GENERIC PROPERTIES OF EQUILIBRIUM SOLUTIONS BY PERTURBATION OF THE BOUNDARY

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1. Introduction. We study equilibrium solutions of

(1) $$u_t = \Delta u + f(x,u,\nabla u) \text{ in } \Omega \subset R^n$$

(which may be a system, $u = \text{col}\,(u_1,\dots,u_p)$), with boundary conditions.

(2) $$u = 0 \quad \text{or} \quad \partial u/\partial N = g(x,u) \text{ on } \partial\Omega.$$

and of the corresponding damped wave equation with "$u_{tt}+r(x)u_t$" in place of "u_t". The equilibrium problem

(3) $\Delta u + f(x,u,\nabla u) = 0$ in Ω, with boundary conditions,

are the same in each case, but the eigenvalue problem for the linearization

(4) $$\Delta v + \sum_{j=1}^{n} b_j(x)\,\partial v/\partial x_j + (c(x) - g(x,\lambda))v = 0 \text{ in } \Omega$$

has $g(x,\lambda)=\lambda$ in the parabolic case, $g(x,\lambda)=\lambda^2+r(x)\lambda$ for the wave equation. Here

$$(b_j, c)(x) = \left(\frac{\partial f}{\partial \beta_j}, \frac{\partial f}{\partial \gamma}\right)(x, u(x), \nabla u(x))$$

where u solves (3) for $f(x,\gamma,\beta_1,\dots,\beta_n)$. If $v(x)$ is a nontri-

NATO ASI Series, Vol. F37
Dynamics of Infinite Dimensional Systems
Edited by S.-N. Chow, and J. K. Hale

vial solution of (4) for some $\lambda \in C$, then $e^{\lambda t}v(x)$ is a nontrivial solution of the linearization of (1), or of the corresponding wave equation, about the equilibrium.

Under various hypotheses about f and g, we can prove that -for most choices of the bounded smooth region Ω - all equilibrium solutions u are simple and (with more restrictive hypotheses), all equilibria are hyperbolic, i.e. (4) has no non-trivial solutions when $Re\lambda = 0$.

The results are far from complete, but seem sufficient to demonstrate that generecity with respect to perturbation of the boundary -holding f and g fixed- is a very strong condition, worthy of further investigation. In many problems, it is also reasonable to require genericity with respect to perturbations of f and g; some studies of this kind are due to Uhlenbeck [5], Saut and Temam [4],Foias and Temam [1]. Some problems -such as the Navier-Stokes equation- are quite rigid, and it seems the only infinite-dimensional class of perturbations naturally allowed is perturbation of the boundary. In any case, we concentrate on perturbing the boundary, though the Navier-Stokes problem is still out of reach.

Results will merely be sketched, with no attempt at proof; details of the argument will be published in [2].

2. Differential calculus of boundary perturbations.

Given a bounded open set $\Omega_o \subset R^n$, consider the collection of all regions C^k-diffeomorphic to Ω_o $(k \geqslant 1)$. We introduce a topology by defining (a sub-basis of the) neighborhoods of a given Ω as

$$\{h(\Omega) \mid h \text{ is in a small } C^k(\Omega,R^n)\text{-neighborhood of the inclusion } i_\Omega : \Omega \subset R^n \} .$$

When $\|h-i_\Omega\|_{C^k}$ is small, h is a C^k imbedding of Ω in R^n, a C^k diffeomorphism to its image $h(\Omega)$. Micheletti [3] shows this topology is metrizable, and the set of regions C^k-diffeomorphic to a given bounded C^k region may be considered a separable complete metric space. We say a function F defined on this space (with values in a Banach space) is of class C^r or C^∞ or analytic if $h \to F(h(\Omega))$ is C^r or C^∞ or analytic as a map of Banach spaces (h near i_Ω in $C^k(\Omega, R^n)$). Thus, for example, a simple eigenvalue of the Laplacian, for the Dirichlet or Neummann, problem in a bounded C^2 region $\Omega \subset R^n$, is an analytic function of Ω (in the space of regions C^2-diffeomorphic to Ω).

In this sense, we may express problems of perturbation of the boundary (or, of the domain of definition) of a boundary-value problem as problems of differential calculus in Banach spaces. Specifically consider a non-linear formal differential operator F_Ω .

$$F_\Omega(u)(x) = f(x,Lu(x)) \text{ for } x \in \Omega,$$

where L is a constant coefficient linear differential operator of order m, say

$$Lu(x) = (u(x);\ \frac{\partial u}{\partial x_j}(x)\ \ (1 \leqslant j \leqslant n);\ \frac{\partial^2 u}{\partial x_j \partial x_k}(1 \leqslant j,k \leqslant n);\text{etc.})$$

and $f(x,\lambda)$ is a given smooth function. We may consider F_Ω as a map from $C^m(\Omega)$ to $C^o(\Omega)$, or from $W_p^m(\Omega)$ to $L_p(\Omega)$ (under appropriate hypotheses). Then if $h:\Omega \to h(\Omega) \subset R^n$ is a C^m imbedding, it induces isomorphisms $h^*:C^k(h(\Omega)) \to C^k(\Omega)$ [or $W_p^k(h(\Omega)) \to W_p^k(\Omega)$] for $0 \leqslant k \leqslant m$, by

$$h^*\varphi = \varphi o h \quad \text{(the pull-back of } \varphi \text{ by h)}$$

and instead of $F_{h(\Omega)}:C^m(h(\Omega)) \to C^o(h(\Omega))$ we study

$h^*F_{h(\Omega)}h^{*-1}:C^m(\Omega) \to C^o(\Omega)$ acting in spaces independent of h. The degree of differentiability of $(h,u) \to h^*F_{h(\Omega)}h^{*-1}(u)$, m appropriate function spaces, follows from the chain rule and if $h(x,t) = x+t\overset{o}{h}(x)+o(t)$, $u(x,t)=u(x)+t\dot{u}(x)+o(t)$, as $t \to o$, we have

$$\frac{\partial}{\partial t}(h^*F_{h(\Omega)}h^{*-1}(u))\Big|_{t=o} = F'_\Omega(u)\dot{u} + \overset{o}{h}\circ\nabla(F_\Omega(u)) - F'_\Omega(u)\overset{o}{h}\circ\nabla u$$

where $F'_\Omega(u)v(x) = \frac{\partial f}{\partial\lambda}(x,Lu(x))Lv(x)$ for $x \in \Omega$, $\overset{o}{h}\circ\nabla = \sum_{j=1}^{n}\overset{o}{h}_j\partial/\partial x_j$.

Note that, when F_Ω is linear, the contribution to the derivative from variation of Ω is simply the commutator of F_Ω and $\overset{o}{h}\circ\nabla$.

3. The transversality theorem.

Our "generic" results are obtained by applying the transversality (or transversal density) theorem. In the usual formulation, for a sufficiently smooth map (C^k) of separable Banach manifolds $f:X\times Y \to Z$, with $\partial f/\partial x\ (x,y):T_xX \times T_yY \to TZ_{f(x,y)}$. Fredholm or semi-Fredholm with index strictly less than k, if $\xi\in Z$ is a regular value of f, then it is also a regular value of $x \to f(x,y)$ for "most" fixed $y \in Y$, the exceptional set being small in the sense of Baire category. The hypothesis says whenever $f(x,y)=\xi$, the derivative $Df(x,y):(\dot{x},\dot{y}) \to \frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial y}\dot{y}$ is surjective. Thus the range $R(\partial f/\partial y)$ must make up any deficit in $R(\partial f/\partial x\ (x,y))$.

However, many cases arise where $\partial f/\partial x$ has index $-\infty$, so $R(\partial f/\partial x)$ has infinite codimension and this hypothesis is difficult to verify. In fact, it is sufficient that the quotient space

$$R(Df(x,y))\ /R(\frac{\partial f}{\partial x}(x,y))$$

has sufficiently high dimension at each point of $f^{-1}(\xi^{\bullet})$ -in practice, we show it is infinite- dimensional (by contradic tion). This extension of the usual transversality theorem is crucial for most of the results below.

4. Generic simplicity of equilibria (scalar case).

For the scalar Dirichlet problem

$$\Delta u + f(x,u,\nabla u) = 0 \text{ in } \Omega \qquad u = 0 \text{ in } \partial\Omega ,$$

given $f:R^n x R x R^n \to R$ of class C^2, for most bounded C^2 regions Ω (in the sense of Baire category), all solutions u are simple; that is, the linerization

$$\Delta v + \sum_{j=1}^{n} \frac{\partial f}{\partial \beta_j}(x,u(x),\nabla u(x)) \frac{\partial v}{\partial x_j} + \frac{\partial f}{\partial \gamma}(xu,\nabla u)v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega$$

has only the trivial solution $v \equiv 0$.

When $f(x,0,0) \equiv 0$, this may be proved with the usual transversality theorem (and was proved by the author and by Saut and Temam [4], who inadvertantly omitted this hypothesis). In the general case, one must show the over determined problem.

$$\Delta u+f(x,u,\nabla u)=0 \text{ in } \Omega, \; u=0 \; \underline{\text{and}} \; \frac{\partial u}{\partial N} =0 \text{ on } \partial\Omega$$

has no solution, if $f(x,0,0) \not\equiv 0$ on $\partial\Omega$. It is easy to construct counter-examples; but one may prove that (for fixed f), there are no solutions for most choices of Ω, excluding only a closed set, of infinite codimension. This suffices to pro ve the result claimed. Note $u \to \Delta u : H^2_o(\Omega) \to L_2(\Omega)$ has index $-\infty$, all harmonic polynomials (for example) being orthogonal to the image. This is a typical example of a problem with Fredholm index $-\infty$.

We have the same conclusion if the boundary condition requires u=0 on certain components of $\partial\Omega$, while (for example) $\frac{\partial u}{\partial N} = g(x,u)$ on other components. It is sufficient to perturb only the "Dirichlet" components of $\partial\Omega$ to obtain simple solutions.

If there are no "Dirichlet" components, the problem is more complicated. We mention only one example, motivated by applications to population genetics. Suppose $h: R \to R$ has only simple zeros, and $S: R^n \to R$ is C^3 with

$$\{x \mid S(x)=0.\ \partial S/\partial x_i (x)=0,\ \partial^2 S/\partial x_j \partial x_k (x)=0,\ \partial^3 S/\partial x_m \partial x_p \partial x_q\ (x)=0\}$$

for all i,j,k,m,p,q

empty, or of dimension $<n-1$.

Then for most bounded connected C^2 regions $\Omega \subset R^n$, all solutions u of

$$\Delta u + S(x)h(u)=0 \text{ in } \Omega,\quad \partial u/\partial N = 0 \text{ on } \partial\Omega$$

are simple. This problem typically has "trivial" solutions $u \equiv$ constant.

5. Generic simplicity of equilibria for a system.

Let $p \geqslant 2$ and suppose $f: R^n \times R^p \to R^p$ is smooth and consider the problem

$$\Delta u + f(x,u)=0 \quad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega$$

where $u = \text{col}\,(u_1, \ldots, u_p)$. Under various hypotheses about f (besides smoothness), it may be proved most solutions u are simple.

Specifically suppose f and f' are at least C^1, where

$$f'(x,u) = [\,\partial f_i(x,u)/\partial u_j\,]_{i,j=1}^{p} \in R^{p\times p}$$

and that $x \to f(x,0)$, $f'(x,0)$ are at least C^2. Define linear spaces $\mathcal{F} \subset C(R^n,R^p)$ and $\mathcal{M} \subset C(R^n,R^{p\times p})$ as follows:

$$\mathcal{F} = \bigcup_{j\geq 0} \mathcal{F}_j \quad , \quad \mathcal{M} = \bigcup_{j\geq 0} \mathcal{M}_j$$

$$\mathcal{F}_o = \text{span}\{f(x,0)\}, \quad \mathcal{M}_o = \text{span}\{I, f'(x,0)\}$$

$$\mathcal{M}_{j+1} = \text{span}\left\{ \mathcal{M}_j;\ \partial A/\partial x_k \text{ for } 1 \leqslant k \leqslant n,\ A \in \mathcal{M}_j \cap C^1;\ f'(x,0)A(x)+A(x)f'(x,0) \text{ for } A \in \mathcal{M}_j \cap C^2 \right\}$$

$$\mathcal{F}_{j+1} = \text{span}\left\{ \mathcal{F}_j;\ \partial a/\partial x_k \text{ for } 1 \leqslant k \leqslant n,\ a \in \mathcal{F}_j \cap C^1;\ A(x)f(x,0) \text{ for } A \in \mathcal{M}_j \cap C^1;\ f'(x,0)a(x) \text{ for } a \in \mathcal{F}_j \cap C^2 \right\}.$$

For example, $\mathcal{M}$ contains (at least) all polynomials in $f'(x,0)$, and $\mathcal{F}$ contains $\{(\text{polynomial in } f'(x,0)).f(x,0)\}$.

(1) If $\{a(x) \mid a \in \mathcal{F}\} = R^p$ for a dense set of $x \in R^n$, then for most bounded C^2 regions $\Omega \subset R^n$, all solutions are simple.

(2) If $f(x,0) \equiv 0$ then $\mathcal{F} = \{0\}$; but suppose for a dense set of $x \in R^n$, that

$$\alpha, \beta \in R^p,\ \alpha \circ A(x)\beta = 0 \text{ for all } A \in \mathcal{M} \Rightarrow |\alpha||\beta| = 0.$$

Then for most bounded C^2 $\Omega \subset R^n$, all solutions are simple.

(3) If $f(x,u) = f(u)$ is independent of x and $f(0) = 0$, then $\mathcal{F} = \{0\}$, $\mathcal{M} = \{ \sum_{j=0}^{p-1} c_j f'(0)^j \mid c_j \in R \}$ and hypothesis (2)

rarely holds. (Perhaps the only exception being when p=2 and f'(0) has complex eigenvalues).

In this case, assume f is C^4, f'(0) has only simple eigenvalues and (writing $\frac{1}{k!} f^{(k)}(0)\beta^k$ for the k-order homogeneous polynomial) in the Taylor expansion for f(β)) assume

$$\alpha,\beta \in R^p, \quad \alpha\circ f'(0)^j\beta = 0, \ \alpha^\circ f'(0)^j(f''(0)\beta^2)=0 \text{ and } \circ f'(0)^j(f'''(0)\beta^3)=0$$

for all $j \geqslant 0$, imply $|\alpha||\beta| = 0$.

Then for most choices of bounded C^2 $\Omega \subset R^n$, all solutions are simple.

For example, in case f(x,u)=f(u) is independent of x but f(0)≠0, hypothesis (1) is verified when $f'(0)^j f(0)$ $(0 \leqslant j \leqslant p-1)$ are linearly independent; and this holds for most choices of f(0) provided f'(0) has only simple eigenvalues. If f(0)=0 and f'(0) has two distinct real eigenvalues, for example, then hypothesis (2) fails: there exist α≠0 and β≠0 in R^p such that

$${}^T f'(0)\alpha=\lambda\alpha, \quad f'(0)\beta = \mu\beta (\lambda\neq\mu) \text{ and then}$$

$$\alpha\circ f'(0)^j\beta = 0 \text{ for all } j \geqslant 0 \text{ but } |\alpha||\beta|\neq 0.$$

Consider the system with p=2

$$\begin{cases} \Delta u_1 + \lambda u_1 + g_1(u^2)+k_1(u^3) + O(|u|^4) = 0 \\ \Delta u_2 + \mu u_2 + g_2(u^2)+k_2(u^3) + O(|u|^4) = 0 \end{cases}$$

with λ≠μ , where the g_j [or k_j] are homogeneous quadratic [or cubic] polynomials. If e_1 = col(1,0), e_2=col(0,1)

$$|g_1(e_2^2)| + |k_1(e_2^3)| \neq 0$$

$$\text{and} \quad |g_2(e_1^2)| + |k_2(e_1^3)| \neq 0$$

Then hypothesis (3) holds and solutions (u_1, u_2) of this Dirichlet problem are simple in most bounded C^2 regions.

If $p \leqslant 3$, the conditions of hypothesis (3) hold for most choices of $f''(0), f'''(0)$, given $f'(0)$; but if $f'(0)$ has four distinct real eigenvalues (for example), the conditions hold on an open, <u>but not dense</u>, set of $f''(0), f'''(0)$.

6. Generic hyperbolicity of equilibria.

In some cases, hyperbolicity follows from simplicity. For example, suppose the (scalar) problem

$$\Delta v + (c(x) - g(x,\lambda))v = 0 \text{ in } \Omega \text{ , } v = 0 \text{ or } \partial v/\partial N = \gamma(x)v \text{ on } \partial\Omega$$

has a non-trivial solution $v(x)$ for some λ with Re $\lambda = 0$. We assume $c(x)$ and $\gamma(x)$ are real-valued, and then multiplication by $\bar{v}$ and integration by parts yields $\int_\Omega (\mathrm{Im} g)|v|^2 = 0$. In the parabolic case $g(x,\lambda) = \lambda$ we conclude $\lambda = 0$; for the damped wave equation $g(x,\lambda) = \lambda^2 + r(x)\lambda$ we conclude either $\lambda = 0$ or $\int_\Omega r(x)|v|^2 = 0$, so if $r(x) > 0$ somewhere in each component Ω, we again conclude that $\lambda = 0$, and simplicity implies hyperbolicity.

Consider now some problems which are not (formally) self-adjoint.

For the linear problem

$$\Delta v + \sum_{j=1}^{n} b_j(x)\ \partial v/\partial x_j + (c(x) - g(x,\lambda))v = 0 \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega$$

with b_j, c, m, r real-valued and $C^2, g(x,\lambda) = m(x)\lambda^2 + r(x)\lambda$, if either $m \equiv 0$ and $r(x) \neq 0$ on a dense set (the parabolic case) or $m(x) \neq 0$ and $r(x) \neq 0$ on a dense set (the damped wave equation), then for most choices of Ω, the above problem has no nontrivial solutions with $\mathrm{Re}\lambda = 0$. This assumes the b_j, c, m, r are given independent of Ω.

The more interesting case is when the equation is obtained by linearization about some equilibrium of a nonlinear problem.

Consider the equilibrium problem

$$\Delta u + \sum_{j=1}^{n} b_j(x,u)\,\partial u/\partial x_j + c(x,u)=0 \quad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega,$$

where the $b_j(x,u), c(x,u)$ are given C^3 functions, so we assume linear dependence on ∇u in $f(x,u,\nabla u)$ - this greatly simplifies the problem, making if sometimes solvable. Specifically if $C(x,0)\neq 0$ on a dense set and $g(x,\lambda)=m(x)\lambda^2+r(x)\lambda$ as before with $r(x)\neq 0$ on a dense set, then for most bounded C^2 regions $\Omega \subset R^n$, all solutions u of the above non-linear Dirichlet problem are hyperbolic, so the linearization of

$$m(x)u_{tt}+r(x)u_t = \Delta u + b(x,u)\circ\nabla u + c(x,u) \quad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega$$

about any equilibrium has no non-trivial solutions of the form $e^{\lambda t}v(x)$. If $c(x,0)\equiv 0$, the zero solution will be hyperbolic for most Ω, but I have no information about the non-zero equilibria (unless we also perturb c). There is also no information about the case when $f(x,u,\nabla u)$ has non-linear dependence on ∇u. Clearly, much remains to be done.

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