Lines of curvature and an integral form of Mainardi-Codazzi Equations

J. SOTOMAYOR

Instituto de Matemática e Estatística, Universidade de São Paulo, C.P. 20.570, 01452-990 São Paulo, SP, Brasil

Manuscript received March 12, 1996, accepted for publication on March 19, 1996

ABSTRACT

Mainardi-Codazzi differential compatibility conditions between quadratic forms, as those appearing in surface theory, are considered in an abstract setting. They are shown to be equivalent to simple integral expressions for the derivatives of the transition maps defined on maximal and minimal principal lines by the foliations of minimal and maximal principal ones. These integral expressions are applied to the study of stability conditions for periodic principal lines.

Key words: Mainardi-Codazzi, Lines of Curvature, Stability.

1. INTRODUCTION

Let M be an open connected set of the (u, v)-plane, endowed with a pair (A, B) of smooth quadratic forms, the first of which is positive definite. Write the following expressions for these forms.

$$A = Edu^2 + 2Fdudv + Gdv^2, \quad EG - F^2 > 0, \quad E > 0, \quad G > 0$$
$$B = edu^2 + 2fdudv + gdv^2.$$

Define the *mean, extrinsic* and *principal curvatures* $H = H(A, B)$, $K = K(A, B)$ and $k_2 = k_2(A, B)$, $k_1 = k_1(A, B)$ of the pair (A, B) by

$$H = (Eg + Ge - 2Ff)/2(EG - F^2), \quad K = (eg - f^2)/(EG - F^2)$$

and

$$k_2 = H + (H^2 - K)^{1/2}, \quad k_1 = H - (H^2 - K)^{1/2}.$$

A point p in M such that $B_p = kA_p$, for some real number $k = k(p)$, is called an *umbilic point* of the pair (A, B). This holds if and only if $H(p)^2 - K(p) = 0$ and gives $k = H(p)$. The set of umbilic points of (A, B) will be denoted by $U = U(A, B)$.

At each point p of M, K_2 is the maximum and k_1 is the minimum of B_p on A_p-unit vectors. These extrema are attained on the eigenspaces $L_2(p)$ and $L_1(p)$ of B_p relative to A_p. On $M \backslash U$, the line fields L_2 and L_1 are well defined, smooth and mutually A-orthogonal. They are called respectively the *maximal and minimal principal line fields* of (A, B). Their integral curves and integral foliations on $M \backslash U$ are called respectively *principal lines* and *principal foliations* of the pair (A, B), the adjective *maximal* or *minimal* is added when necessary.

The author was partially supported by CNPq, Brasil, and Fapesp, São Paulo.

Both principal line fields are expressed by a single quadratic differential equation $W = W(A, B) = 0$, where W is the *lines of curvature* quadratic form:

$$(EG - F^2)W = (Fg - Gf)d\nu^2 + (Eg - Ge)dud\nu + (Ef - Fe)du^2.$$

This form vanishes identically on U; on $M\backslash U$, it locally splits into the product of two non vanishing smooth one forms W_2 and W_1 whose kernels are L_2 and L_1.

The function k_2 and k_1 are called respectively *maximal* and *minimal principal curvatures* of (A, B). They are smooth on Int U and $M\backslash U$.

On Int U, taking any A-orthogonal chart, and on $M\backslash U$ integrating the principal line fields L_2 and L_1 (i.e. the one forms W_2 and W_1), local doubly orthogonal parameters for (A, B) are obtained. This means that, after renaming back the coordinates and coefficient functions, the following expressions hold:

$$A = Edu^2 + Gd\nu^2, \quad B = Ek_1du^2 + Gk_2d\nu^2.$$

The Gauss, $G(A, B)$, and Mainardi-Codazzi, $C(A, B)$, equations for the pair (A, B) adopt a particularly simple form in doubly orthogonal coordinates (u, ν), as follows [1, 5, 6]:

$$G(A,B):\quad 4(EG)^2k_1k_2 = G_u(EG)_u + E_\nu(EG)_\nu - 2EG(G_{uu} + E_{\nu\nu})$$

$$C(A,B):\quad -(k_1)_\nu E = E_\nu(k_2 - k_1)/2, \quad (k_2)_uG = G_u(k_2 - k_1)/2.$$

Bonnet Theorem [1, 6] establishes that Gauss, $G(A, B)$, and Mainardi-Codazzi, $C(A, B)$, compatibility conditions on the pair (A, B) characterize (modulo rigid motions) the immersions of M into the space whose first and second fundamental forms are, respectively, given by A and B.

Condition $G(A, B)$, which is the thesis of Gauss Theorema Egregium, remarkably connects the intrinsic (metric) and extrinsic (shape) Geometries on immersed surfaces.

In section 2, ideas developed in the study of principal foliations on immersed surfaces [2, 3], will be pursued to derive a geometric integral interpretation for $C(A, B)$. This will be applied in section 3 to provide stability conditions for periodic principal lines.

2. AN INTEGRAL CHARACTERIZATION OF $C(A, B)$

Consider a pair (A, B) of smooth quadratic forms on an open plane set M, as in section 1 above.

THEOREM 1. *Conditions* 1 *and* 2 (2.a *and* 2.b)*, below are equivalent.*

1. The pair (A, B) satisfies Mainardi-Codazzi equations on M.

2.a. On the interior of $U(A, B)$, the function $H = H(A, B) = k_i(A, B)$, $i = 1, 2$, is locally constant.

2.b. Along any minimal (resp. maximal) principal arc $\mu = \mu[s, t]$ (resp. $\mathcal{U} = \mathcal{U}[S, T]$) on $M\backslash U$, whose extremes are s, t (resp. S, T), the derivative of the transition map $S \to T$ (resp. $s \to t$) defined on the A-arc lenght parametrized maximal (resp. minimal) principal arcs through s and t (resp. S and T) by the minimal (resp. maximal) principal foliation, is given by

$$\ln(dT/dS) = \int_{\mu[s,t]} dk_2/(k_2 - k_1) \quad \left(\text{resp. } \ln(dt/ds) = -\int_{\mathcal{U}[S,T]} dk_1/(k_2 - k_1) \right).$$

PROOF. The considerably longer full expressions, in arbitrary coordinates, for Mainardi-Codazzi conditions on M can be found in [1, 5, 6]. In this paper, however, only the above ones for $C(A, B)$, in doubly orthogonal coordinates, will be used. In fact, by continuity, to verify Mainardi-Codazzi conditions on M, it suffices to do so on the open dense set $(\text{Int}\, U) \cup (M \backslash U)$, where these coordinates are always available.

On Int U, $C(A, B)$ for $k = k_i(A, B)$, $i = 1, 2$, gives dk identically zero. Conversely if $k = k_i(A, B)$ is locally constant on Int U, Mainardi-Codazzi conditions holds there.

On a smooth doubly orthogonal chart defined on $[s, t] \times [S, T]$ valid locally on $M \backslash U$, as in section 1, the Mainardi-Codazzi equation

$$(k_2)_u G = G_u(k_2 - k_1)/2$$

is equivalent to

$$d(\ln G^{1/2})\partial/\partial u = [dk_2/(k_2 - k_1)]\partial/\partial u.$$

Integration of $d(\ln G^{1/2})$ on the u-arc $[s, t] \times \{\nu\}$ gives

$$\ln\left(G^{1/2}(t, \nu)/G^{1/2}(s, \nu)\right),$$

which coincides with the left hand side of the first integral expression in the conclusion. The additive form of this expression allows its verification on coordinate doubly orthogonal patches, which can be used to cover any principal arc.

Conversely, differentiation, with respect to t at $t = s$, of this integral gives the second Mainardi-Codazzi condition in doubly orthogonal charts and therefore on $M \backslash U$.

Similarly for the equivalence of Mainardi-Codazzi equation

$$-(k_1)_\nu E = E_\nu(k_2 - k_1)/2$$

and the second integral expression in the conclusion.

3. PRINCIPAL CYCLES ON CODAZZI SURFACES

A *Codazzi surface* $M(A, B)$ is an oriented smooth 2-manifold M endowed with a pair of forms (A, B) which on each positive chart (u, ν) are as those (A, B) on an open set M and verify there the Mainardi-Codazzi $C(A, B)$ differential conditions. The notions, functions, line fields and foliations introduced in section 1 make sense and extend in the obvious way to this global setting. Notations, are conserved, with a change into bold characters when necessary.

A compact integral curve γ of L_2 (resp. L_1) in $M(A, B) \backslash U(A, B)$ is called *maximal* (resp. *minimal*) *principal cycle* of $M(A, B)$.

Call π the Poincaré first return map (holonomy) defined by the lines of the foliation to which γ belongs, mapping to itself a segment of a line of the orthogonal foliation through a point p in γ.

A principal cycle is called *hyperbolic* if $\pi' \neq 1$, at p. By choosing the orientation of γ, it can be assumed that $\pi' < 1$, which implies asymptotic (orbital) stability for γ. Hyperbolicity for a principal cycle also implies the local structural stability, around γ, of the corresponding principal foliation. This means that under small perturbations of the form (A, B) the attracting behavior persists for the perturbed foliation. The theory of structural stability of principal lines and umbilic points for immersed surfaces can be found in [2, 3].

THEOREM 2. *Let γ be a principal cycle of a Codazzi surface $M(A, B)$.*

1. The cycle γ is hyperbolic if and only if any of the following expressions, which are equivalent, hold:
$$\int_\gamma dk_1/(k_2-k_1)\neq 0,\quad \int_\gamma dk_2/(k_2-k_1)\neq 0,\quad \int_\gamma dH/(H^2-K)^{1/2}\neq 0.$$
2. Assume that dk_1 (resp. dk_2), restricted along a minimal (resp. maximal) principal cycle γ, does not vanish identically. Then, there is a smooth deformation (A_t, B_t) such that, for small t, the Codazzi surface $M(A_t, B_t)$ has γ as a minimal (resp. maximal) hyperbolic principal cycle.

PROOF. Part 1 follows from direct integration, using Theorem 1.

To prove 2, asume that the cycle is minimal and take a doubly orthogonal chart (u,ν) on the square $[-2,2]$, on which γ is given by $\nu = 0$ where $\partial k_1/\partial u > 0$.

For a non-negative smooth bump function b which is equal to 1 on $[-1,1]$ and vanishes outside $[-2,2]$, define
$$k_{2t} = k_2 - tb(u)b(\nu)\partial k_1/\partial u,\quad k_{1t} = k_1.$$

By Theorem 1, the pair (A_t, B_t) with
$$A_t = E_t du^2 + G_t d\nu^2,\quad B_t = E_t k_{1t} du^2 + G_t k_{2t} d\nu^2$$
satisfies the Mainardi-Codazzi conditions provided
$$E_t = E,\quad G_t(u,\nu) = G(-2,\nu)\exp\left(\int_{[-2,u]} dk_{1t}/(k_{2t}-k_1)\right).$$

The proof of the hyperbolicity follows from part 1, since, for t small,
$$\int_\gamma dk_{1t}/(k_{2t}-k_1) = \int_\gamma dk_1/(k_2-k_1) + t\left(\int_\gamma b(u)(\partial k_1/\partial u)^2/(k_2-k_1)\right) + O(t^2)\neq 0.$$

The case where γ is a maximal cycle is similar, using the other Mainardi-Codazzi equation.

REMARK. In a similar way, by means of small local deformation on the Codazzi pair, it can be shown that the property of a principal curvature to be identically constant along principal lines can be avoided. Therefore, the hypothesis of Theorem 2, part 2, is verified generically.

For the case of surfaces immersed in space, Theorem 2 was first proved in [2]. The presentation in [3], which uses Mainardi-Codazzi equations, suggests already that it is a result of Codazzi (and not Euclidean) Geometry, since it is independent of Gauss condition.

Theorem 1 also gives the following orbital (non-assymptotic) stability result for principal cycles on Codazzi surfaces.

THEOREM 3. *A principal cycle of a Codazzi surface $M(A,B)$ with constant mean curvature $H = H(A,B)$ is contained in the interior of an annulus of principal cycles.*

PROOF. Consider a minimal principal cycle γ, it is enough to verify that its return map π on an arc P of maximal principal curvature through p in γ is the identity. Rewriting the first integral expression in Theorem 1, taking into account that $2H = k_2 + k_1$ is constant and that s and $t = \pi(s)$ are points on P whose A-arc lenght coordinates (with origin at p) are S and T, obtain
$$2\ln(dT/dS) = \int_{\mu[s,t]} dk_2/(k_2-H) = \ln[(k_2-H)(T)/(k_2-H)(S)].$$

This amounts to

$$dT/(k_2 - H)^{1/2}(T) = dS/(k_2 - H)^{1/2}(S).$$

In other words, the return map π on P preserves the 1-form

$$dS/(k_2 - H)^{1/2}(S).$$

Integration of this form in P, with origin at p, gives $S = \pi(S)$.

This proof is inspired in that given in [4] for the case of immersed surfaces with constant mean curvature.

Following the methods of [7], the same conclusion can be obtained for Codazzi surfaces with more general Weingarten conditions.

Other independet developments on the geometry of Codazzi surfaces can be found in [5].

ACKNOWLEDGMENTS

The author is grateful to "Laboratoire Emile Picard", Univ. Paul Sabatier, Toulouse, for its hospitality during the writing of this paper.

REFERENCES

1. Carmo do M., (1976), Differential Geometry of curves and surfaces. Prentice Hall, New Jersey.
2. Gutierrez C., Sotomayor J., (1982), Stable configurations of lines of principal curvature. Asterisque, 98-99.
3. Gutierrez C., Sotomayor J., (1991), Lines of Curvature and Umbilic points on Surfaces. Lecture Notes, 18th Brazilian Math. Colloq., IMPA.
4. Gutierrez C., Sotomayor J., (1986), Lines of Curvature on surfaces immersed with constant mean curvature. *Trans. Amer. Math. Soc.*, **239** (2).
5. Klotz Milnor T., (1980), Abstract Weingarten Surfaces. *Jour. Diff. Geom.*, **15:** 365-380.
6. Spivak M., (1979), A comprehensive introduction to differential geometry, Vol. 3. Publish or Perish.
7. Sotomayor J., (1987), Closed lines of curvature for Weingarten immersions. *Ann. Global Analysis and Geometry*, **5** (1).