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Linear Stieltjes Equation with Generalized Riemann Integral and Existence of Regulated Solutions

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LINEAR STIELTJES EQUATION WITH GENERALIZED RIEMANN INTEGRAL AND EXISTENCE OF REGULATED SOLUTIONS*

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In this work we establish an existence theorem of regulated solutions for a class of Stieltjes equations which involve generalized Riemann kind of integrals. The general method applied consists in considering the continuous-time Stieltjes equation as limit of discrete processes. This approach will be fruitfull into works on the controllability of Stieltjes systems, because it will be possible to get properties on the continuous time equation by transferring properties of the discrete ones.

1. INTRODUCTION

In the last two decades the Riemann generalized integral, having values in B-spaces, has been increasingly studied.

The development in this area concerns mainly with the Henstock-Kurzweil (see e.g. [1], [2]), and the Dushnik and the Young integrals(see e.g. [3]). Recently it appeared in the literature many proper applications in this field (proper here considered in the sense that the results are not disguises of an essentially finite dimensional frame) (see e.g. [4], [5]).

In this work we establish an existence theorem of regulated solutions for a class of Stieltjes equations which involve generalized Riemann kind of integrals.

The general method applied consists in considering the continuous-time Stieltjes equation as limit of discrete processes. This approach will be fruitfull into works on the controllability of Stieltjes systems, because it will be possible to get properties on the continuous time equation by transferring properties of the discrete ones.

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2. TIME DISCRETE STIELTJES EQUATION

2.1 The equation (L_N).

Let be X a B-space, and $N \in \mathbb{Z}^+$ (the set of all positive integers) and X^N the set of the N-uples of elements in X and $L(X)$ the class of all linear bounded operators on X. Let us consider.

$$Q_{m,n} \in L(X) \qquad (m, n \in \mathbb{Z}^+,\ m \leq n \leq N)$$

and

$$J_{p,q} : X^N \to X \tag{1}$$

in this way:

$$J_{p,p} = 0$$

and

$$J_{p,q}\, x = \sum_{p+1 \leq s \leq q} Q_{s-1,s}\, x_s \qquad (p < q)$$

with $x = (x_1, ..., x_N)$. The next result is immediate:

PROPOSITION 2.1. *For all $p \leq r \leq q$ we get:*

$$J_{p,r} + J_{r,q} = J_{p,q} \quad ,$$

and

$$J_{p-1,p}\, x = Q_{p-1,p}\, x_p \quad .$$

Denoting by $\pi_i (1 \leq i \leq N)$ the usual linear projection on X^N:

$$\pi_i(x_1, ..., x_i, ..., x_N) = x_i \qquad (1 \leq i \leq N)$$

we can define for $x,\ u \in X^N$ *the linear Stieltjes equation* (L_N) *on* X^N:

$$(\pi_r + J_{0,r})x = \pi_r u \qquad (1 \leq r \leq N). \tag{2}$$

At this point it is possible to formulate two immediate questions concerning the existence of the (L_N) solutions. The first is:

(P) For a fixed $u \in X^N$, is it possible to synthetize an $x \in X^N$ in such a way to make (L_N) fulfilled?

The second, more general than (P), is the following:

(P_ε) For every fixed $u \in X^N$ and $\varepsilon > 0$ is it possible to have u^ε and x in X^N in such a way: $\| u - u^\varepsilon \| < \varepsilon$ and

$$(\pi_r + J_{0,r})x = \pi_r u^\varepsilon \ ? \qquad (1 \leq r \leq N)$$

The next theorem will give us an answer to the problem (P_ε). After, in a corollary, we will solve (P) too.

THEOREM 2.2. *Consider the* (L_N) *equation (2) and the hypothesis,*

$$Ker(\pi_r + J_{r-1,r})^* = \{0\} \qquad (1 \leq r \leq N), \tag{3}$$

where T^* *denotes the adjoint operator for* $T \in L(X)$. *Then for every* $\varepsilon > 0$ *there exists* $u_\varepsilon,\ x \in X^N$ *with* $\| u^\varepsilon - u \| < \varepsilon$ *and satisfying the equality:*

$$(\pi_r + J_{0,r})x = \pi_r u^\varepsilon \ . \qquad (1 \leq r \leq N)$$

PROOF. The properties showed in the Proposition 2.1 together with hypothesis (3) yield for every $r > 0$:

$$\overline{(I + Q_{r-1,r})X} = X \tag{4}$$

[here $\overline{A}$ denotes the closure of $A \subset X$]. In this way, there exists x_1 and u_1^ε in X satisfying $\| u_1^\varepsilon - u_1 \| < \varepsilon$ with

(5) $$x_1 + Q_{0,1}\, x_1 = u_1^\varepsilon$$

The equality (4) allows us to say that for every $z_r = u_r - u_{r-1} + x_{r-1}$ $(2 \leq r \leq N)$ there exists an x_r such that for some z_r^ε satisfying $\| z_r^\varepsilon - z_r \| < \varepsilon/_N$, the equality

(6) $$(I + Q_{r-1,r})x_r = z_r^\varepsilon \quad .$$

holds.

On other hand, the expression of u_r^ε will be:

(7) $$u_r^\varepsilon = x_r + (u_1^\varepsilon - x_1) + \sum_{2 \leq j \leq r} (z_j^\varepsilon - x_j) \quad .$$

In fact:

$$\begin{aligned} &\| [x_r + (u_1^\varepsilon - x_1) + \sum_{2 \leq j \leq r} (z_j^\varepsilon - x_j)] - u_r \| < \\ &< \| \sum_{2 \leq j \leq r} (z_j^\varepsilon - z_j) \| + \| u_1^\varepsilon - u_r + \sum_{2 \leq j \leq r} (x_j - x_{j-1}) \| < \\ &< \frac{r-1}{N}\varepsilon + \| u_1^\varepsilon - u_1 \| \leq \frac{r}{N}\varepsilon \leq \varepsilon \quad . \end{aligned}$$

Observing that the right hand side term of the equality (7) is exactly the term

$$x_r + J_{0,r}\, x \quad ,$$

and then using (5), we end the proof. ■

Let us give now an answer for the (P) problem: if the operator $\pi_r + J_{r-1,r}$ has closed range (for every $r \geq 1$) we can reproduce the proof in the preceeding theorem making $\varepsilon = 0$, instead of $\varepsilon > 0$. If we have $Ker(\pi_r + J_{r-1,r}) = \{0\}$ and moreover the hypothesis

(8) $$(\pi_r + J_{r-1,r})^{-1} \in L(X) \qquad (1 \leq r \leq N)$$

fullfied, it is possible to state the

COROLARY 2.3. *Under the hypothesis (8), for every $u \in X^N$ there exists an $x \in X^N$ such that (P) is fulfilled.*

2.2 The Stieltjes equation on partitions: the system $(L_{|d|})$.

For the real closed interval $[a, b]$ let $D_{[a,b]}$ – or simply D – be the class of all proper finite divisions of $[a, b]$, and $d = \{t_0 = a < t_1 < ... < t_{|d|} = b\} \in D$.

Considering

$$A : [a, b] \to L(X)$$

let us define, for every $d \in D$,

$$Q^d_{m,n} = A(t_n) - A(t_m) \qquad (0 \leq m \leq n \leq |d|)$$

and analogously, as in (1),

$$J^d_{p,q} : X^{|d|} \to X \quad .$$

The Stieltjes equation, taking into account $J^d_{p,q}$, is in fact an (L_N) type equation: it is enough to make the identification: $n \mapsto t_n$ $(1 \leq n \leq |d|)$. We will denote this equation by $(L_{|d|})[|d| = N]$. Concerning the existence of solutions problem, it is possible finally, in an immediate way, to give a sufficient condition for having (3) fulfilled in every $(L_{|d|})[d \in D]$:

(9) for every $a \leq t_1 \leq t_2 \leq b$ let be $Ker(I - [A(t_2) - A(t_1)])^* = \{0\}$.

3. AN EXAMPLE. EXISTENCE THEOREM FOR CONTINUOUS TIME STIELTJES EQUATION

In this part we will show that a Stieltjes linear equation considered on B-spaces with Riemman-generalized integrals inside, satisfying (9), has a solution – that will be done in a constructive way.

3.1 The Stieltjes equation (L).

Let be $G^{-}([a,b],X)$ the B-space of the left continuous regulated functions, endowed with the sup norm and $\alpha : [a,b] \to L(X)$ a map of bounded semivariation, $[\alpha \in SV([a,b],X)$, see e.g. [3]].

The *linear Stieltjes equation* (L) will be the system,

$$x(t) + {}^{\odot}\!\!\int_a^t d\alpha(s)x(s) = u(t) \qquad a \le t \le b \tag{10}$$

where $x, u \in G^{-}([a,b],X)$ and ${}^{\odot}\!\!\int$ symbolizes an integral satisfying for every $a \le t_1 \le t_2 \le t_3 \le b$ and $y \in X$:

$I_1)$
$$ {}^{\odot}\!\!\int_{t_1}^{t_2} d\alpha(s)x(s) + {}^{\odot}\!\!\int_{t_2}^{t_3} d\alpha(s)x(s) = {}^{\odot}\!\!\int_{t_1}^{t_3} d\alpha(s)x(s)$$

and

$I_2)$
$$ {}^{\odot}\!\!\int_{t_1}^{t_2} d\alpha(s)y = [\alpha(t_2) - \alpha(t_1)]y \ .$$

If in the part ${}^{\odot}\!\!\int_a^t d\alpha(s)x(s)$ of (L) we use, for instance, either the Dushnik or the Henstock-Kurzweil or the Young integrals then I_1 and I_2 will be fulfilled. On the other hand, with the usual Riemann integral we will not have I_1 always true.

For the sake of well-definiteness in (10), we took the map I_α,

$$[I_\alpha y](t) = {}^{\odot}\!\!\int_a^t d\alpha(s)y(s)$$

as an (integral) operator on $G^{-}([a,b],X)$. A sufficient condition to achieve this situation consists in to have α being weak regulated, [see [2] and [3]]. Notice that when the space X is itself a Hilbert space then this condition can be dropped out [see e.g. [6]].

3.2 The Stieltjes equation (L_d) on step function.

Let $d = \{a = t_0 < t_1 < ... < t_{|d|} = b\} \in D_{[a,b]}$. A *step function* u_d *over* d on $G^{-}([a,b],X)$ is a function of the form:

$$u_d(t) = \chi_{[t_0,t_1]}\ y_1 + \sum_{i=2}^{|d|} \chi_{(t_{i-1},t_i]}\ y_i$$

where $y_1, ..., y_{|d|} \in X$ and χ_A is the usual characteristic map on A. For every $d \in D$, it is possible to identify the step function y_d, over d with the element $(y_0, y_1, ..., y_{|d|}) \in X^{|d|}$, making:

$$y_d|_{(t_{i-1},t_i]} = y_i \ . \qquad 1 \leq i \leq |d|$$

Then if we define

$$A : [a,b] \to L(X)$$

as

$$A(t)x = {}^{\odot}\!\!\int_a^t d\alpha(s)x \overset{I_2}{=} [\alpha(t) - \alpha(a)]x \ ,$$

the following system is well-defined on $X^{|d|}$:

(11) $$(\pi_r + J_{0,r})x = \pi_r u \ .$$

3.3 Extending (L_{d_n}) to (L).

There are systems for which it is possible to extend the discrete equation (L_N) to the continuous time equation (L). The system which we are dealing with in this paper is an example of that. The fundamental property that allows us to do so is given in the next theorem

THEOREM 3.1 ([6, Theorem 3]). *Assume (10) and take $\alpha \in SV([a,b], L(X))$ weak regulated. Then there exists a division depending on α*

$$\delta = d_\alpha = \{\tau_0, \tau_1 ... \tau_M\}$$

in $D_{[a,b]}$ in such a way that: if u_δ and x_δ are step functions over δ, for which the equality

$$\| x_\delta(\tau_i) + {}^{\odot}\!\!\int_a^{\tau_i} d\alpha(s)x_\delta(s) - u_\delta(\tau_i) \| < \varepsilon \ ,$$

holds for some $i \in \{1, 2, .., M\}$, then for all $t \in [a, \tau_i]$ we get

$$\| x_\delta(t) + {}^{\odot}\!\!\int_a^{t} d\alpha(s)x_\delta(s) - u_\delta(t) \| < \varepsilon \ .$$

3.4 Existence of solutions for the equation (L).

Let be the equation (L) – as in (10) –

$$x(t) + {}^{\odot}\!\!\int_a^{t} d\alpha(s)x(s) = u(t) \tag{12}$$

with $x, u \in G^-([a,b], X)$, and consider $(u_n)_{n\in\mathbb{N}}$ a sequence of step functions u_n over d_n (hence over $d_n \cup d_\alpha$) such that $u_n \xrightarrow{u.} u$. In this way it is possible to define the sequence of systems:

$$x_n(t) + {}^{\odot}\!\!\int_a^{t} d\alpha(s)x_n(s) = u_n(t) \qquad \text{(on } d_n \cup d_\alpha) \ . \tag{13}$$

Suppose, now, the hypothesis (9) being true. Then we have a sequence $(x_n)_{n\in\mathbb{N}}$ of step functions satisfying

$$x_n + I_\alpha x_n \xrightarrow{u.} u$$

in $G^-([a,b], X)$.

Gathering all the previous results it is possible to state:

THEOREN 3.2. *Suppose that $I + I_\alpha$ (where I is the identity operator on $G^-([a,b], X)$) is an operator of Fredholm type. Then, for every $u \in G^-([a,b], X)$, there exists an $x \in G^-([a,b]X)$ such that (12) is fulfilled.*

PROOF. If $I + I_\alpha$ is Fredholm then it has closed range. Following [7; Ch. V, Th. 1.4], then there exists an $A_0 \in L(X)$ and a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$, in

$G^-([a,b],x)$, in such a way that

$$\hat{x}_n \to A_0 u = x \quad ,$$

for the solutions $x_n = \hat{x}_n + x_n^0$ of (13). The split part x_n^0 corresponds to the part of x_n belonging to the set $Ker\ (I + I_\alpha)$. ■

Furthermore, still according [7; Ch. V], it is possible to replace the condition that $I + I_\alpha$ is of Fredholm type, by a stronger one. In fact, it is enough to have $dim\ Ker(I + I_\alpha) < \infty$ to assure the operator A_0 defined and so, the result as in the above theorem. Note that $(I + I_\alpha)$ is a Fredholm operator if, for instance I_α^k, (i.e. $I_\alpha \circ ... \circ I_\alpha$, k times) is a compact operator for some $k \in \mathbb{Z}^+$.

Finally, if $(I + I_\alpha)$ is a Fredholm operator then the hypothesis (9) can be replaced (see e.g. [8]) by the local equivalent property:

(14) $$Ker(I - (\alpha(t_+) - \alpha(t)))^* = \{0\} \text{ for every } t \in [a,b) \quad ,$$

in where $\alpha(t_+) = \lim_{\varepsilon \to 0^+} \alpha(t + \varepsilon)$.

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