \right\rangle\right| \leq \frac{\mathrm{e}^{-\mu|x-y|}}{\Delta(h,z) - \mathbf{S}(h,\mu)}.$$

Proof. This theorem is an instance of the first part of [27, Theorem 10.5] and is proven in the same way. □

The Combes–Thomas estimate yields the following bound [42, Lemma 3]:

Proposition A.2 *(Bound on differences of resolvents). Let $h = h^* \in \mathcal{B}(\mathfrak{h})$, $\mu \in \mathbb{R}_0^+$ and $\eta \in \mathbb{R}^+$ such that $\mathbf{S}(h,\mu) \leq \eta/2$. Then, for all $x, y \in \mathbb{Z}^d$ and $u \in \mathbb{R}$,*

$$\begin{aligned}&\left|\left\langle \mathfrak{e}_x, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \\ &\quad \leq 12\mathrm{e}^{-\mu|x-y|} \left\langle \mathfrak{e}_x, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_x \right\rangle_{\mathfrak{h}}^{1/2} \left\langle \mathfrak{e}_y, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_y \right\rangle_{\mathfrak{h}}^{1/2}.\end{aligned}$$

We are now in a position to prove the space decay of propagators:

Corollary A.3 *(Space decay of propagators – I). For any self-adjoint operator $h = h^* \in \mathcal{B}(\mathfrak{h})$, $\eta, \mu \in \mathbb{R}^+$, all $x, y \in \mathbb{Z}^d$ and $t \in \mathbb{R}$,*

$$\left|\left\langle \mathfrak{e}_x, \mathrm{e}^{ith}\mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \leq 36 \exp\left(|t\eta| - \mu \min\left\{1, \frac{\eta}{2\mathbf{S}(h,\mu)}\right\} |x-y|\right).$$

Proof. The proof is a simple adaptation of the one from [42, Theorem 3]: Fix all parameters of the lemma and observe that Proposition A.2 combined with Inequality (A.5) yields

$$\begin{aligned}&\left|\left\langle \mathfrak{e}_x, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \\ &\quad \leq 12\mathrm{e}^{-\frac{\mu\eta}{2\mathbf{S}(h,\mu)}|x-y|} \left\langle \mathfrak{e}_x, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_x \right\rangle_{\mathfrak{h}}^{1/2} \left\langle \mathfrak{e}_y, ((h-u)^2+\eta^2)^{-1}\mathfrak{e}_y \right\rangle_{\mathfrak{h}}^{1/2}\end{aligned} \tag{A.6}$$

for $x, y \in \mathbb{Z}^d$, $u \in \mathbb{R}$ and $\eta \in \mathbb{R}^+$. On the other hand, at fixed $\eta \in \mathbb{R}^+$, the function defined by $G(z) \doteq \mathrm{e}^{itz}$ on the stripe

$$\mathbb{R} + i\eta\,[-1,1] \subset \mathbb{C}$$

is analytic and uniformly bounded by $\mathrm{e}^{|t\eta|}$. Using Cauchy's integral formula and translations by $\pm i\eta$ of the integration variable, u, we write the function G as

$$\begin{aligned}G(E) &= \frac{1}{2\pi i}\int_{\mathbb{R}} \left(\frac{G(u-i\eta)}{u-i\eta-E} - \frac{G(u+i\eta)}{u+i\eta-E}\right)\mathrm{d}u \\ &= \frac{\eta}{\pi}\int_{\mathbb{R}} \frac{G(u-i\eta)+G(u+i\eta)}{(E-u)^2+\eta^2}\mathrm{d}u - \frac{2\eta}{\pi}\int_{\mathbb{R}} \frac{G(u)}{(E-u)^2+4\eta^2}\mathrm{d}u\end{aligned} \tag{A.7}$$

for all $E \in \mathbb{R}$ and $\eta \in \mathbb{R}^+$. By spectral calculus, together with (A.6)–(A.7) and the Cauchy–Schwarz inequality, the assertion follows. □

Corollary A.4 *(Space decay of propagators – II). For any self-adjoint operators $h_1, h_2 \in \mathcal{B}(\mathfrak{h})$ and all $x, y \in \mathbb{Z}^d$,*

$$\left|\left\langle \mathfrak{e}_x, \frac{1}{1+\mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2}} \mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \leq 2 \inf_{\mu \in \mathbb{R}_0^+} \exp\left(-\frac{\mu}{2}\mathrm{e}^{-\mathbf{S}_0(h_1,\mu)-2\mathbf{S}_0(h_2,\mu)}|x-y|\right).$$

Proof. By (A.1)–(A.4), note that, for any $\mu \in \mathbb{R}_0^+$,

$$\mathbf{S}(\mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2}, \mu) \leq \mathbf{S}_0(\mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2}, \mu) \leq \mathrm{e}^{\mathbf{S}_0(h_1,\mu)+2\mathbf{S}_0(h_2,\mu)}.$$

Fix $\mu \in \mathbb{R}_0^+$ and define

$$\mu_1 \doteq \frac{\mu}{2}\mathrm{e}^{-\mathbf{S}_0(h_1,\mu)-2\mathbf{S}_0(h_2,\mu)}.$$

By (A.5), $\mathbf{S}(\mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2}, \mu_1) < 1/2$. Meanwhile, by using Theorem A.1 with $h = \mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2} \geq 0$,

$$\left|\left\langle \mathfrak{e}_x, \frac{1}{1+\mathrm{e}^{h_2}\mathrm{e}^{h_1}\mathrm{e}^{h_2}} \mathfrak{e}_y \right\rangle_{\mathfrak{h}}\right| \leq 2\mathrm{e}^{-\mu_1|x-y|}. \quad \square$$

Appendix B. Large deviation formalism

In probability theory, the large deviation (LD) formalism quantitatively describes, for large $n \gg 1$, the probability of finding an empirical mean that differs from the expected value, by more than some fixed amount. That's the reason is why we apply it in Section 3 to prove the exponentially fast convergence of microscopic current densities towards their (classical) macroscopic values. For completeness, in this appendix, we present the main result from LD theory used in the current study, namely, the Gärtner–Ellis theorem (Theorem B.1 below). For more details, see [23,24]. For a historical review of LD in quantum statistical mechanics, see [25, Section 7.1].

Let $\mathcal{X}$ denote a topological vector space. A lower semi-continuous function $\mathrm{I} : \mathcal{X} \to [0, \infty]$ is called a *good rate function* if I is not identically ∞ and has compact level sets, i.e., $\mathrm{I}^{-1}([0, m]) = \{x \in \mathcal{X} : \mathrm{I}(x) \leq m\}$ is compact for any $m \geq 0$. A sequence $(X_L)_{L\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables satisfies the *LD upper bound* with *speed* $(\mathfrak{n}_L)_{L\in\mathbb{N}} \subset \mathbb{R}^+$ (a positive, increasing and divergent sequence) and rate function I if, for any closed subset F of $\mathcal{X}$,

$$\limsup_{L\to\infty} \frac{1}{\mathfrak{n}_L} \ln \mathbb{P}(X_l \in F) \leq -\inf_{x\in F} \mathrm{I}(x), \tag{B.1}$$

and it satisfies the *LD lower bound* if, for any open subset G of $\mathcal{X}$,

$$\liminf_{L\to\infty} \frac{1}{\mathfrak{n}_L} \ln \mathbb{P}(X_l \in G) \geq -\inf_{x\in G} \mathrm{I}(x). \tag{B.2}$$

If both, upper and lower bound, are satisfied, one says that $(X_L)_{L\in\mathbb{N}}$ satisfies an *LD principle* (LDP). The principle is called *weak* if the upper bound in (B.1) holds only for *compact* sets F.

A weak LDP can be strengthened to a full one by showing that the sequence $(X_L)_{L\in\mathbb{N}}$ of distributions is *exponentially tight*, i.e., if for any $\alpha \in \mathbb{R}$, there is a compact subset $\mathcal{G}_\alpha$ of $\mathcal{X}$ such that

$$\limsup_{L\to\infty} \frac{1}{\mathfrak{n}_L} \ln \mathbb{P}(X_L \in \mathcal{X}\backslash\mathcal{G}_\alpha) < -\alpha. \tag{B.3}$$

If $\mathcal{X}$ is a locally compact topological space, i.e., every point possesses a compact neighborhood, then the existence of an LDP with a good rate function I for the sequence $(X_L)_{L\in\mathbb{N}}$ implies its exponential tightness [24, Exercise 1.2.19].

A sufficient condition to ensure that a sequence $(X_L)_{L\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables satisfies an LDP is given by the Gärtner–Ellis theorem. It says [24, Corollary 4.5.27] that an exponentially tight sequence $(X_L)_{L\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables on a Banach space $\mathcal{X}$ satisfies an LDP with the good rate function

$$\mathrm{I}(x) = \sup_{s\in\mathcal{X}^*} \{s(x) - J(s)\}\ , \qquad x \in \mathcal{X}, \tag{B.4}$$

whenever the so-called limiting logarithmic moment generating function

$$J(s) \doteq \lim_{L\to\infty} \frac{1}{\mathfrak{n}_L} \ln \mathbb{E}\left[\mathrm{e}^{\mathfrak{n}_L s(X_L)}\right]\ , \qquad s \in \mathcal{X}^*, \tag{B.5}$$

exists as a Gateaux differentiable and weak* lower semi-continuous (finite-valued) function on the dual space $\mathcal{X}^*$. See also [23, Theorem 2.2.4].

The random variables we study in this paper result from bounded sequences $(A_L)_{L\in\mathbb{N}} \subset \mathcal{U}$ of self-adjoint elements of the CAR C^*-algebra $\mathcal{U}$ along with some fixed state $\rho \in \mathcal{U}^*$. In Section 3, we explain how such a sequence and state naturally define an exponentially tight sequence of random variables on the real line $\mathcal{X} = \mathbb{R}$, via the Riesz–Markov theorem and functional calculus (cf. (22)). The following simple version of the celebrated Gärtner–Ellis theorem of LD theory is sufficient for our purposes:

Theorem B.1 *(Gärtner–Ellis). Take any exponentially tight sequence $(X_L)_{L\in\mathbb{N}}$ of real-valued random variables (i.e., $\mathcal{X} = \mathcal{X}^* = \mathbb{R}$) and assume that the limiting logarithmic moment generating function J defined by (B.5) exists for all $s \in \mathbb{R}$. Then:*

(LD1) *$(X_L)_{L\in\mathbb{N}}$ satisfies the LD upper bound (B.1) with rate function* I *given by (B.4).*
(LD2) *If, additionally,* J *is differentiable for all $s \in \mathbb{R}$ then $(X_L)_{L\in\mathbb{N}}$ satisfies the LD lower bound (B.2) with good rate function* I *given again by (B.4).*

Proof. (LD1) and (LD2) are special cases of [43, Theorem V.6.(a) and (c)], respectively. □

Appendix C. Response of quasi-free fermion systems to electric fields

C.1. Linear response current

Recall that $(\Omega, \mathfrak{A}_\Omega)$ is the measurable space defined in Section 2.1, $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d;\mathbb{C})$ is the one-particle Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and canonical orthonormal basis denoted by $\{\mathfrak{e}_x\}_{x\in\mathbb{Z}^d}$, and the one-particle Hamiltonian of the quasi-free fermion system equals (4), i.e.,

$$h^{(\omega)} \doteq \Delta_{\omega,\vartheta} + \lambda\omega_1\ , \qquad \omega = (\omega_1, \omega_2) \in \Omega,\ \lambda, \vartheta \in \mathbb{R}_0^+,$$

with $\Delta_{\omega,\vartheta}$ being (up to a minus sign) the random discrete Laplacian. See again Section 2.1. The associate (quasi-) free dynamics is thus defined from the (random) unitary group $\{\mathrm{e}^{ith^{(\omega)}}\}_{t\in\mathbb{R}}$.

Then, apply on the fermion system an electromagnetic field resulting[2] from a compactly supported time-dependent space-rescaled vector potential $\eta\mathbf{A}_L$ defined by

[2] We use the Weyl gauge, also named temporal gauge.

$$\eta \mathbf{A}_L(t,x) \doteq \eta \mathbf{A}(t, L^{-1}x), \quad t \in \mathbb{R}, \, x \in \mathbb{R}^d, \, \eta \in \mathbb{R}_0^+, \tag{C.1}$$

where

$$\mathbf{A} \in \mathbf{C}_0^\infty \doteq \bigcup_{l \in \mathbb{R}^+} C_0^\infty(\mathbb{R} \times [-l,l]^d; (\mathbb{R}^d)^*).$$

Here, $(\mathbb{R}^d)^*$ is the set of one-forms[3] on $\mathbb{R}^d$ that take values in $\mathbb{R}$. We see any $\mathbf{A} \in C_0^\infty(\mathbb{R} \times [-l,l]^d; (\mathbb{R}^d)^*) \subseteq \mathbf{C}_0^\infty$, $l \in \mathbb{R}^+$, as a function $\mathbb{R} \times \mathbb{R}^d \to (\mathbb{R}^d)^*$ via the convention $\mathbf{A}(t,x) \equiv 0$ for $x \notin [-l,l]^d$. The main reason for not using (the standard choice) $C_0^\infty(\mathbb{R} \times \mathbb{R}^d; (\mathbb{R}^d)^*)$ instead of $\mathbf{C}_0^\infty$ as a space of vector potentials, is that we need to include (in general non-smooth) functions that are constant for x inside cubes $[-l,l]^d$ and vanish outside. The time derivative of this vector potential is the (time-dependent) electric field. Since we are interested here in the linear response current to electromagnetic fields, we use in (C.1) a real parameter $\eta \in \mathbb{R}_0^+$ to also rescale the strength of the vector potential $\mathbf{A}_L$.

To simplify notation, we consider, without loss of generality, spinless fermions with negative charge. So, such an electromagnetic field leads to a time-dependent Hamiltonian defined by

$$\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda \omega_1, \quad t \in \mathbb{R},$$

where $\Delta_{\omega,\vartheta}^{(\mathbf{A})} \equiv \Delta_{\omega,\vartheta}^{(\mathbf{A}(t,\cdot))} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is the time-dependent self-adjoint operator defined[4] by

$$\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}^{(\mathbf{A})} \mathfrak{e}_y \rangle_{\mathfrak{h}} = \exp\left(i \int_0^1 [\mathbf{A}(t, \alpha y + (1-\alpha)x)] (y-x) \mathrm{d}\alpha \right) \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} \tag{C.2}$$

for $\mathbf{A} \in \mathbf{C}_0^\infty$, $t \in \mathbb{R}$ and $x, y \in \mathbb{Z}^d$. It is (up to a minus sign) the magnetic Laplacian, as explained in [44, Section III, in particular Corollary 3.1]. This yields a dynamics, perturbed by the time-dependent vector potential $\eta \mathbf{A}_L$, given by the (well-defined random) two-parameter family $\{\mathrm{U}_{t,t_0}^{(\omega)}\}_{t_0, t \in \mathbb{R}}$ of unitary operators on $\mathfrak{h}$ satisfying the non-autonomous evolution equation

$$\forall t_0, t \in \mathbb{R}: \qquad \partial_t \mathrm{U}_{t,t_0}^{(\omega)} = -i(\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda \omega_1) \mathrm{U}_{t,t_0}^{(\omega)}, \quad \mathrm{U}_{t_0,t_0}^{(\omega)} \doteq \mathbf{1}_{\mathfrak{h}}. \tag{C.3}$$

In the algebraic formulation, it corresponds to the quasi-free dynamics on the CAR C^*-algebra $\mathcal{U}$, defined by the unique two-parameter group $\{\xi_{t,t_0}^{(\omega)}\}_{t_0, t \in \mathbb{R}}$ of (Bogoliubov) $*$-automorphisms satisfying

$$\xi_{t,t_0}^{(\omega)}(a(\psi)) = a((\mathrm{U}_{t,t_0}^{(\omega)})^* \psi), \qquad t_0, t \in \mathbb{R}, \; \psi \in \mathfrak{h}. \tag{C.4}$$

The above procedure for coupling charged lattice fermions to a vector potential is sometimes called "Peierls coupling".

Additionally to the paramagnetic current observable $I_{(x,y)}^{(\omega)}$ (17), the perturbing vector potential $\mathbf{A} \in \mathbf{C}_0^\infty$ yields a second type of current observable, defined[5] by

$$\tilde{I}_{(x,y)}^{(\omega)} \doteq -2\Im\mathrm{m} \left(\left(\mathrm{e}^{i \int_0^1 [\mathbf{A}(t,\alpha y + (1-\alpha)x)](y-x) d\alpha} - 1 \right) \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle_{\mathfrak{h}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_y) \right) \tag{C.5}$$

[3] In a strict sense, one should take the dual space of the tangent spaces $T(\mathbb{R}^d)_x$, $x \in \mathbb{R}^d$.

[4] Observe that the sign of the coupling between $\mathbf{A} \in \mathbf{C}_0^\infty$ and the Laplacian is wrong in [18, Eq. (2.8)] for negatively charged fermions.

[5] Observe that the sign in the exponent in [21, Eq. (50)] and [7, (4.2)] for negatively charged fermions is wrong, with no consequence on the corresponding results.

for any $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $x, y \in \mathbb{Z}^d$, where we recall that $\Im\mathrm{m}(A) \in \mathcal{U}$ is the imaginary part of $A \in \mathcal{U}$, see (18). We name it *diamagnetic* current observable. The derivation of the paramagnetic and diamagnetic current observables is explained in detail in Appendix C.2. The decomposition of the full current observable

$$\tilde{I}_{(x,y)}^{(\omega)} + I_{(x,y)}^{(\omega)} = -2\Im\mathrm{m}\left(\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_y \rangle_{\mathfrak{h}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_y)\right) \doteq \mathbf{I}_{(x,y)}^{(\omega,\mathbf{A})} \tag{C.6}$$

in so-called paramagnetic and diamagnetic current observables has a physical relevance. First, it comes from the physics literature, see, e.g., [45, Eq. (A2.14)]. Secondly, the paramagnetic current observable is intrinsic to the system and related to a heat production, whereas the diamagnetic one is only non-vanishing in presence of vector potentials and refers to the ballistic accelerations, induced by electromagnetic fields, of charged particles. For more details, see [6,19].

Observe that the time evolution of the KMS state $\varrho^{(\omega)} \in \mathcal{U}^*$ (see (13)–(14)) is given by $\varrho^{(\omega)} \circ \xi_{t,t_0}^{(\omega)}$ for $t, t_0 \in \mathbb{R}$. In [6,18–20][6] we perform a detailed study the behavior of current densities when $\eta \to 0$, *uniformly* w.r.t. the volume $\mathcal{O}\left(L^d\right)$ of the boxes where the vector potential $\mathbf{A}_L$ is non-zero. In [7,21,22], these results are generalized to lattice-fermion systems in disordered media with *very general interactions*[7] and on *passive* states (not necessarily KMS). These mathematically rigorous studies yield an alternative physical picture of Ohm and Joule's laws (at least in the AC-regime), different from usual explanations coming from the Drude model or the Landau theory of Fermi liquids.

To shortly present how the linear response current naturally appears, without requiring a thorough reading of this series of papers, consider a space homogeneous electric fields in the box Λ_L (12) for any $L \in \mathbb{R}^+$. To be more precise, let $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$ and set $\mathcal{E}(t) \doteq -\partial_t \mathcal{A}(t)$ for all $t \in \mathbb{R}$. Therefore, $\mathbf{A}$ is defined to be the vector potential such that the electric field is given by $\mathcal{E}(t) \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$ at time $t \in \mathbb{R}$, for all $x \in [-1,1]^d$, and $(0, 0, \ldots, 0)$ for $t \in \mathbb{R}$ and $x \notin [-1,1]^d$. It yields a rescaled vector potential $\eta \mathbf{A}_L$ for $L \in \mathbb{R}^+$ and $\eta \in \mathbb{R}_0^+$.

Then, by (17) and (C.5), the space-averaged response current observable, or response current density observable, in the box Λ_L and in the direction $\vec{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d$ ($|\vec{w}| = 1$), for any $\omega \in \Omega$, $\lambda, \vartheta, \eta \in \mathbb{R}_0^+$, $L \in \mathbb{R}^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t_0, t \in \mathbb{R} \in \mathbb{R}$, is, by definition, equal to

$$\mathbb{J}_L^{(\omega)}(t, \eta) \doteq \frac{1}{|\Lambda_L|} \sum_{k=1}^{d} w_k \sum_{x \in \Lambda_L} \left(\xi_{t,t_0}^{(\omega)} \left(I_{(x+e_k,x)}^{(\omega)} + \tilde{I}_{(x+e_k,x)}^{(\omega,\eta \mathbf{A}_L)} \right) - I_{(x+e_k,x)}^{(\omega)} \right) \tag{C.7}$$

with $\{e_k\}_{k=1}^d$ being the canonical orthonormal basis of the Euclidean space $\mathbb{R}^d$.

By using the generalization done in [22] of the celebrated Lieb–Robinson bounds (for commutators) to multi-commutators, the full current density observable in the direction $\vec{w} \in \mathbb{R}^d$ ($|\vec{w}| = 1$) satisfies

$$\mathbb{J}_L^{(\omega)}(t, \eta) = \eta \mathbf{J}_L^{(\omega)}(t) + \mathcal{O}\left(\eta^2\right) \tag{C.8}$$

in the CAR C^*-algebra $\mathcal{U}$. The correction terms of order $\mathcal{O}(\eta^2)$ are *uniformly bounded* in $L \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda, t \in \mathbb{R}_0^+$ and ϑ on compacta. By explicit computations, one checks that the linear part is

$$\mathbf{J}_L^{(\omega)}(t) = \sum_{k,q=1}^{d} w_k \int_{-\infty}^{t} \{\mathcal{E}(\alpha)\}_q \left\{ \mathcal{C}_{\Lambda_L}^{(\omega)}(t - \alpha) \right\}_{k,q} \mathrm{d}\alpha, \tag{C.9}$$

which is equal to $\mathbb{I}_{\Lambda_L}^{(\omega,\mathcal{E}_t)}$ (20) for the electric field defined by (21). See also (19) for the definition of $\mathcal{C}_\Lambda^{(\omega)} \in C^1(\mathbb{R}; \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d))$. This current density observable is therefore the *space-averaged linear response current*

[6] In all our papers we use smooth electric fields, but the extension to the continuous case is straightforward.

[7] Sufficiently strong polynomial decays of interactions are necessary. This includes basically standard models of physics that describes interacting fermions in crystal.

observable (or linear response current density observable) in the direction $\vec{w} \in \mathbb{R}^d$ we study in all the paper. Because of (C.9), $\mathcal{C}_{\Lambda_L}^{(\omega)}$ is called the conductivity observable matrix associated with Λ_L. For more details, see also [21, Theorem 3.7].

In [6,7,20,22], for any time $t \in \mathbb{R}$, we prove the existence of the limit $L \to \infty$ of the random linear response current density

$$\varrho^{(\omega)}\left(\mathbf{J}_L^{(\omega)}(t)\right), \qquad L \in \mathbb{R}^+,$$

to a deterministic value, with probability one. At time $t = 0$ this refers to the following assertion:

$$x^{(\mathcal{E})} = \lim_{L \to \infty} \varrho^{(\omega)}\left(\mathbf{J}_L^{(\omega)}(0)\right), \tag{C.10}$$

which is directly related with (23) and (56) at $s = 0$.

C.2. Discrete continuity equation in the CAR algebra

As is usual, the self-adjoint element

$$a(\mathfrak{e}_x)^* a(\mathfrak{e}_x) \in \mathcal{U}$$

represents the particle number observable at the lattice site $x \in \mathbb{Z}^d$. Fixing once for all $\omega \in \Omega$, $\lambda, \vartheta, \eta \in \mathbb{R}_0^+$, $L \in \mathbb{R}^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, its time-evolution by the two-parameter group $\{\xi_{t,t_0}^{(\omega)}\}_{t_0, t \in \mathbb{R}}$ of (Bogoliubov) $*$-automorphisms defined by (C.4) equals

$$\xi_{t,t_0}^{(\omega)}\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_x)\right) = a((\mathrm{U}_{t,t_0}^{(\omega)})^* \mathfrak{e}_x)^* a((\mathrm{U}_{t,t_0}^{(\omega)})^* \mathfrak{e}_x) \tag{C.11}$$

for any $t_0, t \in \mathbb{R}$ and $x \in \mathbb{Z}^d$. Observe that $(\mathrm{U}_{t,t_0}^{(\omega)})^* = \mathrm{U}_{t_0,t}^{(\omega)}$ for any $t_0, t \in \mathbb{R}$ while

$$\forall t_0, t \in \mathbb{R}: \qquad \partial_{t_0} \mathrm{U}_{t,t_0}^{(\omega)} = i \mathrm{U}_{t,t_0}^{(\omega)} (\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda \omega_1), \quad \mathrm{U}_{t_0,t_0}^{(\omega)} \doteq \mathbf{1}_{\mathfrak{h}} \ . \tag{C.12}$$

From standard properties of the so-called fermionic creation/annihilation operators, the time derivative of (C.11) equals

$$\partial_t \left(\xi_{t,t_0}^{(\omega)}\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_x)\right)\right) = \xi_{t,t_0}^{(\omega)}\left(\left(a(i(\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1)\mathfrak{e}_x)^* a(\mathfrak{e}_x) + a(\mathfrak{e}_x)^* a(i(\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1)\mathfrak{e}_x)\right)\right).$$

Recall now that the map $\psi \mapsto a(\psi)^*$ from $\mathfrak{h}$ to $\mathcal{U}$ is linear and, by (3) and (C.2), for any $x \in \mathbb{Z}^d$,

$$(\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1)\mathfrak{e}_x = \lambda\omega_1(x)\,\mathfrak{e}_x + \sum_{z \in \mathbb{Z}^d, |z|=1} \langle \mathfrak{e}_{x+z}, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x \rangle_{\mathfrak{h}} \mathfrak{e}_{x+z}.$$

It follows that

$$\partial_t \left(\xi_{t,t_0}^{(\omega)}\left(a(\mathfrak{e}_x)^* a(\mathfrak{e}_x)\right)\right) = \sum_{z \in \mathbb{Z}^d, |z|=1} \xi_{t,t_0}^{(\omega)}\left(-2\Im\mathrm{m}\left(\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_{x+z} \rangle_{\mathfrak{h}} a(\mathfrak{e}_x)^* a(\mathfrak{e}_{x+z})\right)\right) \tag{C.13}$$

for any $t_0, t \in \mathbb{R}$ and $x \in \mathbb{Z}^d$. Another way to prove this equation is to use [21, Theorem 2.1 (ii) with $\Psi^{\mathrm{IP}} = 0$] together with straightforward computations using the CAR (9). Proceeding in this manner, observe that the quasi-free property of the dynamics is not needed at all. In particular this derivation easily extends to the interacting case. It is not so for the one-particle picture discussed in the next section, which is much more restrictive than the algebraic approach.

Equation (C.13) is interpreted as a *discrete continuity equation*

$$\partial_t \left(\xi_{t,t_0}^{(\omega)}(a(\mathfrak{e}_x)^* a(\mathfrak{e}_x)\right) = \sum_{z\in\mathbb{Z}^d,|z|=1} \xi_{t,t_0}^{(\omega)}\left(\mathbf{I}_{(x,x+z)}^{(\omega,\eta\mathbf{A}_L)}\right)$$

in the CAR C^*-algebra $\mathcal{U}$. The observable $\mathbf{I}_{(x,y)}^{(\omega,\mathbf{A})}$ defined by (C.6) is the observable related to the flow of particles from the lattice site x to the lattice site y or the current from y to x for negatively charged particles. [Positively charged particles can of course be treated in the same way.] In the non-interacting case, this definition of current observable is mathematically equivalent to the usual one in the one-particle picture, like in [8,10,11]. See Equation (C.19).

C.3. The one-particle picture

When dealing with non-interacting fermions, most of the time, the one-particle picture of such a physical system is employed, as for instance in [11]. This is frequently technically convenient. Indeed, note that various important estimates in the current study were obtained in this picture and even all the analysis performed here could have been done in the one-particle Hilbert space $\mathfrak{h}$. However, in many cases, this preference is only subjective and motivated by the fact that, by some reason, people feel more comfortable in dealing with Hilbert spaces than with C^*-algebras. We stress that the algebraic formulation is, from a conceptual point of view, the natural one, as the underlying physical system is many-body. Moreover, it has some advantageous technical aspects, both specific (like the possibility of using Bogoliubov-type inequalities in important estimates) and general ones (like the very powerful theory of KMS states). For convenience of those preferring the one-particle picture of free fermion systems, we establish in the following the precise relation of the "second quantized" objects we used here with this picture.

As in the previous subsection, fix once for all $\omega \in \Omega$, $\lambda, \vartheta, \eta \in \mathbb{R}_0^+$, $L \in \mathbb{R}^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$. Recall that the corresponding KMS state $\varrho^{(\omega)}$ is the gauge-invariant quasi-free state satisfying (14), i.e.,

$$\varrho^{(\omega)}(a^*(\varphi)\, a(\psi)) = \left\langle \psi, \mathbf{d}^{(\omega)}\varphi \right\rangle_{\mathfrak{h}}, \qquad \varphi, \psi \in \mathfrak{h}, \tag{C.14}$$

where

$$\mathbf{d}^{(\omega)} \doteq (1 + \mathrm{e}^{\beta h^{(\omega)}})^{-1} \in \mathcal{B}(\mathfrak{h})$$

and the one-particle Hamiltonian $h^{(\omega)} = (h^{(\omega)})^* \in \mathcal{B}(\mathfrak{h})$ is defined by (4). The positive bounded operator $\mathbf{d}^{(\omega)}$ satisfies $0 \leq \mathbf{d}^{(\omega)} \leq \mathbf{1}_{\mathfrak{h}}$ and is called the symbol, or one-particle density matrix, of the quasi-free state $\varrho^{(\omega)}$. See (15)–(16) for the definition of gauge-invariant quasi-free states.

The time-evolution $\varrho^{(\omega)}$ by the two-parameter group $\{\xi_{t,t_0}^{(\omega)}\}_{t_0,t\in\mathbb{R}}$ of (Bogoliubov) $*$-automorphisms defined by (C.4) is $\varrho^{(\omega)} \circ \xi_{t,t_0}^{(\omega)}$ for any $t_0, t \in \mathbb{R}$. It is again a gauge-invariant quasi-free state and satisfies

$$\varrho^{(\omega)} \circ \xi_{t,t_0}^{(\omega)}(a^*(\varphi)\, a(\psi)) = \left\langle \psi, \mathrm{U}_{t,t_0}^{(\omega)}\mathbf{d}^{(\omega)}(\mathrm{U}_{t,t_0}^{(\omega)})^*\varphi \right\rangle_{\mathfrak{h}}, \qquad \varphi, \psi \in \mathfrak{h}, \tag{C.15}$$

for any $t_0, t \in \mathbb{R}$, by (C.4) and (C.14). Again,

$$\mathbf{d}_{t,t_0}^{(\omega)} \doteq \mathrm{U}_{t,t_0}^{(\omega)}(1 + \mathrm{e}^{\beta h^{(\omega)}})^{-1}(\mathrm{U}_{t,t_0}^{(\omega)})^* \in \mathcal{B}(\mathfrak{h}) \tag{C.16}$$

is a positive bounded operator $\mathbf{d}_{t,t_0}^{(\omega)}$ satisfying $0 \leq \mathbf{d}_{t,t_0}^{(\omega)} \leq \mathbf{1}_{\mathfrak{h}}$. It is the symbol, or one-particle density matrix, of the quasi-free state $\varrho^{(\omega)} \circ \xi_{t,t_0}^{(\omega)}$. Recall that the unitary operators $\mathrm{U}_{t,t_0}^{(\omega)} \in \mathcal{B}(\mathfrak{h})$, $t_0, t \in \mathbb{R}$, are uniquely defined by (C.12).

By (C.3), (C.12) and (C.16) together with $(\mathrm{U}_{t,t_0}^{(\omega)})^* = \mathrm{U}_{t_0,t}^{(\omega)}$, the symbol $\mathbf{d}_{t,t_0}^{(\omega)}$ is the solution of the *Liouville equation*:

$$\forall t_0, t \in \mathbb{R}: \qquad i\partial_t \mathbf{d}_{t,t_0}^{(\omega)} = \left[\left(\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1\right), \mathbf{d}_{t,t_0}^{(\omega)}\right], \qquad \mathbf{d}_{t_0,t_0}^{(\omega)} \doteq \mathbf{d}^{(\omega)}, \tag{C.17}$$

as for instance in [11, Eq. (2.5)]. Then, all the study performed in the current paper for second quantized currents of non-interacting fermions can be translated into the one-particle picture by using the Liouville equation and the fact that the corresponding quasi-free states are completely determined by the one-particle density matrices $\{\mathbf{d}_{t,t_0}^{(\omega)}\}_{t_0,t\in\mathbb{R}}$, solving the above initial value problem.

In this framework, the current observable discussed in Section C.1, and studied along the paper, can be represented by self-adjoint operators on the one-particle Hilbert space $\mathfrak{h}$. See, e.g., (32). In this perspective, note that the full current density observable in a box Λ_L in a fixed direction e_k, $k \in \{1, \ldots, d\}$, in $\mathbb{R}^d$ is the so-called second quantization of the operator defined by

$$\mathfrak{I}_L^{(\omega)} \doteq -\frac{2}{|\Lambda_L|} \sum_{x \in \Lambda_L} \Im\mathrm{m}\{\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x \rangle_{\mathfrak{h}} P_{\{x+e_k\}} s_{e_k} P_{\{x\}}\}, \qquad L \in \mathbb{R}^+, \tag{C.18}$$

using the notation (30) for shift operators. See also (31). In other words, by Definition 4.3,

$$\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \mathbf{I}_{(x+e_k,x)}^{(\omega,\eta \mathbf{A}_L)} = \langle \mathrm{A}, \mathfrak{I}_L^{(\omega)} \mathrm{A} \rangle \ .$$

The one-particle operator $\mathfrak{I}_L^{(\omega)}$ is directly related with the commonly used current observable in the one-particle Hilbert space, like in [8,10,11]. To see this, for $k \in \{1, \ldots, d\}$, define the (unbounded) multiplication operator on $\mathfrak{h}$ with the k^{th} component by

$$X_k(\psi)(x_1, \ldots, x_d) \doteq x_k \psi(x_1, \ldots, x_d),$$

for ψ within the domain of X_k. For any $x \in \mathbb{Z}^d$, remark that

$$\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x = \sum_{z \in \mathbb{Z}^d, |z|=1} \langle \mathfrak{e}_{x+z}, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x \rangle_{\mathfrak{h}} \mathfrak{e}_{x+z}$$

and

$$-i\left[\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)}, X_k\right] \mathfrak{e}_x = i\left(\langle \mathfrak{e}_{x+e_k}, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x \rangle_{\mathfrak{h}} \mathfrak{e}_{x+e_k} - \langle \mathfrak{e}_{x-e_k}, \Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} \mathfrak{e}_x \rangle_{\mathfrak{h}} \mathfrak{e}_{x-e_k}\right).$$

Combining this with (C.18), one checks that

$$\mathfrak{I}_L^{(\omega)} = \frac{1}{|\Lambda_L|} \mathrm{P}_L \left(-i\left[\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1, X_k\right]\right) \mathrm{P}_L + \mathcal{O}(L^{-1}) \ , \qquad L \in \mathbb{R}^+, \tag{C.19}$$

uniformly in $\mathcal{U}$ w.r.t. all parameters, where P_L is the orthogonal projection with range $\mathrm{lin}\{\mathfrak{e}_x : x \in \Lambda_L\}$, that is, the multiplication operator with the characteristic function of the box Λ_L. The term of order $\mathcal{O}(L^{-1})$ results from the existence of $\mathcal{O}(L^{d-1})$ points $x \in \Lambda_L$ such that $x + e_k \notin \Lambda_L$.

We recover from (C.19) the usual description for the current observable as a self-adjoint operator on the one-particle Hilbert space $\mathfrak{h}$, in our case the velocity operator $-i[\Delta_{\omega,\vartheta}^{(\eta \mathbf{A}_L)} + \lambda\omega_1, X_k]$. See, e.g., [8,10,11].

Observe additionally that the quantity obtained by applying the state $\varrho^{(\omega)} \circ \xi_{t,t_0}^{(\omega)}$ on the full current density observable gives, in the large volume limit (i.e., $L \to \infty$), the density of trace of the product of symbol $\mathbf{d}_{t,t_0}^{(\omega)}$ with the velocity operator on the one-particle Hilbert space $\mathfrak{h}$, similar to [11, Equation (2.6)].

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