GRAPHS AND NON ASSOCIATIVE ALGEBRAS

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Resumo

To every simple graph $\Gamma = (V, S)$ we define a non associative algebra (over a fixed field k of characteristic $\neq 2$) $A(\Gamma)$ such that $A(\Gamma) \approx A(\Gamma')$ implies $\Gamma \approx \Gamma'$.

1 Introduction

In [2] the authors have constructed an exceptional Bernstein algebra over a field k, associated to a graph $\Gamma = (V, S)$, where V(resp.S) is the set of points (resp.lines) of Γ. The construction is the following. Let $U = \oplus_{v\in V} kv$ and $Z = \oplus_{(a,b)\in S} kz_a^b$, where z_a^b is the linear operator in U defined by the following rules:

$$\begin{cases} z_a^b = z_b^a \ (\forall a, b \in V) \\ z_a^b(b) = a, z_a^b(a) = b, z_a^b(c) = 0 \text{ where } c \neq a, b; c \in V \end{cases} \tag{1}$$

In the k-vector space $A = A(\Gamma) = U \oplus Z$ we define a commutative multiplication using the second statement in (1) to define products in UZ and imposing that

$$U^2 = Z^2 = 0 \tag{2}$$

It is easily seen that $A(\Gamma)$ satisfies the polynomial identity $(x^2)^2 = 0$ and so we can define on $ke \oplus A(\Gamma)$ the structure of a Bernstein algebra by saying that $2ev = v, \quad (\forall v \in V)$ and $ez_a^b = 0, \quad (\forall a, b \in V)$. We recall, from [3], that the type of this Bernstein algebra is (r, s) where $r = |V|$ and $s = |S|$. It was conjectured in [2] that, for given graphs Γ and Γ', the existence of a baric isomorphism between the corresponding Bernstein algebras associated to Γ and Γ' would imply the existence of an isomorphism between Γ and Γ'. In this paper we prove that even the existence of an isomorphism between $A(\Gamma)$ and $A(\Gamma')$ (hence of an isomorphism between the corresponding Bernstein algebras) will imply that, in fact, Γ and Γ' are isomorphic. We use standard notation on Bernstein algebras as in [3] and groups as in [1]. The proof will appear in a final lemma, which follows from Theorem 1, which, in turn, is a consequence of the Lemmas 1,2 and 3 (below).

2 Some lemmas

We keep the same notations as above. Let us consider the following symmetric bilinear form defined on the bases V of U by

$$(v,w)=1 \text{ if } v=w; (v,w)=0 \text{ if } v\neq w, \text{where } v,w\in V.$$

Lemma 1 *Let $\Gamma=(V,S)$ be convex graph and $|V|>2$, then: (1) For all $v,w\in U$ and $z\in Z$, we have $(vz,w)=(v,wz)$.*
(2) If $<,>$ is another invariant symmetric bilinear form in U such that (1) above holds then $<v,w>=\lambda(v,w)$ for all $v,w\in U$, where $\lambda\in k$ is some constant depending on $(,)$ and $<,>$.

Proof. (1) The proof here is evident, according to (1). In fact, $(vz_v^w,w)=(v,wz_v^w)=1$, that is, $(v,v)=(w,w)$. In case $(p,q)\neq(v,w)$, $(pz_v^w,q)=(p,qz_v^w)=0$.

(2) Suppose we have the identity $<v,v>=\lambda(v,v)=\lambda\in k$, where λ depends, at least in principle, of $v\in V$ and both scalar products $<,>$ and $(,)$. Then for all $t\in k$, we have for any two pairs (v,w_1) and (v,w_2) of S, such that $w_1\neq w_2$; (here we use that $|V|>2$):

$$\begin{aligned}&<(w_1+tw_2)(z_{w_1}^v+tz_{w_2}^v),v>=<v,v>+t^2<v,v>=(1+t^2)\lambda=\\&<w_1+tw_2,v(z_{w_1}^v+tz_{w_2}^v)>=<w_1+tw_2,w_1+tw_2>=\\&<w_1,w_1>+t<w_1,w_2>+t^2<w_2,w_2>\end{aligned}$$

and consequently

$$<w_1,w_1>=\lambda=<w_2,w_2>,<w_1,w_2>=0. \qquad (3)$$

We note that if $J\subset EndU$ is the Jordan algebra generated by Z (that is, the Jordan algebra generated by the operators z_v^w) then $<vx,w>=<v,wx>$ for all $v,w\in U$, $x\in J$. Moreover, $J=\{x\in EndU|(vx,w)=(v,wx),\forall v,w\in U\}$, because the graph Γ is convex. By this and (3) we can write $<v_i,v_j>=\lambda\delta_i^j$.

Definition 1 *A basis Λ of the space U is regular if Z has a basis of the type $\{z_v^w:v,w\in\Lambda\}$.*

It is clear that the mapping $\varphi:v\to tv$ where $v\in\Lambda,t\in k^*$ is an isomorphism. In this way we can always consider regular and orthogonal basis for U.

Definition 2 *Let $\Gamma=(V,S)$ be a graph. Then $v,w\in V$ are called equivalent (denote by $v\sim w$) if $v=w$ or $(v,w)\notin S$ and for every $p\in V\setminus\{v,w\}$, we have $(v,p)\in S$ if and only if $(w,p)\in S$.*

It is clear that this is an equivalence relation in V and $V=\cup_{i=1}^n V_i$ where $\{V_1,...,V_n\}$ are the equivalence classes of V modulo this equivalence relation. Moreovere if U_i is the subspace generated by V_i,

$$U=\oplus_{i=1}^n U_i, U_i=\oplus_{v\in V_i}kv, (U_i,U_j)=0 \text{ if } i\neq j.$$

We denote by $O(U)$ the orthogonal group of the vector space U over k, with the bilinear form above defined.

Lemma 2 $G = \oplus_{i=1}^{n} O(U_i) \oplus k^* \subset Aut A(\Gamma)$.

Proof. Let us fix two elements $v, w \in V_i$, where $v \neq w$, and elements $\alpha, \beta \in k$ with $\alpha^2 + \beta^2 = 1$. Let us show that the mapping φ, defined below, is an automorphism of the algebra A :

$$\begin{cases} v \to \alpha v + \beta w \\ w \to \beta w - \alpha v \\ u \to u \ \ (\text{ for all} \ \ u \in V \setminus \{v, w\}) \\ z_i^v \to \alpha z_i^v + \beta z_i^w \\ z_i^w \to -\beta z_i^v + \alpha z_i^w \end{cases}$$

where z_i^v is a simplified notation to $z_{v_i}^v, \{v_1, ..., v_r\} = \{u \in V | z_u^v \neq 0\}$.

For instance, we have

$$\begin{array}{l} (vz_i^v)^\varphi = v^\varphi (z_i^v)^\varphi = (\alpha v + \beta w)(\alpha z_i^v + \beta z_i^w) = \\ \alpha^2 v z_i^v + \alpha\beta v z_i^w + \beta\alpha w z_i^v + \beta^2 w z_i^w = \\ (\alpha^2 + \beta^2) v_i = v_i = v_i^\varphi. \end{array}$$

As $(v, v) = 1 = (v^\varphi, v^\varphi), (w, w) = 1 = (w^\varphi, w^\varphi))$, etc, we have $\varphi \in O(U_i)$. It is clear that automorphims of the type $\varphi = \varphi(v, w, \alpha, \beta)$ generate the group $O(U_i)$. The lemma is proved.

We denote by $so(U)$ the Lie algebra of the all antisymmetric matrices in a space $EndU$.

Lemma 3 *For every graph* $\Gamma = (V, S), |V| > 2$, *we have*

$$Der A(\Gamma) = \oplus_{i=1}^{n} so(U_i) \oplus kd \quad where \ \ [d, Der A(\Gamma)] = 0.$$

Proof. Choose a regular basis $\Lambda = \{v_1, ..., v_n\}$ of U (for instance V) and let $t \in Der A(\Gamma)$. Suppose that

$$v_i^t = \sum_j x_{ij} v_j, i = 1, ..., n; z_{ij}^t = \sum_{p,q} y_{ij}^{pq} z_{pq},$$

(here we denote by z_{ij} the operator $z_{v_i}^{v_j}$, for convenience of notation). We have

$$(v_i z_{ij})^t = v_i^t z_{ij} + v_i z_{ij}^t \quad \text{for all} \ \ (i, j) \in S.$$

$$v_j^t = \sum_p x_{jp} v_p = x_{ii} v_j + x_{ij} v_i + \sum_{p \neq i} y_{ij}^{ip} v_p.$$

¿From this,

$$y_{ij}^{ij} = x_{jj} - x_{ii}, \ \ (i, j) \in S, \tag{4}$$

$$x_{ij} = x_{ji}, (i, j) \in S, \tag{5}$$

$$x_{jp} = y_{ij}^{ip}, (i,p) \in S, (i,j) \in S, p \neq j, \tag{6}$$

$$x_{jp} = 0, (i,j) \in S, (i,p) \notin S. \tag{7}$$

Now from $(v_p z_{ij})^t = 0$ for $p \notin \{i,j\}, (i,j) \in S$, we have $v_p^t z_{ij} + v_p z_{ij}^t = 0$ or

$$x_{pi} v_j + x_{pj} v_i + \sum_q y_{ij}^{pq} v_q = 0$$

giving

$$x_{pi} + y_{ij}^{pj} = 0, (i,j) \in S, (p,j) \in S, \tag{8}$$

$$y_{ij}^{pq} = 0, (i,j) \in S, (p,q) \in S, \{p,q\} \cap \{i,j\} = \emptyset. \tag{9}$$

Now we have:

1. $(p,j) \in S$. Then if there exists i such that $(i,p) \in S$ and $(j,i) \in S$, we have $x_{jp} = x_{pj} = y_{ij}^{ip} = -x_{pj}$ so that

$$x_{jp} = x_{pj} = 0 \tag{10}$$

(We have used equations (5),(6) and (8) in the above series of equalities.)

If there exists i such that $(i,j) \in S$ but $(p,i) \notin S$ than

$$x_{jp} = 0 \tag{11}$$

by (7). But as the graph Γ is connected, from (10) and (11) we have

$$x_{pj} = 0, (p,j) \in S \tag{12}$$

2. $(p,j) \notin S$. If there exists i such that $(i,j) \in S, (p,j) \notin S$, then, by (4), we have $x_{pj} = 0$. If there exists i such that $(i,j) \in S, (p,i) \in S$ then

$$x_{pj} = -y_{ij}^{pi} = -x_{jp} \tag{13}$$

where we used equalities (8) and (6).

By (4), $x_{jj} - x_{ii} = y_{ij}^{ij} = y_{ji}^{ji} = x_{ii} - x_{jj}$ so that $x_{ii} = x$ (for all i) and $y_{ij}^{ij} = 0$. Now if $i \not\sim j$ then $(i,j) \in S$ or there exists p such that $(i,p) \in S, (j,p \notin S)$. In any case, $x_{ij} = 0$ by (12) and (7). In case $i \sim j$ then $x_{ij} = -x_{ji}$.

Reciprocally if $t : U \oplus Z \to U \oplus Z$ is a linear mapping such that $x_{ij} = 0$ when $i \not\sim j, x_{ij} = -x_{ji}$ if $i \sim j, x_{ii} = x, y_{ij}^{ij} = 0, y_{ij}^{pq} = 0$ when $\{i,j\} \cap \{p,q\} = \emptyset, y_{ij}^{ip} = x_{jp}$ then t is a derivation of $A(\Gamma)$, as it can be easily verified.

Corollary 1 *If* $\Gamma = (V,S), |V| > 2, (AutA(\Gamma))^\circ = k^\star \times \prod_{i=1}^n SO(U_i)$, *where* $(AutA(\Gamma))^\circ$ *is the connected component of the group of automorphisms of* $A(\Gamma)$.

Proof. By Lemma 2, $G_1 = k^\star \times \prod_{i=1}^n SO(U_i) \subset G = (AutA(\Gamma))^\circ$ and by Lemma 3, the Lie algebra L_1 of the algebraic group G is a subalgebra of the Lie algebra $L = \oplus_{i=1}^n so(U_i) \oplus k$. But it is obvious that $L_1 \approx L$, then $G_1 \approx G$.

3 Proof of the two main theorems.

The strategy now is to prove Theorem 1, which is a tecnical one, and then to obtain our main result as a corollary(Theorem 2).

Theorem 1 *Let Γ be a simple graph, $A(\Gamma)$ the corresponding k-algebra, Λ_1 and Λ_2 two regular bases of $U \subset A(\Gamma)$. Then there exists an automorphism $\varphi \in Aut A(\Gamma)$ such that $\Lambda_1^{\varphi} = \Lambda_2$.*

The idea of the proof of this theorem is the following:

1.First we prove that for every regular basis Λ of the space U and every maximal vector space $U_0 \subset P = \{v \in U | \dim vZ = r(\Gamma)\}$ there exists a basis Λ_0 of the space U_0 which has the form $\Lambda_0 \subset \Lambda$. Here $r(\Gamma) = \min\limits_{v \in V}\{l(v)\}$ where $l(v) = |\{w|(w,v) \in S\}|$.

2. Induction by $\dim U, U = U_0 \oplus U_1, (U_0, U_1) = 0$.

Theorem 2 *Let Γ and Γ' be two graphs, simple and connected, such that the k-algebras $A(\Gamma)$ and $A(\Gamma')$ are isomorphic. Then the graphs Γ and Γ' are isomorphic.*

Proof. Let $\Gamma = (V, S)$, $\Gamma' = (V', S')$ and let $\varphi : A(\Gamma) \to A(\Gamma')$ be an isomorphism. Then $U^{\varphi} = U'$ and $Z^{\varphi} = Z'$. Let $V'' = \varphi^{-1}(V')$ be another regular basis of U so that the set $\{z_a^b | a, b \in V''\}$ is another basis of Z. If $S'' = \{(a,b)|z_a^b \neq 0\}$ then $\Gamma'' = (V'', S'')$ is a graph isomorphic to Γ by Theorem 1. But Γ'' and Γ are isomorphic, which ends the proof.

References

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