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RELIABILITY MEASURES OF A RELAY
SYSTEM UNDER DEPENDENCE CONDITIONS

by

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Reliability measure of a relay system under dependence conditions.

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Abstract. In a relay system of dependent components, the failure to close reliability measure is given as a Girsanov transform of the failure to open reliability measure.

Keywords: Martingale methods in reliability theory, compensator process, parallel operation.

1. Introduction.

As in Barlow and Proschan (1981), a complex coherent reliability system is completely characterized by its structure function $\Phi = \Phi(\mathbf{X}(t))$ assuming values in the set $\{0, 1\}$ and where $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_n(t))$ is a stochastic process assuming values in $\{0, 1\}^n$. The stochastic process $(X_i(t))_{t \geq 0}$, represents the state of the i -th component, $1 \leq i \leq n$ and $X_i(t) = 1$ if the component i is working at time t and $X_i(t) = 0$ if the component i is in a failed state at time t . Also, the system state $\Phi(\mathbf{X}(t))$ has such an interpretation, is increasing in each coordinate and each component is relevant, that is, there is a configuration of $(\mathbf{X}(t))_{t \geq 0}$ in which the functioning of the component i is fundamental for the functioning of the system.

A Relay system are subject to two kinds of failure: failure to close and failure to open. Similarly, circuits construct from these relays are subject to the same kinds of failure. If the i -th relay responds correctly to a command to close $X_i(t) = 1$ (that is, closes), and $X_i(t) = 0$ otherwise, $i = 1, 2$ and $\Phi(\mathbf{X}(t)) = 1$ if the circuit responds correctly to a command to close (that is, closes) if, and only if, at least one of its components responds correctly to a command to close, and 0 otherwise, then, $\Phi(\mathbf{X}(t)) = \max\{X_1(t), X_2(t)\}$ is a parallel system. Next, let $Y_i(t) = 1$ if the i -th relay responds correctly to a command to open (that is open), and $Y_i(t) = 0$ otherwise, $i = 1, 2$. Let $\Psi(\mathbf{Y}(t)) = 1$ if the circuit responds correctly to a command to open (that is, open), an 0 otherwise. Then $\Psi(\mathbf{Y}(t)) = \min\{Y_1(t), Y_2(t)\} = \Phi^D(\mathbf{Y}(t))$,

a series system, the dual of Φ , is the system representing the correct system response to a command to open.

Generally, given a structure function Φ , its dual Φ^D is defined by

$$\Phi^D(\mathbf{X}(t)) = 1 - \Phi(1 - \mathbf{X}(t)),$$

where $1 - \mathbf{X}(t) = (1 - X_1(t), \dots, 1 - X_n(t))$.

The concept of dual structure is useful in analyzing system of components subject to two kinds of failures, such as relays system and safety monitoring systems. It is interesting and useful to note that both failure to close and failure to open can be analyzed using the same structure function Φ . In this paper, under dependence conditions, we analyze the dual structure probability measure, through a transform of the failures counting processes compensators in the original probability space. In Section 2 we analyze the dual measure of a parallel system, in Section 3 we do some application and in Section 4 we give the general formulation of the main result.

2. The dual parallel system measure.

As a coherent system always admits a series parallel decomposition and the dual of a module of a coherent system is a module of the dual structure we attained to analyze a parallel system. Without loss of generality we consider a parallel system of two components.

We consider to observe two component lifetimes T and S , which are positive random variables defined in a complete probability space $(\Omega, \mathfrak{F}, P)$ through the family of sub σ -algebras $(\mathfrak{F}_t)_{t \geq 0}$ of $\mathfrak{F}$, where

$$\mathfrak{F}_t = \sigma\{1_{\{S > s\}}, 1_{\{T > s\}}, 0 \leq s \leq t\}$$

satisfies Dellacherie's conditions. In what follows we assume that S and T are totally inaccessible $\mathfrak{F}_t$ -stopping time and that $P(S = T) = 0$, that is, the lifetimes can be dependent but simultaneous failures are ruled out.

The parallel operation of S and T is defined by the maximum between S and T and denoted by

$$S \vee T = \max\{S, T\}.$$

If we denote the survival functions of S and T as $\bar{G}(t) = P(S > t | \mathfrak{F}_t)$ and $\bar{F}(t) = P(T > t | \mathfrak{F}_t)$ respectively, it follows from Arjas and Yashin (1988) that the $\mathfrak{F}_t$ -compensator processes of $N_B(t) = 1_{\{S \leq t\}}$ and $N_A(t) = 1_{\{T \leq t\}}$

are given by $B(t) = -\ln P(S > t | \mathfrak{F}_t) = -\ln(\overline{G}(t \wedge S))$ and $A(t) = -\ln P(T > t | \mathfrak{F}_t) = -\ln(\overline{F}(t \wedge T))$. As S and T are totally inaccessible $\mathfrak{F}_t$ -stopping time the compensator processes are continuous.

Now we calculate

$$P(S \vee T > t | \mathfrak{F}_t) = \overline{F}(t) + \overline{G}(t) - \overline{F}(t)\overline{G}(t) = e^{-[A(t)+B(t)]} \{e^{A(t)} + e^{B(t)} - 1\}$$

and therefore, in the set $\{t < S \vee T\}$ the $\mathfrak{F}_t$ -compensator of $S \vee T$ is

$$-\ln[P(S \vee T > t | \mathfrak{F}_t)] = A(t) + B(t) - \ln[e^{A(t)} + e^{B(t)} - 1].$$

We intend to define compensator transformation of $A(t)$ and $B(t)$ to $A^*(t)$ and $B^*(t)$ in the way that the above expression is the sum $A^*(t) + B^*(t)$ which characterizes uniquely the lifetime of a series system and, therefore, the dual of such a parallel system. As this operation is symmetric on S and T , the idea is to combine compensator transformations in $A(t)$ and $B(t)$.

Firstly, we consider the compensator transform

$$A^*(t) = \int_0^t \frac{e^{B(s)} - 1}{e^{A(s)} + e^{B(s)} - 1} dA(s) = \int_0^t 1 - \left[\frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} \right] dA(s) = A(t) - \int_0^t \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s).$$

To prove the main Theorem of this section we are going to use the following Lemma:

Lemma 2.1 The following process

$$L_A(t) = \left(\frac{e^{B(T)} - 1}{e^{A(T)} + e^{B(T)} - 1} \right)^{1_{\{T \leq t\}}} e^{\int_0^t \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s)}$$

is a nonnegative local $\mathfrak{F}_t$ -martingale and $E[L_A(t)] = 1$.

Proof We consider the $\mathfrak{F}_t$ -stopping time defined by

$$V_n = \inf\{t \geq 0 : A(t) \geq n \text{ or } B(t) \geq n\}.$$

It is sufficient to prove that the process

$$L_A^n(t) = \left(\frac{e^{B(T)} - 1}{e^{A(T)} + e^{B(T)} - 1} \right)^{1_{\{T \leq t \wedge V_n\}}} e^{\int_0^{t \wedge V_n} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s)}$$

is a bounded $\mathfrak{F}_t$ -martingale.

For any $\mathfrak{F}_t$ -stopping time $V \leq V_n$ we can write

$$L_A^n(V) = 1 - \int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} d(N(s) - A(s))$$

where $N(t) = 1_{\{T \leq t\}}$. The procedure is easy:

On the set $\{V < T\}$ we have

$$\begin{aligned} 1 + \int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s) = \\ e^{\int_0^V \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} = L_A^n(V). \end{aligned}$$

Otherwise, on the set $\{V \geq T\}$

$$\begin{aligned} 1 - \int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} d(N(s) - A(s)) = \\ e^{\int_0^T \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \left[1 - \frac{e^{A(T)}}{e^{A(T)} + e^{B(T)} - 1} \right] = L_A^n(V). \end{aligned}$$

As the integrand $e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1}$ is an $\mathfrak{F}_t$ -predictable process and $N(s) - A(s)$ is an $\mathfrak{F}_t$ -martingale, $L_A^n(t)$ is an $\mathfrak{F}_t$ -martingale with $E[L_A^n(t)] = 1$.

Secondly, we consider the compensator transform

$$B^*(t) = \int_0^t \frac{e^{A(s)} - 1}{e^{A(s)} + e^{B(s)} - 1} dB(s) = B(t) - \int_0^t \frac{e^{B(s)}}{e^{A(s)} + e^{B(s)} - 1} dB(s)$$

and with the same argument to prove Lemma 2.1 we can prove Lemma 2.2:

Lemma 2.2 The following process

$$L_B(t) = \left(\frac{e^{A(S)} - 1}{e^{A(S)} + e^{B(S)} - 1} \right) 1_{\{S \leq t\}} e^{\int_0^t \frac{e^{B(s)}}{e^{A(s)} + e^{B(s)} - 1} dB(s)}$$

is a nonnegative local $\mathfrak{F}_t$ -martingale with $E[L_B(t)] = 1$.

Observe that the same expression for the $\mathfrak{F}_t$ -compensator of $1_{\{T \vee S \leq t\}}$ is obtained through the transformation:

$$A^*(t) + B^*(t) = A(t) + B(t) - \int_0^t \frac{e^{A(s)} dA(s) + e^{B(s)} dB(s)}{e^{A(s)} + e^{B(s)} - 1} =$$

$$A(t) + B(t) - \ln[e^{A(t)} + e^{B(t)} - 1].$$

We propose the compensator transforms:

$$A^*(t) = \int_0^t \alpha(s) dA(s), \quad \alpha(s) = \frac{e^{B(s)} - 1}{e^{A(s)} + e^{B(s)} - 1},$$

and

$$B^*(t) = \int_0^t \beta(s) dB(s), \quad \beta(s) = \frac{e^{A(s)} - 1}{e^{A(s)} + e^{B(s)} - 1}.$$

Now, we can write the main Theorem:

Theorem 2.3 The following process

$$L(t) = L_A(t)L_B(t) = (\alpha(T))^{1_{\{T \leq t\}}} (\beta(S))^{1_{\{S \leq t\}}} [e^{A(t)} + e^{B(t)} - 1]$$

is a nonnegative local $\mathfrak{F}_t$ -martingale and $E[L(t)] = 1$.

Proof. Using Lemma 2.1, Lemma 2.2 and the Stieltjes differentiation rule we have

$$L_A(t)L_B(t) - 1 = \int_0^t L_A(s-) dL_B(s) + \int_0^t L_B(s-) dL_A(s) + \sum_{s \leq t} \Delta L_A(s) \Delta L_B(s)$$

As by assumption, $A(t)$ and $B(t)$ are continuous and $P(S = T) = 0$ we have $\sum_{s \leq t} \Delta L_A(s) \Delta L_B(s) = 0$ and therefore $L_A(t)L_B(t)$ is a local $\mathfrak{F}_t$ -martingale with $E[L_A(t)L_B(t)] = 1$ and the theorem is proved.

Therefore, we are looking for a probability measure Q , such that, under Q , $C^*(t) = A^*(t) + B^*(t)$ becomes the $\mathfrak{F}_t$ -compensator of $1_{\{S \vee T \leq t\}}$ with respect to this modified probability measure.

Under certain conditions, it is possible to find Q . Indeed, assume that the process $L(t)$ is uniformly integrable. Then it follows from well known results on point process martingales (Girsanov Theorem, Bremaud (1981)) that the desired measure Q is given by the Radon Nikodyn derivative

$$\frac{dQ}{dP} = L(\infty).$$

Remarks

i) In the case where T and S are identically distributed, we have $A(t) = B(t)$ and the compensator transform is given by

$$A^*(t) = 2 \int_0^t \frac{e^{A(s)} - 1}{2e^{A(s)} - 1} dA(s) = \int_0^t \frac{2 - 2e^{-A(s)}}{2 - e^{-A(s)}} dA(s).$$

which is used in Bueno and Carmo (2007) to define active redundancy operation when the component and the spare are dependent but identically distributed.

ii) The random variable $L(\infty)$ is given by

$$L(\infty) = \left(\frac{e^{B(T)} - 1}{e^{A(T)} + e^{B(T)} - 1} \right) \left(\frac{e^{A(S)} - 1}{e^{A(S)} + e^{B(S)} - 1} \right) [e^{A(T)} + e^{B(S)} - 1] =$$

$$\frac{(e^{B(S \wedge T)} - 1)(e^{A(S \wedge T)} - 1)}{e^{A(S \wedge T)} + e^{B(S \wedge T)} - 1}$$

where $S \wedge T = \min\{S, T\}$.

iii) The Barlow and Proschan reliability importance of a component for the system, in the independent and absolutely continuous case, is the probability that the component causes the system failure. This concept, is extended for dependent components, in Bueno and Menezes (2007) where the importance of the component S for the system is

$$I_P(S) = E_P[(B(S) - B(T))^+]$$

where $a^+ = a \vee 0$.

Follows that the Barlow and Proschan reliability importance of component S for the parallel system $S \vee T$ is

$$\begin{aligned} I_P(S) &= P(S = S \vee T) = P(S \geq S \vee T) = P(S \geq T) = 1 - P(S \leq T) = \\ &1 - E_P[1_{\{S \leq T\}}] = E_P[B(S)] - E_P[B(T)] = E_P[(B(S) - B(T))] = \\ &E_P[(B(S) - B(T))1_{\{S \geq T\}}] = E_P[(B(S) - B(T))^+] \end{aligned}$$

iv) Based in Bueno (2005) we can define the reliability importance of a component with lifetime T for the system reliability, with lifetime τ under a parallel improvement with the lifetime S as

$$\text{cov}(\tau, L(\infty)) = \text{cov}(\tau, \frac{(e^{B(S \wedge T)} - 1)(e^{A(S \wedge T)} - 1)}{e^{A(S \wedge T)} + e^{B(S \wedge T)} - 1})$$

3. Applications.

Parallel operations are very important in reliability theory: the performance of a parallel system are always better than the performance of any coherent system with the same components; it is used in replacement models to optimize system reliability through active redundancy. However, if the component are stochastically dependent the reliability of a coherent system is a difficult and tedious calculation involving multivariate distributions. The calculation becomes more tractable under the assumption of a series system, in which case the reliability is the survival function of a multivariate positive random vector. It is also can be easily done through the compensator processes. We can show this argument easily:

Corollary 3.1 Under the hypothesis of Section 2, in which $\frac{dQ}{dP} = L(\infty)$, we have

$$Q(S \wedge T > t) = P(S \vee T > t), \quad t \geq 0.$$

Proof

$$\begin{aligned} Q(S \wedge T > t) &= E_Q[1_{\{S \wedge T > t\}}] = E_P[L_\infty 1_{\{S \wedge T > t\}}] = \\ &E_P\{E_P[L_\infty 1_{\{S \wedge T > t\}} | \mathfrak{F}_t]\} = E_P\{1_{\{S \wedge T > t\}} E[L_\infty | \mathfrak{F}_t]\} = \\ &E_P[1_{\{S \wedge T > t\}} L_t] = E_P[1_{\{S \wedge T > t\}} (e^{A(t)} + e^{B(t)} - 1)] = \end{aligned}$$

$$(e^{A(t)} + e^{B(t)} - 1)E_P[1_{\{S \wedge T > t\}}] = (e^{A(t)} + e^{B(t)} - 1)P(S \wedge T > t) = (e^{A(t)} + e^{B(t)} - 1)e^{-(A(t)+B(t))} = P(S \vee T > t).$$

Now, we ask how we can use Corollary 3.1 to calculate the Barlow and Proschan reliability importance of component S for the system reliability. In a series representation we have:

$$I_Q(S) = Q(S = S \wedge T) = Q(S \leq S \wedge T) = Q(S \leq T) = E_Q[1_{\{S \leq T\}}] = E_Q[N_B(T)] = E_P[L_\infty N_B(T)] = E_P[N_B(T)] = E_P[1_{\{S \leq T\}}] = 1 - I_P(S).$$

In the above we use that

$$|E_P[L_\infty N_B(T) - N_B(T)]| \leq E_P[(L_\infty - 1)N_B(T)] \leq E_P[L_\infty - 1] = 0$$

We expected this relation since that the reliability of the dual system is

$$P(\Phi^D(\mathbf{X}(t)) = 1) = 1 - E_P[\Phi(1 - \mathbf{X}(t))]$$

and the measure Q is relative to the random vector $(1 - \mathbf{X}(t))$. To clarify this argument, suppose that we can define the *deeptimes*, S' and T' such that the events $\{S' \leq t\}$ and $\{T' \leq t\}$ are equivalent to $\{S > t\}$ and $\{T > t\}$, respectively, then

$$Q(S' \leq S' \wedge T') = Q(S' \leq T') = Q(S > T) = 1 - Q(S \leq T) = I_P(S).$$

At this point we deserve to ask about the reliability properties of the vector lifetime $\mathbf{T} = (T_1, T_2, \dots, T_n)$, under the measure P which are preserved under the measure Q .

3.1. The association property.

In many reliability situations, we encounter structures of coherent systems where components share a load, so that a failure of one component results in increased load on each of the remaining components. Furthermore, the components in a coherent system could be subjected to the same set of stress. In such cases, the random variables of interest are not independent but rather associated.

Therefore, it is very interesting to verify whether the association properties of the lifetimes $T_1, \dots, T_n$ under P are preserved under Q .

We introduce the association definition. For this we give the concept of an upper set in $\mathbb{R}^n$.

A Borel set $U \subset \mathbb{R}^n$ is called an upper set if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \in U$ implies that $\mathbf{y} \in U$. In the univariate case, U is equal to either (u, ∞) or $[u, \infty)$, $u \in \mathbb{R}$.

Definition 3.1.1 The random variables $T_1, \dots, T_n$ (or the corresponding random vector $\mathbf{T}$) are associated, if for all upper sets U_1 and U_2 of $\mathbb{R}^n$

$$P\{\mathbf{T} \in U_1 \cap U_2\} \geq P\{\mathbf{T} \in U_1\}P\{\mathbf{T} \in U_2\}.$$

Note that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are increasing Borel measurable functions such that

$$\text{cov}(f(\mathbf{T}), g(\mathbf{T})) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{cov}(1_{\{f(\mathbf{T}) > s\}}, 1_{\{g(\mathbf{T}) > t\}}) ds dt$$

exists, then Definition 3.1.1 is equivalent to

Definition 3.1.2 The random variables $T_1, \dots, T_n$ (or the corresponding random vector $\mathbf{T}$) are associated if

$$\text{cov}(f(\mathbf{T}), g(\mathbf{T})) \geq 0.$$

In particular, this definition, formulated by Esary, Proschan and Walkup (1967), Esary and Proschan (1970), is useful to produce upper and lower bounds for system reliability.

The association of random variables satisfies the following desirable multivariate properties (Barlow and Proschan (1981)).

- (P1) Any subset of associated random variables is associated.
- (P2) The set consisting of a single random variable is associated.
- (P3) Increasing functions of associated random variables are associated.
- (P4) If two sets of associated random variables are independent, then their union is associated.

Theorem 3.1.3 If $\mathbf{T}$ is associated under P , then also under Q .

Proof We consider the upper sets in $\mathfrak{R}^n$, U_1 and U_2 , and the uniformly integrable $\mathfrak{F}_t$ -martingales, $M_t^1 := E_P[1_{U_1}(T)|\mathfrak{F}_t]$ and $M_t^2 := E_P[1_{U_2}(T)|\mathfrak{F}_t]$.

It follows that there exists a unique $\mathfrak{F}_t$ -predictable process, such that the covariance process $\langle M^1, M^2 \rangle_t$ is increasing, right continuous, with $\langle M^1, M^2 \rangle_0 = 0$ such that $M_t := M_t^1 M_t^2 - \langle M^1, M^2 \rangle_t$ is an $\mathfrak{F}_t$ -martingale.

Since a martingale has a constant expectation, we have

$$E_P[M_t] = E_P[M_0] = E_P[M_0^1 M_0^2] = E_P\{E_P[1_{U_1}(T)|\mathfrak{F}_0]E_P[1_{U_2}(T)|\mathfrak{F}_0]\} = P\{T \in U_1\}P\{T \in U_2\}$$

and

$$E_P[M_\infty] = E_P\{E_P[1_{U_1}(T)|\mathfrak{F}_\infty]E_P[1_{U_2}(T)|\mathfrak{F}_\infty]\} - E_P[\langle M^1, M^2 \rangle_\infty] = P\{T \in U_1\}P\{T \in U_2\}.$$

It follows that T is associated if, and only if,

$$E_P[\langle M^1, M^2 \rangle_\infty] = P\{T \in U_1 \cap U_2\} - P\{T \in U_1\}P\{T \in U_2\} \geq 0.$$

Now,

$$E_Q[\langle M^1, M^2 \rangle_\infty] = E_P L_\infty \langle M^1, M^2 \rangle_\infty \geq 0$$

for $L_\infty \geq 0$ P -a.s..

3.2. The increasing failure rate property.

Classes of non-parametric distributions, such as increasing (decreasing) failure rate (IFR (DFR)) distributions, new better(worse) than used (NBU (NWU)) distributions and others, have been extensively investigated in Reliability Theory. They can be used to achieve the benefit of a maintenance operation or to derive bounds on system reliability.

Several extensions of these concepts appeared in the literature, e.g. Harris (1970), Barlow and Proschan (1981), Marshall (1975) and others. However, they all have in common that they don't order the lifetime vectors in the sense of stochastic order as the univariate concept does. Arjas (1981a), considered to observe the components, continuously in time, based on a family of sub σ -algebras $(\mathfrak{F}_t)_{t \geq 0}$. Arjas (1981a), introduced the notion of increasing

failure rate distribution (new better than used distribution) relative to σ -algebras $(\mathfrak{F}_t)_{t \geq 0}$, $\text{IFR}|\mathfrak{F}_t$ ($\text{NBU}|\mathfrak{F}_t$), generalizing the conventional definition of IFR (NBU) and extending these classes into a multivariate form, denoted by $\text{MIFR}|\mathfrak{F}_t$ ($\text{MNBU}|\mathfrak{F}_t$).

In order to introduce the concepts of Arjas (1981a), we define the residual lifetime of T_i at time t , $1 \leq i \leq n$ as

$$\theta_t T_i := (T_i - t)^+ := \max\{T_i - t, 0\}.$$

Let $\theta_t \mathbf{T} := (\theta_t T_1, \dots, \theta_t T_n)$.

Definition 3.2.1 (Arjas, (1981)) We say that $\mathbf{T}$ is multivariate increasing failure rate relative to $(\mathfrak{F}_t)_{t \geq 0}$, denoted by $\text{MIFR}|\mathfrak{F}_t$, if for all $0 \leq s \leq t$ and all open upper sets $U \in \mathfrak{R}^n$

$$P\{\theta_t \mathbf{T} \in U | \mathfrak{F}_t\} \leq P\{\theta_s \mathbf{T} \in U | \mathfrak{F}_s\}.$$

To generalize the classical results using the series structure, we have to prove that the $\text{MIFR}|\mathfrak{F}_t$ class is preserved under a shift operation.

Theorem 3.2.2 If $\mathbf{T}$ is multivariate IFR relative to $(\mathfrak{F}_t)_{t \geq 0}$, under P , then also under Q .

Proof First, we show that T_i is $\text{IFR}|\mathfrak{F}_t$ under Q . Since, by hypothesis

$$\int_A P\{(T_i - t)^+ > s | \mathfrak{F}_t\} dP = E_P[1_A 1_{\{T_i > t+s\}}] \downarrow t$$

for all $A \in \mathfrak{F}_t$. As L_t is $\mathfrak{F}_t$ -measurable and increasing then

$$Q(A \cap \{T_i > t+s\}) = E_Q[1_A 1_{\{T_i > t+s\}}] = E_P[L_t(1_A 1_{\{T_i > t+s\}})] \downarrow t.$$

Therefore T_i is $\text{IFR}|\mathfrak{F}_t$ under Q . In order to prove that the vector $\mathbf{T}$ is multivariate increasing failure rate relative to $(\mathfrak{F}_t)_{t \geq 0}$, we use the relation

$$\{(T_1 - t)^+ > s, \dots, (T_n - t)^+ > s\} = \bigcap_{1 \leq i \leq n} \{((T_i - t)^+ > s)\}$$

and the equivalent definition of $\text{MIFR}|\mathfrak{S}_t$:

T is $\text{MIFR}|\mathfrak{S}_t$ if, and only if, for all rational $0 \leq s \leq t$ and all $U \in U_Q$

$$P\{\theta_t T \in U|\mathfrak{S}_t\} \leq P\{\theta_s T \in U|\mathfrak{S}_s\}.$$

where U_Q is the class of finite unions of open upper sets with corner point $\mathbf{q} := (q_1, \dots, q_n)$, where $q_i, 1 \leq i \leq n$ are positive rational numbers. An open upper set with corner point $\mathbf{q} := (q_1, \dots, q_n)$ is defined by $U_{\mathbf{q}} = \prod_{i=1}^n (q_i, \infty)$.

4. A general parallel system.

The structural relationship between the lifetime T , of a parallel system and its components lifetime $T_1, T_2, \dots, T_n$ is given by

$$T = \max_{1 \leq i \leq n} T_i = \prod_{1 \leq i \leq n} T_i.$$

We intend to define a compensator transform to characterize the parallel system as a series system through compensator transform. As in Section 2, the idea is to combine compensator transformations on $A_i(t)$ the compensator process of the counting failure process of T_i , the life time of component i , $1 \leq i \leq n$.

For $1 \leq l \leq n$ define the compensator transform $A_l^*(t) = \int_0^t \alpha_l(s) dA_l(s)$ where

$$\alpha_l(s) = \frac{\sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n; i_j \neq l} \exp\left[\sum_{j=1}^{n-k} A_{i_j}(s)\right]}{D(s)}$$

and

$$D(t) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n} \exp\left[\sum_{j=1}^{n-k} A_{i_j}(t)\right].$$

Under the above hypothesis and notation we have

$$\sum_{l=1}^n A_l^* = \sum_{l=1}^n A_l - \ln D(t).$$

The main result follows from an adaptation of Girsanov Theorem.

Theorem 4.1 The following process

$$L(t) = \prod_{l=1}^n (\alpha_l(T_l))^{1_{\{\tau_l \leq t\}}} \exp\left[\sum_{j=1}^n A_j(t) - \sum_{j=1}^n A_j^*(t)\right] =$$

$$\prod_{l=1}^n (\alpha_l(T_l))^{1_{\{\tau_l \leq t\}}} D(t) =$$

$$\prod_{l=1}^n (\alpha_l(T_l))^{1_{\{\tau_l \leq t\}}} \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n} \exp\left[\sum_{j=1}^{n-k} A_{i_j}(t)\right]$$

is a nonnegative $\mathfrak{F}_t$ -martingale with $E[L(\infty)] = 1$.

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