An Introduction to Complex Vector Fields

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This is an account of some aspects, methods questions and ideas related to the theory of vector fields. It doesn't claim to be complete. Rather, it gives a tendentious view of some phenomena known for a long time. The main topic in this introduction will be related to certain complex vector fields over manifolds of dimension 3, and how they can be understood using some complex manifolds techniques. Most of it is based on [F].

Contents

1 Introduction

1.1 Real vector fields.

To start with, consider the physically intuitive concept of real vector field. Let M be a manifold and TM its tangent bundle. A real vector field is simply a differentiable section $L : M \to TM$. At this stage, it is better to suppose everything is in the C^∞ category.

If we choose local coordinates around a point P, say $(x_1, \ldots, x_n)$ in a manifold of dimension n, we are allowed to write a vector field in the following form

$$L(x) = \sum_1^n a^i(x)\frac{\partial}{\partial x^i}$$

It is clear then, that it is a linear first order partial differential operator with real coeficients. The study of such vector fields is multiple in the sense that we can either study equations of the form $Lu=f$, where u and f are functions defined over the manifold, or study integral curves $x(t)$, satisfying locally the equations

$$\frac{dx^i(t)}{dt} = a^i(x(t)).$$

The problems related to the integral curves are properly studied by the theory of dynamical systems. Observe that the field L has the same integral curves as the vector field hL, where h is a nowhere vanishing function, also, the equation $hLu=f$ has the same solutions as $Lu=fh^{-1}$. This leads us to introduce the direction field as the primary object of our study, that is, subbundles of codimension 1 of the tangent bundle, which we will denote also generically by L. A nowhere vanishing vector field L defines clearly a direction field, but there are direction fields which are not defined by any vector field as we see in the following figure of a direction field on $\mathbb{R}^2-\{(0,0)\}$.

We say that two direction fields L on M and L' on M' are equivalent if there exists a diffeomorphism $\phi : M \to M'$ such that $\phi_*(L) = L'$ In terms of vector fields P and P', that might define the direction field, we have that $\phi_*(P) = hP'$, where h is a nowhere vanishing function.

Although the two problems, solving equations and finding integral curves, seem to be of different nature, they are related. Consider the case of a nowhere vanishing vector field L. If $u(x)$ is a solution to $Lu=f$ (*) on an open set A of M, then restricted to the integral curves $x(t)$ we have

$$\frac{du(t)}{dt} = \frac{\partial x^i}{\partial t}\frac{\partial u}{\partial x^i} = a^i(x(t))\frac{\partial u}{\partial x^i} = f. \qquad **$$

Therefore, to solve the partial differential equation *, it is enough to give initial values on a hypersurface of codimension 1 transversal to the vector field and solve the ordinary differential equations **

There is a great local simplification of the problem in this case Locally, all nonvanishing vector fields, on a n-dimensional manifold, are equivalent (up to multiplication by a nonvanishing function) to the vector field $\partial/\partial x_n$ on $\mathbf{R}^n$ In this case we are free to dedicate ourselves to global problems

1.2 Complex vector fields in dimension 2.

A complex vector field on a manifold M is an expression of the form $L = L_1 + iL_2$, where L_1 and L_2 are real vector fields on M. We can define it then as a section $L : M \to CTM$, where CTM is the complexified tangent bundle. We again consider everything within the C^∞ category

The first important observation is that we don't have any more integral curves associated to a general complex vector field. On the other hand we continue to have the equation $Lu = f$ to solve.
The first example to keep in mind is the Cauchy-Riemann field

$$L = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

defined on an open set $\Omega \subset \mathbf{R}^2$.

As in the case of real vector fields, we say that two vector fields L on M and L' on M' are equivalent if there exists a diffeomorphism $\phi : M \to M'$ such that $\phi_*(L) = hL'$, where h is a nowhere vanishing function.

The simplest case to be considered after that, is the generalization of the Cauchy-Riemann field in the sense that L_1 and L_2 are linearly independent at each point. It is surprising that it is not essentially different from the Cauchy-Riemann operator. For a proof of the following theorem and generalizations to higher dimensions see [H]. We assume $L = L_1 + L_2$.

Theorem 1.2.1. *If L_1 and L_2 are linearly independent real vector fields on a manifold of dimension 2, then locally L is equivalent to the Cauchy-Riemann field on the disc $D = \{x^2 + y^2 < 1\}$.*

Corollary 1.2.1. *If L_1 and L_2 are linearly independent real vector fields on a manifold of dimension 2, then L defines a complex structure on M.*

Using the theory of Riemann surfaces, we are able to prove the following

Theorem 1.2.2. *If L is a complex vector field, generating a complex structure on a manifold of dimension 2, and $\Omega \subset M$ is an open non-compact subset, then*

1) the equation $Lu=f$ is always solvable on Ω.
2) the equation $Lu=0$ has many solutions on Ω. Precisely, we can use those solutions to construct a proper holomorphic embedding of Ω in C^n, for some n.

In more general cases, that is when L_1 and L_2 are not linearly independent, we don't have the complex structure to guide us. It might happen that the equation $Lu = f$ is not locally solvable for some C^∞-functions f, even if it has analytic coefficients and is nonvanishing at every point. Also, the homogeneous equation might have only constant solutions. The phenomenon of local nonsolvability was discovered by Lewy in 1957 [Lw] in complex vector fields over manifolds of dimension 3. Later it was shown that this phenomenon ocurred also in dimension 2.([G]) The simplest example is the Mizohata operator

$$L = \frac{\partial}{\partial x} + ix\frac{\partial}{\partial y}$$

For this example see [N1]. For the example of a field with no nonconstant solutions to the homogeneous equation see also [N1]. For further results on general complex vector fields in two dimensions see [T].

1.3 Complex vector fields in dimension 3.

A complex vector field in dimension 3 has a disadvantage with respect to the previous two cases. While the intuition for real vector fields is the strongest, complex vector fields in two dimension manifolds have the well developed theory of Riemann surfaces as an experimented guide.
Our first need is to find examples where such vector fields naturally arise. Those models will be 3-real-dimensional submanifolds (hypersurfaces) of complex manifolds of complex dimension 2.

Consider the complex manifold $C^2 = \{ (z_1,z_2) \mid z_1,z_2 \in C \}$. The complexified tangent bundle CTC^2 has a special subbundle of complex dimension 2, that is the space $T^{1,0}C^2$ generated by the vectors $\partial/\partial z_1$ and $\partial/\partial z_2$ (observe that nonzero vectors in this space have linearly independent real and imaginary parts). Consider now a hypersurface $M \subset C^2$. The intersection $\Delta = CTM \cap T^{1,0}C^2$ are the vectors tangent to M which are linear combinations of $\partial/\partial z_1$ and $\partial/\partial z_2$. Observe that $\dim\Delta = \dim CTM + \dim T^{1,0}C^2 - \dim(CTM \cup T^{1,0}C^2) \geq 3+2-4 = 1$. On the other hand $\dim\Delta \leq 1$ because Δ is properly contained in $T^{1,0}C^2$. We conclude that $\dim\Delta = 1$, that is, locally, Δ is generated by a complex vector field with linearly independent real and complex parts

Examples.

1) Consider $\mathbf{R}^3 = \{(z_1,t) \mid z_1 \in \mathbf{C}, t \in \mathbf{R}\} \subset \mathbf{C}^2$. In this case Δ is generated by the field $\partial/\partial z_1$. We have, then, a foliation of $\mathbf{R}^3$ by complex planes.

2) Consider the sphere $S^3 = \{(z_1,z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. The distribution Δ is generated by the field

$$z_1\partial/\partial\bar{z}_2 - z_2\partial/\partial\bar{z}_1$$

We have given an interpretation to vector fields over 3-dimensional manifolds. We now give the more general definition of a Cauchy-Riemann structure, which is the analogous of the direction field in the case of real vector fields.

Definition 1.3.1. *A CR-structure over a manifold M is a subbundle Δ of the complexified tangent bundle CTM such that*

1) $\Delta \cap \bar{\Delta} = \{0\}$

2) Δ is involutive, that is, if L and L' are local sections of Δ, then so is $[L,L']$

Some comments are due concerning this definition. The case we are interested is when the subbundle is of rank 1, that is, it is generated locally by a complex vector field. In this case condition 2 is vacuous, and condition 1 simply means that this complex vector field has linearly independent real and imaginary parts. We will mostly deal with manifolds M of real dimension 3. The two examples above are opposite in the sense that the first manifold is foliated by Riemann surfaces and in the second manifold does not contain any Riemann surface. To exclude the first case we will assume a condition called pseudoconvexity

Definition 1.3.2. *A CR-structure Δ over a 3-dimensional manifold M is strictly pseudoconvex if for any local section L generating Δ we have that L, $\bar{L}$ and $[L,\bar{L}]$ are linearly independent*

Example Consider S^3 with the CR-structure generated by the field

$$L = z_1\partial/\partial\bar{z}_2 - z_2\partial/\partial\bar{z}_1 \quad \text{Where } z_1=(x_1,y_1),\ z_2=(x_2,y_2)$$

Then $[L,\bar{L}] = iL_3$, where $L_3 = x_1\partial/\partial y_1 - y_1\partial/\partial x_1 + x_2\partial/\partial y_2 - y_2\partial/\partial x_2$

It is easy to see that L, L and L_3 are linearly independent, therefore L defines a strictly pseudoconvex CR-structure on S^3.

We associated a CR-structure to any hypersurface in $\mathbf{C}^2$, the natural question to ask is whether a CR-structure is always obtained in this way We will frequently confuse the notion of a CR-structure Δ on a manifold and a complex field L which generates the subbundle.

Definition 1.3.3. *1) A CR-structure Δ over M is equivalent to a CR-structure Δ' over M' if there exists a diffeomorphism $F: M \to M'$ such that $F_*(\Delta) = \Delta'$.*

2) We say that a CR-structure L on a manifold M is embeddable if there exists an embedding $F: M \to \mathbf{C}^n$ for some n, where $f_*(L) \in T^{1,0}\mathbf{C}^n$.

3) A CR-structure on M is locally embeddable at a given $p \in M$, if there exists a neighborhood of p which is embeddable.

4) A complex function $f: M \to \mathbf{C}$ on a manifold with CR-structure L is said to be a CR-function if $Lf=0$.

Of course, the existence of many solutions to the homogeneous equation is a necessary condition for the existence of an embedding

CR-structures as boundaries of complex manifolds were first studied by Poincaré in 1907 [P], where he showed that the polydisc is not biholomorphic to the ball in two complex dimensions. The abstract definition of the boundary as a CR structure on a manifold is essentially in E. Cartan [C], where it is given a classification of the 3-dimensional homogeneous structures. We will not deal in these notes with the problem of classification (see [C-M]), rather, we will analyse the problem of embedding, or in other words, solving the homogeneous equation. The following example will be essencial in further development.

Example. The quadric in $\mathbf{C}^2$ is the real 3-dimensional manifold defined as $Q = \{ (z_1,z_2) \in \mathbf{C}^2 \ / \ \operatorname{Im} z_2 = |z_1|^2 \}$ and its CR-structure is given by the complex vector field

$$Z = i\frac{\partial}{\partial z_1} - 2\bar{z}_1 \frac{\partial}{\partial z_2}$$

The quadric Q is CR-equivalent to $S^3-\{p\} \subset \mathbf{C}^2$, where $p=(1,0) \in \mathbf{C}^2$, by the following transformations

$$\eta = z_2-i \ / \ z_2+i \qquad\qquad \zeta = 2z_1 \ / \ z_2+i$$

If we choose coordinates $(z,t)=(z_1,\operatorname{Re} z_2)$ we get the following CR-structure on $\mathbf{R}^3$ equivalent to the quadric

$$L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}$$

We will start giving the fundamental examples of global non-embeddability and local non-embeddability.

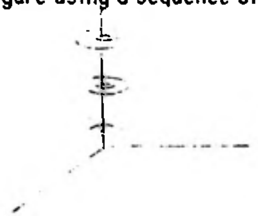
1.4 Local solvability

We will follow [JT1, Lw] closely in this section. Consider the vector field L of the previous example over $\mathbf{R}^3$

Theorem 1.4.1. *L is not locally solvable.*

Proof Observe that L is real analytic, therefore the equation Lh=f is always solvable if f is analytic by Cauchy-Kowalevski. To prove the theorem we will construct a C^∞-function f which rends the equation with no C^1-solution

We observe that the functions z and $w = t+i|z|^2$ are solutions of the homogeneous equation Lh=0, and that the forms dz and dw are linearly independent at every point. As Lh=0 is equivalent to dh(L)=0 and the dimension of the space which annihilates L has dimension 2, we must have dh = pdz + qdw. Fixing $w = t+i|z|^2$ we get a circle in the z-plane which we denote by $\Gamma(w)$. Consider a sequence T_n of disjoint tori converging to the origin obtained as in the figure using a sequence of circles in the w-plane.



Let f be a C^∞-function which vanishes in the complement of the tori and is strictly positive the interior of each torus

We claim that, if $\Gamma(w)$ doesn't intersect any torus

$$\int_{\Gamma(w)} hdz = 0$$

We first prove that it is holomorphic in w in the complement of the full circles, we parametrize $\Gamma(w)$ and note that if h is a CR-function

$$\frac{\partial}{\partial \bar{w}}\int_{\Gamma(w)} hdz' = \frac{\partial}{\partial \bar{w}}\int_0^{2\pi} hz_\theta d\theta$$

$$= \int_0^{2\pi}(h_{\bar{w}} z_\theta + hz_{\theta\bar{w}})d\theta = \int_0^{2\pi}(pz_{\bar{w}} z_\theta - h_\theta z_{\bar{w}})d\theta + \int_0^{2\pi}(hz_{\bar{w}})_\theta d\theta$$

$$= \int_0^{2\pi}(pz_{\bar{w}} z_\theta - pz_\theta z_{\bar{w}})d\theta = 0$$

Therefore, $\int_{\Gamma(w)} hdz'$ is holomorphic in w.

Observe now that in the real axis of the w-plane $\Gamma(w)$ degenerates to a point By Rado's theorem we obtain that

$$\int_{\Gamma(w)} hdz = 0$$

Inside the tori T_n we have $Lh=f > 0$. We compute

$$0 < \int_{T_n}(Lh)dzd\bar{z}dt = \int_{T_n}(h_{\bar{z}} - izh_t)dzd\bar{z}dt = -i\int_{T_n} d(hdzdt) + \int_{T_n} d(zhdzd\bar{z})$$

$$= -i\int_{\partial T_n} hdzdt + \int_{\partial T_n} zhdzd\bar{z} = -i\int_{\partial T_n} hdz(dt + izd\bar{z} + i\bar{z}dz) = -i\int_{\partial T_n} hdzdw$$

Parametrizing each torus using two circles with parameters μ for the circle in the w-plane and σ for the circle in the z-plane, we get

$$= \int_{\partial T_n} hdzdw = \int_{S^1\times S^1} hz_\sigma d\sigma w_\mu d\mu = \int_S\left(\int_S hz_\sigma d\sigma\right)w_\mu d\mu = \int_S\left(\int_{\Gamma(w)} hdz\right)w_\mu d\mu = 0$$

a contradiction. We conclude that there is no C^1-solution.

1.5 Local non-embeddability

We will follow [JT 1] to show that a small deformation of L as above gives a CR-structure on $\mathbf{R}^3$ which is not locally embeddable at 0.

Consider the same sequence of tori T_n as in the section above and the same function f. Define the vector field L_f which is equal to the standard one outside the tori and is defined as

$$L_f = \frac{\partial}{\partial \bar z} - (iz + f)\frac{\partial}{\partial t} \qquad \text{on } T_n \text{ when n is even}$$

$$L_f = f\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z} - iz\frac{\partial}{\partial t} \qquad \text{on } T_n \text{ when n is odd}$$

Theorem 1.5.1. *L_f is not locally embeddable at the origin.*

Proof. Suppose, by contradiction, that the CR-structure, in a neighbourhood of the origin, defined by this vector field is embeddable in $\mathbb{C}^n$, for some n. There should exist a CR-function h, such that the gradient at the origin is nonvanishing. Suppose $h_t(0) \neq 0$. Then, using the integral computed above, for even n

$$0 = \int_{T_n} (Lh) dz d\bar z dt = \int_{T_n} (h_{\bar z} - izh_t - fh_t) dz d\bar z dt = -\int_{T_n} fh_t dz d\bar z dt) \neq 0 \text{ for large n}$$

Suppose that $h_x(0)$ or $h_y(0)$ are nonvanishing. this means that $h_z(0) \neq 0$. Computing now for n odd

$$0 = \int_{T_n} (Lh) dz d\bar z dt = \int_{T_n} fh_z dz d\bar z dt \neq 0 \text{ for large n.}$$

This contradiction shows that this CR_structure is not locally embeddable at the origin.

This surprising phenomenom was shown for the first time by Nirenberg [N], although an example of a globally non-embeddable CR-structure was known before (see bellow). This situation was further analysed by Jacobowitz and Treves [JT1,2] to show that, in fact, non-embeddable CR-structures are, in some sense, dense in the space of CR-structures over a 3-dimensional manifold. In the higher dimensional case, it is known that local embeddability occurs always in dimensons ≥ 7 (for CR-structures of real codimension 1)[K] [A], but the 5-dimensional case is unsettled.

1.6 Global non-embeddability

In [BM] Boutet de Monvel showed that any compact, orientable d-dimensional, $d \geq 5$, strictly pseudoconvex CR-structure (of real codimension 1) is embeddable in some $\mathbb{C}^N$. By a result of Harvey and Lawson [HL], this manifold is the boundary of a subvariety in $\mathbb{C}^N$. On the other hand 3-dimensional compact orientable CR-manifolds are not necessarily embeddable (they are even not locally embeddable in general as we've seen above), the first example is attributed to Andreotti in [R]. This example is actually in the list of homogeneous structures of Cartan, but by that time it was unclear whether it was embeddable or not.

In this section we follow [B] to show that Rossi's example is not embeddable.

Example. Consider the field $L = z_1\partial/\partial\bar{z}_2 - z_2\partial/\partial\bar{z}_1$ defined over the sphere S^3. This is a strictly pseudoconvex CR-structure on the sphere. We define a 1-parameter family of fields by $Z_t = L + t\bar{L}$, where t is a complex number in the unit complex disc. It is easy to see that this new field defines a strictly pseudoconvex CR-structure.

Theorem 1.6.1. *Z_t defines nonembeddable CR-structures for $t \neq 0$. Precisely, solutions to $Z_t h = 0$ are even functions, that is, $h(z_1, z_2) = h(-z_1, -z_2)$.*

Proof. Observe that S^3 is diffeomorphic to the group SU(2). As a matrix group we can write

$$SU(2) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \text{ such that } |z_1|^2 + |z_2|^2 = 1 \right\}$$

and this gives a diffeomorphism with S^3. This is also a representation of SU(2) acting on $\mathbb{C}^2$. We define the action of SU(2) on $L^2(S^3)$ by $g.f(x) = f(g^{-1}x)$. It is easy to see that the operator L defined above is invariant by this action, in the sense that it commutes with the action. Therefore Z_t also is invariant. To analyse the equation $Z_t h=0$ we decompose $L^2(S^3)$ in irreducible factors and take advantage of the invariance of the operator Z_t.

It is known that $L^2(S^3) = \sum_{p+q\geq 0} H^{p,q}$, that is, a direct sum of irreducible representation spaces of SU(2) of dimension p+q+1(the spherical harmonics), where each $H^{p,q}$ is the restriction of harmonic polynomials on $\mathbb{C}^2$ of type (p,q) to S^3. Noting that $L(H^{p,q}) \subset H^{p-1,q+1}$ and is nonvanishing, we conclude using Schur's lemma, that $L : H^{p,q} \to H^{p-1,q+1}$ is an isomorphism for $p-1 \geq 0$ and $q \geq 0$. We have, by the same reason, $\bar{L}$ $H^{p,q} \to H^{p+1,q-1}$ an isomorphism for $q-1 \geq 0$ and $p \geq 0$

As $H^k = \sum_{p+q=k} H^{p,q}$ is an invariant subspace of $L^2(S^3)$, to solve the equation $Z_t h=0$, it is enough to solve for $h \in H^k$. Assume $h = \sum_{p+q=k} h^{p,q}$, where $h^{p,q} \in H^{p,q}$

We show that for k odd h = 0. Decomposing the equation $Z_t h=0$ we get

$$
\begin{array}{lrl}
(k,0) & 0 & = L(h^{k-1,1}) \\
(k-1,1) & -tL(h^{k,0}) & = L(h^{k-2,2}) \\
(k-2,2) & -tL(h^{k-1,1}) & = L(h^{k-3,3}) \\
 & & \\
(1,k-1) & -tL(h^{2,k-2}) & = L(h^{0,k}) \\
(0,k) & -tL(h^{1,k-1}) & = 0
\end{array}
$$

If k is odd, clearly the only solution to this sistem of equations is h = 0. In section 3 we will give a different proof of this theorem, and will relate it to the local non-embeddable examples.

2 Stein Manifolds

2.1 Topology.

We saw the phenomena of local and global non-embeddability. It is not clear that the examples we showed have a common source, especially because the global example is in the analytic category, while the local example is a C^∞ phenomenon. Our goal, from now on is to unify both phenomena in a common source. In particular, we will obtain local examples by pasting global examples. In some sense, we gave priority to global examples and tried to identify a possible mechanism for it.

Strictly pseudoconvex CR-manifolds arise naturally as boundaries of Stein spaces (see bellow). By a theorem of Andreotti-Frankel-Milnor [M1], if a Stein manifold is simply connected its boundary is also simply connected in dimensions > 2. This is not the case in complex dimension 2. We will give a construction of non-embeddable compact and non-compact 3-dimensional CR-structures arising from covers of boundaries of simply connected Stein manifolds of dimension 2. The construction arises naturally from surface singularities whose links are not simply connected. Specifically, we show that there are deformations of the CR-structures on the covers of those links such that the CR-functions are invariant by the covering group. We also show that the series A_n of singularities gives rise to the existence of locally non-embeddable structures. The construction is based on the possibility of pasting CR-structures analogously as to what happens in the theory of Riemann surfaces.

Stein manifolds are generalizations of open Riemann surfaces to higher dimensions, in the sense that they have plenty of holomorphic functions.

Definition 2.1.1. A complex manifold S is a Stein manifold if there exists a proper imbedding $\iota : S \to \mathbb{C}^n$, for some n.

We recall that a a map is proper if inverse images of compact sets are compact. It is important to have an intrinsic characterization of Stein manifolds, because the existence of an embedding is difficult to verify. This is the result of Grauert's deep theorem [Gr]. We will need some definitions before stating it.

Definition 2.1.2

1. A C^∞ function $f \quad \Omega \to \mathbf{R}$, where $\Omega \subset \mathbf{C}^n$ is a domain, is strictly plurisubharmonic (spsh), if for any point in Ω, the complex hessian

$$\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \quad \text{is positive definite.}$$

2. A C^∞ function $f : M \to \mathbf{R}$, defined on a complex manifold is strictly plurisubharmonic, iff in local coordinates it is strictly plurisubharmonic.
3 A strictly plurisubharmonic function f on M is an exhaustion function, iff the set $\{ x \in M \,/\, f(x) < c \}$ is compact, for any $c \in \mathbf{R}$.

Examples

1. Every open Riemann surface is a Stein manifold.
2. The function $f=|z_1|^2+ \ldots +|z_n|^2$ defined on $\mathbf{C}^n$ is strictly plurisubharmonic More generally the distance function from a fixed point P_0 to a proper embedding of a complex manifold $M \subset \mathbf{C}^n$ is a strictly plurisubharmonic exhaustion function for each fixed P_0 in a dense subset of $\mathbf{C}^n - M$. The converse of this result is Grauert's theorem.

Theorem 2.1.1.[Gr] *If S is a complex manifold which has a strictly plurisubharmonic exhaustion function, then S is a Stein manifold.*

The strictly plurisubharmonic exhaustion function can be aproximatedby a Morse function, that is, a function with no degenerate critical points. This gives us imediately an information on the topology of a Stein manifold.

Theorem 2.1.2.[M1] *If S is a Stein manifold, then S is homotopic to a CW-complex of dimention n.*

Proof. Consider a splush exhaustion function $f: S \to \mathbf{R}$, which is furthermore a Morse function. If we introduce local complex coordinates $z_i=x_i+iy_i$, the real hessian can be written as

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\alpha_i\alpha_j + 2\frac{\partial^2 f}{\partial x_i \partial y_j}\alpha_i\beta_j + \frac{\partial^2 f}{\partial y_i \partial y_j}\beta_i\beta_j = 2\mathrm{Re}\left\{ \frac{\partial^2 f}{\partial z_i \partial z_j} w_i w_j \right\} + 2\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j$$

where $w_i=\alpha_i+i\beta_i$. The last term on the right is the complex hessian which we suppose positive It remains to analyse the first term on the right. Observe now that if $(w_1,\ldots,w_n)$ is an eigenvector of this term with negative eigenvalue, then $(iw_1,\dots,iw_n)$ will have positive eigenvalue. We therefore have at most n linearly independent eigenvectors over **R** with negative eigenvalues, that is, the index of f in all critical points is less then or equal to n. From Morse theory we conclude that S is homotopic to a CW-complex of dimention n.

The problem of classifying by homotopy types Stein manifolds is not completely solved. Some partial results were already obtained in [G] and recently in [E1].

In the next section we will examine a phenomenon which occurs at neighborhoods of boundaries of Stein manifolds, we then give background on the topology of the boundaries.

Let X be a topological space. Suppose X is a relatively compact subset of a larger space Y. We say that an open set $V \subset X$ is a neighborhood of ∂X if it is the intersection of a neighborhood of ∂X in Y and X. A distinguished neighborhood of ∂X is defined to be such a neighborhood which doesn't have relatively compact components in X. If X is not a subspace of a larger space we still can define a distinguished neighborhood of the boundary of X as an open set V such that $V = X - K$, where K is a compact in X and there are no relatively compact components of $X - K$. We say then that V is a neighborhood of ∂X, as we will consider only distinguished neighborhoods.

Observe that this definition doesn't imply that the neighborhood is connected. As an example, consider the cylinder $S^1 x(0,1)$ which clearly has neighborhoods of the boundary which have two connected components. As any open Riemann surface is a Stein space, there aren't restrictions on the possible boundaries of 1-dimensional Stein spaces. In particular, as in the case of the cylinder, the boundary might not be connected. The situation changes completly in higher dimensions. In fact we have the following theorem, where we identify, by abuse of language, ∂S with a neighbourhood of ∂S.

Theorem 2.1.3.[M1] *If S is a Stein manifold of dimension n, then $\pi_r(S,\partial S) = 0$ for $r < n$.*

Proof. Consider on S a strictly plurisubharmonic exhaustion function f with no degenerate critical points and nonvanishing everywhere. If the dimension of S is n, the function 1/f doesn't have also degenerate critical points and the index of these is $\geq n$. Therefore given any neighborhood V of ∂S, by Morse theory, S is homotopic to a space obtained by attaching a finite number of cells of dimension $\geq n$ to V. This implies that $\pi_r(S) = \pi_r(V)$ for $r < n$.

2.2 Covers of boundaries of Stein manifolds

In this section we will examine a phenomenon which occurs at neighborhoods of boundaries of Stein manifolds. Most of our notation is taken from [GR] which will be used also as basic reference.

As we will work with spaces with singularities we recall now the definition of a complex space (for a complete treatment see [GR]). Let $A \subset \mathbb{C}^n$ be a domain. A subset $V \subset A$ is an analytic subvariety if every point $a \in A$ has a neighbourhood N_a such that $N_a \cap V = \{ z \in N_a \mid f_1(z) = f_2(z) = \ldots = f_n(z) = 0 \}$, where the $f_i(z)$ are holomorphic functions. A continuous, complex valued function on an open subset of V is said to be holomorphic if it is locally the restriction of a function holomorphic in A. A complex space is a space modeled locally as an analytic subvariety of $\mathbb{C}^n$. A Hausdorff topological space X is called a complex space if every point has a neighbourhood such that it is a an analytic subvariety in some open subset of $\mathbb{C}^n$. Moreover, we consider holomorphic functions on X, locally, as pullbacks of holomorphic functions on these analytic subsets. A complex space is said to be reducible if it can be written as a union of two complex spaces. A point in a complex space is called regular if the there is a neighbourhood of it which is a manifold, and is called singular if it is not regular.

We are interested on boundaries of Stein spaces. We have already seen that the topology of the boundary is related to the topology of the interior of a Stein manifold. Although we deduced this fact from the existence of a strictly plurisubharmonic function, it is interesting to get those results in a more direct way, relating them to complex function theoretic properties. One of the fundamental results responsible for this is Hartogs' extension theorem for which we give the following version

Theorem 2.2.1. *(Gunning - Rossi, th. VII D 4) Let S be an irreducible Stein space of dimension n > 1. Let h be a locally bounded function defined and holomorphic on the regular set of a neighborhood of ∂S. Then, there is a locally bounded continuous function defined on the set of irreducible points of S and holomorphic on the regular points.*

An immediate consequence of this result is the following corollary, a special case of theorem 2.1.3.

Corollary 2.2.1 *Let S be a connected Stein manifold of dimension strictly greater than 1. Then a neighborhood of ∂S is connected.*

Definition 2.2.1. If M is a subvariety of a domain in $\mathbf{C}^n$ we say that a continuous function on M is strictly plurisubharmonic if it has a continuous extension to a strictly plurisubharmonic function on the domain.

Patching together this definition we define strictly plurisubharmonic functions on complex spaces as we defined on complex manifolds.

Definitions 2.2.2

Let X be an analytic space and M an open subset of X. Let $x_0 \in \partial M$. We say that M is strictly pseudoconvex (strictly pseudoconcave) at x_0 if there exists a neighborhood U of x_0 and a strictly plurisubharmonic function f defined on U such that

$$M \cap U = \{ x \in U / f(x) < f(x_0) \}$$

$$(M \cap U = \{ x \in U / f(x) > f(x_0) \})$$

If at every point a boundary component B is strictly pseudoconvex (strictly pseudoconcave) we call B strictly pseudoconvex (strictly pseudoconcave).

Let X be an analytic space and V a component of a neighborhood of ∂X. We say that the boundary at V is pseudoconvex (pseudoconcave) if there exists a strictly plurisubharmonic exhaustion function f defined on V, that is, for every $c \in \mathbf{R}$, the closure of $\{x / f(x) < c\}$ ($\{x / f(x) > c\}$) in X is compact.

The two definitions are related as the following lemma shows. On the other hand, there are domains in $\mathbf{C}^n$ which are pseudoconvex at the boundary but do not have a strictly pseudoconvex boundary.

Lemma 2.2.1 [R]. Let X be an analytic space and M an open subset of X. Let B be a strictly pseudoconvex boundary component of M then there exists a neighborhood of B with a strictly pseudoconvex exhaustion function.

Definition 2.2.3. Filling a hole. Let M be a complex space and suppose V is a component of a neighborhood of ∂M. If the boundary at V is not pseudoconvex we say that there is a hole at V. We say that the hole at V can be filled if M is biholomorphic to an open set, which is denoted M by abuse of language, of a complex space $\underline{M}$ such that the closure of $\underline{M}$ - U is compact, where U is the complement of V in M. Observe, therefore, that a pseudoconvex boundary is not considered to be a hole.

Function theory on spaces which have boundaries with many connected components which are not pseudoconvex is not completely understood. One approach is to try to fill the holes and get a simpler manifold. This, however, is not always possible as is shown in the following proposition which is essentially in [A-S], but for convenience we give a version adapted to our purposes. First, we collect some definitions.

Definition 2.2.3. A surjective holomorphic mapping $\pi : X \to Y$ of complex spaces is called a cover if each $y \in Y$ has a neighborhood U such that π maps every connected component of $\pi^{-1}(U)$ biholomorphically onto U.

Definition 2.2.4. An open, proper and discrete holomorphic mapping $\pi : X \to Y$ between analytic spaces is called a finite branched cover if there exists a nowhere dense analytic set $A \subset Y$ such that $\pi^{-1}(A)$ is nowhere dense in X and $\pi : X - \pi^{-1}(A) \to Y - A$ is a cover. The set A is called the branch locus.

Proposition 2.2.2 *Let S be a simply connected Stein manifold of dimension greater than or equal to two.*

Let V be a neighborhood of ∂S.

Suppose that $\pi : M \to V$ is a nontrivial branched cover by a complex space with discrete branch locus.

Then the holes of M cannot be filled.

Proof. Suppose we could fill the holes of M. We can suppose then the existence of an embeding $i: M \to \underline{M}$ such that $\underline{M} - M$ is compact. Observe that $\underline{M}$ is a modification of a Stein space with isolated singularities (Gunning - Rossi, th. IX C 4). We may then as well suppose that $\underline{M}$ is a Stein space with a discrete set of singular points. Using Hartogs' extension theorem we obtain the extension of π; $\Pi : \underline{M} \to S$. We claim that Π is proper. To see this, let F be a compact in S. In particular it is closed and is contained in the complement a neighborhood of ∂S. Therefore its inverse image is a closed set in $\underline{M}$ which is contained in a compact set of $\underline{M}$, namely, the complement of a neighborhood of the strictly pseudoconvex boundary of $\underline{M}$. As Π is proper by the proper mapping theorem its image is an analytic space and therefore it is surjective because its image contains an open set. We conclude that Π is a non ramified cover of S outside a subvariety **A**. But observe that in a neighborhood of the boundary of $\tilde{M}$ the cover is non-ramified outside a discrete set by hypotheses. This implies that **A** is a discrete set. Deleting those points from S we get a non-ramified cover of S - **A** which is still simply connected, a contradiction.

We compare now proposition 2.2.2 with the following proposition by Rossi[R]

Proposition 2.2.3. *Let M be a relatively compact subspace of an analytic space of pure dimension n > 2 with boundaries B_1 and B_2 where B_1 is strictly pseudoconvex and B_2 is strictly pseudoconcave. Then the pseudoconcave holes of M can be filled.*

The proof involves essentially two ingredients. First, the fact that holomorphic functions separate points near a strictly pseudoconvex hypersurface of an analytic space of dimension > 2. Second, the finiteness of the cohomology group $H^1(M,O)$ in dimensions > 2. Apparently, it is not known whether this proposition is valid in dimension 2 if we assume that holomorphic functions separate points in M. We then get the following corollary generalizing corollary 2.2.1.

Corollary 2.2.2. *Let S be a simply connected Stein manifold of dimension strictly greater than 2.*

Let V be a neighborhood of ∂S such that V = S - K with ∂K strictly pseudoconvex.

Then $\pi_1(V)$ doesn't have subgroups of finite index.

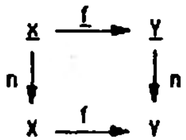
This corollary is a special case of theorem 2.1.3. In fact, $\pi_1(V)$ is trivial in this case, and generaly $\pi_1(S) = \pi_1(V)$ for dimensions > 2.

We conclude that the situation occuring in proposition 2.2.2 happens only in complex dimension 2. In complex dimension 2 we can also obtain a more general version of proposition 1, namely, if $\pi_1(V)$ is diferent from $\pi_1(S)$ and M covers V with covering group G such that no subgroup of $\pi_1(S)$ has the same index then the holes of M can not be filled.

The fact that we cannot fill the holes of M as in proposition 2.2.2

implies, in particular, that M is not an open set of a Stein space. We might suspect, then, that holomorphic functions don't separate points of M. As the next proposition shows this is indeed what is happening. As a preparation, we need the following lemma

Lemma 2.2.1. ([KK] prop. 71.15) If $f : X \to Y$ is a holomorphic mapping between analytic spaces and $f^{-1}(\{y \in Y; {}_Y O_y$ is not normal $\})$ is a nowhere dense subset of X, then there exists exactly one holomorphic mapping $\underline{f}$ between the normalizations such that the diagram below commutes.



Proposition 2.2.4. *Let S be a simply connected Stein manifold of dimension 2.*

Let V be a neighborhood of ∂S.

Suppose that $\pi : M \to V$ is a finite nonramified cover of complex manifolds. Then holomorphic functions of M are pull-backs of holomorphic functions on V.

Proof. Consider a holomorphic function $f : M \to \mathbb{C}$ and assume that f separates 2 points over a point in V. We will construct a ramified cover of S using f. Let n be the degree of the cover $\pi : M \to V$. Over each point of V there are, therefore, counting multiplicities, n values of f. In other words over each point $z \in V$ the values of f satisfy a polynomial $P(z,w)$ of degree n in the variable w. Using Hartogs extension theorem we obtain a polynomial $P(z,w)$ defined on S.

Consider now the set

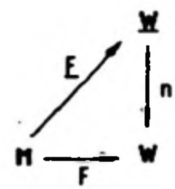
$W=\{(z,w)\in S\times\mathbb{C} \ / \ P(z,w)=0\}$

with the obvious projection $p:W\to S$

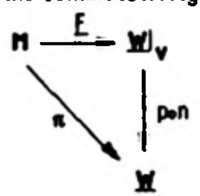
W is a subvariety of $S\times\mathbb{C}$ therefore it has a natural structure of an analytic space. Moreover it is a Stein space which might have singularities. Let $n:\underline{W}\to W$ be the normalization of W. The singularity set A of $\underline{W}$ is discrete (Laufer, th. 3.12).

Observe now that we have a natural map $F: M \to W$; $F(x) = (\pi(x), f(x))$.

The singularity set of W is a proper subvariety $T \subset W$. We will prove that $F^{-1}(T)$ is nowhere dense. Observe that $p(T)$ is a proper subvariety by the proper mapping theorem, therefore $\pi^{-1}(p(T))$ is a proper subvariety because π is a local biholomorphism. But clearly $\pi^{-1}(p(T)) \subset F^{-1}(T)$, so that we conclude that $F^{-1}(T)$ is nowhere dense. By lemma 2.2.1 applied to our case we get the map $\underline{F}: M \to \underline{W}$ such that the following diagram is commutative



Therefore, over V we have also the commutativity of the diagram below



Observe that $F : M \to W|_V$ is surjective and as $\underline{F} : M \to \underline{W}|_V$ is proper, we conclude that $\underline{F} : M \to \underline{W}|_V$ is also surjective.

As π is a nonramified cover, we conclude that $p \circ n$ is a branched cover with ramification set at most a discrete set contained in A over V. We arrived at a situation contradicting theorem 2.2.2 because $\underline{W}$ fills the holes of $\underline{W}|_V$ which has only isolated singularities. This implies that f can not separate points.

We will need a more general form for proposition 2.2.4. The idea is to cut open neighborhoods of the boundary and construct new manifolds by pasting them together. The following version will allow us to carry on this construction by showing that the same conclusion of the proposition above holds if we have a cover of just a large enough piece of V.

Proposition 2.2.5. *Let S be a simply connected Stein manifold of dimension 2.*

Let V be a neighborhood of ∂S.

Let T ⊂ V be a closed set such that

1) holomorphic functions on V - T extend to V-F, where F ⊂ T is closed.

2) V-F contains the boundary of a Stein manifold.

3) S-F is simply connected.

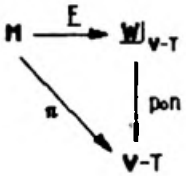
Suppose that π : M → V-T is a finite nonramified cover.

Then holomorphic functions of M are pull-backs of holomorphic functions on V-T.

Proof. The proof follows the same lines as proposition 2.2.4. Consider a holomorphic function f:M→C and assume that f separates 2 points over a point in V-T. As before we construct the polynomial P(z,w), where now z ∈ V-T. By condition 1 we can extend it to be defined over V - F. Using Hartogs extension theorem and condition 2 we obtain again a polynomial P(z,w) defined on S - F. We consider again the set

W = { (z,w) ∈ (S - F)×C / P(z,w) = 0 } with projection p : W → S - F, and its normalization n : $\underline{W}$ → W.

Therefore, over V-T we have the commutativity of the diagram below



As π is a nonramified cover over V-T, we conclude that p∘n is also a nonramified cover over V-T outside the isolated singularities of $\underline{W}$. The ramification set R ⊂ S should therefore pass through T, that is, R∩V ⊂ T. By hypothesis, holomorphic functions on V - T extend to V-F. This implies that R∩V ⊂ F. Also, by condition 2, R doesn't meet the boundary of a Stein manifold in V, therefore R ⊂ F. We conclude that p∘n is nonramified over S-F which is simply connected by assumption 3, a contradiction.

To illustrate the situation occuring in proposition 2.2.5 we will start with the following lemma by Hartogs, which is a consequence of the Laurent series expansion. See [S] or section 4.1 for a more general formulation.

Lemma 2.2.2. *Let $f(z,w)$ be a holomorphic function on $(\Delta(r)-\Delta)\times\Omega$, $r>1$, and $\Omega\subset C$ open connected. Suppose B is an open set in Ω such that if $w\in B$, $f(z,w)$ extends to a holomorphic function on $\Delta(r)$. Then f extends to a holomorphic function on $\Delta(r)\times\Omega$.*

Proof. Consider the Laurent series expansion

$$f(z,w) = \Sigma\, c_n(w)\, z^n \qquad 1 < |z| < r\,,\ n\in Z$$

where $c_n(w)$ is holomorphic on Ω. For w in the open set B, f extends to $\Delta(r)$, this means $c_n(w) = 0$ for $n < 0$. As $c_n(w)$ is holomorphic and B is open we conclude that it vanishes in Ω, and therefore f is holomorphic on $\Delta(r)\times\Omega$.

Examples

2.2.1) Consider the set $V=\{(x_1,y_1,x_2,y_2)\in C^2 \,/\, -1< y_2 <1\}$ and

let $\Omega=\{(x_2,y_2)\in C \,/\, -1< y_2 <1\}$

$A=\{(x_2,y_2)\in C \,/\, -\epsilon < x_2 < \epsilon$ and $-1< y_2 <1\}$

$T= \Delta\times A$

According to the lemma if a holomorphic function is defined on $V-T$ then it extends to V.

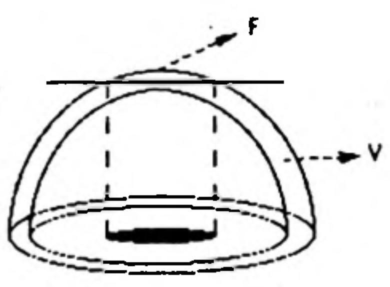
2.2.2) Consider the set

$V=\{(x_1,y_1,x_2,y_2)\in C^2 \,/\, 1 < x_1^2+y_1^2+x_2^2+y_2^2 < 2\}$

let $T= \{(x_1,y_1,x_2,y_2)\in V \,/\, x_1^2+y_1^2+x_2^2 < \delta\,,\ y_2 > 0\}$

Then using example 1 we show that any holomorphic function defined on $V-T$ extends to $V-F$ where

$F= \{(x_1,y_1,x_2,y_2)\in V \,/\, y_2 > \sqrt{2(1-\delta)}\}$



2.2.3) In general, let S be a Stein manifold and $\phi : S \to \mathbf{R}$ a strictly plurisubharmonic exaustion function without degenerate critical points. The set $V = \{ z \in S \,/\, a < \phi(z) < b \}$ is relatively compact. Let P be a point of the strictly pseudoconvex boundary of V. There is then a coordinate chart (x_1, y_1, x_2, y_2) around P such that the boundary of V is given by the graph of a convex function $y_2 = f(x_1, y_1, x_2)$. As in example 2, we consider $T = \{ (x_1, y_1, x_2, y_2) \in V \,/\, f(x_1, y_1, x_2) > f(0,0,0) - \delta$ for small $\delta > 0 \}$. Locally in this coordinate chart any holomorphic function on $V - T$ extends to $V - F$ where $F = \{ (x_1, y_1, x_2, y_2) \in V \,/\, y_2 > f(0,0,0) - \delta \}$. Observe that we've defined T only on a neighborhood of P, but that will be enough and we can extend T to any form we want with the only requirement that it stay below the level sets which pass through F.

Corollary 2.2.3. *Let $L \subset \mathbf{R}^3$ be a knot. Let $T_L = L \times \mathbf{R} \subset \mathbf{C}^2$. (Note that $\mathbf{C}^2 - T_L$ is homotopically equivalent to $\mathbf{R}^3 - L$)*

If $\pi : M \to \mathbf{C}^2 - T_L$ is a finite cover then holomorphic functions on M are the pull-backs of holomorphic functions on $\mathbf{C}^2 - T_L$.

Proof: Use proposition 2.2.5 in the case where $V = S = \mathbf{C}^2$ $(K = \emptyset)$ and $T = T_L$. Observe that in this case $F = \emptyset$

A less explicit proof can be given of proposition 2.2.4 using the existence of the envelope of holomorphy. The proof should give a more general result

Definition A Riemann domain over a Stein manifold S is a pair (M, π) where $\pi : M \to S$ is a local biholomorphism.

Definition. The envelope of holomorphy of (M, π) is another Riemann domain $(E(M), \underline{\pi})$ together with a map $p : M \to E(M)$ such that

1) the diagram below is comutative.

$$\begin{array}{ccc} M & \xrightarrow{\;p\;} & E(M) \\ \pi \downarrow & & \downarrow \underline{\pi} \\ S & \xrightarrow{\;Id\;} & S \end{array}$$

2) it is maximal, in the sense that if ($E'(M)$, π') is another such pair, then there exists a map $e : E'(M) \to E(M)$ satisfying $\pi' = \pi \circ e$

With those definitions, the content of proposition 2.2.4 is that the envelope of holomorphy of the Riemann domain (M, π) which is the cover of a neighborhood of the boundary of S is S itself

Suppose now that N is a Riemann domain which is a tubular neighborhood of a closed strictly pseudoconvex hypersurface contained in M and which divides it in 2 components as in the figure

We want to show that holomorphic functions on N are the pull-backs of holomorphic functions on $\pi(N)$. Or, what is equivalent, that $E(N) = E(\pi(N))$

Conjecture. *Let S be a simply connected Stein manifold of dimension 2*

Let V be a neighborhood of ∂S.

Suppose that π $M \to V$ is a finite nonramified cover of complex manifolds. Furthermore, suppose that N is a tubular neighborhood of a closed strictly pseudoconvex hypersurface contained in M as above.

Then $E(N) = E(\pi(N))$.

A special case of this conjecture is proved in [F]. Another approach to this problem is being developed in [F2].

3 Singularities

The question now arises as to whether the situation described in proposition 2.2.4 can be encountered in "nature". That is, the problem is to find simply connected Stein manifolds of dimension 2 with non-simply connected boundaries. In this section we will gather results about surface singularities which show that the situation we are looking for is not exceptional but, on the contrary, the rule in singularity theory.

Let $A \subset \mathbb{C}^N$ be an analytic variety of dimension 2 with $p \in A$ an isolated singular point. Consider a ball $B_\epsilon \subset \mathbb{C}^N$ of radius ϵ centered at p. Then $S^\epsilon = B_\epsilon \cap A$ is a Stein neighborhood of p in A.

In [M2-th 2.10] it is proved that, for small ϵ, S^ϵ is homeomorphic to the cone over ∂S^ϵ. On the other hand, Mumford showed [Mm] that for every normal isolated singularity $\pi_1(\partial S^\epsilon)$ is non-trivial. We therefore have that for any normal isolated surface singularity S^ϵ is a simply connected Stein space with non-simply connected boundary. The situation looks very promising except for the singularity in S^ϵ. What we need is to find a smooth deformation of S^ϵ and hope that the first homotopy group remains trivial.

We will first prove a proposition which shows that the hypothesis in proposition 2.2.4 that S be a manifold is essential.

Theorem 3.1. *Let S be a small Stein neighborhood of an isolated singularity p as above.*
Suppose ∂S has a nontrivial finite cover.
Then, then there exists an analytic space $\underline{M}$ and a branched cover $\pi : \underline{M} \to S$ with branch locus p.

Proof. As the Stein neighborhood is small, it is homeomorphic to the cone over the boundary ∂S. If there exists a finite cover ∂M of ∂S define $\underline{M}$ to be the cone over ∂M. Define in the obvious way a map π extending the cover at the boundary. We constructed a topological space and a cover with branch locus p. It remains to show that it can be given the structure of an analytic space such that the cover is holomorphic. This will follow from the theorem by Grauert and Remmert [GR] that analytic covers are analytic spaces. Choose S small and a map $c : S \to U$ to be an analytic cover [G-R theorem III C10]. then the map $c \circ \pi$ from $\underline{M}$ to U is also an analytic cover; the only nontrivial condition to verify is the existence of a thin branch locus and that the complement of its inverse image is dense, but this follows easily because the map c has these properties and the map π is branched only at one point. Using [GR] we conclude that $\underline{M}$ is an analytic space. The fact that the projection is holomorphic follows easily if we substitute $\underline{M}$ by a normalization.

Definition 3.1. Smoothing of a singularity. Let $S \to \mathbb{C}^N$ be a pure n-dimensional complex space with an isolated singularity at the origin. The singularity is said to be smoothable if there exists an open ball $B \to \mathbb{C}^N$ and a small disc $D \to \mathbb{C}$ with centers at the origin, a closed nonsingular subspace $X \to B \times D$ and a holomorphic map $f : X \to D$ which is a submersion outside the origin and the restriction to X of the projection $p : B \times D \to D$ such that $f^{-1}(0) = S$. A map $f : X \to D$ satisfying these conditions is called a smoothing.

Given a smoothing of a singularity, if we choose B and D small enough, define S to be the closure of S in $B \times D$ and S_c the closure of $S_c = f^{-1}(c)$ in $B \times \{c\}$. Then $S - \{0\}$ is a manifold with boundary ∂S which is diffeomorphic to the boundary ∂S_c of the manifold S_c for all $c \neq 0$. Moreover $f : X - S \to D - \{0\}$ is a smooth fiber bundle.

Definition 3.2. In the situation above, we call $B \times D$ a Milnor tube and S_c a Milnor fiber of the smoothing. ∂S_c is the link of the Milnor fiber

Not every singularity is smoothable, and even if it is, the Milnor fibers might not be simply connected. For a review see [G-S]. We quote here a positive result which will suffice for our applications.

Theorem 3.2[H]. Any isolated singular point of a complete intersection is smoothable, moreover the Milnor fibers of a smoothing are simply connected.

Therefore, holomorphic functions on finite covers of a neighborhood of the link of a Milnor fiber are pull-backs of holomorphic functions on the neighborhood.

The simplest case of the proposition above occurs when the complete intersection is actually an hypersurface [M2]:

Consider a polynomial $f(x_1,x_2,x_3)$ which vanishes at the origin and **A** the algebraic set with an isolated singularity at the origin

$$A = \{ (x_1,x_2,x_3) \in \mathbf{C}^3 \ / \ f(x_1,x_2,x_3) = 0 \}$$

As above we define the Stein neighborhood $S^\epsilon = A \cap B_\epsilon$ where B_ϵ is a small ball centered at the origin. In the following we will fix an arbitrarily small ϵ and denote the Stein neighborhood simply by S droping the superscript. A smoothing of S is the family $S_c = \{ (x_1,x_2,x_3) \in B_\epsilon \ / \ f(x_1,x_2,x_3) = c \}$.

In particular we will consider the quotient singularities. Those are precisely the ones for which $\pi^1(\partial S)$ is finite. Let $G \subset SU(2)$ be a finite subgroup. We consider $SU(2)$ acting on $\mathbf{C}^2$ as a matrix group. It is a classical result that $G\backslash\mathbf{C}^2$ can be given a structure of an analytic space with one isolated singularity. In fact, the ring of polynomials in $\mathbf{C}[z_1,z_2]$ which are invariant by G is generated by 3 polynomials and those can be used to construct a map $\pi : \mathbf{C}^2 \to \mathbf{C}^3$ that induces an isomorphism on the quotient to an hypersurface in $\mathbf{C}^3$. Observe then that $G\backslash S^3$ is diffeomorphic to ∂S, the boundary of a Stein neighborhood of the isolated singularity.

Example 3.1. As a specific example let G be the cyclic subgroup of order k given by

$$G_k = \{ \begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix} \in SU(2) \mid g^k = 1 \}$$

The invariant polynomials are generated by z_1^k, z_2^k and $z_1 . z_2$, so by the observations above $G_k\backslash\mathbf{C}^2$ is isomorphic to the hypersurface defined by the equation $x_1.x_2 - x_3^k = 0$

$$A_{k-1} = \{ (x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ x_1.x_2 - x_3^k = 0 \}$$

The map $\pi : \mathbf{C}^2 \to \mathbf{C}^3$ defines a non-ramified cover outside the origin, that is $\pi : \mathbf{C}^2-\{0\} \to A_{k-1}-\{0\}$ is a k-sheeted holomorphic cover. Analogous descriptions can be given for the groups D_k, E_6, E_7, E_8. See [L].

If we construct S_c as indicated above then a neighborhood V_c of ∂S_c will have a differential cover diffeomorphic to the space between two concentric spheres in $\mathbf{C}^2$. Therefore on a tubular neighborhood of $S^3 \subset \mathbf{C}^2$ there exists a family of complex structures and we denote by M_c the neighborhood with complex structure structure such that $\pi : M_c \to V_c$ is a holomorphic cover. Observe that V_c and S_c satisfy the hypotheses of proposition 3 , therefore we get the following

Theorem 3.3. *For each finite subgroup G of SU(2) we get families of complex structures on a tubular neighborhood of $S^3 \subset \mathbb{C}^2$ such that holomorphic functions are invariant under G.*

Observe however that the central fiber of the family is the tubular neighborhood with the standard complex structure induced as an open set of $\mathbb{C}^2$, so holomorphic functions separate points. Besides, the complex structures on nearby fibers can be made arbitrarily close to the standard one. This is an example of the " symmetry breaking phenomenon "; we can think of the family of complex structures on a tubular neighborhood of $S^3 \subset \mathbb{C}^2$ as given by a family $\bar\partial_c$ of Cauchy-Riemann operators invariant under the group G with the property that holomorphic functions are also invariant. The symmetry is broken at the central fiber because the operator is still invariant but holomorphic functions are not any more invariant.

Example 3.2. This example is attributed to Andreotti in [R], see also [B] and section 1.6. It corresponds to the proposition above in the case of the quadratic quotient singularity

$$A_1 = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 \ / \ x_1 . x_2 - x_3^2 = 0 \}$$

Let $M = \{ (z_1, z_2) \in \mathbb{C}^2 \ / \ 1-\varepsilon < |z_1|^2 + |z_2|^2 < 1+\varepsilon \}$ be a tubular neighborhood of the sphere $S^3 \subset \mathbb{C}^2$. For a small complex number c define $t = c / (|z_1|^2 + |z_2|^2)^2$ and the map $\pi_c : M \to A_1$ given by

$$x_1 = z_1^2 + t\,\bar z_2^2$$
$$x_2 = z_2^2 + t\,\bar z_1^2$$
$$x_3 = z_1 z_2 - t\,\bar z_1 \bar z_2$$

It is then clear that the map has its image onto a neighborhood of the image of S^3 inside the manifold

$$S_c = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 \ / \ x_1 . x_2 - x_3^2 = c \}$$

Observe that the image of S^3 is precisely the intersection

$$S_c \cap \{ (x_1, x_2, x_3) \in \mathbb{C}^3 \ / \ |x_1|^2 + |x_2|^2 + 2|x_3|^2 = 1 \}.$$

4 CR-structures

4.1 Extension and Pasting

We were looking for simply connected Stein manifolds S of dimension 2 with a non-simply connected neighborhood V of ∂S. In the case when S is a manifold with boundary we can think of V as a tubular neighborhood of a real codimension one hypersurface. This introduces naturally the theory of CR manifolds (see 1.3) into the picture.

The fundamental result we will use is an extension theorem, due to Lewy, for CR-holomorphic functions defined on a CR manifold N embedded in a complex manifold M. We will follow the proof in [Lw] and also [JT1]. Hartogs proved the extension theorem for holomorphic functions and Lewy proved essentialy the same theorem for extension of CR-functions.
A defining function for a hypersurface H defined in an open set $U \subset \mathbb{C}^n$ is a function $\rho : U \to \mathbb{R}$ such that $d\rho|_H \neq 0$ and $H = \{ z \in U \mid \rho(z) = 0 \}$. We will suppose that ρ can always be chosen to be a C^2 function.

Definition 4.1.1. A hypersurface $H \subset \mathbb{C}^n$ is pseudoconvex at P if it has a defining function ρ such that

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) v_j \bar{v}_k \geq 0 \quad \text{for every } v \in \mathbb{C}^n \text{ satisfying } \frac{\partial \rho}{\partial z_j}(P) v_j = 0.$$

If strict inequality holds the hypersurface is said to be strictly pseudoconvex.

Definition 4.1.2. A hypersurface $H \subset \mathbb{R}^n$ is convex at P if it has a defining function ρ such that

$$\frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) v_j v_k \geq 0 \quad \text{for every } v \in T_P(H)$$

If strict inequality holds the hypersurface is said to be strictly convex

Lemma 4.1.1 Let $P \in H$ be a strongly pseudoconvex point. Then, there exists a neighborhood $U \subset \mathbb{C}^n$ of P and a biholomorphic coordinate change ϕ, so that $\phi(U \cap H)$ is strongly convex.

Proof By a complex linear transformation and a translation of coordinates, we may assume that P = 0 and (1, 0, ..., 0) is the unit outward normal to H at P Choose a defining function ρ such that

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) v_j \bar{v}_k \geq C|v|^2 \quad \text{for every } v \in \mathbb{C}^n$$

Then we have the expansion

$$\rho(v) = \rho(0) + \frac{\partial\rho}{\partial z_j} v_j + \frac{\partial\rho}{\partial \bar{z}_j} \bar{v}_j + \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} v_j \bar{v}_k + \frac{\partial^2 \rho}{\partial z_j \partial z_k} v_j v_k + \frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_k} \bar{v}_j \bar{v}_k + O(|v|^2)$$

$$\rho(v) = 2\mathrm{Re}\left(\frac{\partial\rho}{\partial z_j} v_j + \frac{\partial^2 \rho}{\partial z_j \partial z_k} v_j v_k \right) + \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} v_j \bar{v}_k + O(|v|^2)$$

Define then, in a small neighborhood U of P, the following biholomorphic transformation

$$v'_1 = \phi_1(v) = \frac{\partial\rho}{\partial z_j}(P) v_j + \frac{\partial^2 \rho}{2\partial z_j \partial z_k}(P) v_j v_k$$

$$v'_2 = \phi_2(v) = v_2$$

$$\vdots$$

$$v'_n = \phi_n(v) = v_n$$

Remenbering that $\frac{\partial\rho}{\partial z_j}(P) = \delta_{1j}$, in the new coordinates we have

$$\rho(v') = 2\mathrm{Re}(v'_1) + \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) v'_j \bar{v}'_k + O(|v'|^2)$$

Thus the real Hessian at P is precisely the Levi form which is positive definite and that proves the lemma.

If we diagonalyse the Levi form by a change of coordinates we can further simplify the expression of the defining equation of a strictly pseudoconvex hypersurface. To keep notation simple and as the considerantions in this work don't involve higher dimensions, we will restrict the further discussion to the two dimensional case. From the last expression above, a strictly pseudoconvex hypersurface H in $\mathbb{C}^2$, defined by an equation $\rho(z, w) = 0$, can be supposed, without loss of generality, to be written in appropriate coordinates, locally, as a graph of a strictly convex function

$$v = H(x, y, u) \quad \text{where we suppose } w = v + iu \text{ and } z = x + iy$$

that to second order is

$$v = x^2 + y^2 + axu + byu + cu^2.$$

Consider the sets

$$H_{\alpha\beta} = \{ (z,w) \in H \mid \alpha < v < \beta \} \qquad \text{and}$$

$$\underline{H}_{\alpha\beta} = \{ (z,w) \in \mathbb{C}^2 \mid v \geq H \ \text{ and } \ \alpha < v < \beta \}$$

We can, then, formulate Hartog's-Lewy extension theorem

Theorem 4.1.1. *A CR-function defined on $H_{\alpha\beta}$ extends to a holomorphic function on $\underline{H}_{\alpha\beta}$*

Proof. Fixing v_0, the set $S_{v_0} = \{ (x,y,u) \in \mathbb{R}^3 \mid H(x,y,u) = v_0 \}$ is a surface which, for small v_0, is diffeomorphic to S^2. Naturally, we can suppose it to be convex. Define

$$\pi_{v_0} : S_{v_0} \longrightarrow \mathbb{R} \qquad \text{by} \quad \pi_{v_0}(x,y,u) = u$$

For each u in the image of π, we define also the set $\Gamma_{v_0} = \Gamma_{(v_0 u_0)} = \pi_{v_0}^{-1}(u_0)$ which is a closed curve. Let f be a CR-function defined on $H_{\alpha\beta}$. The extension will be defined by the function

$$F(z,w) = \int_{\Gamma(w)} \frac{f(z',w)}{z' - z} dz'$$

Clearly, this function is holomorphic in z To prove that it is holomorphic in w, we proceed as in sections 1.4 and further. We parametrize $\Gamma(w)$ and note that if h is a CR-function, then $dh = p\,dz + q\,dw$ so

$$\frac{\partial}{\partial \bar w}\int_{\Gamma(w)} h\,dz' = \frac{\partial}{\partial \bar w}\int_0^{2\pi} h z_\theta\, d\theta$$

$$= \int_0^{2\pi}(h_{\bar w} z_\theta + h z_{\theta\bar w})\,d\theta = \int_0^{2\pi}(p z_{\bar w} z_\theta - h_\theta z_{\bar w})\,d\theta + \int_0^{2\pi}(h z_{\bar w})_\theta\, d\theta$$

$$= \int_0^{2\pi}(p z_{\bar w} z_\theta - p z_\theta z_{\bar w})\,d\theta = 0$$

Therefore, $\int_{\Gamma(w)} h\,dz'$ is holomorphic in w.

We conclude that F is holomorphic. To prove that F is an extension of f, we observe that there is a curve in the w-plane where $\Gamma(w)$ degenerates to a point. By Rado's theorem we obtain that

$$\int_{\Gamma(w)} h\,dz = 0$$

Therefore F(z,w) is 0 for z in the interior of $\Gamma(w)$ and this implies that F(z,w), defined in the interior of $\Gamma(w)$, is the extension of f.

Remark 4.1.1. We will apply the theorem above in the situation where N is a CR-manifold contained in a complex manifold $N \subset M$. Then there exists an open set V of M such that $N \subset \partial V$ and any CR function on N extends continuously to an holomorphic function on V.

Remark 4.1.2 We will encounter situations where it might happen that only part of N is embedded in a complex manifold. For instance, as in example 2.2.3, consider a coordinate chart (x_1,y_1,x_2,y_2) in $\mathbb{C}^2$ and N given by the graph of a strictly convex function $y_2=f(x_1,y_1,x_2)$ with $0=f(0,0,0)$. Let $M_\delta = \{(x_1,y_1,x_2,y_2) \,/\, -\delta > y_2$ for $\delta > 0\}$. In principle, we could change the CR-structure on the piece of N in the complement of M_δ, so that it is not anymore embedded. In any case it is still true that a CR function on $N \cap M_\delta$ extends to the convex hull in $\mathbb{R}^4$.

Observe also that if $f_\epsilon(x_1,y_1,x_2)$ defines a smooth family of strictly convex hypersurfaces with $f_0 = f$, choosing δ, for small ϵ we have that $N_\epsilon \cap M_\epsilon$ is homeomorphic to N with a 3-ball deleted.

Theorem 4.1.2. *Let ϕ be a strictly plurisubharmonic exaustion function of a simply connected Stein manifold S of dimension 2.*
Let V be a neighborhood of ∂S and $\pi: M \to V$ a finite nonramified cover.
Then the compact connected level sets of $\pi^*(\phi)$ are non-embeddable CR-manifolds. In fact CR functions are pull-backs of CR functions of the level set of ϕ in V.

Proof. The CR functions on a compact level set of $\pi^*(\phi)$ extend to an holomorphic function between 2 level sets by the proposition 4.1. We then are in the situation of proposition 2.2.3, and that proves the proposition.

Remark 4.1.3. Observe that if we delete a small enough tube T from V, such that still $\pi: M - \pi^*(T) \to V-T$ is a cover, then by proposition 2.2.4 and the remark above, the same result is true. That is, CR functions defined on a compact level set of $\pi^*(\phi)$ and outside $\pi^*(T)$ are pull-backs of CR functions defined on the level set of ϕ in V and otside T. This allows us to change the CR-structure of the level set inside $\pi^*(T)$.

To be able to deal with limits of CR-structures, we introduce in the following a concept of distance between CR-structures.

Let X_1, X_2, X_3 be a parallelization of a 3-dimensional orientable manifold N. Given two vector fields L_1 and L_2, we express them using the basis and then, using the coeficients and their derivatives up to order k define a metric $\| L_1 - L_2 \|_k$. We can also define a C^∞ metric $\| L_1 - L_2 \|_\infty$. A different choice of parallelization gives rise to an equivalent metric. If a relatively compact domain B 1 N has C^∞ boundary, we can extend vector fields defined on its closure to N, therefore the metric defined above is complete for the space of vector fields defined on the closure of B.

Fix a Riemannian metric on N. Given two CR-structures Δ_1 and Δ_2 We define their distance to be the infimum of $\| L_1 - L_2 \|$ taken over all L_1 and L_2 normalized global sections of the CR-structures. If there are no global sections we use a partition of unity.

Let S be a Stein neighborhood of a smoothable isolated singularity as in section 3. The link ∂S is diffeomorphic to the links ∂S_c of the Milnor fibers and the diffeomorphisms can be given by a family F_c such that F_0 is the identity. We pull-back the CR-structures on the links of the Milnor fibers by the diffeomorphisms to get a family of CR-structures on the link of the singularity. It should be clear that these CR-structures can be made arbitrarily close to the original CR-structure on the link in the metric defined above.

From those observations and proposition 3.2 we get the following construction of non-embeddable CR-structures

Theorem 4.1.3. *For each finite subgroup G of SU(2) there is a family $L^G{}_c$ of G-invariant CR-structures on S^3, deformation of the standard structure, such that CR-functions are invariant under G for $c \neq 0$.*

One of the most fundamental facts in 1 dimensional complex manifolds is that we can make connected sums. this fact is essentially a consequence of the fact that a disc in $\mathbf{C}$ is biholomorphic to the complement of any disc centered at the compactification point of $\mathbf{C}$, and is reflected by the observation that there exist inversions of the annulus mapping one boundary component to the other.

This situation is no longer true in higher dimensions. Essentially because there are no one point compactifications of $\mathbf{C}^n$, $n > 1$. Fortunately, things are different for CR-structures. Analogously to the 1-dimensional complex case, the quadric, which is the standard open model, admits a 1-point compactification which is $S^3 \subset \mathbf{C}^2$ and an open ball in the quadric is CR-biholomorphic to a complement of a neighborhood of the compactification point.

We recall that the quadric in $\mathbf{C}^2$ is the real 3-dimensional manifold defined as $Q = \{ (z_1,z_2) \in \mathbf{C}^2 \ / \ \operatorname{Im} z_2 = |z_1|^2 \}$ and its CR-structure is given by the complex vector field

$$Z = i\frac{\partial}{\partial z_1} - 2\bar{z}_1 \frac{\partial}{\partial z_2}$$

The quadric Q is CR-biholomorphic to $S^3-\{p\} \subset C^2$, where $p=(1,0)\in C^2$, by the following transformations

$$\eta = z_2-i / z_2+i \qquad \zeta = 2z_1 / z_2+i$$

Consider the point $(0,0)\in Q$ and the neighborhood

$B_\delta = \{ (z_1,z_2)\in Q / |z_1|^2 + x_2^2 < \delta \}$.

By the transformation $(z_1,z_2) \to (tz_1, t^2z_2)$, $t \in R^+$, the neighborhood B_δ is dilated. Given a neighborhood of p it is clear then that, for a suficiently large t, B_δ is CR-biholomorphic to the complement of a closed set contained in this neighborhood.

Theorem 4.1.4. *Let M and M' be CR-manifolds which have balls B and B' with the standard structure. Then they admit a pasting which doesn't change the structure in the complement of these balls.*

Proof. Without loss of generality, we assume the balls to be CR-biholomorphic to B_δ. We think of B' as a neighborhood of $p\in S^3$ and B, by the observations above is CR-biholomorphic to an open set $U \subset S^3$ such that $\partial U \subset B'$. Call $S^3-B' = M$ and $S^3-U=N$. Then $U-M \approx B_\delta-N$, or $B_\delta-M \approx B_\delta-N$. But observe that by the CR-biholomorphism the boundaries components are interchanged, therefore we can paste the structures by the 2 annuli without changing the structure elsewhere.

Now, pasting together the examples of non-embeddable CR-structures on S^3 of proposition 4.4 we will obtain an example of a CR-structure which is not locally embeddable. Before that, we prove the following lemma.

Lemma 4.1.2. Let L^G_c denote a family of vector fields defining the CR-structures on S^3 obtained in propositon 4.3. Then, there exists an open set U, such that, for sufficiently small c, we can deform L^G_c to agree with the standard structure on an open subset $D \subset U$ and remain unchanged on the complement of U, and such that the CR-functions continue to be G-invariant in S^3 - G(U). Moreover these new vector fields can be chosen arbitrarily near the standard one in the C^∞ metric.

Proof. To find U, note that, locally, those CR-structures are induced, via a family of diffeomorphisms, from an embedded one given by the intersection in $\mathbb{C}^3$ of a family of subvarieties $f(x,y,z)=c$ and S^5. Choosing a point P in the intersection for $c=0$ such that x and y are local coordinates of the subvariety, we see that the family of CR-structures is given, locally, by a family of hypersurfaces in $\mathbb{C}^2$. If we choose coordinates so that the hypersurfaces be strictly convex, by remark 4.2 and 4.3 we can choose an open set U uniformly such that, for small $c\neq 0$, CR-functions continue to be G-invariant in $S^3 - G(U)$.

Choose a support function ϕ with value 1 on D and equal to zero on the complement of U. Define then the deformation to be $\mathbb{L}_c = (1-\phi) L_c + \phi L$, where L denotes the standard vector field. It is clear that, if c is sufficiently small, it defines a CR-structure and moreover $\| L - \mathbb{L}_c \| = \| (1-\phi)(L - L_c) \|$ so that also $\mathbb{L}_c$ can be chosen arbitrarily close to L.

Theorem 4.1.5. *There exists a CR-structure on B_4 arbitrarily near the standard one, such that, any germ of CR-function at the origin has vanishing first derivatives.*

Proof. For n given we constructed examples of CR-structures on S^3 arbitrarily near the standard one such that CR-functions don't separate n points. By lemma 4.1 we can choose a small ball in S^3 and modify those CR-structures inside it to agree with the standard one so that, moreover, CR-functions still don't separate n points. Choose a sequence of balls in B_4 converging to the origin on each of the coordinate axis and paste the above structures choosing representative vector fields such that n goes to infinity as the balls aproach the origin. Moreover, we paste those vector fields such that their distance to the standard one goes to zero as we approach the origin. As the metric defined above on the space of vector fields is complete on compact sets, there exists a limit vector field and it defines a CR-structure. A CR-function defined on a neighborhood of the origin will assume the same value at n different points for each n and each ball containing the origin. This clearly implies the proposition.

4.2 Dilations and Deformations of embeddable structures

The general case of non embeddable deformations of any embeddable CR-structure can be treated in a similar fashion by using the dilation to pass from a local situation to a situation which aproximates the global situation of proposition 4.1.3. The essential observation that allows us to carry on this case is the following

Remark 4.2.1. Let H be a small neighborhood around a point S of an embeddable CR manifold. As we saw in section 4.1 we can consider it to be given locally, as a strictly convex hypersurface, by a graph of a function, which up to second order is

$$v = x^2 + y^2 + axu + byu + cu^2$$

By a translation, we can arrange it so that $S = (-1, 0)$. Furthermore, by a convenient dilation, and perhaps after taking a smaller neighborhood H, we can suppose that H is between two spheres; one of them being S^3 and the other one of smaller radius r and center $(-1+r, 0)$, which we denote by S_r^3.

As in the case of the standard structure on the sphere, we analyse what happens to a small neighborhood of the origin when we apply a dilation. With this objective in mind, we recall that the dilation on the quadric $Q = \{ (z_1,z_2) \in \mathbb{C}^2 \ / \ \operatorname{Im} z_2 = |z_1|^2 \}$ is given by the transformation

$$(z_1,z_2) \to (tz_1, t^2 z_2) \,,\ t \in \mathbb{R}^+ .$$

The quadric Q is CR-biholomorphic to $S^3-\{p\} \subset \mathbb{C}^2$, where $p = (1,0) \in \mathbb{C}^2$, by the following Cayley transformations

$$\eta = z_2-1 \ / \ z_2+1 \qquad\qquad \zeta = 2z_1 \ / \ z_2+1$$

And the dilation becomes

$$(\eta,\zeta) \to \left(\frac{\eta + \frac{t^2-1}{t^2+1}}{\frac{t^2-1}{t^2+1}\eta + 1} \,,\ \frac{2t}{t^2+1} \frac{\zeta}{\frac{t^2-1}{t^2+1}\eta + 1} \right)$$

By the transformation $t = tg\theta$ we can write the dilation as $T_\theta : B^3 \to B^3$ given by

$$(\eta, \zeta) \quad \to \quad \left(\frac{\eta - \cos\theta}{1 - \eta\cos\theta}, \frac{\zeta\sin\theta}{1 - \eta\cos\theta} \right)$$

What happens when $\theta \to \pi$?

As a first step to analyse this question, consider the Möbius transformation $T_\theta : D^1 \to D^1$ given by

$$T_\theta(\eta) = \frac{\eta - \cos\theta}{1 - \eta\cos\theta} \qquad \frac{\pi}{2} < \theta < \pi$$

If $0 < \alpha < 1$ then the image of the line $\{\eta \mid \text{Re}\,\eta = -\alpha\}$ by T_θ is the circle with center in the real axis and intersecting it at the points

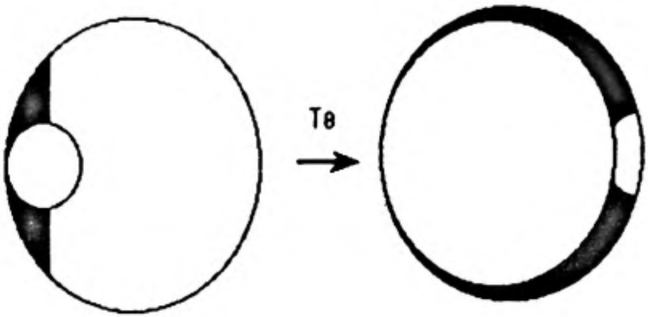
$$-(\cos\theta)^{-1} \text{ and } \beta = -\frac{\cos\theta + \alpha}{1 + \alpha\cos\theta}$$

If $r < 1$, the image of the circle $\{\eta \mid |\eta - (-1+r)| = r\}$ is the circle with center in the real axis and intersecting it at the points

$$(-1, 0) \text{ and } \frac{-1 + 2r + \cos\theta}{1 - (-1+2r)\cos\theta}$$

So that the image of the set

$V = \{\eta \mid \text{Re}\,\eta < -\alpha\} \cap \{\eta \mid |\eta - (-1+r)| > r\}$ is, as shown in the figure for θ near π, almost a "complete" neighborhood of S^1



A similar result is true in higher dimensions. We will make it explicit in 2 dimensions, as we will use this case in the construction of non-embeddable 3-dimensional CR-structures.

For $0 < a < 1$, we define the neighborhood $H_{-a} = \{(\eta,\zeta) \mid \mathrm{Re}\,\eta < -a\}$ It's image will contain the neighborhood

$$H_b = \{(\eta,\zeta) \mid \mathrm{Re}\,\eta < b\} \text{ with } b = \frac{-a-\cos\theta}{1+a\cos\theta}$$

To estimate the thickness of the image of the set

$$\{(\eta,\zeta) \mid |(\eta,\zeta)-(-1+r,0)| \ge r\} \qquad r = 1-\delta$$

observe that $|(\eta+\delta,\zeta)| \ge r$ implies that $|\eta|^2 \ge r^2 - |\eta+\delta|^2$

So

$$\left|\frac{\eta-\cos\theta}{1-\eta\cos\theta}\right|^2 + \left|\frac{\zeta\sin\theta}{1-\eta\cos\theta}\right|^2 = \frac{|\eta|^2 - \cos\theta(\eta+\bar{\eta}) + \cos^2\theta + \sin^2\theta|\zeta|^2}{|1-\eta\cos\theta|^2}$$

If we call A the expression above, we obtain the following estimate

$$\frac{|\eta|^2 - \cos\theta(\eta+\bar{\eta}) + \cos^2\theta + \sin^2\theta\,|(1-\delta)^2 - |\eta+\delta|^2|}{|1-\eta\cos\theta|^2} \le A \le 1$$

Simplifying this formula, we get

$$1 - \frac{\sin^2\theta(2+\eta+\bar{\eta})\delta}{|1-\eta\cos\theta|^2} \le A \le 1$$

Let $\eta = x + iy$, then

$$\frac{\sin^2\theta(2+\eta+\bar{\eta})\delta}{|1-\eta\cos\theta|^2} = \frac{2\delta\sin^2\theta(1+x)}{1-2x\cos\theta+\cos^2\theta(x^2+y^2)} \le \frac{2\delta\sin^2\theta(1+x)}{1-2x\cos\theta+\cos^2\theta x^2} = \frac{2\delta\sin^2\theta(1+x)}{|1-x\cos\theta|^2}.$$

Call the last term $R(\theta,x)$. To find the maximum of $R(\theta,x)$ we compute its derivatives

$$\frac{\partial R(\theta,\zeta)}{\partial x} = \frac{2\delta\sin^2\theta(1+x\cos\theta+\cos\theta)}{|1-x\cos\theta|^3} > 0$$

So, if $\pi/2 \le \theta < \pi$ and a are fixed, the maximum of $R(\theta,x)$ occurs at $x = -a$. In this case

$$R(\theta, a) = \frac{2\delta\sin^2\theta(1-a)}{|1+a\cos\theta|^2} < \frac{2\delta\sin^2\theta}{1-a}$$

Now it is clear that, for fixed a, R tends to zero as θ goes to π. We resume the discussion above in the following lemma.

Lemma 4.2.1. Let $0 < \delta < 1$ and $0 < a < 1$ be fixed. Given ε and b, there exists a θ_0 such that, for $\theta_0 < \theta < \pi$, the image by the Möbius transformation T_θ of the set

$$\{(\eta,\zeta) \mid \mathrm{Re}\,\eta < -a\} \cap \{(\eta,\zeta) \mid |\eta+\delta|^2 + |\zeta|^2 \ge (1-\delta)^2\}$$

is outside a ball of radius $1-\varepsilon$, and such that $\mathrm{Re}\,\eta_{T_\theta} > b$ for $\mathrm{Re}\,\eta = -a$.

Proof. By the above estimates we just need to choose θ_0 such that

$$b > \frac{-a-\cos\theta_0}{1+a\cos\theta_0}$$

$$1-\varepsilon < \frac{2\delta\sin^2\theta_0}{1-a}$$

With this lemma, we are ready now to apply theorem 2.2.5 and proposition 3.3 to get non-embeddable deformations of any given embeddable CR-manifold.

Theorem 4.2.2 *Given any neighborhood of a point P in an embeddable CR-manifold, there exists a deformation of the CR-structure in this neighborhood such that CR-functions don't separate at least 2 points.*

Proof. By remark 4.2.1, without loss of generality, we can suppose that this neighborhood is contained in the domain between two spheres S^3 and S_r^3, with $P = (-1, 0)$ the intersection point of those two spheres. We will intersect this neighborhood with a sphere of radius $1-\gamma$, call it $S^3(1-\gamma)$, with the same center as S^3 and arrange the deformations so that the intersection points between this sphere and the neighborhood are not separated.
To see this, restrict attention to the plane $\zeta = 0$. Call C it's intersection with with the CR-manifold neighborhood. If we intersect C with a circle of radius $1-\gamma$ centered at the origin the angle between the points of intersection decreases continuously as γ decreases to zero. If γ is small enough there will exist a nontrivial intersection. Fix, then, an angle of $2\pi/m$ so that m is large enough to insure the existence of intersections with that angle. We now apply T_a, with a large enough so that $T_a(\mathrm{Re}\,\eta=-a)$ is contained in a small region comprised in a total angle much less than $2\pi/m$. In this way we will be able to use remark 4.1.2. By applying T_a the angle between the points of intersection increases, so by taking a circle of larger radius we can make it intersect C again at, say C_1 and C_2, with an angle between them of $2\pi/m$. Using now proposition 3.3 so that CR-functions are invariant by the group of rotations of angle $2\pi/m$ we conclude that CR-functions won't separate points C_1 and C_2 in the deformed CR-structures.

As in proposition 4.1.5 we can use the above result to show the following

Theorem 4.2.3 *Given any point P in an embeddable CR-structure of dimension 3, there exists a deformation so that CR-functions have vanishing first derivatives at P.*

Finally, to complete the picture we should mention the result of [JT2] that, whenever proposition 4.2.3 is true, we can show the existence of aberrant structures by using a Baire category argument. Dilations were also used in [F1] to obtain different examples of non-embeddable CR-structures.

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